



Exact G_2 -structures on compact quotients of Lie groups

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Abstract

We show that the compact quotient $\Gamma \backslash G$ of a seven-dimensional simply connected Lie group G by a co-compact discrete subgroup $\Gamma \subset G$ does not admit any exact G_2 -structure which is induced by a left-invariant one on G .

Keywords Exact G_2 -structure · Lie algebra · Lattice

Mathematics Subject Classification 53C10 · 53C30

1 Introduction

A G_2 -structure on a 7-manifold M is a reduction of the structure group of its frame bundle from the linear group $GL(7, \mathbb{R})$ to the compact exceptional Lie group G_2 .

The existence of a G_2 -structure on M is characterized by the existence of a 3-form $\varphi \in \Omega^3(M)$ satisfying a certain nondegeneracy condition. This 3-form induces a Riemannian metric g_φ and an orientation on the manifold, and thus a Hodge star operator $*_\varphi$.

When φ is closed and co-closed, namely $d\varphi = 0$ and $d*_\varphi\varphi = 0$, the intrinsic torsion of the G_2 -structure vanishes identically, the Riemannian metric g_φ is Ricci-flat, and $\text{Hol}(g_\varphi) \subseteq G_2$, see [3, 11]. In this case, the G_2 -structure is said to be *torsion-free*. A G_2 -structure defined by a 3-form φ satisfying the weaker condition $d\varphi = 0$ is said to be *closed*. A closed G_2 -structure is called *exact* if $[\varphi] = 0 \in H^3_{\text{dR}}(M)$, namely if $\varphi = d\alpha$ for some $\alpha \in \Omega^2(M)$.

Currently, many examples of compact manifolds admitting closed G_2 -structures are available, see [6, 16, 17, 19, 20] for examples admitting holonomy G_2 metrics, Fernández

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et al. [9] for an example obtained resolving the singularities of an orbifold, and [1, 4, 5, 7, 8, 12, 18] for examples on compact quotients of Lie groups. However, it is still not known whether exact G_2 -structures may occur on compact 7-manifolds. A negative answer to this problem was given in [10, 13] in some special cases. In [10], M. Fernández and the first and third named author of this paper proved that there are no compact examples of the form $(\Gamma \backslash G, \varphi)$, where G is a simply connected solvable Lie group with $(2, 3)$ -trivial Lie algebra \mathfrak{g} , namely $b_2(\mathfrak{g}) = 0 = b_3(\mathfrak{g})$, $\Gamma \subset G$ is a cocompact discrete subgroup (lattice), and φ is an invariant exact G_2 -structure on $\Gamma \backslash G$, namely it is induced by a left-invariant exact G_2 -structure on G . In [13], Freibert and Salamon showed that the same conclusion holds, more generally, when the Lie algebra of G admits a codimension-one nilpotent ideal.

Motivated by these results, in this article we investigate the existence of invariant exact G_2 -structures on compact quotients of Lie groups, without considering any extra assumption on the properties of the group. In particular, we prove the following result.

Theorem 1.1 *A potential compact 7-manifold M with an exact G_2 -structure φ cannot be of the form $M = \Gamma \backslash G$, where G is a seven-dimensional simply connected Lie group, $\Gamma \subset G$ is a cocompact discrete subgroup, and the exact G_2 -structure φ on M is invariant.*

The proof of this theorem will be divided into two parts: in Sect. 2 we focus on the case when G is non-solvable, while we investigate the solvable case in Sect. 3. We shall deal only with Lie groups that are unimodular, as this is a necessary condition for the existence of lattices [21].

There is a one-to-one correspondence between left-invariant exact G_2 -structures on G and G_2 -structures on the Lie algebra $\mathfrak{g} = \text{Lie}(G)$ that are exact with respect to the Chevalley–Eilenberg differential. This allows us to investigate the existence of exact G_2 -structures at the Lie algebra level. We recall that a 3-form φ on \mathfrak{g} defines a G_2 -structure if and only if the symmetric bilinear map

$$b_\varphi : \mathfrak{g} \times \mathfrak{g} \rightarrow \Lambda^7 \mathfrak{g}^*, \quad b_\varphi(v, w) = \frac{1}{6} \iota_v \varphi \wedge \iota_w \varphi \wedge \varphi, \tag{1.1}$$

satisfies the condition $\det(b_\varphi)^{1/9} \neq 0 \in \Lambda^7 \mathfrak{g}^*$ and the symmetric bilinear form

$$g_\varphi := \det(b_\varphi)^{-1/9} b_\varphi : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathbb{R}$$

is positive definite, see e.g. [15]. In particular, for any choice of orientation on \mathfrak{g} , the map $b_\varphi : \mathfrak{g} \times \mathfrak{g} \rightarrow \Lambda^7 \mathfrak{g}^* \cong \mathbb{R}$ has to be positive or negative definite.

By [12], there are 4 non-solvable unimodular Lie algebras admitting closed G_2 -structures, up to isomorphism. Three of these Lie algebras are decomposable, and a direct computation with the aid of the software Maple 21 shows that b_φ is never definite for every exact 3-form $\varphi = d\alpha$ on each one of them (see Proposition 2.1). The remaining Lie algebra is indecomposable, and we show that the corresponding simply connected Lie group does not admit any lattice (see Proposition 2.1). These results prove Theorem 1.1 in the case when G is non-solvable.

We then focus on the solvable case. Here, there is a further constraint that has to be taken into account. Indeed, a solvable Lie group admits lattices only if it is strongly unimodular [14] (see Sect. 3 for the definition). The proof of Theorem 1.1 when G is solvable follows then from Theorem 3.2, where we show that a seven-dimensional strongly unimodular solvable Lie algebra \mathfrak{g} does not admit any exact G_2 -structure. To achieve this result, we first observe that every such Lie algebra is a semidirect product $\mathfrak{g} \cong \mathfrak{s} \rtimes_D \mathbb{R}$, for

some codimension-one unimodular ideal \mathfrak{g} of \mathfrak{g} , which must be solvable and non-nilpotent by [13], and some derivation $D \in \text{Der}(\mathfrak{g})$. The strongly unimodular condition on \mathfrak{g} is then encoded into the derivation D , while the existence of an exact G_2 -structure on \mathfrak{g} implies the existence of a certain type of $SU(3)$ -structure on \mathfrak{g} . Using these constraints together with the classification of six-dimensional unimodular solvable non-nilpotent Lie algebras, we show that none of these Lie algebras can occur as an ideal of a strongly unimodular solvable Lie algebra admitting exact G_2 -structures. As in the proof of Proposition 2.1, the computations are done with the aid of the software Maple 21.

1.1 Notation

Given an n -dimensional Lie algebra \mathfrak{g} , its structure equations with respect to a basis $\mathcal{B}^* = (e^1, \dots, e^n)$ of \mathfrak{g}^* are specified by the n -tuple (de^1, \dots, de^n) , where d denotes the Chevalley–Eilenberg differential of \mathfrak{g} . The basis of \mathfrak{g} with dual basis \mathcal{B}^* is denoted by (e_1, \dots, e_n) .

The shortening $e^{ijk\dots}$ for the wedge product of covectors $e^i \wedge e^j \wedge e^k \wedge \dots$ is used throughout the paper.

2 The non-solvable case

In this section, we deal with the case when the simply connected unimodular Lie group G is non-solvable. We claim that, in such a case, there are no compact 7-manifolds of the form $\Gamma \backslash G$ admitting invariant exact G_2 -structures.

By [12], we know that G admits left-invariant closed G_2 -structures if and only if its Lie algebra \mathfrak{g} is isomorphic to one of the following

$$\begin{aligned} \mathfrak{q}_1 &= \left(-e^{23}, -2e^{12}, 2e^{13}, 0, -e^{45}, \frac{1}{2}e^{46} - e^{47}, \frac{1}{2}e^{47}\right); \\ \mathfrak{q}_2 &= \left(-e^{23}, -2e^{12}, 2e^{13}, 0, -e^{45}, -\mu e^{46}, (1 + \mu)e^{47}\right), \quad -1 < \mu \leq -\frac{1}{2}; \\ \mathfrak{q}_3 &= \left(-e^{23}, -2e^{12}, 2e^{13}, 0, -\mu e^{45}, \frac{\mu}{2}e^{46} - e^{47}, e^{46} + \frac{\mu}{2}e^{47}\right), \quad \mu > 0; \\ \mathfrak{q}_4 &= \left(-e^{23}, -2e^{12}, 2e^{13}, -e^{14} - e^{25} - e^{47}, e^{15} - e^{34} - e^{57}, 2e^{67}, 0\right). \end{aligned}$$

The first three Lie algebras appearing in the previous list decompose as $\mathfrak{q} \cong \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{r}$, where \mathfrak{r} is a four-dimensional unimodular centerless solvable Lie algebra, while the Lie algebra \mathfrak{q}_4 is indecomposable.

The proof of our claim follows from the next two propositions.

Proposition 2.1 *A seven-dimensional unimodular non-solvable Lie algebra \mathfrak{g} does not admit any exact G_2 -structure if it is decomposable.*

Proof By [12], \mathfrak{g} is isomorphic to one of $\mathfrak{q}_1, \mathfrak{q}_2, \mathfrak{q}_3$. For each one of these Lie algebras, we consider the generic 2-form $\alpha = \sum_{1 \leq i < j \leq 7} a_{ij} e^{ij}$, where $a_{ij} \in \mathbb{R}$, and we use the structure equations to compute the expression of the generic exact 3-form $\varphi = d\alpha$. In detail, we obtain

- Lie algebra \mathfrak{q}_1

$$\begin{aligned} \varphi = & -2a_{24} e^{124} - 2a_{25} e^{125} - 2a_{26} e^{126} - 2a_{27} e^{127} + 2a_{34} e^{134} + 2a_{35} e^{135} + 2a_{36} e^{136} \\ & + 2a_{37} e^{137} + a_{15} e^{145} - \frac{1}{2}a_{16} e^{146} + \left(a_{16} - \frac{1}{2}a_{17}\right)e^{147} - a_{14} e^{234} - a_{15} e^{235} \\ & - a_{16} e^{236} - a_{17} e^{237} + a_{25} e^{245} - \frac{1}{2}a_{26} e^{246} + \left(a_{26} - \frac{1}{2}a_{27}\right)e^{247} + a_{35} e^{345} \\ & - \frac{1}{2}a_{36} e^{346} + \left(a_{36} - \frac{1}{2}a_{37}\right)e^{347} - \frac{1}{2}a_{56} e^{456} + a_{67} e^{467} - \left(a_{56} + \frac{1}{2}a_{57}\right)e^{457}; \end{aligned}$$

- Lie algebra \mathfrak{q}_2

$$\begin{aligned} \varphi = & -2a_{24} e^{124} - 2a_{25} e^{125} - 2a_{26} e^{126} - 2a_{27} e^{127} + 2a_{34} e^{134} + 2a_{35} e^{135} + 2a_{36} e^{136} \\ & + 2a_{37} e^{137} + a_{15} e^{145} + a_{16}\mu e^{146} - a_{17}(1 + \mu) e^{147} - a_{14} e^{234} - a_{15} e^{235} \\ & - a_{16} e^{236} - a_{17} e^{237} + a_{25} e^{245} + a_{26}\mu e^{246} - a_{27}(1 + \mu) e^{247} + a_{35} e^{345} \\ & + a_{36}\mu e^{346} - a_{37}(1 + \mu) e^{347} - a_{56}(1 + \mu) e^{456} + a_{57}\mu e^{457} + a_{67} e^{467}; \end{aligned}$$

- Lie algebra \mathfrak{q}_3

$$\begin{aligned} \varphi = & -2a_{24} e^{124} - 2a_{25} e^{125} - 2a_{26} e^{126} - 2a_{27} e^{127} + 2a_{34} e^{134} + 2a_{35} e^{135} + 2a_{36} e^{136} \\ & + 2a_{37} e^{137} + a_{15}\mu e^{145} - \left(\frac{1}{2}a_{16}\mu + a_{17}\right)e^{146} - \left(\frac{1}{2}a_{17}\mu - a_{16}\right)e^{147} - a_{14} e^{234} \\ & - a_{15} e^{235} - a_{16} e^{236} - a_{17} e^{237} + a_{25}\mu e^{245} - \left(\frac{1}{2}a_{26}\mu + a_{27}\right)e^{246} - \left(\frac{1}{2}a_{27}\mu - a_{26}\right)e^{247} \\ & + a_{35}\mu e^{345} - \left(\frac{1}{2}a_{36}\mu + a_{37}\right)e^{346} - \left(\frac{1}{2}a_{37}\mu - a_{36}\right)e^{347} - \left(\frac{1}{2}a_{56}\mu - a_{57}\right)e^{456} \\ & - \left(\frac{1}{2}a_{57}\mu + a_{56}\right)e^{457} + a_{67}\mu e^{467}. \end{aligned}$$

Now, a direct computation with the aid of the software Maple 21 shows that in each case the bilinear map b_φ defined in (1.1) satisfies $b_\varphi(e_i, e_i) = 0$, for $i = 5, 6, 7$. Consequently, $\varphi = d\alpha$ does not define a G_2 -structure on \mathfrak{g}_k , for $k = 1, 2, 3$. □

Proposition 2.2 *Let Q_4 be the simply connected Lie group with Lie algebra \mathfrak{q}_4 . Then, Q_4 does not admit any lattice.*

Proof The Lie algebra \mathfrak{q}_4 is isomorphic to a semi-direct product of the form $\mathfrak{sl}(2, \mathbb{R}) \ltimes \mathfrak{r}$, where the semisimple part is spanned by e_1, e_2, e_3 , and the four-dimensional radical $\mathfrak{r} = \mathbb{R} \ltimes_D \mathbb{R}^3$ is almost abelian, with $\mathbb{R} = \langle e_7 \rangle$, $\mathbb{R}^3 = \langle e_4, e_5, e_6 \rangle$ and

$$D := \text{ad}_{e_7}|_{\mathbb{R}^3} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}.$$

In particular, the radical of Q_4 is the almost abelian Lie group $\mathbb{R} \rtimes_{\mu} \mathbb{R}^3$, where the one-parameter group $\mu : \mathbb{R} \rightarrow \text{Aut}(\mathbb{R}^3)$ is defined by the condition $d\mu(t) = \exp(tD)$.

