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Vertex transitivity and distance metric of the quad-cube

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Abstract
The quad-cube is a special case of the metacube that itself is derivable from the hypercube. It is amenable to an application as a network topology, especially when the node size exceeds several million. This paper presents the following welcome properties of the graph, relating to its structure: (1) vertex transitivity that facilitates the working of an algorithm meant for a "local" context in the global context as well, and (2) an exact formula for the distance metric, which leads to a precise result on the distance-wise vertex distribution of the graph and an exact formula for the average vertex distance. Remarkably, the vertex distribution of the quad-cube resembles, to a large extent, the vertex distribution of the twin copies of a hypercube. In a parallel study, the author recently reported similar results with respect to the dual-cube (Jha in J Supercomput 78:17758–17775, 2022)

Keywords Quad-cube · Metacube · Hypercube · Interconnection network · Network topology · Vertex transitivity · Shortest distance

1 Introduction

The *quad-cube* is a *network topology* that is a special case of the *metacube* [9] that itself is obtainable from the *hypercube*. The objective behind its introduction has been to mitigate the problem of the rapid increase in the degree of the hypercube when the node size exceeds several million. With the same node degree n, the quad-cube has 2^{3n-6} as many nodes as the hypercube, where $n \geq 3$, and with the same number of nodes, the quad-cube has about 75% fewer edges than the hypercube, yet its diameter is practically equal to that of the latter.

This paper presents results relating to the vertex transitivity, distance metric and distance-wise node distribution of the quad-cube that significantly enhance its importance from both theoretical point of view and the engineering point of view. Interestingly, the node distribution of CQ_m parallels, to a large extent, the node

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distribution of $2Q_{4m+1}$, i.e., a set of twin copies of Q_{4m+1}. Meanwhile, CQ_m admits a 1-perfect code under a certain condition [5]. (The formal definitions appear below.)

In a parallel study, the author [6] recently presented analogous results relating to the dual-cube that is another (relatively simpler) derivative of the hypercube.

1.1 Definitions and preliminaries

A graph connotes a finite, simple, undirected and connected graph. Let G be a graph, and let dist(u, v) denote the (shortest) distance between vertices u and v in G [12]. Further, let dia(G) denote the *diameter* of G.

For n-bit binary strings x and y, let H(x, y) denote the *Hamming distance* between the two. The n-dimensional hypercube Q_n (also called the n-cube) is the graph on the vertex set $\{0, 1\}^n$, where nodes x and y are adjacent iff $\text{H}(x, y) = 1$.

Let xy denote the concatenation of the strings x and y, and let $\overline{a} := 1 - a$, where $a \in \{0, 1\}$.

Definition 1.1 For an n-bit integer $x = b_{n-1} \dots b_0$ (so $0 \le x \le 2^n - 1$ in decimal), let $x^{(a)}$ be the n-bit integer obtainable from x by replacing b_a by $\overline{b}_a$, where $0 \le a \le n - 1$. □

It is easy to see that $x^{(a)} = x \veebar 2^a$, where $\veebar$ is the XOR operation. (See Definition 1.4.)

Definition 1.2 For $m \ge 1$, the quad-cube CQ_m is a spanning subgraph of Q_{4m+2}. Its edge set is given by $E_0 \cup E_1 \cup E_2 \cup E_3 \cup E_4$, where

1. $E_0 = \{\{ux00, ux^{(0)}00\}, \dots, \{ux00, ux^{(m-1)}00\} \mid u \in \{0, 1\}^{3m} \text{ and } x \in \{0, 1\}^m\}$
2. $E_1 = \{\{uvx01, uv^{(0)}x01\}, \dots, \{uvx01, uv^{(m-1)}x01\} \mid u \in \{0, 1\}^{2m}$ and $v, x \in \{0, 1\}^m\}$
3. $E_2 = \{\{uvx10, uv^{(0)}x10\}, \dots, \{uvx10, uv^{(m-1)}x10\} \mid u, v \in \{0, 1\}^m \text{ and } x \in \{0, 1\}^{2m}\}$
4. $E_3 = \{\{ux11, u^{(0)}x11\}, \dots, \{ux11, u^{(m-1)}x11\} \mid u \in \{0, 1\}^m \text{ and } x \in \{0, 1\}^{3m}\}$, and
5. $E_4 = \{\{u00, u01\}, \{u00, u10\}, \{u01, u11\}, \{u10, u11\} \mid u \in \{0, 1\}^{4m}\}$. □

Let $e \in E(CQ_m)$. Call e an edge of Type i if $e \in E_i$, $0 \le i \le 3$, and call e a *cross edge* if $e \in E_4$. See Fig. 1 for a depiction of the five edge types. Meanwhile, a node of the hypercube/quad-cube is viewable both as a binary string, say, x and as the corresponding nonnegative integer denoted by x.

It is easy to see that CQ_m is a regular graph of degree $m + 2$. Accordingly, $|V(CQ_m)| = 2^{4m+2}$ and $|E(CQ_m)| = (m + 2)2^{4m+1}$.

Definition 1.3 The nodes of CQ_m are distinguishable into four types, as follows:

- Type 0: those that are of the form $u00$ (binary) or $4i + 0$ (decimal)
- Type 1: those that are of the form $u01$ (binary) or $4i + 1$ (decimal)

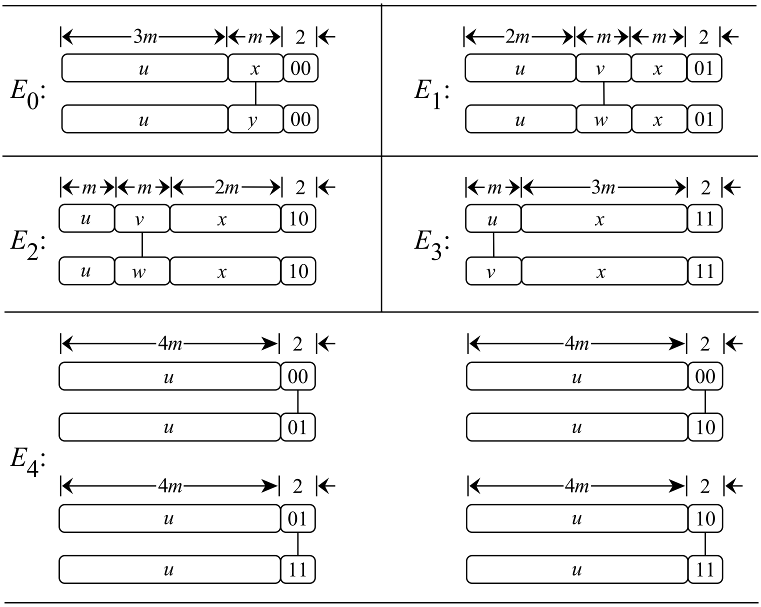

Fig. 1 Five edge types of CQ_m, vide Definition 1.2

- Type 2: those that are of the form $u10$ (binary) or $4i + 2$ (decimal) and
- Type 3: those that are of the form $u11$ (binary) or $4i + 3$ (decimal). □

A node z of CQ_m (in binary) is of the form $\boxed{u \mid v \mid w \mid x \mid a \mid b}$, where $|u| = |v| = |w| = |x| = m$, and $a, b \in \{0, 1\}$.

