



On a class of Cheeger inequalities

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Abstract

We study a general version of the Cheeger inequality by considering the shape functional $\mathcal{F}_{p,q}(\Omega) = \lambda_p^{1/p}(\Omega) / \lambda_q^{1/q}(\Omega)$. The infimum and the supremum of $\mathcal{F}_{p,q}$ are studied in the class of all domains Ω of \mathbb{R}^d and in the subclass of convex domains. In the latter case the issue concerning the existence of an optimal domain for $\mathcal{F}_{p,q}$ is discussed.

Keywords Cheeger constant · Principal eigenvalue · Shape optimization · p -Laplacian

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1 Introduction

The starting point of this research is the celebrated *Cheeger inequality*:

$$\frac{\lambda(\Omega)}{h^2(\Omega)} \geq \frac{1}{4}, \quad (1.1)$$

here $\lambda(\Omega)$ denotes the first eigenvalue of the Laplace operator $-\Delta$ on the open set Ω , with Dirichlet boundary conditions, and $h(\Omega)$ denotes the *Cheeger constant*

$$h(\Omega) = \inf \left\{ \frac{P(E)}{|E|} : E \Subset \Omega, |E| > 0 \right\}, \quad (1.2)$$

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where the symbol $E \Subset \Omega$ indicates that the closure of E is contained in Ω . Here $P(E)$ denotes the *perimeter* of E in the De Giorgi sense, and $|E|$ the Lebesgue measure of E . Equivalently $h(\Omega)$ can be defined through the expression

$$h(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla u| \, dx}{\int_{\Omega} |u| \, dx} : u \in C_c^\infty(\Omega) \setminus \{0\} \right\}.$$

With some additional regularity assumption on Ω , in (1.2) the infimum can be equivalently evaluated on the whole class of subsets $E \subset \Omega$. For instance, it is enough to require that Ω coincides with its essential interior; we refer the reader to [16] and [19] for a survey on the Cheeger constant. We recall that if Ω is a ball of radius r in \mathbb{R}^d we have $h(\Omega) = d/r$.

In this paper we consider, for every $1 < p < +\infty$, the p -Laplace operator

$$-\Delta_p u = -\operatorname{div} (|\nabla u|^{p-2} \nabla u)$$

and the corresponding principal eigenvalue

$$\lambda_p(\Omega) = \inf \left\{ \frac{\int_{\Omega} |\nabla u|^p \, dx}{\int_{\Omega} |u|^p \, dx} : u \in C_c^\infty(\Omega) \setminus \{0\} \right\}. \tag{1.3}$$

Any minimizer of (1.3) solves, in the weak sense, the Dirichlet problem:

$$\begin{cases} -\Delta_p u = \lambda |u|^{p-2} u & \text{in } \Omega, \\ u \in W_0^{1,p}(\Omega); \end{cases}$$

The following properties are well-known:

- the *monotonicity property* with respect to the sets' inclusion, namely

$$\lambda_p(\Omega) \leq \lambda_p(\Omega'), \quad h(\Omega) \leq h(\Omega'), \quad \text{if } \Omega' \subset \Omega; \tag{1.4}$$

- the *scaling property*

$$\lambda_p(t\Omega) = t^{-p} \lambda_p(\Omega), \quad h(t\Omega) = t^{-1} h(\Omega), \quad \text{for all } t > 0; \tag{1.5}$$

- the *asymptotics*

$$\lim_{p \rightarrow +\infty} \lambda_p^{1/p}(\Omega) = \rho^{-1}(\Omega), \quad \lim_{p \rightarrow 1^+} \lambda_p(\Omega) = h(\Omega), \tag{1.6}$$

where $\rho(\Omega)$ denotes the so-called *inradius* of Ω , corresponding to the maximal radius of a ball contained in Ω (see [13] and [15]). Equivalently, $\rho(\Omega)$ can be defined as

$$\rho(\Omega) = \|d_\Omega\|_{L^\infty(\Omega)},$$

where d_Ω is the distance function from $\partial\Omega$

$$d_\Omega(x) = \inf \{ |x - y| : y \in \partial\Omega \}.$$

Taking into account (1.6) we define

$$\lambda_p^{1/p}(\Omega) = \begin{cases} h(\Omega) & \text{if } p = 1; \\ \rho(\Omega)^{-1} & \text{if } p = +\infty. \end{cases} \tag{1.7}$$

Inequality (1.1) can be then seen as a particular case of the more general inequality

$$\frac{\lambda_p^{1/p}(\Omega)}{\lambda_q^{1/q}(\Omega)} \geq \frac{q}{p} \quad \text{for every } 1 \leq q \leq p \leq +\infty, \tag{1.8}$$

that can be also rephrased as a monotonicity property:

the map $p \mapsto p\lambda_p^{1/p}(\Omega)$ is monotonically nondecreasing.

Although this result is already known for $1 < q \leq p < +\infty$ (see [17]), for the sake of completeness we recall its proof in Proposition 2.2.

Our goal is to study from the shape optimization point of view the functionals

$$\mathcal{F}_{p,q}(\Omega) = \frac{\lambda_p^{1/p}(\Omega)}{\lambda_q^{1/q}(\Omega)}.$$

From the properties listed above $\mathcal{F}_{p,q}$ is *scaling free*, that is,

$$\mathcal{F}_{p,q}(t\Omega) = \mathcal{F}_{p,q}(\Omega) \quad \text{for all } t > 0.$$

We consider the minimization/maximization problem of $\mathcal{F}_{p,q}$ in the classes

$$\begin{aligned} \mathcal{A}^d &= \{ \Omega \subset \mathbb{R}^d : \Omega \text{ open, } 0 < |\Omega| < +\infty \}, \\ \mathcal{A}_c^d &= \{ \Omega \in \mathcal{A}^d : \Omega \text{ convex} \}. \end{aligned}$$

For the sake of brevity we denote by $m_d(p, q), M_d(p, q)$ the quantities

$$m_d(p, q) = \inf_{\Omega \in \mathcal{A}^d} \mathcal{F}_{p,q}(\Omega), \quad M_d(p, q) = \sup_{\Omega \in \mathcal{A}^d} \mathcal{F}_{p,q}(\Omega).$$

Similarly, for the convex case, we use the notation

$$\bar{m}_d(p, q) = \inf_{\Omega \in \mathcal{A}_c^d} \mathcal{F}_{p,q}(\Omega), \quad \bar{M}_d(p, q) = \sup_{\Omega \in \mathcal{A}_c^d} \mathcal{F}_{p,q}(\Omega).$$

The study of the functionals $\mathcal{F}_{p,q}$ has been proposed in [20], where the author focused on the case $p = 2, q = 1$. Recently some developments have been made in [9], again in the case $p = 2, q = 1$.