Now, by [23, Prop. 1.3], if Q_4 has a lattice, then also its radical does. By [2], in such a case there must be some $t' \in \mathbb{R} \setminus \{0\}$ such that

$$\exp(t'D) = \begin{pmatrix} e^{-t'} & 0 & 0 \\ 0 & e^{-t'} & 0 \\ 0 & 0 & e^{2t'} \end{pmatrix}$$

is conjugate to a matrix in $SL(3, \mathbb{Z})$. This is not possible by [2, Lemma B.4]. □

3 The solvable case

We now assume that G is solvable. As shown in [14], a simply connected solvable Lie group admitting lattices must be *strongly unimodular* according to the following.

Definition 3.1 Let G be a simply connected solvable Lie group with Lie algebra \mathfrak{g} and nilradical \mathfrak{n} . For each positive integer $i \geq 1$, let $\mathfrak{n}^i := [\mathfrak{n}, \mathfrak{n}^{i-1}]$ denote the i^{th} term in the descending central series of \mathfrak{n} , where $\mathfrak{n}^0 = \mathfrak{n}$. The Lie algebra \mathfrak{g} is *strongly unimodular* if for all $X \in \mathfrak{g}$ the restriction of ad_X to each space $\mathfrak{n}^i/\mathfrak{n}^{i+1}$ is traceless. In this case, the Lie group G is said to be *strongly unimodular*.

As the name suggests, strongly unimodular Lie groups are unimodular, but the converse does not hold in general, see for instance [10].

The proof of Theorem 1.1 in the case when G is solvable follows from the next result.

Theorem 3.2 *A seven-dimensional strongly unimodular solvable Lie algebra \mathfrak{g} does not admit any exact G_2 -structure.*

Before describing the strategy of the proof, we discuss some preliminary results. Let \mathfrak{g} be a unimodular solvable Lie algebra endowed with a G_2 -structure φ . Then, it has a codimension-one unimodular ideal \mathfrak{z} , and we can consider the g_φ -orthogonal decomposition $\mathfrak{g} = \mathfrak{z} \oplus \mathbb{R}$, where \mathbb{R} denotes the orthogonal complement of \mathfrak{z} . As a Lie algebra, \mathfrak{g} is then a semidirect product of the form $\mathfrak{g} = \mathfrak{z} \rtimes_D \mathbb{R}$, for some derivation D of \mathfrak{z} . The G_2 -structure φ on \mathfrak{g} can be written as follows

$$\varphi = \omega \wedge \eta + \psi,$$

where $\eta := z^\flat$ is the metric dual of a unit vector $z \in \mathbb{R}$, and the pair (ω, ψ) defines an $SU(3)$ -structure on \mathfrak{z} . In detail (see also [15]):

- a) $\omega \in \Lambda^2 \mathfrak{z}^*$ is a non-degenerate 2-form, i.e., $\omega^3 = \omega \wedge \omega \wedge \omega \neq 0$;

- b) $\psi \in \Lambda^3 \mathfrak{g}^*$ is a *negative stable* 3-form, namely a stable 3-form whose associated quartic polynomial satisfies $\lambda(\psi) < 0$. Here, $\lambda(\psi) := \frac{1}{6} \text{tr}(K_\psi^2)$, where $K_\psi \in \text{End}(\mathfrak{g})$ is defined as follows. Let $A: \Lambda^5 \mathfrak{g}^* \rightarrow \mathfrak{g} \otimes \Lambda^6 \mathfrak{g}^*$ be the isomorphism induced by the wedge product $\wedge: \Lambda^5 \mathfrak{g}^* \otimes \mathfrak{g}^* \rightarrow \Lambda^6 \mathfrak{g}^*$, then $K_\psi(v) \otimes \omega^3 = A(\iota_v \psi \wedge \psi)$, for all $v \in \mathfrak{g}$. In particular, $K_\psi^2 = \lambda(\psi) \text{Id}_{\mathfrak{g}}$, so that (ω, ψ) determines an almost complex structure

$$J : \mathfrak{g} \rightarrow \mathfrak{g}, \quad J = \frac{1}{\sqrt{-\lambda(\psi)}} K_\psi; \tag{3.1}$$

- c) ψ is primitive with respect to ω , i.e., $\psi \wedge \omega = 0$. This is equivalent to ω being of type $(1, 1)$ with respect to J , namely $\omega(J \cdot, J \cdot) = \omega$;
- d) the symmetric bilinear form $g := \omega(\cdot, J \cdot)$ is positive definite.

Remark 3.3 More generally, given a stable 3-form ψ on \mathfrak{g} , one can define the endomorphism K_ψ by choosing *any* volume form Ω on \mathfrak{g} in place of ω^3 , and the sign of $\lambda(\psi)$ does not depend on this choice. Moreover, if $\lambda(\psi) < 0$, the almost complex structure J depends only on ψ and on the orientation of \mathfrak{g} . Changing the orientation, one obtains the almost complex structure $-J$. Finally, we recall that ψ is a negative stable 3-form if and only if the contraction $\iota_v \psi$ has rank four for every non-zero vector $v \in \mathfrak{g}$.

Using the definition of the Chevalley–Eilenberg differential d of \mathfrak{g} , we also see that $d\eta = 0$. Indeed, for every $x, y \in \mathfrak{g}$ we have

$$d\eta(x, y) = -\eta([x, y]) = -g_\varphi(z, [x, y]) = 0,$$

since $[x, y] \in \mathfrak{g} = \langle z \rangle^{\perp_{g_\varphi}}$.

Assume now that φ is an exact G_2 -structure on $\mathfrak{g} = \mathfrak{g} \rtimes_D \mathbb{R}$, namely $\varphi = d\tilde{\alpha}$ for some $\tilde{\alpha} \in \Lambda^2 \mathfrak{g}^*$. By [13], we know that if \mathfrak{g} is strongly unimodular, then the solvable ideal \mathfrak{g} is not nilpotent.

We can write $\tilde{\alpha} = \alpha + \beta \wedge \eta$, where $\alpha \in \Lambda^2 \mathfrak{g}^*$ and $\beta \in \mathfrak{g}^*$. Then,

$$\varphi = \hat{d}\alpha + D^* \alpha \wedge \eta + \hat{d}\beta \wedge \eta = (\hat{d}\beta + D^* \alpha) \wedge \eta + \hat{d}\alpha,$$

where \hat{d} denotes the Chevalley–Eilenberg differential of \mathfrak{g} , and the action of $D \in \text{Der}(\mathfrak{g})$ on $\Lambda^2 \mathfrak{g}^*$ is defined as follows

$$D^* \alpha(x_1, x_2) = -\alpha(Dx_1, x_2) - \alpha(x_1, Dx_2),$$

for all $x_1, x_2 \in \mathfrak{g}$. From this, we see that \mathfrak{g} has an $SU(3)$ -structure defined by the pair

$$\omega := \hat{d}\beta + D^* \alpha, \quad \psi := \hat{d}\alpha. \tag{3.2}$$

In particular, ψ is an exact stable 3-form on \mathfrak{g} .

The previous discussion highlights some necessary conditions imposed by the existence of an exact G_2 -structure on a seven-dimensional (strongly) unimodular solvable Lie algebra $\mathfrak{g} = \mathfrak{g} \rtimes_D \mathbb{R}$. To show Theorem 3.2, we can then proceed as follows. The ideal \mathfrak{g} is a six-dimensional unimodular solvable non-nilpotent Lie algebra. The Lie algebras

satisfying these properties are classified up to isomorphism, so we can investigate each case separately. First, we determine which of these Lie algebras do not admit any negative stable exact 3-form, and we rule them out. For each one of the remaining Lie algebras, we consider the generic derivation $D \in \text{Der}(\mathfrak{g})$ and we determine the conditions guaranteeing that the extension $\mathfrak{g} = \mathfrak{g} \rtimes_D \mathbb{R}$ is strongly unimodular. Then, we investigate whether a generic pair (ω, ψ) of the form (3.2) can define an $SU(3)$ -structure on \mathfrak{g} . If this is not the case, then we rule \mathfrak{g} out. As we will see, none of the six-dimensional unimodular solvable non-nilpotent Lie algebras passes both tests. From this, the proof of Theorem 3.2 follows.

Remark 3.4 If the derivation $D \in \text{Der}(\mathfrak{g})$ is not nilpotent, then the nilradical of $\mathfrak{g} = \mathfrak{g} \rtimes_D \mathbb{R}$ coincides with the nilradical \mathfrak{n} of \mathfrak{g} . Otherwise, it is given by $\mathfrak{n} \rtimes_D \mathbb{R}$.

The structure equations of all six-dimensional unimodular solvable non-nilpotent Lie algebras can be found in the literature. Here, we consider the list given in [2, Appendix A], where the classification results of various preceding works have been meticulously collected. The structure equations of the decomposable unimodular Lie algebras can be determined from the tables A.1, A.3, A.4, A.5, A.6, A.7 in [2], and they are listed in Table 1 of Appendix A, where the unimodular Lie algebra $\mathfrak{g}_{4,2}^{-2}$ not appearing in Table A.1 of [2] is also included (see [22]). The structure equations of the unimodular indecomposable Lie algebras are given in the tables A.9–A.19 of [2], and we refer the reader to it for the list.

In what follows, the non-abelian Lie algebras are denoted as in [2], namely we use the symbol $\mathfrak{g}_{n,k}$ to denote the k^{th} Lie algebra of dimension n appearing in the list of non-isomorphic n -dimensional solvable Lie algebras. Moreover, superscripts like $\mathfrak{g}^{p,q,\dots}$ denote the values of the real parameters on which a Lie algebra depends. Finally, we denote the n -dimensional abelian Lie algebra $n\mathfrak{g}_1$ by \mathbb{R}^n .

We will investigate the decomposable and the indecomposable case separately.

3.1 The decomposable case

We begin considering the Lie algebras listed in Table 1 of Appendix A. The next result shows that most of them cannot occur as an ideal of a unimodular solvable Lie algebra admitting exact G_2 -structures.

Proposition 3.5 *A six-dimensional unimodular decomposable solvable non-nilpotent Lie algebra \mathfrak{g} admits negative stable exact 3-forms if and only if it is isomorphic to one of the following: $\mathfrak{g}_{3,4}^{-1} \oplus \mathfrak{g}_{3,4}^{-1} \oplus \mathfrak{g}_{5,30}^{-4/3} \oplus \mathbb{R}, \mathfrak{g}_{5,33}^{-1,-1} \oplus \mathbb{R}, \mathfrak{g}_{5,35}^{-2,0} \oplus \mathbb{R}.$*

Proof Let \mathfrak{g} be one of the Lie algebras appearing in Table 1, and let (e^1, \dots, e^6) be the basis of \mathfrak{g}^* used to describe the structure equations. We consider a generic 2-form $\alpha = \sum_{1 \leq i < j \leq 6} a_{ij} e^{ij} \in \Lambda^2 \mathfrak{g}^*$, where $a_{ij} \in \mathbb{R}$, and we compute its Chevalley–Eilenberg differential $\hat{d}\alpha$ using the structure equations. Then, we determine the quartic polynomial $\lambda(\hat{d}\alpha)$ as explained before. Notice that we are free to choose the 6-form e^{123456} in place of a generic non-zero element in $\Lambda^6 \mathfrak{g}^*$ representing ω^3 , since the sign of $\lambda(\hat{d}\alpha)$ does not depend on the choice of orientation for \mathfrak{g} . The Lie algebras for which $\lambda(\hat{d}\alpha) \geq 0$ are the following

$$\begin{aligned}
 \mathfrak{g}_{3,4}^{-1} \oplus \mathfrak{g}_{3,1} &: \lambda(\hat{d}\alpha) = 4a_{14}^2 a_{24}^2; \\
 \mathfrak{g}_{3,4}^{-1} \oplus \mathfrak{g}_{3,5}^0 &: \lambda(\hat{d}\alpha) = 4(a_{14}^2 + a_{15}^2)(a_{24}^2 + a_{25}^2); \\
 \mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_{3,1} &: \lambda(\hat{d}\alpha) = (a_{14}^2 + a_{24}^2)^2; \\
 \mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_{3,5}^0 &: \lambda(\hat{d}\alpha) = ((a_{14} + a_{25})^2 + (a_{15} - a_{24})^2)((a_{14} - a_{25})^2 + (a_{15} + a_{24})^2); \\
 \mathfrak{g}_{5,19}^{p,-2p-2} \oplus \mathbb{R} &: \lambda(\hat{d}\alpha) = 4(1+p)^2 a_{14}^2 a_{16}^2; \\
 \mathfrak{g}_{5,23}^{-4} \oplus \mathbb{R} &: \lambda(\hat{d}\alpha) = 16 a_{14}^2 a_{16}^2; \\
 \mathfrak{g}_{5,25}^{4,4p} \oplus \mathbb{R} &: \lambda(\hat{d}\alpha) = 16 p^2 a_{14}^2 a_{16}^2; \\
 \mathfrak{g}_{5,28}^{-3/2} \oplus \mathbb{R} &: \lambda(\hat{d}\alpha) = a_{14}^2 a_{16}^2.
 \end{aligned}$$