See Fig. 2 that presents three drawings of CQ_1. Among other things, it shows that the graph admits (a) a vertex partition into sixteen copies of the four cycles, (b) an embedding on the torus without any edge crossing and (c) an edge decomposition into a Hamiltonian cycle and a perfect matching. Meanwhile, Brouwer et al. ([1], p. 27) present the drawing in Fig. 2a to depict the graph as a vertex-transitive induced subgraph of Q_8.

Definition 1.4 For n-bit strings x and y, let $x \veebar y$ denote the n-bit string obtainable by the bitwise XOR operation between x and y. □

It is easy to see that $\veebar$ is both commutative and associative.

Definition 1.5 A graph is said to be vertex-transitive if for every pair of vertices u and v, it admits an automorphism that sends u to v.

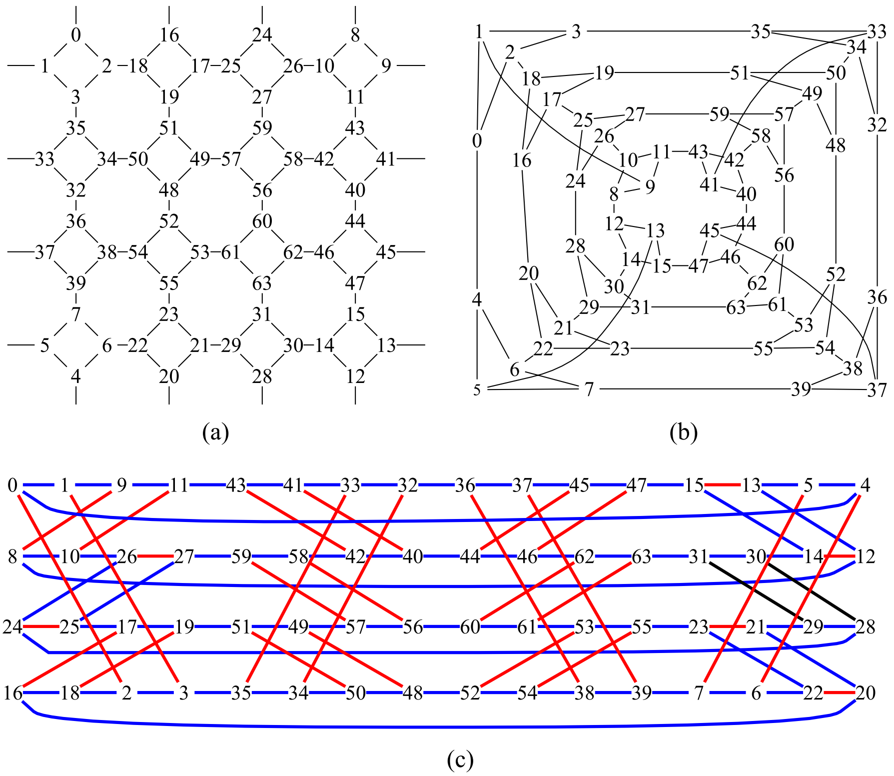

Fig. 2 Three drawings of CQ_1

Let $k \geq 0$. For a set S of integers, let $(S + k)$ denote the set $\{i + k \mid i \in S\}$, and for a graph G, let $(G + k)$ be the graph on the vertex set $\{x + k \mid x \in V(G)\}$, where $\{x + k, y + k\} \in E(G + k)$ iff $\{x, y\} \in E(G)$. Further, let

$$C(n, k) = \begin{cases} \binom{n}{k} & \text{if } n \geq k \geq 0 \\ 0 & \text{if } k > n or k < 0 \end{cases} \tag{1}$$

where $\binom{n}{k}$ denotes the binomial coefficient. For any undefined term, see West [12].

1.2 Literature review

The concepts of vertex transitivity and distance metric are of prime importance in the design and working of a viable network topology.

Informally, a graph is vertex-transitive if every vertex in it has the same local environment, so no vertex is distinguishable from any other based on the vertices and edges surrounding it. In that light, a vertex-transitive graph (that is necessarily regular) offers a huge advantage, viz., “local” algorithms on it work globally as well, since all vertices hold equivalent roles in the global context. Another plus is that a vertex-transitive graph is more strongly connected than other regular graphs ([2], p. 33). Not surprisingly, most network topologies in use today (notably the hypercube, the torus and the circulants) are vertex-transitive. See Heydemann [4] and the references therein for applications of these graphs in interconnection networks.

The importance of the distance metric in a graph is well-understood and well documented in the literature [12]. In particular, given two nodes, say, u and v, a shortest $u - v$ path has the least total cost among all $u - v$ paths. Not surprisingly, several routing protocols base their decisions on a shortest path to a given destination. Examples include (i) RIP (Routing Information Protocol) that is widely used for routing traffic in the global internet and (ii) IGRP (Interior Gateway Routing Protocol) that is a Cisco standard routing protocol. In that light, the results of this paper are directly relevant to the construction of smart routing algorithms around the quad-cube.

See Saad and Schultz [11], and Hayes and Mudge [3] for certain similar results relating to the hypercube, and see Loh et al. [10] for those relating to the exchanged hypercube.

1.3 Quad-cube vis-à-vis dual-cube

Although the quad-cube and the dual-cube have the same lineage, the former is qualitatively more complex than the latter [9]. Therefore, results relating to the dual-cube do not automatically carry over to those relating to the quad-cube. In particular, the author [6] recently studied vertex transitivity and distance metric of the dual-cube. In each category, there are far fewer cases than those in the present study. Indeed, the issues addressed here are lot more challenging.

Probably because of its relative simplicity, the dual-cube has been an object of study by many. See the references in [6]. The results in the present paper are likely to stimulate deeper studies around the quad-cube. Possible areas of investigation include collective communications [8], effective fault tolerance, Hamiltonian decomposability and the design of efficient routing algorithms.