The paper is organized as follows. In Section 2 we discuss the optimization problem in the class \mathcal{A}^d . In particular we prove that (1.8) becomes sharp when $d \rightarrow +\infty$ (Theorem 2.6) and we characterize the behavior of $M_d(p, q)$ in varying p, q , showing that it remains finite if and only if $q > d$ (Theorem 2.9). The optimization problems in the class \mathcal{A}_c^d are discussed in Section 3. After recalling some known estimates we prove that $\bar{M}_d(p, q)$ is always finite (Proposition 3.2) and that, in some cases, the minimization problems for $\mathcal{F}_{p,q}$ among planar convex open sets, admits a solution (Theorem 3.8). In Section 4, we collect some open problems that in our opinion can be interesting for future researches. At last, we conclude the paper with a small appendix, where we give self contained proofs of some known facts in shape optimization, which are useful for our purpose.

2 Optimization in \mathcal{A}^d

As it often happens in shape optimization, the one-dimensional case is simpler. Indeed in this case the functional $\mathcal{F}_{p,q}$ turns out to be constant. Hereinafter we denote by π_p the Poincaré–Sobolev constant

$$\pi_p = \lambda_p^{1/p}(0, 1). \tag{2.1}$$

Explicit computations, see for instance [14], show that, when $1 < p < +\infty$, it holds

$$\pi_p = 2\pi \frac{(p - 1)^{1/p}}{p \sin(\pi/p)}$$

which implies, in particular, $\pi_2 = \pi$; moreover (1.7) gives $\pi_1 = \pi_\infty = 2$.

Proposition 2.1 *Let $1 \leq q \leq p \leq +\infty$. Then, for every $\Omega \in \mathcal{A}^1$ we have*

$$\mathcal{F}_{p,q}(\Omega) = \frac{\pi_p}{\pi_q}.$$

Proof It is enough to notice that if $\Omega \in \mathcal{A}^1$ is the disjoint union of a family of open intervals $(\Omega_i)_{i \in I}$, then, for every $1 \leq p \leq +\infty$, we have

$$\lambda_p^{1/p}(\Omega) = \inf_{i \in I} \lambda_p^{1/p}(\Omega_i), \tag{2.2}$$

Indeed, when $p = +\infty$ (2.2) is straightforward by (1.7), while, when $1 \leq p < +\infty$, we notice that for every $u \in C_c^\infty(\Omega)$ it holds

$$\int_\Omega |\nabla u|^p dx \geq \sum_{i \in I} \int_{\Omega_i} |\nabla u|^p dx \geq \sum_{i \in I} \lambda_p(\Omega_i) \int_{\Omega_i} |u|^p dx \geq \inf_{i \in I} \lambda_p(\Omega_i) \sum_{i \in I} \int_{\Omega_i} |u|^p dx,$$

which implies

$$\lambda_p(\Omega) \geq \inf_{i \in I} \lambda_p(\Omega_i).$$

By (1.4), the latter inequality easily leads to (2.2). Taking into account that, by (1.5) and (2.1), we have

$$\inf_{i \in I} \lambda_p^{1/p}(\Omega_i) = \inf_{i \in I} |\Omega_i|^{-1} \pi_p,$$

we achieve the thesis. □

From now on we always assume $d \geq 2$. The next proposition provides a lower bound to $m_d(p, q)$ and generalizes inequality (1.1).

Proposition 2.2 *Let $\Omega \in \mathcal{A}^d$. Then, the function $p \mapsto \lambda_p^{1/p}(\Omega)$ is nondecreasing in $[1, +\infty]$. In particular we have*

$$m_d(p, q) \geq q/p. \tag{2.3}$$

Proof By (1.6) it is enough to consider the case $1 < q < p < \infty$. Let $u \in C_c^\infty(\Omega)$ and let $v = u^{p/q}$. Then, by Hölder inequality, we get

$$\begin{aligned} \lambda_q(\Omega) &\leq \frac{\int_\Omega |\nabla v|^q dx}{\int_\Omega |v|^q dx} = \left(\frac{p}{q}\right)^q \frac{\int_\Omega |\nabla u|^q |u|^{p-q} dx}{\int_\Omega |u|^p dx} \\ &\leq \left(\frac{p}{q}\right)^q \frac{(\int_\Omega |\nabla u|^p dx)^{q/p}}{\int_\Omega |u|^p dx} \left(\int_\Omega |u|^p dx\right)^{1-q/p} = \left(\frac{p}{q}\right)^q \left(\frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx}\right)^{q/p}. \end{aligned}$$

Since u is arbitrary we obtain

$$q\lambda_q^{1/q}(\Omega) \leq p\lambda_p^{1/p}(\Omega)$$

as required. □

In general, we do not expect the bound given in (2.3) to be sharp. For instance, as $p \rightarrow +\infty$, the right-hand side in (2.3) tends to zero, while it is easy to prove that the minimum of $\mathcal{F}_{\infty,q}$ is strictly positive and attained at any ball. Indeed, since any Ω contains a sequence of balls of radii converging to $\rho(\Omega)$, by (1.4), (1.5) and (1.7), we have

$$\lambda_q^{1/q}(\Omega) \leq \rho^{-1}(\Omega)\lambda_q^{1/q}(B_1) \quad \text{for every } 1 \leq q \leq +\infty, \tag{2.4}$$

which clearly implies

$$m_d(\infty, q) = \mathcal{F}_{\infty,q}(B_1), \quad \text{for every } 1 \leq q < +\infty;$$

here we denote by B_r^d the ball in \mathbb{R}^d of radius r centered at the origin, and we omit the dependence on d when there is no ambiguity.