As for the Lie algebras of Table 1 that are not isomorphic to any of the previous ones nor to one of $\mathfrak{g}_{3,4}^{-1} \oplus \mathfrak{g}_{3,4}^{-1} \oplus \mathfrak{g}_{5,30}^{-4/3} \oplus \mathbb{R}$, $\mathfrak{g}_{5,33}^{-1,-1} \oplus \mathbb{R}$, $\mathfrak{g}_{5,35}^{-2,0} \oplus \mathbb{R}$, we have $\lambda(\hat{d}\alpha) = 0$. On the remaining Lie algebras, there exist exact 3-forms $\hat{d}\alpha$ such that $\lambda(\hat{d}\alpha) < 0$. The expression of $\lambda(\hat{d}\alpha)$ for these Lie algebras will be given in the proofs of the next propositions. \square

We are left with the decomposable Lie algebras $\mathfrak{g}_{3,4}^{-1} \oplus \mathfrak{g}_{3,4}^{-1}$, $\mathfrak{g}_{5,30}^{-4/3} \oplus \mathbb{R}$, $\mathfrak{g}_{5,33}^{-1,-1} \oplus \mathbb{R}$, $\mathfrak{g}_{5,35}^{-2,0} \oplus \mathbb{R}$. We divide the discussion into three propositions, as we use different strategies to rule them out.

Proposition 3.6 *The Lie algebras $\mathfrak{g}_{5,30}^{-4/3} \oplus \mathbb{R}$ and $\mathfrak{g}_{5,35}^{-2,0} \oplus \mathbb{R}$ cannot occur as an ideal of a strongly unimodular solvable Lie algebra admitting exact G_2 -structures.*

Proof Let $\mathfrak{g} = \mathfrak{g}_{5,30}^{-4/3} \oplus \mathbb{R}$, and consider the basis (e^1, \dots, e^6) of \mathfrak{g}^* for which the structure equations are those given in Table 1, namely

$$\mathfrak{g}_{5,30}^{-4/3} \oplus \mathbb{R} = \left(-e^{24} - \frac{2}{3}e^{15}, -e^{34} + \frac{1}{3}e^{25}, \frac{4}{3}e^{35}, -e^{45}, 0, 0 \right).$$

Let $\mathcal{B} = (e_1, \dots, e_6)$ be the basis of \mathfrak{g} with dual basis (e^1, \dots, e^6) . The generic derivation $D \in \text{Der}(\mathfrak{g})$ has the following matrix representation with respect to the basis \mathcal{B}

$$D = \begin{pmatrix} a_1 & a_2 & 0 & a_3 & a_4 & 0 \\ 0 & a_5 & a_2 & a_6 & -\frac{1}{3}a_3 & 0 \\ 0 & 0 & 2a_5 - a_1 & 0 & -\frac{4}{3}a_6 & 0 \\ 0 & 0 & 0 & a_1 - a_5 & -a_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_7 & a_8 \end{pmatrix},$$

where $a_1, \dots, a_8 \in \mathbb{R}$.

The nilradical of \mathfrak{g} is $\mathfrak{n} = \langle e_1, e_2, e_3, e_4, e_6 \rangle$, and it has the following descending central series

$$\mathfrak{n}^0 = \mathfrak{n}, \quad \mathfrak{n}^1 = \langle e_1, e_2 \rangle, \quad \mathfrak{n}^2 = \langle e_1 \rangle, \quad \mathfrak{n}^k = \{0\}, \quad k \geq 3.$$

From this, we see that the Lie algebra $\mathfrak{g} = \mathfrak{g} \rtimes_D \mathbb{R}$ is strongly unimodular only when $a_1 = a_5 = a_8 = 0$.

We now consider a generic 2-form $\alpha = \sum_{1 \leq i < j \leq 6} \alpha_{ij} e^{ij}$ and a generic 1-form $\beta = \sum_{i=1}^6 \beta_i e^i$ on \mathfrak{g} , and we compute the forms

$$\omega = \hat{d}\beta + D^* \alpha, \quad \psi = \hat{d}\alpha.$$

Then, for the values of the parameters α_{ij} and b_k for which $\omega^3 \neq 0$ and

$$\lambda(\hat{d}\alpha) = \frac{4}{9} (4\alpha_{13}^2 \alpha_{16}^2 + 2\alpha_{16} \alpha_{26} \alpha_{12} \alpha_{13} - 10\alpha_{12} \alpha_{16}^2 \alpha_{23} - 4\alpha_{12}^2 \alpha_{16} \alpha_{36} + \alpha_{12}^2 \alpha_{26}^2) < 0,$$

we determine the almost complex structure J using the formula (3.1). Notice that the sign of J depends on ω^3 being a positive or negative multiple of the volume form e^{123456} . Now, a direct computation shows that $\omega(e_i, J e_i) = 0$, for $i = 1, 2, 3$. Therefore, the pair (ω, ψ) cannot define an $SU(3)$ -structure on \mathfrak{g} .

Similar computations for the Lie algebra

$$\mathfrak{g}_{5,35}^{-2,0} \oplus \mathbb{R} = (2e^{14}, -e^{24} - e^{35}, e^{25} - e^{34}, 0, 0, 0)$$

show that the derivation D must have the following matrix representation

$$D = \begin{pmatrix} a_1 & 0 & 0 & a_2 & 0 & 0 \\ 0 & a_3 & a_4 & a_5 & a_6 & 0 \\ 0 & -a_4 & a_3 & a_6 & -a_5 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_7 & a_8 & -a_1 - 2a_3 \end{pmatrix},$$

and that whenever

$$\lambda(\hat{d}\alpha) = 4(\alpha_{26}^2 + \alpha_{36}^2)(\alpha_{12}^2 + \alpha_{13}^2) + 16\alpha_{23}\alpha_{16}(\alpha_{12}\alpha_{36} - \alpha_{13}\alpha_{26}) < 0,$$

we have

$$\omega(e_1, J e_1) \omega(e_6, J e_6) = \frac{1}{\lambda(\hat{d}\alpha)} 16\alpha_{16}^2 (\alpha_{26}^2 + \alpha_{36}^2) (\alpha_{12}^2 + \alpha_{13}^2) (a_1 + 2a_3)^2 \leq 0,$$

whence the thesis follows. □

Proposition 3.7 *The Lie algebra $\mathfrak{g}_{3,4}^{-1} \oplus \mathfrak{g}_{3,4}^{-1}$ cannot occur as an ideal of a strongly unimodular solvable Lie algebra admitting exact G_2 -structures.*

Proof Let $\mathfrak{g} = \mathfrak{g}_{3,4}^{-1} \oplus \mathfrak{g}_{3,4}^{-1}$, and let $\mathcal{B}^* = (e^1, \dots, e^6)$ be the basis of \mathfrak{g}^* for which the structure equations are

$$\mathfrak{g}_{3,4}^{-1} \oplus \mathfrak{g}_{3,4}^{-1} = (-e^{13}, e^{23}, 0, -e^{46}, e^{56}, 0).$$

Let $\mathcal{B} = (e_1, \dots, e_6)$ be the basis of \mathfrak{g} with dual basis \mathcal{B}^* . Then, the nilradical of \mathfrak{g} is the abelian ideal $\mathfrak{n} = \langle e_1, e_2, e_4, e_5 \rangle$, and the generic derivation $D \in \text{Der}(\mathfrak{g})$ for which $\mathfrak{g} \rtimes_D \mathbb{R}$ is strongly unimodular has the following matrix representation with respect to \mathcal{B}

$$D = \begin{pmatrix} a_1 & 0 & a_2 & 0 & 0 & 0 \\ 0 & a_3 & a_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_5 & 0 & a_6 \\ 0 & 0 & 0 & 0 & -a_5 - a_3 - a_1 & a_8 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $a_i \in \mathbb{R}$.

We consider a generic 2-form $\alpha = \sum_{1 \leq i < j \leq 6} \alpha_{ij} e^{ij}$, a generic 1-form $\beta = \sum_{i=1}^6 \beta_i e^i$ on \mathfrak{g} , and the forms $\omega = \hat{\alpha}\beta + D^*\alpha$ and $\psi = \hat{\alpha}\alpha$. We have

$$\begin{aligned} \omega = & -\alpha_{12}(a_1 + a_3)e^{12} + (-a_1\alpha_{13} - a_4\alpha_{12} - \beta_1)e^{13} - \alpha_{14}(a_1 + a_5)e^{14} + \alpha_{15}(a_5 + a_3)e^{15} \\ & + (-a_1\alpha_{16} - a_6\alpha_{14} - a_8\alpha_{15})e^{16} + (a_2\alpha_{12} - a_3\alpha_{23} + \beta_2)e^{23} - \alpha_{24}(a_5 + a_3)e^{24} \\ & + \alpha_{25}(a_1 + a_5)e^{25} + (-a_3\alpha_{26} - a_6\alpha_{24} - a_8\alpha_{25})e^{26} + (-a_2\alpha_{14} - a_4\alpha_{24} - a_5\alpha_{34})e^{34} \\ & + (a_1\alpha_{35} - a_2\alpha_{15} + a_3\alpha_{35} - a_4\alpha_{25} + a_5\alpha_{35})e^{35} + (-a_2\alpha_{16} - a_4\alpha_{26} - a_6\alpha_{34} - a_8\alpha_{35})e^{36} \\ & + \alpha_{45}(a_1 + a_3)e^{45} + (-a_5\alpha_{46} - a_8\alpha_{45} - \beta_4)e^{46} + (a_1\alpha_{56} + a_3\alpha_{56} + a_5\alpha_{56} + a_6\alpha_{45} + \beta_5)e^{56}, \end{aligned}$$

and

$$\begin{aligned} \psi = & -\alpha_{14}e^{134} - \alpha_{15}e^{135} - \alpha_{16}e^{136} + \alpha_{14}e^{146} - \alpha_{15}e^{156} + \alpha_{24}e^{234} + \alpha_{25}e^{235} \\ & + \alpha_{26}e^{236} + \alpha_{24}e^{246} - \alpha_{25}e^{256} + \alpha_{34}e^{346} - \alpha_{35}e^{356}. \end{aligned}$$

Choosing the volume form e^{123456} , we compute

$$\lambda(\psi) = 16 \alpha_{14} \alpha_{15} \alpha_{24} \alpha_{25}.$$

Assuming that $\lambda(\psi) < 0$, we determine the almost complex structure J induced by ψ and the chosen orientation. We now show that there exists a nonzero vector $x \in \mathfrak{g}$ such that $\omega(x, Jx) = 0$. From this, the thesis follows.

Let us consider the family of nilpotent derivations $S \in \text{Der}(\mathfrak{g})$ having the following matrix representation with respect to \mathcal{B} :

$$S = \begin{pmatrix} 0 & 0 & s_1 & 0 & 0 & 0 \\ 0 & 0 & s_2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & s_3 \\ 0 & 0 & 0 & 0 & 0 & s_4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $s_i \in \mathbb{R}$, and let $F := \exp(S) \in \text{Aut}(\mathfrak{g})$. From the pair (ω, ψ) , we obtain the pair $(F^*\omega, F^*\psi)$ with associated almost complex structure $J_{F^*\psi} = F^{-1} \circ J \circ F$.