1.4 What follows

Section 2 establishes vertex transitivity of the quad-cube, while Sect. 3 derives a formula for the (shortest) distance between Node 0 and a given node z in the graph. (Vertex transitivity ensures an easy generalization of the formula.) Section 4 makes an effective use of the distance formula to develop the distance-wise node distribution of the graph that, in turn, leads to an exact formula for the average node distance of the graph. The paper ends with certain concluding remarks in Sect. 5.

2 Vertex transitivity

Lemma 2.1 *CQ_m admits an automorphism that carries a given node of Type* 0 *to Node* 0.

Proof Let $z = uvwx00$ (binary) be an arbitrary but fixed node of Type 0, where $|u| = |v| = |w| = |x| = m$, and consider the mapping $\phi_z : V(CQ_m) \to V(CQ_m)$ given by $pqrsab \mapsto (p \veebar u)(q \veebar v)(r \veebar w)(s \veebar x)ab$, where $|p| = |q| = |r| = |s| = m$ and $a, b \in \{0, 1\}$. It is easy to see that ϕ_z is well defined. Further, it maps a node of a particular type to one of the same types, and $\phi_z(z) = 0$ (decimal). Note next that $\phi_z(y_1) = y_2$ iff $\phi_z(y_2) = y_1$, so the mapping is total and invertible, hence a bijection.

Consider a node $pqrsab$, and first assume that it is of Type 0, so $a = b = 0$. Two of its neighbors are $pqrs01$ and $pqrs10$, while the remaining are $pqr(s \veebar 2^0)00, \ldots, pqr(s \veebar 2^{m-1})00$. It is easy to see that $\phi_z(pqrs00)$ is adjacent to each of $\phi_z(pqrs01)$ and $\phi_z(pqrs10)$. Consider next $pqr(s \veebar 2^j)00$, $0 \le j \le m - 1$. By virtue of the fact that $\mathrm{H}(p_1, p_2) = 1$ iff $\mathrm{H}(p_1 \veebar t, p_2 \veebar t) = 1$ (where p_1, p_2 and t are m-bit strings), it is easy to see that $\phi_z(pqrs00)$ is adjacent to $\phi_z(pqr(s \veebar 2^j)00)$, $0 \le j \le m - 1$.

By an analogous argument, the foregoing conclusion is reachable with respect to nodes of Type 1 (resp. Type 2 and Type 3) as well. Finally, there exists a one-to-one correspondence between the $m + 2$ neighbors of $pqrsab$ and those of $\phi_z(pqrsab)$.

□

Figure 3 illustrates the proof of Lemma 2.1 where $m = 1$ and $z = 4$ and where the i-th node in a particular row to the left maps to the i-th node on the same row to the right. (See the dotted arcs.)

Corollary 2.2 *The following holds with respect to the mapping ϕ_z in the proof of Lemma* 2.1:

- Every quadrilateral $tab - ta\bar{b} - t\bar{a}\bar{b} - t\bar{a}b - tab$ maps to a (not necessarily distinct) quadrilateral induced by the corresponding nodes $\phi_z(tab)$, $\phi_z(ta\bar{b})$, $\phi_z(t\bar{a}\bar{b})$ and $\phi_z(t\bar{a}b)$, where $t \in \{0, 1\}^{4m}$ and $a, b \in \{0, 1\}$. □

Lemma 2.3 *There exists an automorphism ϕ_z on CQ_m that carries a given node z of Type t to the Node t, where $t \in \{1, 2, 3\}$.* □

Proof Similar to that of Lemma 2.1. □

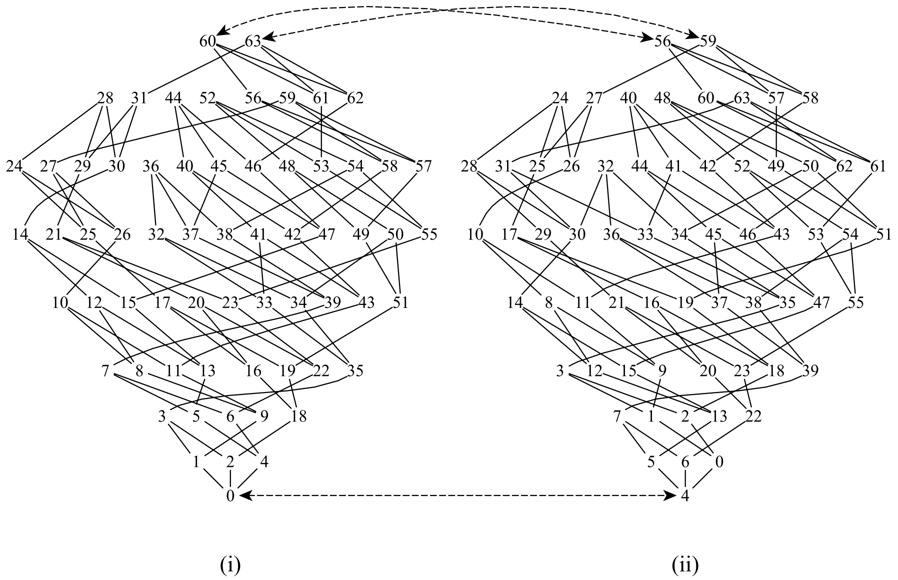

Fig. 3 An automorphism on CQ_1, $(4 \leftrightarrow 0)$, vide Lemma 2.1

Lemma 2.4 *There exists an automorphism on CQ_m that carries Node i to Node 0, where $i = 1, 2, 3$.*

Proof Let ϕ_1, ϕ_2 and ϕ_3 be the mappings, each from $V(CQ_m)$ to $V(CQ_m)$, whose definitions appear in Table 1. Figure 4 depicts the mappings themselves. It is easy to see that $\phi_i(y_1) = y_2$ iff $\phi_i(y_2) = y_1$, $i = 1, 2, 3$; hence, each is a well-defined bijection.

Consider ϕ_1 first, and let $z_1 = u_1v_1w_1x_1a_1b_1$ and $z_2 = u_2v_2w_2x_2a_2b_2$, so $\phi_1(z_1) = v_1u_1x_1w_1a_1\overline{b}_1$ and $\phi_1(z_2) = v_2u_2x_2w_2a_2\overline{b}_2$.