Recently, by exploiting the fact that $\lambda_2(B_1^d) = j_{d/2-1,1}$, where $j_{d/2-1,1}$ denotes the first root of the d -th Bessel function of first kind, Ftouhi (see [9]) has noticed that

$$\lim_{d \rightarrow +\infty} m_d(2, 1) = 1/2. \tag{2.5}$$

Our next goal is to generalize the limit (2.5) to every p, q . With this aim we introduce the quantity

$$B(s, t) = \int_0^1 \tau^{s-1}(1 - \tau)^{t-1} d\tau$$

and recall that, in terms of the Euler’s function Γ , we have

$$B(s, t) = \frac{\Gamma(s)\Gamma(t)}{\Gamma(s + t)}. \tag{2.6}$$

Lemma 2.3 *Let $\Omega \in \mathcal{A}^d$ and $s \geq 1$. Then,*

$$\lambda_p(B_1) \leq s^p \left(\frac{\Gamma(sp + d + 1)\Gamma(sp - p + 1)}{\Gamma(sp + 1)\Gamma(sp + d - p + 1)} \right). \tag{2.7}$$

Proof Let $s \geq 1$ and $\phi(x) = (1 - |x|)^s$. Clearly $\phi \in W_0^{1,p}(B_1)$ and

$$\int_{B_1} |\phi(x)|^p dx = d\omega_d \int_0^1 (1 - t)^{sp} t^{d-1} dt = d\omega_d B(d, sp + 1).$$

Similarly we have

$$\int_{B_1} |\nabla\phi(x)|^p dx = d\omega_d s^p \int_0^1 (1 - t)^{(s-1)p} t^{d-1} dt = d\omega_d s^p B(d, sp - p + 1).$$

Now, using ϕ as a test function in (1.3), we obtain

$$\lambda_p(B_1) \leq s^p \left(\frac{B(d, sp - p + 1)}{B(d, sp + 1)} \right).$$

Finally, (2.7) follows from (2.6). □

Lemma 2.4 Let $1 \leq p < +\infty$, $L > 0$, and $\omega \in \mathcal{A}_{all}^{d-1}$. Denote by $\Omega_L = \omega \times (-L/2, L/2)$. Then

$$\begin{cases} \lambda_p(\omega) + \pi_p^p/L^p \leq \lambda_p(\Omega_L) \leq (\lambda_p^{2/p}(\omega) + \pi_p^2/L^2)^{p/2} & \text{if } p \geq 2, \\ (\lambda_p^{2/p}(\omega) + \pi_p^2/L^2)^{p/2} \leq \lambda_p(\Omega_L) \leq \lambda_p(\omega) + \pi_p^p/L^p & \text{if } p < 2. \end{cases} \tag{2.8}$$

In particular

$$\lim_{L \rightarrow +\infty} \lambda_p^{1/p}(\Omega_L) = \lambda_p^{1/p}(\omega). \tag{2.9}$$

Proof We denote by (x,y) the points in $\mathbb{R}^{d-1} \times \mathbb{R}$. Let $u \in C_c^\infty(\Omega_L)$, then for every $(x, y) \in \Omega_L$ we have

$$u(\cdot, y) \in C_c^\infty(\omega), \quad u(x, \cdot) \in C_c^\infty(-L/2, L/2).$$

If $p \geq 2$, using the super-additivity of the function $t \rightarrow t^{p/2}$ and Fubini theorem, we have

$$\begin{aligned} \int_{\Omega_L} |\nabla u|^p dx dy &= \int_{-L/2}^{L/2} \int_{\omega} (|\nabla_x u|^2 + |\partial_y u|^2)^{p/2} dx dy \\ &\geq \left(\lambda_p(\omega) + \frac{\pi_p^p}{L^p} \right) \int_{\Omega_L} |u|^p dx dy \end{aligned}$$

where the last inequality follows by (2.1) and (1.5). Similarly, if $p < 2$, using Fubini theorem together with the reverse Minkowski inequality

$$\|f + g\|_{L^{p/2}(\Omega_L)} \geq \|g\|_{L^{p/2}(\Omega_L)} + \|f\|_{L^{p/2}(\Omega_L)},$$

we obtain

$$\begin{aligned} \int_{\Omega_L} |\nabla u|^p dx dy &= \int_{-L/2}^{L/2} \int_{\omega} (|\nabla_x u|^2 + |\partial_y u|^2)^{p/2} dx dy \\ &\geq \left\{ \left(\int_{\Omega_L} |\nabla_x u|^p dx dy \right)^{2/p} + \left(\int_{\Omega_L} |\partial_y u|^p dx dy \right)^{2/p} \right\}^{p/2} \\ &\geq \left(\lambda_p(\omega)^{2/p} + \frac{\pi_p^2}{L^2} \right)^{p/2} \int_{\Omega_L} |u|^p dx dy. \end{aligned}$$

In both cases, the arbitrariness of u proves the left hand side inequalities in (2.8).