We claim that there exists a choice of the real numbers s_i for which $F^*\omega(e_3, e_6) = 0$ and $J_{F^*\psi}(e_3) \in \langle e_3, e_6 \rangle$. This implies that $\omega(x, Jx) = 0$ for $x = Fe_3$, as

$$\omega(Fe_3, JFe_3) = F^*\omega(e_3, J_{F^*\psi}e_3) = 0.$$

Comparing

$$\begin{aligned}
 F^*\psi &= \alpha_{25}e^{235} - \alpha_{25}e^{256} + \alpha_{24}e^{234} + \alpha_{24}e^{246} - \alpha_{15}e^{135} - \alpha_{15}e^{156} - \alpha_{14}e^{134} + \alpha_{14}e^{146} \\
 &\quad - (\alpha_{14}s_3 + \alpha_{15}s_4 + \alpha_{16})e^{136} + (\alpha_{24}s_3 + \alpha_{25}s_4 + \alpha_{26})e^{236} \\
 &\quad + (\alpha_{14}s_1 + \alpha_{24}s_2 + \alpha_{34})e^{346} - (\alpha_{15}s_1 + \alpha_{25}s_2 + \alpha_{35})e^{356},
 \end{aligned}$$

and

$$\begin{aligned}
 F^*\omega(e_3, e_6) &= -a_1s_1(\alpha_{14}s_3 + \alpha_{16}) + a_1s_4(\alpha_{25}s_2 + \alpha_{35}) + a_3s_4(\alpha_{15}s_1 + \alpha_{35}) - a_3s_2(\alpha_{24}s_3 + \alpha_{26}) \\
 &\quad - (a_5s_3 + a_6)(\alpha_{14}s_1 + \alpha_{24}s_2 + \alpha_{34}) + (a_5s_4 - a_8)(\alpha_{15}s_1 + \alpha_{25}s_2 + \alpha_{35}) \\
 &\quad - a_2(\alpha_{14}s_3 + \alpha_{15}s_4 + \alpha_{16}) - a_4(\alpha_{24}s_3 + \alpha_{25}s_4 + \alpha_{26}),
 \end{aligned}$$

we see that the latter is zero if the coefficients of $e^{136}, e^{236}, e^{346}, e^{356}$ in the expression of $F^*\psi$ vanish, namely if the following linear system in s_1, s_2, s_3, s_4 is compatible

$$\begin{cases} \alpha_{14}s_3 + \alpha_{15}s_4 = -\alpha_{16}, \\ \alpha_{24}s_3 + \alpha_{25}s_4 = -\alpha_{26}, \\ \alpha_{14}s_1 + \alpha_{24}s_2 = -\alpha_{34}, \\ \alpha_{15}s_1 + \alpha_{25}s_2 = -\alpha_{35}. \end{cases}$$

Under the assumption $\lambda(\psi) = 16 \alpha_{14} \alpha_{15} \alpha_{24} \alpha_{25} < 0$, the system has a unique solution $(\bar{s}_1, \bar{s}_2, \bar{s}_3, \bar{s}_4)$. Let $\bar{F} \in \text{Aut}(\mathfrak{g})$ be the automorphism corresponding to the choice $s_i = \bar{s}_i$, for $i = 1, 2, 3, 4$. Then,

$$\bar{F}^*\psi = \alpha_{25}e^{235} - \alpha_{25}e^{256} + \alpha_{24}e^{234} + \alpha_{24}e^{246} - \alpha_{15}e^{135} - \alpha_{15}e^{156} - \alpha_{14}e^{134} + \alpha_{14}e^{146},$$

and $\bar{F}^*\omega(e_3, e_6) = 0$. A computation then shows that $F_{\bar{F}^*\psi}e_3 \in \langle e_3, e_6 \rangle$, and the claim follows. □

Proposition 3.8 *The Lie algebra $\mathfrak{g}_{5,33}^{-1,-1} \oplus \mathbb{R}$ cannot occur as an ideal of a strongly unimodular solvable Lie algebra admitting exact G_2 -structures.*

Proof Let $\mathfrak{g} = \mathfrak{g}_{5,33}^{-1,-1} \oplus \mathbb{R}$ and let $\mathcal{B}^* = (e^1, \dots, e^6)$ be the basis of \mathfrak{g}^* for which the structure equations are

$$(-e^{14}, -e^{25}, e^{34} + e^{35}, 0, 0, 0).$$

Let $\mathcal{B} = (e_1, \dots, e_6)$ be the basis of \mathfrak{g} with dual basis \mathcal{B}^* . Then, the nilradical of \mathfrak{g} is the abelian ideal $\mathfrak{n} = \langle e_1, e_2, e_3, e_6 \rangle$, and the generic derivation $D \in \text{Der}(\mathfrak{g})$ for which $\mathfrak{g} = \mathfrak{g} \rtimes_D \mathbb{R}$ is strongly unimodular must have the following matrix representation with respect to \mathcal{B}

$$D = \begin{pmatrix} a_1 & 0 & 0 & a_2 & 0 & 0 \\ 0 & a_3 & 0 & 0 & a_4 & 0 \\ 0 & 0 & a_5 & a_6 & a_6 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & a_7 & a_8 & -a_5 - a_3 - a_1 \end{pmatrix},$$

where $a_i \in \mathbb{R}$.

We consider a generic 2-form $\alpha = \sum_{1 \leq i < j \leq 6} \alpha_{ij} e^{ij}$ and a generic 1-form $\beta = \sum_{i=1}^6 \beta_i e^i$ on \mathfrak{g} , and we let $\omega = \hat{d}\beta + D^*\alpha$ and $\psi = \hat{d}\alpha$. Then

$$\begin{aligned} \omega = & -\alpha_{12}(a_1 + a_3)e^{12} - \alpha_{13}(a_1 + a_5)e^{13} + (-\alpha_{14}a_1 - \alpha_{13}a_6 - \alpha_{16}a_7 - \beta_1)e^{14} \\ & + (-\alpha_{15}a_1 - \alpha_{12}a_4 - \alpha_{13}a_6 - \alpha_{16}a_8)e^{15} + \alpha_{16}(a_3 + a_5)e^{16} - \alpha_{23}(a_3 + a_5)e^{23} \\ & + (\alpha_{12}a_2 - \alpha_{24}a_3 - \alpha_{23}a_6 - \alpha_{26}a_7)e^{24} + (-\alpha_{25}a_3 - \alpha_{23}a_6 - \alpha_{26}a_8 - \beta_2)e^{25} \\ & + \alpha_{26}(a_1 + a_5)e^{26} + (\alpha_{13}a_2 - \alpha_{34}a_5 - \alpha_{36}a_7 + \beta_3)e^{34} + \alpha_{36}(a_1 + a_3)e^{36} \\ & + (\alpha_{23}a_4 - \alpha_{35}a_5 - \alpha_{36}a_8 + \beta_3)e^{35} + (\alpha_{46}a_1 - \alpha_{16}a_2 + \alpha_{46}a_3 + \alpha_{46}a_5 - \alpha_{36}a_6)e^{46} \\ & + (-\alpha_{15}a_2 + \alpha_{24}a_4 + \alpha_{34}a_6 - \alpha_{35}a_6 + \alpha_{56}a_7 - \alpha_{46}a_8)e^{45} \\ & + (\alpha_{56}a_1 + \alpha_{56}a_3 - \alpha_{26}a_4 + \alpha_{56}a_5 - \alpha_{36}a_6)e^{56}, \end{aligned}$$

and

$$\begin{aligned} \psi = & \alpha_{12}(e^{125} + e^{124}) - \alpha_{13}e^{135} - \alpha_{15}e^{145} - \alpha_{16}e^{146} - \alpha_{23}e^{234} + \alpha_{24}e^{245} - \alpha_{26}e^{256} \\ & + (\alpha_{35} - \alpha_{34})e^{345} + \alpha_{36}(e^{346} + e^{356}). \end{aligned}$$

We will prove that there are no values of the parameters $a_r, \alpha_{ij}, \beta_k$ for which the pair (ω, ψ) defines an $SU(3)$ -structure.

Let us assume that $\omega^3 \neq 0$ and

$$\lambda(\psi) = 4(\alpha_{26}\alpha_{36}\alpha_{12}\alpha_{13} - \alpha_{23}\alpha_{36}\alpha_{12}\alpha_{16} + \alpha_{13}\alpha_{16}\alpha_{23}\alpha_{26}) < 0.$$

Up to changing the sign of ω , we can assume that the corresponding orientation is e^{123456} . We can then determine the almost complex structure J induced by the pair (ω, ψ) and consider the bilinear form $g = \omega(\cdot, J\cdot)$. Let $g_{ij} := g(e_i, e_j)$ be the components of the matrix associated with g with respect to the basis \mathcal{B} . We must have

i) $\alpha_{12} \alpha_{13} \alpha_{16} \alpha_{23} \alpha_{26} \alpha_{36} \neq 0,$

as otherwise $g_{ii} = 0$, for at least one $i \in \{1, 2, 3, 6\}$.

We now focus on the compatibility condition $\omega \wedge \psi = 0$, which is equivalent to a system of five polynomial equations in the variables $a_r, \alpha_{ij}, \beta_k$. We compute

$$\begin{aligned} (\omega \wedge \psi)(e_1, e_2, e_3, e_4, e_6) &= 2(\alpha_{16}\alpha_{23}(a_3 + a_5) - \alpha_{12}\alpha_{36}(a_1 + a_3)) =: z_1, \\ (\omega \wedge \psi)(e_1, e_2, e_3, e_5, e_6) &= -2(\alpha_{13}\alpha_{26}(a_1 + a_5) + \alpha_{12}\alpha_{36}(a_1 + a_3)) =: z_2, \end{aligned}$$

and we use the equalities

$$\begin{aligned} \frac{g_{12}}{\alpha_{12}} = -\frac{g_{21}}{\alpha_{12}} = \frac{g_{63}}{\alpha_{36}} = -\frac{g_{36}}{\alpha_{36}} &= (z_1 - z_2)\sqrt{-\lambda(\psi)}, \\ \frac{g_{13}}{\alpha_{13}} = -\frac{g_{31}}{\alpha_{13}} = \frac{g_{62}}{\alpha_{26}} = -\frac{g_{26}}{\alpha_{26}} &= -z_1\sqrt{-\lambda(\psi)}, \\ \frac{g_{16}}{\alpha_{16}} = -\frac{g_{61}}{\alpha_{16}} = \frac{g_{32}}{\alpha_{23}} = -\frac{g_{23}}{\alpha_{23}} &= -z_2\sqrt{-\lambda(\psi)}, \end{aligned}$$

to conclude that every solution of the system must give

ii) $g(e_i, e_j) = 0$, for $i, j \in \{1, 2, 3, 6\}$ with $i \neq j$.

The three equations

$$(\omega \wedge \psi)(e_2, e_3, e_4, e_5, e_6) = 0, \quad (\omega \wedge \psi)(e_1, e_3, e_4, e_5, e_6) = 0, \quad (\omega \wedge \psi)(e_1, e_2, e_4, e_5, e_6) = 0,$$

determine a linear system in the variables $\beta_1, \beta_2, \beta_3$, which has a unique solution under the constraint (i). The remaining equations

$$\begin{cases} (\omega \wedge \psi)(e_1, e_2, e_3, e_4, e_5) = 0, \\ (\omega \wedge \psi)(e_1, e_2, e_3, e_4, e_6) = 0, \\ (\omega \wedge \psi)(e_1, e_2, e_3, e_5, e_6) = 0, \end{cases} \tag{3.3}$$

do not contain the variables β_k 's, and we can solve the system (3.3) in the following cases:

- (A) $\alpha_{12}\alpha_{13}\alpha_{26}\alpha_{36} + \alpha_{12}\alpha_{16}\alpha_{23}\alpha_{36} + \alpha_{13}\alpha_{16}\alpha_{23}\alpha_{26} \neq 0$, namely (3.3) is a compatible linear system in the variables a_1, a_2, a_5 ;
- (B) $\alpha_{12}\alpha_{13}\alpha_{26}\alpha_{36} - \alpha_{12}\alpha_{16}\alpha_{23}\alpha_{36} - \alpha_{13}\alpha_{16}\alpha_{23}\alpha_{26} \neq 0$, namely (3.3) is a compatible linear system in the variables a_1, a_2, a_3 .

Indeed, if the two polynomials above are both zero, then $\alpha_{12}\alpha_{13}\alpha_{26}\alpha_{36} = 0$, which contradicts the condition (i). In both cases (A) and (B), we can use the constraint to solve the system (3.3). Then, we have $\omega \wedge \psi = 0$, and the bilinear form $g = \omega(\cdot, J\cdot)$ is symmetric. We now show that g is never positive definite.

To simplify the computations, we can proceed as follows. Let us consider the derivation $S \in \text{Der}(\mathfrak{g})$ whose matrix with respect to the basis \mathcal{B} is

$$\begin{pmatrix} 0 & 0 & 0 & s_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & s_2 & 0 \\ 0 & 0 & 0 & s_3 & s_3 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} s_1 &= \frac{\alpha_{12}\alpha_{34} - \alpha_{12}\alpha_{35} + \alpha_{13}\alpha_{24} - \alpha_{15}\alpha_{23}}{2\alpha_{12}\alpha_{13}}, \\ s_2 &= -\frac{\alpha_{12}\alpha_{34} - \alpha_{12}\alpha_{35} - \alpha_{13}\alpha_{24} + \alpha_{15}\alpha_{23}}{2\alpha_{12}\alpha_{13}}, \\ s_3 &= \frac{\alpha_{12}\alpha_{34} - \alpha_{12}\alpha_{35} - \alpha_{13}\alpha_{24} - \alpha_{15}\alpha_{23}}{2\alpha_{12}\alpha_{13}}. \end{aligned}$$

Then, the automorphism $F = \exp(S) \in \text{Aut}(\mathfrak{g})$ is such that

$$F^*\psi = \alpha_{12}(e^{124} + e^{125}) - \alpha_{13}e^{135} - \alpha_{16}e^{146} - \alpha_{23}e^{234} - \alpha_{26}e^{256} + \alpha_{36}(e^{346} + e^{356}).$$

Notice that $\lambda(F^*\psi) = \lambda(\psi)$. Moreover, this choice of F guarantees that the condition (ii) is satisfied also by the bilinear form $F^*g = F^*\omega(\cdot, J_{F^*\psi}\cdot)$, where $J_{F^*\psi} = F^{-1} \circ J \circ F$. In addition, the subspaces $V_1 = \langle e_1, e_2, e_3, e_6 \rangle$ and $V_2 = \langle e_4, e_5 \rangle$ are $J_{F^*\psi}$ -invariant.