1. Let z_1 and z_2 be adjacent via a cross edge, so $u_1 = u_2$, $v_1 = v_2$, $w_1 = w_2$, $x_1 = x_2$ and $\mathrm{H}(a_1b_1, a_2b_2) = 1$. That $\mathrm{H}(a_1b_1, a_2b_2) = 1$ ensures that $\mathrm{H}(a_1\overline{b}_1, a_2\overline{b}_2) = 1$; hence,

Table 1 Mappings ϕ_1, ϕ_2 and ϕ_3, vide Lemma 2.4

i	z	$\phi_i(z)$
1	u | v | w | x | a | b	v | u | x | w | a | $\overline{b}$
2	u | v | w | x | a | b	w | x | u | v | $\overline{a}$ | b
3	u | v | w | x | a | b	x | w | v | u | $\overline{a}$ | $\overline{b}$
$u, v, w, x \in \{0,1\}^m$ and $a, b \in \{0,1\}$		

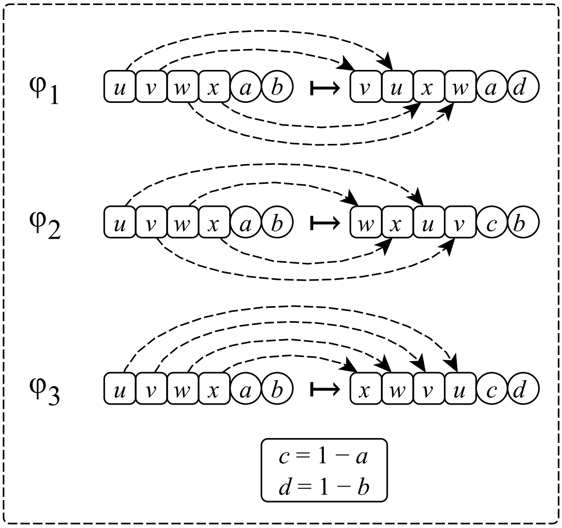

Fig. 4 Mappings ϕ_1, ϕ_2 and ϕ_3, vide Lemma 2.4

$\phi_1(z_1)$ and $\phi_1(z_2)$ are adjacent via a cross edge. Further, the converse is equally true.

2. Let z_1 and z_2 be both of Type 0 and adjacent, so $z_1 = u_1v_1w_1x_100$ and $z_2 = u_1v_1w_1(x_1 \veebar 2^j)00$, where $0 \le j \le m-1$. In this case, $\phi_1(z_1) = v_1u_1x_1w_101$ and $\phi_1(z_2) = v_1u_1(x_1 \veebar 2^j)w_101$. It is clear that $\phi_1(z_1)$ and $\phi_1(z_2)$ are both of Type 1, and they are adjacent by virtue of $\mathrm{H}(x_1, (x_1 \veebar 2^j))$ being equal to one. Further, the converse is equally true.

The other cases, where z_1 and z_2 are adjacent, both of Type 1 (resp. Type 2 and Type 3), are similar. Finally, it is not difficult to check that the mappings ϕ_2 and ϕ_3 admit characteristics that are analogous to those of ϕ_1. □

Remark It is easy to compute the inverses of the automorphisms that appear in Lemmas 2.1, 2.3 and 2.4.

Theorem 2.5 *CQ_m is a vertex-transitive graph.*

Proof Recall that automorphisms are closed under composition and that the inverse of an automorphism is again such. In that light, let $y, z \in V(CQ_m)$, where $y \neq z$. An automorphism that takes y to z is obtainable by using the constructions in Lemmas 2.1, 2.3 and 2.4, as follows.

1. If y and z are both of the same type, say i, then use the construction in the proof of Lemma 2.1/2.3, and compose the automorphism that takes y to i with the one that takes i to z.

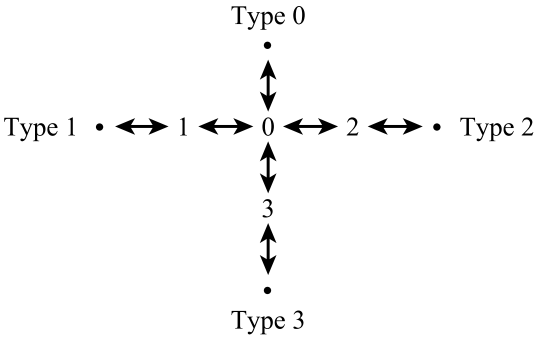

Fig. 5 Automorphisms that take a node of one type to one of another

2. If y and z are of types i and j, respectively, where $i \neq j$, then build an automorphism that takes y to z as per the schematic that appears in Fig. 5. □

Remark CQ_m is not edge-transitive. For example, observe from Fig. 3(i) that the edge $\{0, 1\}$ of CQ_1 lies on a four-cycle, whereas the edge $\{0, 4\}$ does not lie on any four-cycle.

Table 2 Distance between Node 0 and Node z in CQ_m

Case	Node z	Type of z	Predicate	Dist(0, z)
1(a)	$0^m0^m0^mx00$	0	–	$\|x\|_1$
1(b)	0^m0^mwx00	0	$\|w\|_1 > 0$	$\|w\|_1+\|x\|_1+2$
1(c)	0^mv0^mx00	0	$\|v\|_1 > 0$	$\|v\|_1+\|x\|_1+2$
1(d)	0^mvwx00	0	$\|v\|_1 > 0$ and $\|w\|_1 > 0$	$\|v\|_1+\|w\|_1+\|x\|_1+4$
1(e)	$uvwx00$	0	$\|u\|_1 > 0$	$\|u\|_1+\|v\|_1+\|w\|_1+\|x\|_1+4$
2(a)	0^m0^mwx01	1	–	$\|w\|_1+\|x\|_1+1$
2(b)	0^mvwx01	1	$\|v\|_1 > 0$	$\|v\|_1+\|w\|_1+\|x\|_1+3$
2(c)	$uvwx01$	1	$\|u\|_1 > 0$	$\|u\|_1+\|v\|_1+\|w\|_1+\|x\|_1+3$
3(a)	0^mv0^mx10	2	–	$\|v\|_1+\|x\|_1+1$
3(b)	0^mvwx10	2	$\|w\|_1 > 0$	$\|v\|_1+\|w\|_1+\|x\|_1+3$
3(c)	$uvwx10$	2	$\|u\|_1 > 0$	$\|u\|_1+\|v\|_1+\|w\|_1+\|x\|_1+3$
4(a)	$u0^mwx11$	3	–	$\|u\|_1+\|w\|_1+\|x\|_1+2$
4(b)	$uv0^mx11$	3	–	$\|u\|_1+\|v\|_1+\|x\|_1+2$
4(c)	$uvwx11$	3	$\|v\|_1 > 0$ and $\|w\|_1 > 0$	$\|u\|_1+\|v\|_1+\|w\|_1+\|x\|_1+4$

$u, v, w, x \in \{0, 1\}^m$

$|u|_1$ stands for the number of 1's in the binary string u, etc.