The upper estimates in (2.8) can be proved with the same argument, once chosen a suitable test function. More precisely, when $1 < p < +\infty$, we take $u(x)$ and $v_L(y)$ optimal functions, respectively, for $\lambda_p(\omega)$ and $\lambda_p(-L/2, L/2)$, both with unitary L^p norm, that is, (taking also (1.5) into account) we require:

$$\|\nabla u\|_{L^p(\omega)}^p = \lambda_p(\omega), \quad \|v'_L\|_{L^p(-L/2, L/2)}^p = \pi_p^p / L^p. \tag{2.10}$$

Now, the product function $\phi(x, y) = u(x)v_L(y)$ is admissible in the computation of $\lambda_p(\Omega_L)$ and gives

$$\lambda_p(\Omega_L) \leq \int_{\Omega_L} |\nabla \phi(x, y)|^p dx dy = \int_{\Omega_L} (|\nabla_x u(x)v(y)|^2 + |u(x)v'(y)|^2)^{p/2} dx dy.$$

If $p \leq 2$, by the sub-additivity of the function $t \rightarrow t^{p/2}$, (2.10) and Fubini theorem, we get

$$\lambda_p(\Omega_L) \leq \lambda_p(\omega) + \frac{\pi_p^p}{L^p}.$$

Similarly if $p \geq 2$, by (2.10), Fubini theorem and Minkowski inequality we have that

$$\lambda_p(\Omega_L) \leq \left(\lambda_p^{2/p}(\omega) + \frac{\pi_p^2}{L^2} \right)^{p/2}.$$

The case $p = 1$ follows by using an approximation argument. □

Remark 2.5 The limit (2.9) is clearly true also when $p = +\infty$, since in this case $\rho(\Omega_L) = \rho(\omega)$, as soon as L is large enough.

We may now prove the general form of limit (2.5).

Theorem 2.6 *Let $1 \leq q < p \leq +\infty$. Then the sequence $d \mapsto m_d(p, q)$ is nonincreasing and*

$$\lim_{d \rightarrow +\infty} m_d(p, q) = \inf_{d \geq 1} m_d(p, q) = q/p. \tag{2.11}$$

In particular,

$$\frac{q}{p} \leq m_d(p, q) \leq m_1(p, q) = \frac{\pi_p}{\pi_q}.$$

Proof The monotonicity of the sequence follows at once by (2.9), hence the limit above exists as well. By (2.3) we get the inequality

$$\lim_{d \rightarrow +\infty} m_d(p, q) = \inf_{d \geq 1} m_d(p, q) \geq q/p.$$

In order to prove the opposite inequality, first we suppose $q = 1$. By applying (2.7) with $s = \sqrt{d}$, we get

$$\mathcal{F}_{p,1}(B_1^d) = \frac{\lambda_p^{1/p}(B_1^d)}{d} \leq \frac{1}{\sqrt{d}} \cdot \left(\frac{\Gamma(\sqrt{d}p + d + 1)\Gamma(\sqrt{d}p - p + 1)}{\Gamma(\sqrt{d}p + 1)\Gamma(\sqrt{d}p + d - p + 1)} \right)^{1/p}.$$

Moreover, using the fact that $\Gamma(s + t) \approx \Gamma(s)s^t$ as $s \rightarrow \infty$, we obtain that, as $d \rightarrow \infty$,

$$\frac{1}{\sqrt{d}} \left(\frac{\Gamma(\sqrt{d}p + d + 1)\Gamma(\sqrt{d}p - p + 1)}{\Gamma(\sqrt{d}p + 1)\Gamma(\sqrt{d}p + d - p + 1)} \right)^{1/p} \approx \frac{1}{\sqrt{d}} \left(1 + \frac{\sqrt{d}}{p} \right).$$

Hence we obtain

$$\lim_{d \rightarrow \infty} m_d(p, 1) \leq \limsup_{d \rightarrow \infty} \mathcal{F}_{p,1}(B_1^d) \leq 1/p.$$

To achieve the general case we notice that, for all $\Omega \subset \mathbb{R}^d$, we have

$$m_d(p, q) \leq \mathcal{F}_{p,q}(\Omega) = \mathcal{F}_{p,1}(\Omega)\mathcal{F}_{q,1}^{-1}(\Omega) \leq q\mathcal{F}_{p,1}(\Omega),$$

where the last inequality follows again by (2.3). Then

$$\lim_{d \rightarrow \infty} m_d(p, q) \leq q \lim_{d \rightarrow \infty} m_d(p, 1) = q/p$$

as required. Finally, the last statement is an easy consequence of (1.8), of the monotonicity proved above and of Proposition 2.1. □

We now turn our attention to the quantity $M_d(p, q)$ and we notice that limit (2.9) also implies that the sequence $d \mapsto M_d(p, q)$ is nondecreasing and hence

$$\frac{\pi_p}{\pi_q} = M_1(p, q) \leq M_d(p, q).$$

Our next result deals with the upper bound for $M_d(p, q)$. We recall that, for $1 < p < +\infty$, the (relative) p -capacity of a set $E \subset \Omega$ is defined as

$$\text{cap}_p(E; \Omega) = \inf \left\{ \int_{\Omega} |\nabla u|^p dx : u \in W_0^{1,p}(\Omega), u \geq 1 \text{ a.e. in a neighborhood of } E \right\}.$$

For a comprehensive introduction to p -capacity we refer the reader to [10] and [18]. A set $E \subset \mathbb{R}^d$ is said to be of zero p -capacity if

$$\text{cap}_p(E \cap \Omega; \Omega) = 0 \quad \text{for all } \Omega \in \mathcal{A}^d;$$

in this case we simply write $\text{cap}_p(E) = 0$. We recall that countable union of zero capacity sets has zero capacity. Moreover, given E a relatively closed subset of Ω , then

$$\text{cap}_p(E) = 0 \implies \lambda_p(\Omega \setminus E) = \lambda_p(\Omega).$$

Finally, a single point has zero p -capacity if and only if $1 < p \leq d$. This last property is employed in [21] to show the following result.

Theorem 2.7 *Let $d \in \mathbb{N}$, $d \geq 1$ and $d < p < +\infty$. There exists a positive constant $C_{p,d}$, depending on p and d , such that for every bounded open set $\Omega \subset \mathbb{R}^d$ it holds*

$$\lambda_p^{1/p}(\Omega) \geq C_{d,p} \rho^{-1}(\Omega).$$

Remark 2.8 Theorem 2.7 can be extended to the whole class \mathcal{A}^d by means of a simple approximation argument. Indeed, it is sufficient to note that, if $\Omega \in \mathcal{A}^d$ is unbounded, and we set $\Omega_n := \Omega \cap B_n$, it holds

$$\lim_{n \rightarrow +\infty} \rho(\Omega_n) = \rho(\Omega), \quad \lambda_p(\Omega) = \lim_{n \rightarrow +\infty} \lambda_p(\Omega_n),$$

and, by Theorem (2.7),

$$\lambda_p(\Omega_n) \geq C_{d,p} \rho^{-p}(\Omega_n) \quad \text{for every } n \in \mathbb{N}.$$

Passing to the limit as $n \rightarrow +\infty$ in the inequality above gives the conclusion.