Let Q be the matrix associated to F^*g with respect to the basis \mathcal{B} . We will show that there are no values of the parameters $a_r, \alpha_{ij}, \beta_k$ for which Q is symmetric and positive definite. If that was the case, then it would be possible to construct a F^*g -orthonormal basis (v_1, \dots, v_6) starting from \mathcal{B} in such a way that

$$v_i = h_{ii} e_i, \quad i = 1, 2, 3, 6, \quad v_4 = \sum_{i=1}^4 h_{i4} e_i + h_{64} e_6, \quad v_5 = \sum_{i=1}^6 h_{i5} e_i,$$

where $h_{kk} > 0$, for $1 \leq k \leq 6$. Consequently, there would exist an invertible 6×6 matrix $P = (p_{ij})$, given by the inverse of $H = (h_{ij})$, such that

$$Q = P^t P,$$

and whose entries satisfy the following conditions

$$p_{kk} > 0, \quad p_{54} = 0, \quad p_{ki} = 0, \quad \text{for } 1 \leq k \leq 6 \text{ and } i \neq k \text{ in } \{1, 2, 3, 6\}. \tag{3.4}$$

Moreover, the following quantities should all be positive

$$\frac{F^*g(e_1, e_1)}{F^*g(e_6, e_6)} = -\frac{\alpha_{12}\alpha_{13}}{\alpha_{26}\alpha_{36}}, \quad \frac{F^*g(e_2, e_2)}{F^*g(e_6, e_6)} = \frac{\alpha_{12}\alpha_{23}}{\alpha_{16}\alpha_{36}}, \quad \frac{F^*g(e_3, e_3)}{F^*g(e_6, e_6)} = -\frac{\alpha_{13}\alpha_{23}}{\alpha_{16}\alpha_{26}}. \tag{3.5}$$

Let $\mathcal{E} \subset \mathbb{R}^6$ be the (non-empty) open subset where all of the previous conditions hold. Notice that condition (i) is satisfied by every 6-tuple $(\alpha_{12}, \alpha_{13}, \alpha_{16}, \alpha_{23}, \alpha_{26}, \alpha_{36}) \in \mathcal{E}$.

Since $(J_{F^*\psi})^t Q = -Q J_{F^*\psi}$, we determine all 6×6 invertible matrices $P = (p_{ij})$ whose entries satisfy the conditions (3.4) and for which

$$Z := (J_{F^*\psi})^t (P^t P) + (P^t P) J_{F^*\psi}$$

is the zero matrix. On \mathcal{E} , this boils down to solving a system of 17 equations in the unknowns p_{ij} under the constraints (3.4). The sub-system $\{Z_{ij} = 0 \mid i, j = 1, 2, 3, 6, i < j\}$ has the following solution

$$p_{11} = p_{66} \sqrt{-\frac{\alpha_{12}\alpha_{13}}{\alpha_{26}\alpha_{36}}}, \quad p_{22} = p_{66} \sqrt{\frac{\alpha_{12}\alpha_{23}}{\alpha_{16}\alpha_{36}}}, \quad p_{33} = p_{66} \sqrt{-\frac{\alpha_{13}\alpha_{23}}{\alpha_{16}\alpha_{26}}},$$

for any choice of $p_{66} > 0$. The positivity of the quantities in (3.5) together with conditions (3.4) ensure that the sub-system $\{Z_{i4} = 0 \mid i = 1, 2, 3, 6\}$ can be solved with respect to the unknowns p_{i4} , for $i = 1, 2, 3, 6$, and one has that the solution also solves the sub-system $\{Z_{i5} = 0 \mid i = 1, 2, 3, 6\}$. We are then left with the equations $Z_{44} = 0, Z_{45} = 0, Z_{55} = 0$. The equation $Z_{44} = 0$ has the following solution

$$p_{44} = \left(1 + \frac{\alpha_{16}\alpha_{23}}{\alpha_{12}\alpha_{36}} \right) p_{45},$$

and by the positivity of the quantities in (3.5) we must have $p_{45} > 0$. Finally, the equations $Z_{45} = 0 = Z_{55}$ hold if and only if

$$(\alpha_{12}\alpha_{36})^2 p_{55}^2 + \frac{\lambda(F^*\psi)}{4} p_{45}^2 = 0,$$

and we thus obtain

$$p_{55} = \frac{\sqrt{-\lambda(F^*\psi)}}{2|\alpha_{12}\alpha_{36}|} p_{45},$$

Summing up, when $(\alpha_{12}, \alpha_{13}, \alpha_{16}, \alpha_{23}, \alpha_{26}, \alpha_{36}) \in \mathcal{E}$, then all 6×6 matrices P satisfying the conditions (3.4) and $(J_{F^*\psi})^t(P^tP) + (P^tP)J_{F^*\psi} = 0$ constitute a family \mathcal{P} of matrices depending on two positive real parameters p_{66} and p_{45} , and on real parameters $p_{15}, p_{25}, p_{35}, p_{65}$.

Now, if Q was symmetric and positive definite, then there would exist a matrix $P \in \mathcal{P}$ such that $Q = P^tP$. In particular, the following identity should hold

$$QJ_{F^*\psi} = P^tPJ_{F^*\psi}. \tag{3.6}$$

Assume that $(\alpha_{12}, \alpha_{13}, \alpha_{16}, \alpha_{23}, \alpha_{26}, \alpha_{36}) \in \mathcal{E}$ is given. In the cases (A) and (B), which ensure that Q is symmetric, we consider the system of equations corresponding to the matrix identity (3.6), where P is any matrix in \mathcal{P} . This consists in 15 equations in the unknowns $\alpha_{15}, \alpha_{24}, \alpha_{34}, \alpha_{35}, \alpha_{46}, \alpha_{56}, a_4, a_6, a_7, a_8$, and a_3 , in case A), or a_5 , in case B). With the aid of a computer algebra system, it is possible to show that there are no values of the unknowns for which the system can be solved. This gives a contradiction.

For the reader’s convenience, we now describe the relevant steps leading to the conclusion. Let $M := QJ_{F^*\psi} - P^tPJ_{F^*\psi}$. In case (A), the entries M_{ij} , for $i, j \in \{1, 2, 3, 6\}$ with $i < j$, are all proportional to the same polynomial and one has $M_{ij} = 0$ if and only if

$$a_3 = \frac{p_{66}^2}{\alpha_{16}\alpha_{26}\alpha_{36}} \frac{\alpha_{12}\alpha_{13}\alpha_{26}\alpha_{36} + \alpha_{12}\alpha_{16}\alpha_{23}\alpha_{36} + \alpha_{13}\alpha_{16}\alpha_{23}\alpha_{26}}{\lambda(F^*\psi)}.$$

We now consider the equations $M_{i4} = 0$ and $M_{i5} = 0$, $i \in \{1, 2, 3, 6\}$, which can be seen as a linear system of eight equations in the unknowns a_4, a_6, a_7 and a_8 . This system admits a unique solution on \mathcal{E} . We are left with the equation $M_{45} = 0$. On \mathcal{E} , M_{45} can be seen as a second degree polynomial in the unknowns $\alpha_{15}, \alpha_{24}, \alpha_{34}, \alpha_{35}, \alpha_{46}, \alpha_{56}$. We claim that M_{45} is always non-zero. Thinking of M_{45} as a polynomial in α_{15} , we first compute its discriminant Δ_1 , which is a second degree polynomial in the remaining unknowns. To show the claim, we think of Δ_1 as a second degree polynomial in α_{24} and we prove that it is always negative. First, we observe that the leading coefficient of Δ_1 is negative on \mathcal{E} . Indeed, its sign is determined by

$$-\alpha_{16}\alpha_{23}(\alpha_{12}\alpha_{36} - \alpha_{13}\alpha_{26} + \alpha_{16}\alpha_{23}),$$

and the quantity inside the brackets has the same sign of $\alpha_{16}\alpha_{23}$ on \mathcal{E} . The discriminant Δ_2 of Δ_1 can be seen as a quadratic form in $p_{15}, p_{25}, p_{35}, p_{45}, p_{65}$, which is negative definite on \mathcal{E} since $p_{45} > 0$. Therefore, $\Delta_1 < 0$. An analogous discussion shows the thesis also in case B). □

3.2 The indecomposable case

We now consider six-dimensional *indecomposable* unimodular solvable non-nilpotent Lie algebras. Their structure equations with respect to a suitable basis (X_1, \dots, X_6) can be

found in the tables A.9–A.19 of [2], where the Lie algebras are gathered together according to their nilradical. Notice that there are a few misprints in [2] that must be corrected as follows:

- Lie algebra $\mathfrak{g}_{6,55}^{-4}$ of Table A.12: $[X_3, X_6] = 5X_3, [X_5, X_6] = -4X_5$;
- Lie algebra $\mathfrak{g}_{6,83}^{0,l}$ of Table A.15: $[X_2, X_6] = lX_2 + X_3$;
- Lie algebra $\mathfrak{g}_{6,135}^{0,-4}$ of Table A.19: $[X_3, X_5] = X_2$.

In the following, we will keep on denoting the basis of a Lie algebra by (e_1, \dots, e_6) and the corresponding dual basis by (e^1, \dots, e^6) .

The next general result rules out the Lie algebras listed in tables A.9, A.10, A.15 of [2].

Proposition 3.9 *Let \mathfrak{s} be a six-dimensional unimodular solvable non-nilpotent Lie algebra, and denote by \mathfrak{n} its nilradical. Then, every exact 3-form $\hat{\alpha} \in \Lambda^3 \mathfrak{s}^*$ is not stable, whenever \mathfrak{n} is isomorphic to one of the Lie algebras $\mathbb{R}^5, \mathfrak{g}_{3,1} \oplus \mathbb{R}^2, \mathfrak{g}_{5,4}$, where $\mathfrak{g}_{3,1}$ and $\mathfrak{g}_{5,4}$ denote the three-dimensional and the five-dimensional Heisenberg Lie algebra, respectively.*

Proof We have $\mathfrak{s} \cong \mathfrak{n} \rtimes_S \mathbb{R}$, where $S \in \text{Der}(\mathfrak{n})$ is a derivation of the five-dimensional nilpotent ideal \mathfrak{n} . We choose a basis (e_1, \dots, e_6) of \mathfrak{s} so that $\mathfrak{n} = \langle e_1, \dots, e_5 \rangle$ and $\mathbb{R} = \langle e_6 \rangle$. Then, we can write every 2-form $\alpha \in \Lambda^2 \mathfrak{s}^*$ with respect to the dual basis (e^1, \dots, e^6) as follows

$$\alpha = \alpha_{\mathfrak{n}} + \alpha' \wedge e^6,$$

where $\alpha_{\mathfrak{n}} \in \Lambda^2 \mathfrak{n}^*$ and $\alpha' \in \mathfrak{n}^*$. We will prove that $\hat{\alpha}$ is never stable by showing the existence of a non-zero vector $x \in \mathfrak{n}$ for which the 2-form $\iota_x \hat{\alpha}$ has rank at most two.

When \mathfrak{s} is almost abelian, namely $\mathfrak{n} \cong \mathbb{R}^5$, the Chevalley–Eilenberg differential of α is given by

$$\hat{\alpha} = S^* \alpha_{\mathfrak{n}} \wedge e^6.$$

Thus, for every non-zero vector $x \in \mathfrak{n}$ we have $\iota_x \hat{\alpha} = (\iota_x S^* \alpha_{\mathfrak{n}}) \wedge e^6$. Consequently, $\iota_x \hat{\alpha} \wedge \iota_x \hat{\alpha} = 0$ and the claim follows.

Let $d_{\mathfrak{n}}$ denote the Chevalley–Eilenberg differential of \mathfrak{n} . When $\mathfrak{n} \cong \mathfrak{g}_{3,1} \oplus \mathbb{R}^2$ or $\mathfrak{n} \cong \mathfrak{g}_{5,4}$, we can choose the basis (e^1, \dots, e^5) of \mathfrak{n}^* in such a way that $d_{\mathfrak{n}} e^k = 0$, for $k = 2, 3, 4, 5$, and $d_{\mathfrak{n}} e^1 = e^{23}$ when $\mathfrak{n} \cong \mathfrak{g}_{3,1} \oplus \mathbb{R}^2$ while $d_{\mathfrak{n}} e^1 = e^{24} + e^{35}$ when $\mathfrak{n} \cong \mathfrak{g}_{5,4}$. In both cases, we then obtain

$$\iota_{e_1} \hat{\alpha} = (\iota_{e_1} S^* \alpha_{\mathfrak{n}}) \wedge e^6,$$

and from this we see that $\iota_{e_1} \hat{\alpha}$ has rank at most two. □

For every Lie algebra \mathfrak{s} not isomorphic to one of those considered in the previous proposition, we first have to compute the expression of the generic derivation $D \in \text{Der}(\mathfrak{s})$, consider the extension $\mathfrak{g} = \mathfrak{s} \rtimes_D \mathbb{R}$, and determine for which derivations D it is strongly unimodular. Then, we have to show that there are no pairs (ω, ψ) of the form (3.2) defining an $SU(3)$ -structure on \mathfrak{s} . We shall deal with this problem in the next propositions.