3 Distance metric

Theorem 3.1 *Table 2 presents the distance between Node* 0 *and Node* z *in* CQ_m.

Proof A (shortest) path between the respective nodes appears below. Check to see that, in each case, any other path is at least as long. (As stated earlier, $|u|_1$ stands for the number of 1's in the binary string u.)

1. (a) $0^{4m}00 \overset{|x|_1}{\rightsquigarrow} 0^{3m}x00$.

   (b) $0^{4m}00 \overset{1}{\to} 0^{4m}01 \overset{|w|_1}{\rightsquigarrow} 0^{2m}w0^m01 \overset{1}{\to} 0^{2m}w0^m00 \overset{|x|_1}{\rightsquigarrow} 0^{2m}wx00$.

   (c) $0^{4m}00 \overset{1}{\to} 0^{4m}10 \overset{|v|_1}{\rightsquigarrow} 0^m v0^m0^m10 \overset{1}{\to} 0^m v0^m0^m00 \overset{|x|_1}{\rightsquigarrow} 0^m v0^m x00$.

   (d) $0^{4m}00 \overset{|x|_1}{\rightsquigarrow} 0^{3m}x00 \overset{1}{\to} 0^{3m}x01 \overset{|w|_1}{\rightsquigarrow} 0^{2m}wx01 \overset{1}{\to} 0^{2m}wx00 \overset{1}{\to} 0^{2m}wx10 \overset{|v|_1}{\rightsquigarrow} 0^m vwx10 \overset{1}{\to}$ $0^m vwx00$.

   (e) $0^{4m}00 \overset{|x|_1}{\rightsquigarrow} 0^{3m}x00 \overset{1}{\to} 0^{3m}x01 \overset{|w|_1}{\rightsquigarrow} 0^{2m}wx01 \overset{1}{\to} 0^{2m}wx11 \overset{|u|_1}{\rightsquigarrow} u0^m wx11 \overset{1}{\to} u0^m wx10$ $\overset{|v|_1}{\rightsquigarrow} uvwx10 \overset{1}{\to} uvwx00$.

The remaining cases are similar. □

Evidently, the parity of the distance between two nodes in the quad-cube is equal to that of the Hamming distance between the two. Meanwhile, it is easy to see from Table 2 that $|z|_1 \le \text{dist}(0, z) \le |z|_1 + 4$. In that light, there are three possibilities: (i) $\text{dist}(0, z) = |z|_1$, (ii) $\text{dist}(0, z) = |z|_1 + 2$ and (iii) $\text{dist}(0, z) = |z|_1 + 4$. Interestingly, it is possible to enumerate the nodes in each category. See Lemmas 3.2, 3.3 and 3.4.

Table 3 Various cases relating to the proof of Lemma 3.2 ($|u| = |v| = |w| = |x| = m$)

Relevant case, vide Table 2	Node z	Predicate	Number of nodes of that form
1(a)	$0^m0^m0^mx00$	–	2^m
2(a)	0^m0^mwx01	–	2^{2m}
3(a)	0^mv0^mx10	–	2^{2m}
4(a) and 4(b)	$uvwx11$	$\|v\|_1 \cdot \|w\|_1 = 0$	$2^{3m+1} - 2^{2m}$
			Total: $2^{3m+1} + 2^{2m} + 2^m$

Table 4 Various cases relating to the proof of Lemma 3.3 ($|u| = |v| = |w| = |x| = m$)

Relevant case, vide Table 2	Node z	Predicate	Number of nodes of that form
1(b)	0^m0^mwx00	$\|w\|_1 > 0$	$(2^m - 1)2^m$
1(c)	0^mv0^mx00	$\|v\|_1 > 0$	$(2^m - 1)2^m$
2(b)	0^mvwx01	$\|v\|_1 > 0$	$(2^m - 1)2^{2m}$
2(c)	$uvwx01$	$\|u\|_1 > 0$	$(2^m - 1)2^{3m}$
3(b)	0^mvwx10	$\|w\|_1 > 0$	$(2^m - 1)2^{2m}$
3(c)	$uvwx10$	$\|u\|_1 > 0$	$(2^m - 1)2^{3m}$
4(c)	$uvwx11$	$\|v\|_1 \cdot \|w\|_1 > 0$	$2^m(2^m - 1)(2^m - 1)2^m$
			Total: $3 \times 2^{4m} - 2^{3m+1} + 2^{2m} - 2^{m+1}$

Table 5 Various cases relating to the proof of Lemma 3.4 ($|u| = |v| = |w| = |x| = m$)

Relevant case, vide Table 2	Node z	Predicate	Number of nodes of that form
1(d)	0^mvwx00	$\|v\|_1 \cdot \|w\|_1 > 0$	$(2^m - 1)(2^m - 1)2^m$
1(e)	$uvwx00$	$\|u\|_1 > 0$	$(2^m - 1)2^{3m}$
			Total: $2^{4m} - 2^{2m+1} + 2^m$

Lemma 3.2 *There are a total of* $2^{3m+1} + 2^{2m} + 2^m$ *nodes* z *for which* $dist(0, z) = |z|_1$.

Proof Refer to Table 3, and first note that Cases 1(a), 2(a) and 3(a) are easy. For Cases 4(a) and 4(b) together, there are 2^{4m} nodes of the form $uvwx11$, out of which $2^m(2^m - 1)(2^m - 1)2^m$ are such that $|v|_1 \cdot |w|_1 > 0$, so there are $2^{4m} - 2^m(2^m - 1)(2^m - 1)2^m = 2^{3m+1} - 2^{2m}$ nodes of that form. Finally, the four cases are mutually exclusive, hence the result. □

Lemma 3.3 *There are a total of* $3 \times 2^{4m} - 2^{3m+1} + 2^{2m} - 2^{m+1}$ *nodes* z *for which* $dist(0, z) = |z|_1 + 2$.

Proof See Table 4. □

Lemma 3.4 *There are a total of* $2^{4m} - 2^{2m+1} + 2^m$ *nodes* z *for which* $dist(0, z) = |z|_1 + 4$.

Proof See Table 5. □

Remark All nodes z for which $\text{dist}(0, z) = |z|_1 + 4$ are of Type 0.

4 Distance-wise node distribution

Let n_d denote the number of nodes at a distance of d from Node 0 in CQ_m. As stated earlier, $C(n, k)$ denotes the binomial coefficient $\binom{n}{k}$, if $n \geq k \geq 0$, and $C(n, k) = 0$, if $k > n$ or $k < 0$.