Theorem 2.9 *Let $1 \leq q < p \leq \infty$. Then*

$$\begin{cases} M_d(p, q) \leq \frac{\lambda_p^{1/p}(B_1)}{C_{d,q}} & \text{if } d < q, \\ M_d(p, q) = +\infty & \text{otherwise,} \end{cases}$$

where $C_{d,q}$ is the constant given by Theorem 2.7.

Proof The case when $d < q$ follows by combining Theorem 2.7 (applied to λ_q) and inequality (2.4) (applied to λ_p).

The case $1 \leq q \leq d < p \leq \infty$ is a consequence of the fact that if $1 < q \leq d$ then a single point has zero q -capacity. More precisely, let (x_n) be a dense sequence in a ball $B \subset \mathbb{R}^d$ and define

$$\Omega_n := B \setminus \bigcup_{i=1}^n \{x_i\}.$$

Since $\text{cap}_q(\bigcup_{i=1}^n \{x_i\}) = 0$, we have $\lambda_q(\Omega_n) = \lambda_q(B)$ for every $n \in \mathbb{N}$. Taking into account (1.6), we have also $h(\Omega_n) = h(B)$ for every $n \in \mathbb{N}$. Moreover, since $\rho(\Omega_n) \rightarrow 0$, by Theorem 2.7 or (1.7), we have that $\lambda_p^{1/p}(\Omega_n) \rightarrow +\infty$. Therefore $\mathcal{F}_{p,q}(\Omega_n) \rightarrow +\infty$ for every $1 \leq q < d < p$.

The case when $1 \leq q < p \leq d$ is more delicate. Our argument is inspired by the example exhibited in the Appendix A of [5]. Given $1 < p \leq d$ we construct a sequence of open bounded sets $\Omega_n \subset \mathbb{R}^d$ such that for every $q < p$

$$\lim_{n \rightarrow \infty} \mathcal{F}_{p,q}(\Omega_n) = +\infty.$$

Let $Q = (-1/2, 1/2)^d$. Being $1 < p \leq d$, it is well known that there exists a compact set $E_p \subset [0, 1]^d$ such that $\text{cap}_p(E_p) > 0$ and $\text{cap}_q(E_p) = 0$ when $1 < q < p$ (see Lemma 7.1 in

[17]; for instance, E_p can be constructed as a Cantor set). By translating and rescaling the compact set E_p , we can assume that $E_p \subset [-1/4, 1/4]^d$. Then we consider the open sets

$$\Omega_n = (-(n + 1/2), n + 1/2)^d \setminus \bigcup_{z \in \mathbb{Z}_n^d} (E_p + z),$$

where $\mathbb{Z}_n^d = \mathbb{Z}^d \cap [-n, n]^d$ and

$$E = \bigcup_{n \in \mathbb{N}} \Omega_n = \mathbb{R}^d \setminus \bigcup_{z \in \mathbb{Z}^d} (E_p + z).$$

Being $\text{cap}_p(E_p) > 0$, by Theorem 10.1.2 in [18], we have that

$$\min \left\{ \frac{\int_Q |\nabla u|^p \, dx}{\int_Q |u|^p \, dx} : u \in W^{1,p}(Q), u = 0 \text{ on } E_p \right\} = C(d, p, E_p) > 0. \tag{2.12}$$

Now, since any function $u \in C_c^\infty(E)$ when restricted to $Q + z$, with $z \in \mathbb{Z}^d$, vanishes on a translated copy of E_p , (2.12) readily implies

$$\lambda_p(E) \geq C(d, p, E_p).$$

Then, by monotonicity we have

$$\lambda_p(\Omega_n) \geq \lambda_p(E) \geq C(d, p, E_p) > 0, \tag{2.13}$$

for every $n \in \mathbb{N}$. Moreover, for every $q > 1$, being $\text{cap}_q(E_p) = 0$, we have that

$$\lambda_q(\Omega_n) = \lambda_q((-(n + 1/2), n + 1/2)^d)$$

and hence

$$h(\Omega_n) = h((-(n + 1/2), n + 1/2)^d)$$

as well. This gives

$$\lambda_q^{1/q}(\Omega_n) = (2n + 1)^{-1} \lambda_q^{1/q}(Q) \rightarrow 0 \quad \text{as } n \rightarrow +\infty \tag{2.14}$$

for every $q \geq 1$. By combining (2.13) and (2.14) the thesis is easily achieved. □

Remark 2.10 The case $1 \leq q < p \leq d$ in the previous theorem can be also proved by constructing a sequence Ω_n satisfying:

$$\lambda_q(\Omega_n) \rightarrow \lambda_q(B_1), \quad \lambda_p(\Omega_n) \rightarrow +\infty. \tag{2.15}$$

To do this, one can consider the sequence Ω_n obtained by removing from the unit ball a periodic array of spherical holes of size r_n , where

$$\begin{cases} n^{d/(p-d)} \ll r_n \ll n^{d/(q-d)} & \text{if } p < d; \\ e^{-n^{d/(d-1)}} \ll r_n \ll n^{d/(q-d)} & \text{if } p = d. \end{cases}$$

Then classical results of shape optimization theory can be used to get (2.15) (see [7] and references therein). We devote Appendix to give a self-contained proof.