Proposition 3.10 *The indecomposable unimodular solvable non-nilpotent Lie algebras listed in [2, Table A.11] (nilradical $\mathfrak{g}_{4,1} \oplus \mathbb{R}$) cannot occur as an ideal of a strongly unimodular solvable Lie algebra admitting exact G_2 -structures.*

Proof Let \mathfrak{s} denote one of the Lie algebras listed in Table A.11 of [2], and let (e_1, \dots, e_6) be the basis of \mathfrak{s} for which the structure equations are those given in that table. The nilradical of \mathfrak{s} is $\mathfrak{n} = \langle e_1, e_2, e_3, e_4, e_5 \rangle \cong \mathfrak{g}_{4,1} \oplus \mathbb{R}$, and its descending central series is

$$\mathfrak{n}^0 = \mathfrak{n}, \quad \mathfrak{n}^1 = \langle e_1, e_2 \rangle, \quad \mathfrak{n}^2 = \langle e_2 \rangle, \quad \mathfrak{n}^i = \{0\}, i \geq 3.$$

Let D be a generic derivation of \mathfrak{s} , and consider the extension $\mathfrak{g} = \mathfrak{s} \rtimes_D \mathbb{R}$. Then, a computation shows that \mathfrak{g} is strongly unimodular only when $e_2 \in \ker D$ and the image of the restriction of D to the subspace $\langle e_1, e_3 \rangle \subset \mathfrak{s}$ is $\langle e_2 \rangle$. Now, we determine the expression of the almost complex structure J induced by a generic negative stable exact 3-form $\hat{d}\alpha$ and the volume form e^{123456} , and we observe that $Je_2 \in \langle e_1, e_2, e_3 \rangle$. Since $[e_1, e_2] = 0 = [e_2, e_3]$, we see that

$$\omega(e_2, Je_2) = (\hat{d}\beta + D^*\alpha)(e_2, Je_2) = -\beta([e_2, Je_2]) - \alpha(De_2, Je_2) - \alpha(e_2, DJe_2) = 0.$$

Since the previous discussion holds for every $\alpha \in \Lambda^2 \mathfrak{s}^*$ and $\beta \in \mathfrak{s}^*$ such that $\omega^3 \neq 0$ and $\hat{d}\alpha$ is stable, the thesis follows. □

Proposition 3.11 *The six-dimensional indecomposable unimodular solvable non-nilpotent Lie algebras listed in Table A.12 (nilradical $\mathfrak{g}_{5,1}$), A.13 (nilradical $\mathfrak{g}_{5,2}$), A.14 (nilradical $\mathfrak{g}_{5,3}$), A.16 (nilradical $\mathfrak{g}_{5,5}$), A.18 and A.19 (nilradical $\mathfrak{g}_{3,1} \oplus \mathbb{R}$) of [2] cannot occur as an ideal of a strongly unimodular solvable Lie algebra admitting exact G_2 -structures.*

Proof Among the Lie algebras mentioned in the statement, $\mathfrak{g}_{6,54}^{2+2l,l}$, $\mathfrak{g}_{6,70}^{4p,p}$ and $\mathfrak{g}_{6,65}^{4l,l}$ (all with nilradical $\mathfrak{g}_{5,1}$) are the only ones depending on a real parameter. When \mathfrak{s} is one of these Lie algebras, then $\mathfrak{g} = \mathfrak{s} \rtimes_D \mathbb{R}$ is strongly unimodular only for a certain value of the parameter. As we will see, this fact will be relevant to rule out two of them, namely $\mathfrak{g}_{6,54}^{2+2l,l}$ and $\mathfrak{g}_{6,70}^{4p,p}$. Thus, we begin assuming that \mathfrak{s} is any of the Lie algebras considered in the statement, with the exception of $\mathfrak{g}_{6,54}^{2+2l,l}$ and $\mathfrak{g}_{6,70}^{4p,p}$.

For each Lie algebra \mathfrak{s} , we consider the basis (e_1, \dots, e_6) for which the structure equations are those given in [2], and the generic derivation $D \in \text{Der}(\mathfrak{s})$ such that $\mathfrak{g} = \mathfrak{s} \rtimes_D \mathbb{R}$ is strongly unimodular. We fix the volume form e^{123456} , and we compute the almost complex structure J induced by the generic negative stable 3-form $\psi = \hat{d}\alpha$. Then, arguing as in the proof of Proposition 3.10, we consider the generic non degenerate 2-form $\omega = \hat{d}\beta + D^*\alpha$, and we observe that the properties of D and J allow us to single out (at least) one basis vector e_i in the nilradical of \mathfrak{s} such that $\omega(e_i, Je_i) = 0$. For the sake of clarity, we give the details in the case when \mathfrak{s} is isomorphic to the Lie algebra $\mathfrak{g}_{6,55}^{-4}$. The discussion in the remaining cases is similar, but it depends both on the specific expressions of D and J one obtains and on the structure equations of the nilradical of \mathfrak{s} .

The nilradical of $\mathfrak{g}_{6,55}^{-4}$ is $\mathfrak{g}_{5,1}$, and we can choose a basis (e_1, \dots, e_5) of it in such a way that the only non-zero brackets are $[e_3, e_5] = e_1$ and $[e_4, e_5] = e_2$. Now, a generic derivation D of $\mathfrak{s} = \mathfrak{g}_{6,55}^{-4}$ for which $\mathfrak{g} = \mathfrak{s} \rtimes_D \mathbb{R}$ is strongly unimodular satisfies $De_3 \in \langle e_1 \rangle$ and $\langle e_1, e_2 \rangle \subset \ker D$. Moreover, the almost complex structure J induced by a generic negative

stable exact 3-form $\hat{d}\alpha$ and the volume form e^{123456} satisfies $Je_1 \in \langle e_1, e_2, e_3 \rangle$. Therefore, we have

$$\omega(e_1, Je_1) = -\beta([e_1, Je_1]) - \alpha(De_1, Je_1) - \alpha(e_1, DJe_1) = 0,$$

and the claim follows.

We are then left with the Lie algebras $\mathfrak{g}_{6,70}^{4p,p}$ and $\mathfrak{g}_{6,54}^{2+2l,l}$, where a different type of approach is needed in order to rule them out.

If $\mathfrak{g} = \mathfrak{g}_{6,70}^{4p,p}$, we consider the basis (e^1, \dots, e^6) of \mathfrak{g}^* for which the structure equations are the following

$$(e^{26} - p e^{16} - e^{35}, -e^{16} - p e^{26} - e^{45}, 3p e^{36} + e^{46}, 3p e^{46} - e^{36}, -4p e^{56}, 0), \quad p \in \mathbb{R}.$$

The nilradical of \mathfrak{g} is $\mathfrak{n} = \langle e_1, e_2, e_3, e_4, e_5 \rangle \cong \mathfrak{g}_{5,1}$, and the only non-trivial terms in its descending central series are

$$\mathfrak{n}^0 = \mathfrak{n}, \quad \mathfrak{n}^1 = \langle e_1, e_2 \rangle.$$

Moreover, the generic derivation of \mathfrak{g} has the following matrix representation

$$D = \begin{pmatrix} a_1 & a_2 & a_3 & 0 & a_4 & a_5 \\ -a_2 & a_1 & 0 & a_3 & a_6 & a_7 \\ 0 & 0 & a_8 & a_2 & 0 & -3pa_4 - a_6 \\ 0 & 0 & -a_2 & a_8 & 0 & -3pa_6 + a_4 \\ 0 & 0 & 0 & 0 & -a_8 + a_1 & -4pa_3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

In order to obtain a strongly unimodular extension $\mathfrak{g} = \mathfrak{g} \rtimes_D \mathbb{R}$, we must have $a_1 = 0 = a_8$ and $0 = \text{tr}(\text{ad}_{e_6}|_{\mathfrak{n}^1}) = 2p$ (notice that the nilradical of \mathfrak{g} coincides with \mathfrak{n} when $a_1 = 0 = a_8$). We can then conclude observing that the quartic polynomial associated with the generic exact 3-form $\hat{d}\alpha$ on \mathfrak{g} is

$$\begin{aligned} \lambda(\hat{d}\alpha) &= 4\alpha_{12}^2 \left((\alpha_{13} - \alpha_{24})^2 + (\alpha_{14} + \alpha_{23})^2 \right) \\ &\quad - 16\alpha_{12}^2 p^2 (3\alpha_{12}\alpha_{34} + \alpha_{13}\alpha_{24} - \alpha_{14}^2 + \alpha_{14}\alpha_{23} - \alpha_{23}^2), \end{aligned}$$

and thus $\lambda(\hat{d}\alpha) \geq 0$ when $p = 0$.

Finally, we consider $\mathfrak{g} = \mathfrak{g}_{6,54}^{2+2l,l}$, whose structure equations with respect to a basis (e^1, \dots, e^6) of \mathfrak{g}^* are the following

$$(-e^{16} - e^{35}, -l e^{26} - e^{45}, (1 + 2l)e^{36}, (2 + l)e^{46}, -(2 + 2l)e^{56}, 0), \quad l \in \mathbb{R}.$$

As in the previous case, the nilradical of \mathfrak{g} is $\mathfrak{n} = \langle e_1, e_2, e_3, e_4, e_5 \rangle \cong \mathfrak{g}_{5,1}$. Let $D \in \text{Der}(\mathfrak{g})$ be a generic derivation and consider the extension $\mathfrak{g} = \mathfrak{g} \rtimes_D \mathbb{R}$. Requiring \mathfrak{g} to be strongly unimodular gives

$$D = \begin{pmatrix} a_1 & 0 & a_2 & 0 & a_3 & a_4 \\ 0 & -a_1 & 0 & a_2 & a_6 & a_7 \\ 0 & 0 & a_1 & 0 & 0 & a_3(-1-2l) \\ 0 & 0 & 0 & -a_1 & 0 & a_6(-2-l) \\ 0 & 0 & 0 & 0 & 0 & a_2(-2-2l) \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $0 = \text{tr}(\text{ad}_{e_6}|_{\mathfrak{n}_1}) = l + 1$. We now compute the generic 2-form $\omega = \hat{\alpha} \beta - D^* \alpha$ and the generic 3-form $\psi = \hat{\alpha} \alpha$, and we consider all possible values of the parameters a_r, α_j, β_k for which $\omega^3 \neq 0$ and

$$\lambda(\psi) = 16\alpha_{12}^2 \alpha_{24} \alpha_{13} < 0.$$

We then determine the almost complex structure J induced by (ω, ψ) , and we conclude observing that $\omega(e_i, J e_i) = 0$, for $i = 1, 2, 3, 4$. Notice that, when $l + 1 \neq 0$, the Lie algebra \mathfrak{g} is not strongly unimodular and $\omega(e_i, J e_i)$ is proportional to $l + 1$, for $i = 1, 2, 3, 4$. \square

We still have to examine the indecomposable Lie algebras listed in [2, Table A.17], namely those with abelian nilradical \mathbb{R}^4 . In the next result, we rule out all of them but $\mathfrak{g}_{6,101}^{a,b,-1-a,-1-b}$. This last Lie algebra will be considered in Proposition 3.13.

Proposition 3.12 *Let \mathfrak{s} be a Lie algebra that is isomorphic to one of those listed in Table A.17 of [2] but $\mathfrak{g}_{6,101}^{a,b,-1-a,-1-b}$. Then, \mathfrak{s} cannot occur as an ideal of a strongly unimodular solvable Lie algebra admitting exact G_2 -structures.*

Proof Assume first that \mathfrak{s} is not isomorphic to one of $\mathfrak{g}_{6,114}^{a,-1,-a/2}, \mathfrak{g}_{6,115}^{-1,b,c,-c}, \mathfrak{g}_{6,118}^{0,b,-1}$. Let (e_1, \dots, e_6) be a basis of \mathfrak{s} for which the structure equations are those given in Table A.17 of [2], and consider the generic derivation $D \in \text{Der}(\mathfrak{s})$ such that $\mathfrak{g} = \mathfrak{s} \rtimes_D \mathbb{R}$ is strongly unimodular. Then, arguing as in the proof of Proposition 3.10, we obtain that for every non-degenerate 2-form $\omega = \hat{\alpha} \beta + D^* \alpha$ and every negative stable 3-form $\psi = \hat{\alpha} \alpha$ there exists (at least) one basis vector e_i in the nilradical \mathbb{R}^4 of \mathfrak{s} such that $\omega(e_i, J e_i) = 0$. As before, this depends on the expressions of D and J in each case under exam.