Lemma 4.1 *If* $m \geq 2$ *and* $0 \leq d \leq 4m + 4$, *then* n_d *is equal to*

$$2C(4m + 1, d - 3)$$

$$+\big(2C(3m, d - 2) - 2C(3m, d - 4)\big)$$

$$+\big(C(2m + 1, d - 1) - C(2m + 1, d - 3) + C(2m, d - 1) - C(2m, d - 3)\big)$$

$$+\big(C(m, d) - 2C(m, d - 2) + C(m, d - 4)\big).$$

Proof Refer to Table 2. □

1. (a) Case 1(a) contributes $C(m, d)$ nodes.
   (b) Case 1(b) contributes $C(2m, d - 2) - C(m, d - 2)$ nodes, where $C(m, d - 2$ denotes the number of nodes of the form $0^m 0^m wx00$ with $|w|_1 = 0$.
   (c) Case 1(c) is identical to Case 1(b), so the answer in this case, too, is $C(2m, d - 2) - C(m, d - 2)$.
   (d) Case 1(d) contributes $C(3m, d - 4) - 2C(2m, d - 4) + C(m, d - 4)$ nodes, where $C(2m, d - 4)$ denotes the number of nodes of the form $0^m vwx00$ in which $|v|_1 = 0$ (resp. $|w|_1 = 0$), and $C(m, d - 4)$ denotes the number of nodes of that form in which $|v|_1$ and $|w|_1$ are both zero.
   (e) Case 1(e) contributes $C(4m, d - 4) - C(3m, d - 4)$ nodes, where $C(3m, d - 4)$ denotes the number of nodes of the form $uvwx00$ in which $|u|_1 = 0$.

   It is easy to see that the foregoing cases are mutually exclusive.

2. (a) Relative to Case 2(a), $|w|_1 + |x|_1 = d - 1$, so this case contributes $C(2m, d - 1)$ nodes.
   (b) Relative to Case 2(b), $|v|_1 + |w|_1 + |x|_1 = d - 3$, where $|v|_1 > 0$, so the answer in this case is $C(3m, d - 3) - C(2m, d - 3)$.
   (c) Check to see that Case 2(c) contributes $C(4m, d - 3) - C(3m, d - 3)$ nodes.

3. (a) Case 3(a) is similar to Case 2(a), so the answer is $C(2m, d - 1)$.

(b) Case 3(b) is similar to Case 2(b), so the answer is $C(3m, d-3) - C(2m, d-3)$.
(c) Case 3(c) is similar to Case 2(c), so the answer is $C(4m, d-3) - C(3m, d-3)$.

4. (a) Case 4(a) and Case 4(b) are not entirely exclusive, since strings of the form $u0^m0^mx$ belong to each. Accordingly, these two together contribute $2C(3m, d-2) - C(2m, d-2)$ nodes.
(b) Relative to Case 4(c), $|u|_1 + |v|_1 + |w|_1 + |x|_1 = d-4$, where $|v|_1 \cdot |w|_1 > 0$, so the answer in this case is $C(4m, d-4) - 2C(3m, d-4) + C(2m, d-4)$, where $C(3m, d-4)$ denotes the number of nodes in which $|v|_1 = 0$ (resp. $|w|_1 = 0$), and $C(2m, d-4)$ denotes the number of nodes in which $|v|_1$ and $|w|_1$ are both zero.

See Table 6 that summarizes the foregoing, and establishes the claim. □

Corollary 4.2 *Every vertex in CQ_m admits exactly two diametrical vertices, and the radius as well as the diameter of the graph is equal to* $4m + 4$.

Proof By Table 2, there are exactly two nodes, viz., $1^{4m}00$ and 1^{4m+2} (binary), arising out of Cases 1(e) and 4(c), that are at the distance of $4m + 4$ from Node 0. Indeed,

Table 6 Computing n_d, $0 \le d \le 4m+4$, vide Lemma 4.1, where $m \ge 2$

Case 1(a)				$C(m, d)$
Case 1(b)			$C(2m, d-2)$	$-C(m, d-2)$
Case 1(c)			$C(2m, d-2)$	$-C(m, d-2)$
Case 1(d)		$C(3m, d-4)$	$-2C(2m, d-4)$	$C(m, d-4)$
Case 1(e)	$C(4m, d-4)$	$-C(3m, d-4)$		
Case 2(a)			$C(2m, d-1)$	
Case 2(b)		$C(3m, d-3)$	$-C(2m, d-3)$	
Case 2(c)	$C(4m, d-3)$	$-C(3m, d-3)$		
Case 3(a)			$C(2m, d-1)$	
Case 3(b)		$C(3m, d-3)$	$-C(2m, d-3)$	
Case 3(c)	$C(4m, d-3)$	$-C(3m, d-3)$		
Case 4(a) & 4(b)		$2C(3m, d-2)$	$-C(2m, d-2)$	
Case 4(c)	$C(4m, d-4)$	$-2C(3m, d-4)$	$C(2m, d-4)$	
Algebraic sum	$= 2C(4m+1, d-3)$ $+ 2C(3m, d-2) - 2C(3m, d-4)$ $+ C(2m+1, d-1) - C(2m+1, d-3)$ $+ C(2m, d-1) - C(2m, d-3)$ $+ C(m, d) - 2C(m, d-2) + C(m, d-4)$			

$4m+4$ is the largest such integer. This fact and the vertex transitivity of the graph together lead to the claim. □

Corollary 4.3 *Let* $m \geq 2$.

1. *If* $3m+5 \leq d \leq 4m+4$, *then* $n_d = 2\,C(4m+1, d-3)$.
2. *If* $m \geq 3$ *and* $2\,m+5 \leq d \leq 3\,m+4$, *then* $n_d = 2C(4m+1, d-3) - 2C(3m, d-4) + 2C(3m, d-2)$.
3. *If* $m \geq 3$ *and* $m+5 \leq d \leq 2\,m+4$, *then* $n_d = 2C(4m+1, d-3) - 2C(3m, d-4) + 2C(3m, d-2)$ $+\,C(2\,m+1, d-1) - C(2\,m+1, d-3) + C(2\,m, d-1) - C(2\,m, d-3)$. □

4.1 CQ_m versus twin copies of Q_{4m+1}

Let $2Q_n$ denote a set of disjoint twin copies of Q_n, and assume that the two copies are laid out in the plane in parallel in such a way that Node 0 of each appears at the zeroth level, and the nodes at the distance of d from Node 0 appear at the d-th level, where $0 \leq d \leq n$. It is easy to see that there are $2\,C(n,d)$ nodes at the d-th level of the graph. See Fig. 6 that depicts $2Q_3$.