3 Optimization in \mathcal{A}_c^d

In this section, we consider the optimization problems in the class \mathcal{A}_c^d . We remark that also in this case Lemma 2.4 provides the monotonicity properties:

$$d \mapsto \overline{m}_d(p, q) \text{ is nonincreasing and } d \mapsto \overline{M}_d(p, q) \text{ is nondecreasing.}$$

To carry on our analysis we use two fundamental inequalities which hold for every $1 < p < +\infty$ and for every $\Omega \in \mathcal{A}_c^d$:

- the *Hersch–Protter inequality*:

$$\rho(\Omega)\lambda_p^{1/p}(\Omega) > \frac{\pi_p}{2}; \tag{3.1}$$

- the *Buser inequality*:

$$\frac{\lambda_p^{1/p}(\Omega)}{h(\Omega)} < \frac{\pi_p}{2}. \tag{3.2}$$

Inequality (3.1) was first proved in [12] and [22] when $p = 2$, and then extended to general case in [3], while inequality (3.2) is proved in [20] in the planar linear case, and in [4] in the general one. Both inequalities are sharp, as one can verify by taking a sequence of thin slab domains $\Omega_n := (-n, n)^{d-1} \times (0, 1)$, see for instance [3] and [4]. As a consequence one has that

$$\overline{M}_d(p, 1) = \frac{\pi_p}{2}, \quad \overline{M}_d(\infty, q) = \frac{2}{\pi_q},$$

so that the following conjecture formulated by Parini in [20], is satisfied in the particular cases $p = +\infty$ or $q = 1$.

Conjecture 3.1 *Let $1 \leq q < p \leq +\infty$. Then we have*

$$\overline{M}_d(p, q) = \frac{\pi_p}{\pi_q},$$

and no maximizer set exists.

Although we are not able prove the conjecture we show the following estimates.

Proposition 3.2 *Let $1 \leq q < p \leq +\infty$. Then, for all $\Omega \in \mathcal{A}_c^d$ we have*

$$\max \left\{ \frac{q}{p}, \frac{\pi_p}{d\pi_q} \right\} \leq \overline{m}_d(p, q) \leq \overline{M}_d(p, q) \leq \pi_p \min \left\{ \frac{q}{2}, \frac{d}{\pi_q} \right\}.$$

Proof We first notice that, being $h(B_1) = d$, inequality (2.4) with $p = 1$ provides

$$h(\Omega)\rho(\Omega) \leq d.$$

Hence, by using (3.1) (with q) and (3.2) (with p), we obtain

$$\mathcal{F}_{p,q}(\Omega) = \frac{\lambda_p^{1/p}(\Omega)}{\lambda_q^{1/q}(\Omega)} \leq \frac{\pi_p}{\pi_q} h(\Omega)\rho(\Omega) \leq \frac{d\pi_p}{\pi_q}. \tag{3.3}$$

By interchanging the role of p and q , we get

$$\mathcal{F}_{p,q}(\Omega) = \frac{\lambda_p^{1/p}(\Omega)}{\lambda_q^{1/q}(\Omega)} \geq \frac{\pi_p}{\pi_q} \frac{1}{h(\Omega)\rho(\Omega)} \geq \frac{\pi_p}{d\pi_q}. \tag{3.4}$$

Inequalities (3.4) and (1.8) prove that

$$\max \left\{ \frac{q}{p}, \frac{\pi_p}{d\pi_q} \right\} \leq \bar{m}_d(p, q),$$

while, using (3.2) and (1.8), we have

$$\lambda_p^{1/p}(\Omega) \leq \frac{\pi_p}{2} h(\Omega) \leq q \frac{\pi_p}{2} \lambda_q^{1/q}(\Omega),$$

which, together with (3.3), implies

$$\bar{M}_d(p, q) \leq \pi_p \min \left\{ \frac{q}{2}, \frac{d}{\pi_q} \right\}$$

as required. □

In [20] it is proved that the functional $\mathcal{F}_{2,1}$ admits a minimizing set in the class of bounded convex planar domains. Recently in [9], the author discussed the existence of minimizers for $\mathcal{F}_{2,1}$ in \mathcal{A}_c^d for $d \geq 3$, which, up to our knowledge, remains open. In Theorem 3.8 below we show the existence of a minimizer for $\mathcal{F}_{p,q}$ in the class \mathcal{A}_c^2 when $q \leq 2 \leq p$. Before proving the theorem we need some preliminary results, that we state in the general case of dimension d .

Lemma 3.3 *Let $1 \leq p \leq +\infty$ and $\Omega \in \mathcal{A}_c^d$. Let $a = (0, \dots, 0)$, $b = (0, \dots, \text{diam}(\Omega))$, and suppose $a, b \in \partial\Omega$. Then there exists $0 < t < \text{diam}(\Omega)$ such that*

$$\lambda_p^{1/p}(\Omega) \geq \lambda_p^{1/p}(\Omega \cap \{x_d = t\}),$$

where in the right-hand side $\lambda_p(\Omega \cap \{x_d = t\})$ is intended in the $d - 1$ dimensional sense.

Proof The case when $p = +\infty$ is trivial and hence we can suppose $1 \leq p < \infty$. We notice that there exists $t \in (0, \text{diam}(\Omega))$ such that

$$\lambda_p(\Omega \cap \{x_d = t\}) = \inf_{\tau \in (0, \text{diam}(\Omega))} \lambda_p(\Omega \cap \{x_d = \tau\}).$$

Indeed the map

$$\tau \mapsto \{\Omega \cap \{x_d = \tau\}\} \subset \mathbb{R}^d,$$

is continuous with respect to the Hausdorff distance, and thus, thanks to the well-known continuity properties for λ_p with respect to Hausdorff metrics on the class of bounded convex sets (see [6] and [11], for details about this fact), the map

$$\tau \mapsto \lambda_p(\Omega \cap \{x_d = \tau\}),$$

is continuous as well. Moreover both $\Omega \cap \{x_d = 0\}$ and $\Omega \cap \{x_d = \text{diam}(\Omega)\}$ are empty, so that

$$\lim_{t \rightarrow 0^+} \lambda_p(\Omega \cap \{x_d = \tau\}) = \lim_{\tau \rightarrow \text{diam}(\Omega)^-} \lambda_p(\Omega \cap \{x_d = \tau\}) = +\infty.$$