For the remaining Lie algebras of Table A.17, we obtain different types of contradictions. If \mathfrak{s} is one of $\mathfrak{g}_{6,115}^{-1,b,c,-c}, \mathfrak{g}_{6,118}^{0,b,-1}$, we can proceed as above and conclude observing that

$$\omega(e_3, J e_3) \omega(e_4, J e_4) \leq 0.$$

Finally, if $\mathfrak{s} = \mathfrak{g}_{6,114}^{a,-1,-a/2}$, we consider the basis (e^1, \dots, e^6) of \mathfrak{s}^* for which the structure equations are

$$\mathfrak{g}_{6,114}^{a,-1,-a/2} = \left(a e^{15} - e^{16}, e^{26}, -\frac{a}{2} e^{35} - e^{45}, e^{35} + \frac{a}{2} e^{45}, 0, 0 \right), \quad a \neq 0.$$

Then, we choose the volume form e^{123456} , and we compute the quartic polynomial $\lambda(\hat{\alpha} \alpha)$, for a generic $\alpha = \sum_{1 \leq i < j \leq 6} \alpha_{ij} e^{ij} \in \Lambda^2 \mathfrak{s}^*$, obtaining

$$\lambda(\hat{\alpha} \alpha) = (a(\alpha_{13} \alpha_{24} + \alpha_{14} \alpha_{23}) - 2\alpha_{13} \alpha_{23} - 2\alpha_{14} \alpha_{24})^2 + 4(-\alpha_{13} \alpha_{24} + \alpha_{14} \alpha_{23})^2 \geq 0.$$

\square

Proposition 3.13 *The indecomposable Lie algebra $\mathfrak{g}_{6,101}^{a,b,-1-a,-1-b}$ cannot occur as an ideal of a strongly unimodular solvable Lie algebra admitting exact G_2 -structures.*

Proof Let $\mathfrak{g} = \mathfrak{g}_{6,101}^{a,b,-1-a,-1-b}$, and consider the basis $\mathcal{B}^* = (e^1, \dots, e^6)$ of \mathfrak{g}^* for which the structure equations are

$$(ae^{15} + be^{16}, -(a + 1)e^{25} - (b + 1)e^{26}, e^{36}, e^{45}, 0, 0),$$

with $ab \neq 0$ and $(-a - 1)^2 + (-b - 1)^2 \neq 0$. Let $\mathcal{B} = (e_1, \dots, e_6)$ be the basis of \mathfrak{g} with dual basis \mathcal{B}^* . We will study the cases $a = -1$ and $a \neq -1$ separately.

Assume that $a = -1$. Then, the nilradical of \mathfrak{g} is the abelian ideal $\mathfrak{n} = \langle e_1, e_2, e_3, e_4 \rangle$, and the generic derivation $D \in \text{Der}(\mathfrak{g})$ for which $\mathfrak{g} = \mathfrak{g} \rtimes_D \mathbb{R}$ is strongly unimodular has the following matrix representation with respect to \mathcal{B}

$$D = \begin{pmatrix} a_1 & 0 & 0 & 0 & a_2 & -a_2 b \\ 0 & a_3 & 0 & 0 & 0 & a_4 \\ 0 & 0 & a_5 & 0 & 0 & a_6 \\ 0 & 0 & 0 & -a_5 - a_3 - a_1 & a_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $a_i \in \mathbb{R}$.

Let $\alpha = \sum_{1 \leq i < j \leq 6} \alpha_{ij} e^{ij}$ and $\beta = \sum_{i=1}^6 \beta_i e^i$ be a generic 2-form and a generic 1-form on \mathfrak{g} , respectively. Then, the forms (ω, ψ) given by (3.2) have the following expressions

$$\begin{aligned} \omega &= -\alpha_{12}(a_1 + a_3)e^{12} - \alpha_{13}(a_1 + a_5)e^{13} + \alpha_{14}(a_3 + a_5)e^{14} + (-a_1\alpha_{15} - a_8\alpha_{14} - \beta_1)e^{15} \\ &\quad + (b\beta_1 - a_1\alpha_{16} - a_4\alpha_{12} - a_6\alpha_{13})e^{16} - \alpha_{23}(a_3 + a_5)e^{23} + \alpha_{24}(a_1 + a_5)e^{24} \\ &\quad + (a_2\alpha_{12} - a_3\alpha_{25} - a_8\alpha_{24})e^{25} + (-ba_2\alpha_{12} - b\beta_2 - a_3\alpha_{26} - a_6\alpha_{23} - \beta_2)e^{26} \\ &\quad + \alpha_{34}(a_1 + a_3)e^{34} + (a_2\alpha_{13} - a_5\alpha_{35} - a_8\alpha_{34})e^{35} \\ &\quad + (-ba_2\alpha_{13} + a_4\alpha_{23} - a_5\alpha_{36} + \beta_3)e^{36} + (a_1\alpha_{45} + a_2\alpha_{14} + a_3\alpha_{45} + a_5\alpha_{45} + \beta_4)e^{45} \\ &\quad + (-ba_2\alpha_{15} - a_2\alpha_{16} + a_4\alpha_{25} + a_6\alpha_{35} - a_8\alpha_{46})e^{56} \\ &\quad + (-ba_2\alpha_{14} + a_1\alpha_{46} + a_3\alpha_{46} + a_4\alpha_{24} + a_5\alpha_{46} + a_6\alpha_{34})e^{46} \end{aligned}$$

and

$$\begin{aligned} \psi &= \alpha_{12}(e^{125} + e^{126}) + \alpha_{13}(e^{135} - (1 + b)e^{136}) - \alpha_{14}be^{146} + \alpha_{24}(-e^{245} + (1 + b)e^{246}) \\ &\quad + \alpha_{23}be^{236} - \alpha_{34}(e^{345} + e^{346}) + (-ba_{15} - \alpha_{16})e^{156} + \alpha_{25}(1 + b)e^{256} - \alpha_{35}e^{356} + \alpha_{46}e^{456}. \end{aligned}$$

Assume that $\lambda(\psi) < 0$, and let J be the almost complex structure induced by ψ and the orientation e^{123456} . We will show that there exists a nonzero vector $x \in \mathfrak{g}$ such that $\omega(x, Jx) = 0$. To prove this, we consider the nilpotent derivation $S \in \text{Der}(\mathfrak{g})$ with associated matrix

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & s_1 & -bs_1 \\ 0 & 0 & 0 & 0 & 0 & s_2 \\ 0 & 0 & 0 & 0 & 0 & s_3 \\ 0 & 0 & 0 & 0 & s_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and the automorphism $F = \exp(S)$. With similar computations as in the proof of Proposition 3.7, we see that there exist certain s_i , $1 \leq i \leq 4$, such that $F^*\omega(e_5, e_6) = 0$ and $J_{F^*\psi}(e_5) \in \langle e_5, e_6 \rangle$, where $J_{F^*\psi} = F^{-1} \circ J \circ F$. This follows comparing

$$\begin{aligned} F^*\psi &= \alpha_{12}(e^{125} + e^{126}) + \alpha_{13}(e^{135} - (1 + b)e^{136}) - b\alpha_{14}e^{146} + b\alpha_{23}e^{236} \\ &\quad + \alpha_{24}(-e^{245} + (1 + b)e^{246}) - \alpha_{34}(e^{345} + e^{346}) \\ &\quad - (b\alpha_{14}s_4 + b\alpha_{15} + \alpha_{12}s_2 + \alpha_{13}s_3 + \alpha_{16})e^{156} \\ &\quad + (1 + b)(-\alpha_{12}s_1 + \alpha_{24}s_4 + \alpha_{25})e^{256} \\ &\quad + (\alpha_{13}s_1 - \alpha_{34}s_4 - \alpha_{35})e^{356} + (b\alpha_{14}s_1 - \alpha_{24}s_2 - \alpha_{34}s_3 + \alpha_{46})e^{456}, \end{aligned}$$

with

$$\begin{aligned} F^*\omega(e_5, e_6) &= -a_1s_1(b\alpha_{15} + \alpha_{12}s_2 + \alpha_{13}s_3 + \alpha_{16}) + a_1s_4(\alpha_{46} - \alpha_{34}s_3 - \alpha_{24}s_2) \\ &\quad + a_3s_2(-\alpha_{12}s_1 + \alpha_{25}) + a_3s_4(b\alpha_{14}s_1 - \alpha_{34}s_3 + \alpha_{46}) \\ &\quad + a_5s_3(-\alpha_{13}s_1 + \alpha_{35}) + a_5s_4(b\alpha_{14}s_1 - \alpha_{24}s_2 + \alpha_{46}) \\ &\quad - a_2(b\alpha_{14}s_4 + b\alpha_{15} + \alpha_{12}s_2 + \alpha_{13}s_3 + b_5) + a_4(-\alpha_{12}s_1 + \alpha_{24}s_4 + \alpha_{25}) \\ &\quad + a_6(-\alpha_{13}s_1 + \alpha_{34}s_4 + \alpha_{35}) + a_8(-b\alpha_{14}s_1 + \alpha_{24}s_2 + \alpha_{34}s_3 - \alpha_{46}), \end{aligned}$$

and observing that $F^*\omega(e_5, e_6) = 0$ if the coefficients of e^{156} , e^{256} , e^{356} and e^{456} in the expression of $F^*\psi$ vanish. This last condition gives the linear system

$$\begin{cases} b\alpha_{14}s_4 + b\alpha_{15} + \alpha_{12}s_2 + \alpha_{13}s_3 = -\alpha_{16}, \\ -\alpha_{12}s_1 + \alpha_{24}s_4 = -\alpha_{25}, \\ \alpha_{13}s_1 - \alpha_{34}s_4 = \alpha_{35}, \\ b\alpha_{14}s_1 - \alpha_{24}s_2 - \alpha_{34}s_3 = -\alpha_{46}, \end{cases}$$

which has a unique solution $(\bar{s}_1, \bar{s}_2, \bar{s}_3, \bar{s}_4)$ under the constraint $\lambda(\psi) < 0$. The choice $(\bar{s}_1, \bar{s}_2, \bar{s}_3, \bar{s}_4)$ gives

$$\begin{aligned} F^*\psi &= \alpha_{12}(e^{125} + e^{126}) + \alpha_{13}(e^{135} - (1 + b)e^{136}) - \alpha_{14}be^{146} + \alpha_{23}be^{236} \\ &\quad + \alpha_{24}(-e^{245} + (1 + b)e^{246}) - \alpha_{34}(e^{345} + e^{346}), \end{aligned}$$

and $F^*\omega(e_5, e_6) = 0$. From the expression of $F^*\psi$, we observe that $J_{F^*\psi}e_5 \in \langle e_5, e_6 \rangle$. In detail,

$$\frac{1}{2} \sqrt{-\lambda(F^*\psi)} J_{F^*\psi}e_5 = -((b + 1)\alpha_{13}\alpha_{24} + \alpha_{12}\alpha_{34})e_5 + (\alpha_{12}\alpha_{34} - \alpha_{13}\alpha_{24})e_6.$$

We then have that $\omega(x, Jx) = 0$ for $x = Fe_5$.

Let us now focus on the case $a \neq -1$. The nilradical of \mathfrak{g} is still the abelian ideal $\mathfrak{n} = \langle e_1, e_2, e_3, e_4 \rangle$, and the generic derivation $D \in \text{Der}(\mathfrak{g})$ for which $\mathfrak{g} = \mathfrak{g} \rtimes_D \mathbb{R}$ is strongly unimodular must have the following matrix representation with respect to \mathcal{B}

$$D = \begin{pmatrix} a_1 & 0 & 0 & 0 & a_2 & \frac{a_2 b}{a+b} \\ 0 & a_3 & 0 & 0 & a_4 & \frac{a_4(1+b)}{a+1} \\ 0 & 0 & a_5 & 0 & 0 & a_6 \\ 0 & 0 & 0 & -a_5 - a_3 - a_1 & a_8 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where $a_i \in \mathbb{R}$.

Let

$$t_2 := -\frac{1}{a(1+a)}(a^2 b^2 \alpha_{13}^2 \alpha_{24}^2 - 2a^2 b^2 \alpha_{13} \alpha_{14} \alpha_{23} \alpha_{24} + a^2 b^2 \alpha_{14}^2 \alpha_{23}^2 - 2a^2 b \alpha_{12} \alpha_{13} \alpha_{24} \alpha_{34} - 2a^2 b \alpha_{12} \alpha_{14} \alpha_{23} \alpha_{34} + 2a^2 b \alpha_{13}^2 \alpha_{24}^2 - 2a^2 b \alpha_{13} \alpha_{14} \alpha_{23} \alpha_{24} + 2ab^2 \alpha_{12} \alpha_{13} \alpha_{24} \alpha_{34} + 2ab^2 \alpha_{12} \alpha_{14} \alpha_{23} \alpha_{34} - 2ab^2 \alpha_{13} \alpha_{14} \alpha_{23} \alpha_{24} + 2ab^2 \alpha_{14}^2 \alpha_{23}^2 + a^2 \alpha_{12}^2 \alpha_{34}^2 - 2a^2 \alpha_{12} \alpha_{13} \alpha_{24} \alpha_{34} + a^2 \alpha_{13}^2 \alpha_{24}^2 - 2ab \alpha_{12}^2 \alpha_{34}^2 + 2ab \alpha_{12} \alpha_{13} \alpha_{24} \alpha_{34} - 2ab \alpha_{12} \alpha_{14} \alpha_{23} \alpha_{34} - 2aba \alpha_{13} \alpha_{14} \alpha_{23} \alpha_{24} + b^2 \alpha_{12}^2 \alpha_{34}^2 + 2b^2 \alpha_{12} \alpha_{14} \alpha_{23} \alpha_{34} + b^2 \alpha_{14}^2 \alpha_{23}^2).$$

We will study the cases $t_2 \neq 0$ and $t_2 = 0$ separately.