Interestingly enough, the distance-wise node distribution of CQ_m closely parallels that of $2Q_{4m+1}$. To that end, let diff denote the difference between the number of nodes at the d-th level of CQ_m and the number of nodes at the $(d-3)$-rd level of $2Q_{4m+1}$, where $3 \leq d \leq 4m+4$. See Table 7, Fig. 7 and Fig. 8.

Here are important observations on diff.

1. If $4m+4 \geq d \geq 3m+5$, then diff $= 0$, vide Corollary 4.3.
2. If $3m+4 \geq d \geq 2m+5$, then diff is equal to $-2C(3m, d-4) + 2C(3m, d-2)$ that is negative in this range.
3. If $2m+4 \geq d \geq m+5$, then diff is equal to $-2C(3m, d-4) + 2C(3m, d-2) - C(2m+1, d-3) + C(2m+1, d-1) - C(2m, d-3) + C(2m, d-1)$. Notice that $-2C(3m, d-4) + 2C(3m, d-2) := D$ (say) constitutes the dominant term in the foregoing expression.

    (a) If m is even, and $d = \frac{3m}{2} + 3$, then $D = 0$.

Fig. 6 Twin copies of Q_3

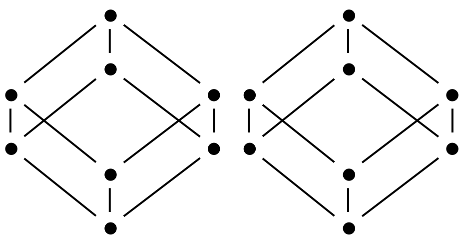

Table 7 CQ_m versus $2Q_{4m+1}$, $m = 5$

d	Number n_d of nodes at level d of CQ_m	Number q_d of nodes at level $d-3$ of $2Q_{4m+1}$	Diff $= n_d - q_d$	$\frac{\text{Diff}}{q_d} \times 100$
$4m+4=24$	2	2	0	0.000
23	42	42	0	0.000
22	420	420	0	0.000
21	2660	2660	0	0.000
$3m+5=20$	11,970	11,970	0	0.000
$3m+4=19$	40,696	40,698	−2	−0.005
18	108,498	108,528	−30	−0.028
17	232,352	232,560	−208	−0.089
16	406,100	406,980	−880	−0.216
$2m+5=15$	585,340	587,860	−2520	−0.429
$2m+4=14$	700,335	705,432	−5097	−0.723
13	698,140	705,432	−7292	−1.033
12	580,932	587,860	−6928	−1.179
11	403,922	406,980	−3058	−0.751
$m+5=10$	235,035	232,560	2475	1.064
$m+4=9$	114,931	108,528	6403	5.900
8	47,719	40,698	7021	17.251
7	17,206	11,970	5236	43.743
6	5609	2660	2949	–
$m=5$	1726	420	1306	–
4	500	42	458	–
3	130	2	128	–
2	31	–	31	–
1	7	–	7	–
0	1	–	1	–
–	Total = 4,194,304	Total = 4,194,304	–	–

Number of nodes in CQ_5 = Number of nodes in $2Q_{21} = 2^{22} = 4194304$

(b) If m is even, then D is positive at $d = \frac{3m}{2} + 3 - i$, and it is negative at $d = \frac{3m}{2} + 3 + i$, yet the absolute value is the same at each, where $1 \le i \le \frac{m}{2} + 1$.

(c) If m is odd, then D is positive at $d = \frac{3m+1}{2} + 2 - i$, and it is negative at $d = \frac{3m+1}{2} + 3 + i$, yet the absolute value is the same at each, where $0 \le i \le \frac{m+1}{2}$.