Now, let $\varepsilon > 0$ and $\phi \in C^1(\Omega)$ be such that $\|\phi\|_p = 1$ and $\varepsilon + \lambda_p(\Omega) \geq \|\nabla\phi\|_p^p$. Then, by denoting $x = (x', \tau) \in \mathbb{R}^{d-1} \times \mathbb{R}$ and using Fubini Theorem, we get

$$\begin{aligned} \varepsilon + \lambda_p(\Omega) &\geq \int_{\Omega} |\nabla\phi|^p dx = \int_0^{\text{diam}(\Omega)} \left(\int_{\Omega \cap \{x_d = \tau\}} |\nabla\phi|^p dx' \right) d\tau \\ &\geq \int_0^{\text{diam}(\Omega)} \left(\int_{\Omega \cap \{x_d = \tau\}} |\nabla_{x'}\phi|^p dx' \right) d\tau \\ &\geq \int_0^{\text{diam}(\Omega)} \left(\lambda_p(\Omega \cap \{x_d = \tau\}) \int_{\Omega \cap \{x_d = \tau\}} |\phi|^p dx' \right) d\tau \\ &\geq \lambda_p(\Omega \cap \{x_d = t\}), \end{aligned}$$

which, by the arbitrariness of ε , implies the thesis. □

Lemma 3.4 *Let $1 \leq p \leq +\infty$ and $(\Omega_n) \subset \mathcal{A}_c^d$ with $|\Omega_n| = 1$ for every $n \in \mathbb{N}$. Suppose that*

- $(0, \dots, 0), (0, \dots, \text{diam}(\Omega_n)) \in \partial\Omega_n,$
- $\inf_{n \in \mathbb{N}} \text{diam}(\Omega_n) > 0.$

Then, there exists $c > 0$ such that

$$\inf_{n \in \mathbb{N}} \inf_{\tau \in (0, \text{diam}(\Omega_n))} \lambda_p^{1/p}(\Omega_n \cap \{x_d = \tau\}) \geq c.$$

Proof For any $n \in \mathbb{N}$ and any $t \in (0, \text{diam}(\Omega_n))$, we denote

$$\omega_n(t) = \Omega_n \cap \{x_d = t\} \in \mathcal{A}_c^{d-1}.$$

By (3.1) and by using an approximation argument, for every $1 \leq p \leq +\infty$ we have

$$\lambda_p^{1/p}(\omega_n(t)) \geq \frac{\pi_p}{2\rho(\omega_n(t))}.$$

Between the two right cones of basis $\omega_n(t)$ and vertexes the origin or $b_n = (0, \dots, \text{diam}(\Omega_n))$, we can select the one with the greatest height $l_n(t)$, so that $l_n(t) \geq \text{diam}(\Omega_n)/2$. By convexity, such a cone is contained in Ω_n and we have

$$\frac{\rho^{d-1}(\omega_n(t)) |B_1^{d-1}| l_n(t)}{d} \leq |\Omega_n| = 1.$$

In particular

$$\rho(\omega_n(t)) \leq \left(\frac{2d}{|B_1^{d-1}| \operatorname{diam}(\Omega_n)} \right)^{1/(d-1)}.$$

Since $\inf_{n \in \mathbb{N}} \operatorname{diam}(\Omega_n) > 0$, we obtain

$$\rho(\omega_n(t)) \leq \left(\frac{2d}{|B_1^{d-1}| \inf_n \operatorname{diam}(\Omega_n)} \right)^{1/(d-1)},$$

and the thesis easily follows. □

Proposition 3.5 *Let $1 \leq q < p \leq +\infty$ and $(\Omega_n) \subset \mathcal{A}_c^d$ with $|\Omega_n| = 1$ for every $n \in \mathbb{N}$. If $\operatorname{diam}(\Omega_n) \rightarrow +\infty$, then*

$$\bar{m}_{d-1}(p, q) \leq \liminf_{n \rightarrow +\infty} \mathcal{F}_{p,q}(\Omega_n) \leq \limsup_{n \rightarrow +\infty} \mathcal{F}_{p,q}(\Omega_n) \leq \bar{M}_{d-1}(p, q).$$

Proof Let $a_n = (0, \dots, 0)$ and $b_n = (0, \dots, \operatorname{diam}(\Omega_n))$. Being the functional $\mathcal{F}_{p,q}$ rotation and translation invariant we can suppose, without loss of generality, that $a_n, b_n \in \partial\Omega_n$. By applying Lemma 3.3 there exists $t_n \in (0, \operatorname{diam}(\Omega_n))$ such that

$$\lambda_p^{1/p}(\Omega_n) \geq \lambda_p^{1/p}(\omega_n), \tag{3.5}$$

where $\omega_n = \Omega_n \cap \{x_d = t_n\}$. Moreover, up to rotations, we can also suppose $t_n \geq \operatorname{diam}(\Omega_n)/2$. Let $\alpha \in (0, 1)$, and define U_α^n to be the cylinder with basis $\alpha\omega_n$ and height $(1 - \alpha)t_n$. More precisely we consider

$$U_\alpha^n := \{(x, y) : x \in \alpha\omega_n, y \in (\alpha t_n, t_n)\}.$$

Then, by the convexity of Ω_n , we have $U_\alpha^n \subset \Omega_n$ so that

$$\lambda_q(U_\alpha^n) \geq \lambda_q(\Omega_n). \tag{3.6}$$

Since

$$\lambda_q(U_\alpha^n) = \alpha^{-q} \lambda_q\left(\omega_n \times \left(0, \frac{(1 - \alpha)t_n}{\alpha}\right)\right), \tag{3.7}$$

by (3.6), (3.5) and (3.7), we obtain

$$\mathcal{F}_{p,q}(\Omega_n) \geq \frac{\lambda_p^{1/p}(\Omega_n)}{\lambda_q^{1/q}(U_\alpha^n)} \geq \frac{\lambda_p^{1/p}(\omega_n)}{\lambda_q^{1/q}(\omega_n)} \left(\frac{\lambda_q(\omega_n)}{\alpha^{-q} \lambda_q(\omega_n \times (0, (1 - \alpha)\alpha^{-1}t_n))} \right)^{1/q}$$

and thus

$$\mathcal{F}_{p,q}(\Omega_n) \geq \bar{\alpha} \bar{m}_{d-1}(p, q) \left(\frac{\lambda_q(\omega_n)}{\lambda_q(\omega_n \times (0, (1 - \alpha)\alpha^{-1}t_n))} \right)^{1/q}. \tag{3.8}$$