If $t_2 \neq 0$, we claim that there exists a nonzero vector $x \in \mathfrak{g}$ such that $\omega(x, Jx) = 0$. The discussion is similar to the previous case. Here, we consider the nilpotent derivation S with matrix representation

$$S = \begin{pmatrix} 0 & 0 & 0 & 0 & s_1 & \frac{bs_1}{a+b} \\ 0 & 0 & 0 & 0 & s_2 & \frac{s_2(1+b)}{1+a} \\ 0 & 0 & 0 & 0 & 0 & s_3 \\ 0 & 0 & 0 & 0 & s_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

and the automorphism $F = \exp(S)$. Requiring the coefficients of $e^{156}, e^{256}, e^{356}, e^{456}$ in the expression of $F^* \psi$ to be zero gives the following linear system in the variables $s_i, 1 \leq i \leq 4$,

$$\begin{cases} \alpha_{13} a s_3 - \alpha_{14} b s_4 - \frac{1+b}{1+a} \alpha_{12} s_2 + \alpha_{12} s_2 = -a \alpha_{16} + b \alpha_{15}, \\ -\alpha_{23} s_3 a + \alpha_{24} s_4 b + \frac{b}{a} \alpha_{12} s_1 - \alpha_{12} s_1 - \alpha_{23} s_3 + \alpha_{24} s_4 = -(1+b) \alpha_{25} + (1+a) \alpha_{26}, \\ \frac{1+b}{1+a} a \alpha_{23} s_2 - \alpha_{23} b s_2 + \frac{1+b}{1+a} \alpha_{23} s_2 + \alpha_{13} s_1 \alpha_{34} s_4 = \alpha_{35}, \\ \frac{1+b}{1+a} a \alpha_{24} s_2 - \alpha_{24} s_2 b - \frac{b}{a} \alpha_{14} s_1 - \alpha_{24} s_2 - \alpha_{34} s_3 = -\alpha_{46}. \end{cases}$$

This system has a unique solution $(\bar{s}_1, \bar{s}_2, \bar{s}_3, \bar{s}_4)$ under the constraint $t_2 \neq 0$. In such a case, we have $F^* \omega(e_5, e_6) = 0$ and $J_{F^* \psi} e_5 \in \langle e_5, e_6 \rangle$, whence the claim follows.

If $t_2 = 0$, we claim that $g = \omega(\cdot, J \cdot)$ cannot be definite. To prove this, suppose by contradiction that g is positive (or negative) definite, and consider $g' := \sqrt{-\lambda(\psi)} g$. Then,

$$\begin{aligned} g'_{11} &= 4\alpha_{12} \alpha_{13} \alpha_{14} ((a_1 + a_3 + a_5)a - ba_5 + a_1), \\ g'_{33} &= 4\alpha_{13} \alpha_{23} \alpha_{34} ((a_1 + a_3 + a_5)a - ba_5 + a_1), \\ g'_{44} &= -4\alpha_{14} \alpha_{24} \alpha_{34} ((a_1 + a_3 + a_5)a - ba_5 + a_1). \end{aligned}$$

Therefore, $\alpha_{12}\alpha_{13}\alpha_{14}\alpha_{23}\alpha_{24}\alpha_{34} \neq 0$, and the polynomials $p_1 := \alpha_{12}\alpha_{34}$, $p_2 := \alpha_{14}\alpha_{23}$ and $p_3 := -\alpha_{13}\alpha_{24}$ must have the same sign.

Since $b(a + 1)\alpha_{23} \neq 0$, the condition $t_2 = 0$ can be seen as a second order equation in the variable α_{14} . We can solve it provided that $a(b + 1)\alpha_{13}\alpha_{24}(a - b)b_1\alpha_{34} \geq 0$, obtaining the solutions:

$$\alpha_{14}^\pm = \frac{1}{\alpha_{23}b(a + 1)} \left(a(b + 1)\alpha_{13}\alpha_{24} + (a - b)\alpha_{12}\alpha_{34} \pm 2\sqrt{a(b + 1)\alpha_{13}\alpha_{24}(a - b)\alpha_{12}\alpha_{34}} \right).$$

A case by case analysis ensures that the condition $a(b + 1)\alpha_{13}\alpha_{24}(a - b)b_1\alpha_{34} \geq 0$ is not compatible with the constraint on p_1, p_2, p_3 . To check this, assume that $a(b + 1)\alpha_{13}\alpha_{24}(a - b)\alpha_{12}\alpha_{34} \geq 0$ and that p_1 and p_3 have the same sign. Then, we can distinguish the four cases:

- (1) $\alpha_{12}\alpha_{34} > 0, \alpha_{13}\alpha_{24} < 0, a - b \leq 0, a(b + 1) \geq 0,$
- (2) $\alpha_{12}\alpha_{34} > 0, \alpha_{13}\alpha_{24} < 0, a - b \geq 0, a(b + 1) \leq 0,$
- (3) $\alpha_{12}\alpha_{34} < 0, \alpha_{13}\alpha_{24} > 0, a - b \leq 0, a(b + 1) \geq 0,$
- (4) $\alpha_{12}\alpha_{34} < 0, \alpha_{13}\alpha_{24} > 0, a - b \geq 0, a(b + 1) \leq 0.$

Under the assumptions of case (1),

$$\alpha_{14}^\pm \alpha_{23} = -\frac{1}{b(a + 1)} \left(\sqrt{-a(b + 1)\alpha_{13}\alpha_{24}} \mp \sqrt{-(a - b)\alpha_{12}\alpha_{34}} \right)^2.$$

If in addition p_1 and p_2 have the same sign, then $b(a + 1) < 0$. This condition is incompatible with the inequalities $a \leq b$ and $a(b + 1) \geq 0$, since these conditions imply $a \leq b$ and $-a \leq ab < -b$. In the remaining cases, we can proceed in a similar way. □

Appendix A.

In this appendix, we list the structure equations of all six-dimensional unimodular decomposable solvable non-nilpotent Lie algebras that exist up to isomorphism.

Remark A.1 As for the Lie algebras appearing in Table 1 and depending on some parameters, with the exception of $\mathfrak{g}_{5,9}^{p,-2-p}$, all of the corresponding simply connected Lie groups admit a lattice for certain values of the parameters, see [2].

Table 1 Isomorphism classes of six-dimensional unimodular decomposable solvable non-nilpotent Lie algebras

| \mathfrak{g} | $(de^1, de^2, de^3, de^4, de^5, de^6)$ |
|--|--|
| $\mathfrak{g}_{3,4}^{-1} \oplus \mathbb{R}^3$ | $(-e^{13}, e^{23}, 0, 0, 0, 0)$ |
| $\mathfrak{g}_{3,4}^{-1} \oplus \mathfrak{g}_{3,1}$ | $(-e^{13}, e^{23}, 0, -e^{56}, 0, 0)$ |
| $\mathfrak{g}_{3,4}^{-1} \oplus \mathfrak{g}_{3,4}^{-1}$ | $(-e^{13}, e^{23}, 0, -e^{46}, e^{56}, 0)$ |
| $\mathfrak{g}_{3,4}^{-1} \oplus \mathfrak{g}_{3,5}^0$ | $(-e^{13}, e^{23}, 0, -e^{56}, e^{46}, 0)$ |
| $\mathfrak{g}_{3,5}^0 \oplus \mathbb{R}^3$ | $(-e^{23}, e^{13}, 0, 0, 0, 0)$ |
| $\mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_{3,1}$ | $(-e^{23}, e^{13}, 0, -e^{56}, 0, 0)$ |
| $\mathfrak{g}_{3,5}^0 \oplus \mathfrak{g}_{3,5}^0$ | $(-e^{23}, e^{13}, 0, -e^{56}, e^{46}, 0)$ |
| $\mathfrak{g}_{4,2}^{-2} \oplus \mathbb{R}^2$ | $(-2e^{14}, e^{24} + e^{34}, e^{34}, 0, 0)$ |
| $\mathfrak{g}_{4,5}^{p,-p-1} \oplus \mathbb{R}^2$ | $(-e^{14}, -pe^{24}, (p+1)e^{34}, 0, 0, 0), p \in [-\frac{1}{2}, 0)$ |
| $\mathfrak{g}_{4,6}^{-2p,p} \oplus \mathbb{R}^2$ | $(2pe^{14}, -pe^{24} - e^{34}, e^{24} - pe^{34}, 0, 0, 0), p > 0$ |
| $\mathfrak{g}_{4,8}^{-1} \oplus \mathbb{R}^2$ | $(-e^{23}, -e^{24}, e^{34}, 0, 0, 0)$ |
| $\mathfrak{g}_{4,9}^0 \oplus \mathbb{R}^2$ | $(-e^{23}, -e^{34}, e^{24}, 0, 0, 0)$ |
| $\mathfrak{g}_{5,7}^{p,q,r} \oplus \mathbb{R}$ | $(-e^{15}, -pe^{25}, -qe^{35}, -re^{45}, 0, 0), -1 \leq r \leq q \leq p \leq 1, pqr \neq 0, p+q+r = -1$ |
| $\mathfrak{g}_{5,8}^{-1} \oplus \mathbb{R}$ | $(-e^{25}, 0, -e^{35}, e^{45}, 0, 0, 0)$ |
| $\mathfrak{g}_{5,9}^{p,-2-p} \oplus \mathbb{R}$ | $(-e^{15} - e^{25}, -e^{25}, -pe^{35}, (2+p)e^{45}, 0, 0), p \geq -1$ |
| $\mathfrak{g}_{5,11}^{-3} \oplus \mathbb{R}$ | $(-e^{15} - e^{25}, -e^{25} - e^{35}, -e^{35}, 3e^{45}, 0, 0)$ |
| $\mathfrak{g}_{5,13}^{-1-2q,q,r} \oplus \mathbb{R}$ | $(-e^{15}, (1+2q)e^{25}, -qe^{35} - re^{45}, re^{35} - qe^{45}, 0, 0), q \in [-1, 0], q \neq -\frac{1}{2}, r \neq 0$ |
| $\mathfrak{g}_{5,14}^0 \oplus \mathbb{R}$ | $(-e^{25}, 0, -e^{45}, e^{35}, 0, 0)$ |
| $\mathfrak{g}_{5,15}^{-1} \oplus \mathbb{R}$ | $(-e^{15} - e^{25}, -e^{25}, e^{35} - e^{45}, e^{45}, 0, 0)$ |
| $\mathfrak{g}_{5,16}^{-1,q} \oplus \mathbb{R}$ | $(-e^{15} - e^{25}, -e^{25}, e^{35} - qe^{45}, qe^{35} + e^{45}, 0, 0), q \neq 0$ |
| $\mathfrak{g}_{5,17}^{p,-p,r} \oplus \mathbb{R}$ | $(-pe^{15} - e^{25}, e^{15} - pe^{25}, pe^{35} - re^{45}, re^{35} + pe^{45}, 0, 0), r \neq 0$ |
| $\mathfrak{g}_{5,18}^0 \oplus \mathbb{R}$ | $(-e^{25} - e^{35}, e^{15} - e^{45}, -e^{45}, e^{35}, 0, 0)$ |
| $\mathfrak{g}_{5,19}^{p,-2p-2} \oplus \mathbb{R}$ | $(-e^{23} - (1+p)e^{15}, -e^{25}, -pe^{35}, (2p+2)e^{45}, 0, 0), p \neq -1$ |
| $\mathfrak{g}_{5,20}^{-1} \oplus \mathbb{R}$ | $(-e^{23} - e^{45}, -e^{25}, e^{35}, 0, 0, 0)$ |
| $\mathfrak{g}_{5,23}^{-4} \oplus \mathbb{R}$ | $(-e^{23} - 2e^{15}, -e^{25}, -e^{25} - e^{35}, 4e^{45}, 0, 0)$ |
| $\mathfrak{g}_{5,25}^{4,4p} \oplus \mathbb{R}$ | $(-e^{23} - 2pe^{15}, -pe^{25} + e^{35}, -e^{25} - pe^{35}, 4pe^{45}, 0, 0), p \neq 0$ |
| $\mathfrak{g}_{5,26}^{0,\epsilon} \oplus \mathbb{R}$ | $(-e^{23} - \epsilon e^{45}, e^{35}, -e^{25}, 0, 0, 0), \epsilon = \pm 1$ |
| $\mathfrak{g}_{5,28}^{-3/2} \oplus \mathbb{R}$ | $(-e^{23} + \frac{1}{2}e^{15}, \frac{3}{2}e^{25}, -e^{35}, -e^{35} - e^{45}, 0, 0)$ |
| $\mathfrak{g}_{5,30}^{-4/3} \oplus \mathbb{R}$ | $(-e^{24} - \frac{2}{3}e^{15}, -e^{34} + \frac{1}{3}e^{25}, \frac{4}{3}e^{35}, -e^{45}, 0, 0)$ |
| $\mathfrak{g}_{5,33}^{-1,-1} \oplus \mathbb{R}$ | $(-e^{14}, -e^{25}, e^{34} + e^{35}, 0, 0, 0)$ |
| $\mathfrak{g}_{5,35}^{-2,0} \oplus \mathbb{R}$ | $(2e^{14}, -e^{24} - e^{35}, e^{25} - e^{34}, 0, 0, 0)$ |

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