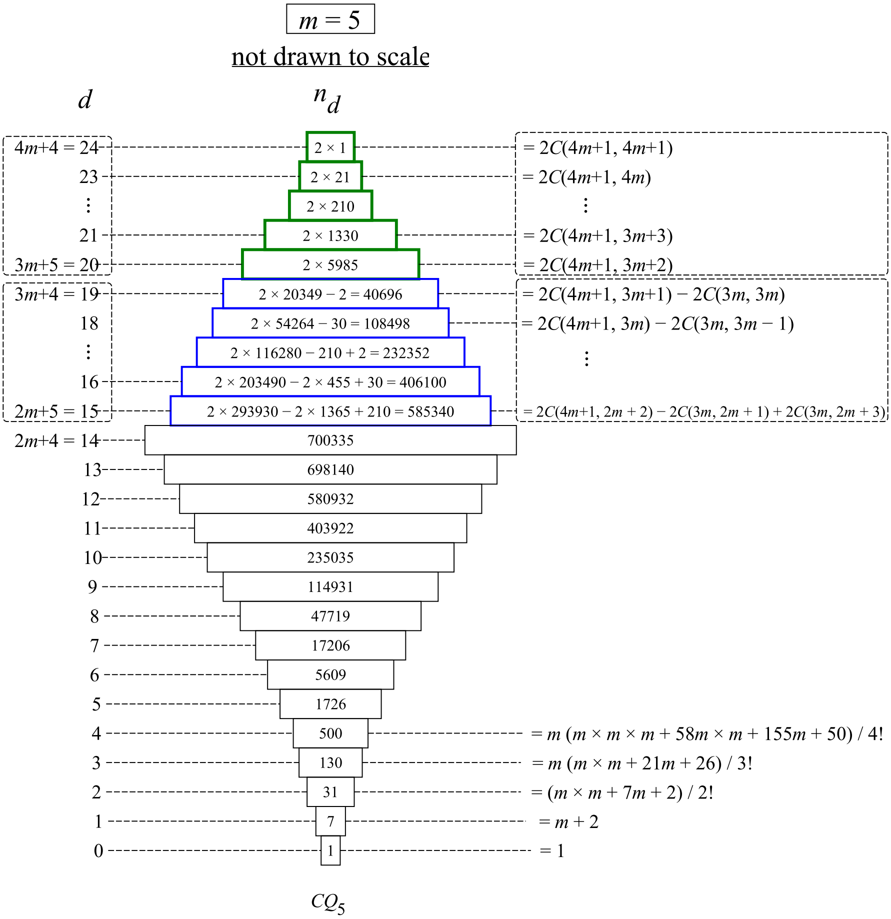

Fig. 7 Distance-wise node distribution of CQ_5

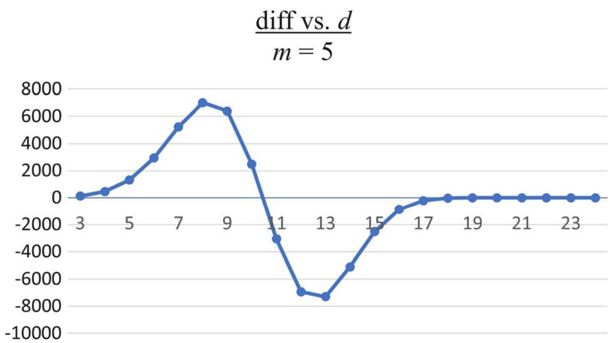

Fig. 8 Diff versus d for $m = 5$, $3 \le d \le 4m + 4$, vide Table 7

For $2m + 4 \geq d \geq m + 2$, therefore, D is symmetric about the point $d = \frac{3m}{2} + 3$, and diff is practically equal to $-2C(3m, d-4) + 2C(3m, d-2)$ whose absolute value is a small percentage of $2C(4m+1, d-3)$ in this range.

4. For $m + 1 \geq d \geq 3$, diff remains positive, and its value progressively declines. At the same time, the value relative to $2C(4m+1, d-3)$ is no longer negligible, particularly for small values of m.

Table 8 presents n_d for certain small values of d and certain large values of d. It relies on the results in Table 6.

4.2 Average node distance of CQ_m

Table 9 computes the average distance of CQ_m. It relies on the expression for n_d developed in Table 6. The identities used are: $\sum_{i=0}^{n} C(n,i) = 2^n$ and $\sum_{i=0}^{n} i\, C(n,i) = n2^{n-1}$.

Table 8 Computing n_d for $1 \leq d \leq 4$ and $4\,m \leq d \leq 4\,m + 4$, where $m \geq 2$

d	n_d	
0	$C(m,0)$	$O(1)$
1	$C(2m+1,0) + C(2m,0) + C(m,1)$	$O(m)$
	$= m + 2$	
2	$2C(3m,0) + C(2m+1,1) + C(2m,1) + C(m,2) - 2C(m,0)$	$O(m^2)$
	$= \frac{1}{2!}(m^2 + 7m + 2)$	
3	$2C(4m+1,0) + 2C(3m,1) + C(2m+1,2) - C(2m+1,0)$	$O(m^3)$
	$+C(2m,2) - C(2m,0) + C(m,3) - 2C(m,1)$	
	$= \frac{1}{3!}m(m^2 + 21m + 26)$	
4	$2C(4m+1,1) + 2C(3m,2) - 2C(3m,0) + C(2m+1,3)$	$O(m^4)$
	$-C(2m+1,1) + C(2m,3) - C(2m,1) + C(m,4) - 2C(m,2) + C(m,0)$	
	$= \frac{1}{4!}m(m^3 + 58m^2 + 155m + 50)$	
$4m$	$2C(4m+1, 4m-3) = 2C(4m+1, 4)$	$O(m^4)$
$4m+1$	$2C(4m+1, 4m-2) = 2C(4m+1, 3)$	$O(m^3)$
$4m+2$	$2C(4m+1, 4m-1) = 2C(4m+1, 2)$	$O(m^2)$
$4m+3$	$2C(4m+1, 4m) = 2C(4m+1, 1)$	$O(m)$
$4m+4$	$2C(4m+1, 4m+1) = 2C(4m+1, 0)$	$O(1)$

Table 9 Computing the average distance

$2\sum_{d=3}^{4m+4}(d\cdot C(4m+1,d-3))=2\sum_{i=0}^{4m+1}((i+3)\cdot C(4m+1,i))=(4m+7)2^{4m+1}$
$\sum_{d=2}^{3m+2}(d\cdot C(3m,d-2))=2\sum_{i=0}^{3m}((i+2)\cdot C(3m,i))=(3m+4)2^{3m}$
$-2\sum_{d=4}^{3m+4}(d\cdot C(3m,d-4))=-2\sum_{i=0}^{3m}((i+4)\cdot C(3m,i))=-(3m+8)2^{3m}$
$\sum_{d=1}^{2m+2}(d\cdot C(2m+1,d-1))=\sum_{i=0}^{2m+1}((i+1)\cdot C(2m+1,i))=(2m+3)2^{2m}$
$-\sum_{d=3}^{2m+4}(d\cdot C(2m+1,d-3))=-\sum_{i=0}^{2m+1}((i+3)\cdot C(2m+1,i))=-(2m+7)2^{2m}$
$\sum_{d=1}^{2m+1}(d\cdot C(2m,d-1))=\sum_{i=0}^{2m}((i+1)\cdot C(2m,i))=(m+1)2^{2m}$
$-\sum_{d=3}^{2m+3}(d\cdot C(2m,d-3))=-\sum_{i=0}^{2m}((i+3)\cdot C(2m,i))=-(m+3)2^{2m}$
$\sum_{d=0}^{m}d\cdot C(m,d)=m2^{m-1}$
$-2\sum_{d=2}^{m+2}(d\cdot C(m,d-2))=-2\sum_{i=0}^{m}((i+2)C(m,i))=-(m+4)2^{m}$
$\sum_{d=4}^{m+4}(d\cdot C(m,d-4))=\sum_{i=0}^{m}((i+4)\,C(m,i))=m2^{m-1}+2^{m+2}$
Algebraic sum $=(4m+7)\,2^{4m+1}-2^{3m+2}-3\cdot 2^{2m+1}$
Average distance $=\dfrac{(4m+7)2^{4m+1}-2^{3m+2}-3\cdot 2^{2m+1}}{2^{4m+2}}=\dfrac{4m+7}{2}-\dfrac{1}{2^{m}}-\dfrac{3}{2^{2m+1}}$

5 Concluding remarks

The quad-cube is a special case of a network topology called the metacube [9] that belongs to the family of networks derivable from the hypercube. It has emerged as a viable topology for a system in which the number of nodes exceeds several million. Among other things, it admits a 1-perfect code under a certain condition [5]. This paper presents results relating to its vertex transitivity, distance metric and distance-wise node distribution that significantly enhance its importance from both theoretical point of view and the engineering point of view. Interestingly, the node distribution of CQ_m closely parallels that of $2Q_{4m+1}$, i.e., a set of twin copies of Q_{4m+1}.

Graph invariants like Wiener index and surface area [7] of the quad-cube are easily obtainable from the distance-wise node distribution of the graph that appears in Sect. 4.

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