Now, suppose that $q \leq 2$. By Lemma 2.4 we have

$$\lambda_q(\omega_n \times (0, 1 - \alpha)\alpha^{-1}t_n) \leq \lambda_q(\omega_n) + \left(\frac{\alpha\pi_q}{(1 - \alpha)t_n} \right)^q. \tag{3.9}$$

Since $\text{diam}(\Omega_n) \rightarrow +\infty$, we can assume $\inf_n \text{diam}(\Omega_n) > 0$; by applying Lemma 3.4 we have that $\lambda_q^{1/q}(\omega_n) \geq c$ for some constant $c > 0$. Being $t_n \geq \text{diam}(\Omega_n)/2$, we get

$$\lim_{n \rightarrow +\infty} \frac{\lambda_q(\omega_n)}{\lambda_q(\omega_n) + \frac{\alpha^q \pi_q^q}{(1 - \alpha)^q t_n^q}} = 1. \tag{3.10}$$

By passing to the limit for $n \rightarrow +\infty$ in (3.8) and using (3.9) and (3.10), we infer that

$$\liminf_{n \rightarrow +\infty} \mathcal{F}_{p,q}(\Omega_n) \geq \alpha \bar{m}_{d-1}(p, q).$$

Letting $\alpha \rightarrow 1^-$, we conclude that

$$\liminf_{n \rightarrow +\infty} \mathcal{F}_{p,q}(\Omega_n) \geq \bar{m}_{d-1}(p, q).$$

The case when $q > 2$ is similar. Indeed, (2.8) ensures that

$$\lambda_q(\omega_n \times (0, 1 - \alpha)\alpha^{-1}t_n) \leq \lambda_q(\omega_n) \left(1 + \frac{\alpha^2 \pi_q^2}{(1 - \alpha)^2 \lambda_q^{2/q}(\omega_n) t_n^2} \right)^{q/2},$$

and again Lemma 3.4 applies leading to the analog of the limit (3.10).

Finally, if we choose ω_n to be such that $\lambda_q(\omega_n) \leq \lambda_q(\Omega_n)$, by using the fact that $\lambda_p^{1/p}(\Omega_n) \leq \lambda_p^{1/p}(U_\alpha^n)$ and by applying (3.7) with p in the place of q , we obtain:

$$\mathcal{F}_{p,q}(\Omega_n) \leq \frac{\lambda_p^{1/p}(U_\alpha^n)}{\lambda_q^{1/q}(\Omega_n)} \leq \frac{\bar{M}_{d-1}(p, q)}{\alpha} \left(\frac{\lambda_p(\omega_n \times (0, \alpha^{-1}(1 - \alpha)t_n))}{\lambda_p(\omega_n)} \right)^{1/p}.$$

By the same argument as above (distinguishing the case $p \leq 2$ and $p > 2$, when we use Lemma 2.4), passing to the limit, as $n \rightarrow \infty$, we have

$$\limsup_{n \rightarrow \infty} \mathcal{F}_{p,q}(\Omega_n) \leq \alpha^{-1} \bar{M}_{d-1}(p, q),$$

and, letting $\alpha \rightarrow 1^-$, we conclude that

$$\limsup_{n \rightarrow \infty} \mathcal{F}_{p,q}(\Omega_n) \leq \bar{M}_{d-1}(p, q)$$

as required. □

Theorem 3.6 *Let $1 \leq q < p \leq +\infty$. If $\bar{m}_d(p, q) < \bar{m}_{d-1}(p, q)$, then there exists $\Omega_\star^d \in \mathcal{A}_c^d$ such that*

$$\bar{m}_d(p, q) = \mathcal{F}_{p,q}(\Omega_\star^d).$$

Proof Let (Ω_n) be such that $\mathcal{F}_{p,q}(\Omega_n) \rightarrow \bar{m}_d(p, q)$ with $|\Omega_n| = 1$ for every $n \in \mathbb{N}$. Then, by Proposition 3.5, we have

$$\sup_n \text{diam}(\Omega_n) < +\infty.$$

Hence, up to translations, the whole sequence (Ω_n) is contained in a compact set. By applying Blaschke Selection Theorem (see [11]), we can extract a subsequence (Ω_{n_k}) which converges in the Hausdorff distance to some Ω_\star^d . Using the continuity properties for λ_p with respect to Hausdorff metrics on the class of bounded convex sets, we have

$$\bar{m}_d(p, q) = \lim_{n \rightarrow \infty} \mathcal{F}_{p,q}(\Omega_n) = \mathcal{F}_{p,q}(\Omega_\star^d)$$

as required. □

Lemma 3.7 *Let $1 < p < \infty$, $p \neq 2$ and let $Q = (0, 1)^d$ be the unitary cube of \mathbb{R}^d . Then*

$$\begin{aligned} d^{1/p} \pi_p &\leq \lambda_p^{1/p}(Q) < d^{1/2} \pi_p && \text{for every } p > 2; \\ d^{1/2} \pi_p &< \lambda_p^{1/p}(Q) \leq d^{1/p} \pi_p && \text{for every } p < 2. \end{aligned}$$

Proof By Lemma 2.4 (applied d times and with $L = 1$) we need only to prove the two strict inequalities. With this aim we define

$$v_p(Q) = \inf_{\phi \in C_c^\infty(Q) \setminus \{0\}} \frac{\int_Q \sum_{i=1}^d \left| \frac{\partial \phi}{\partial x_i} \right|^p dx}{\int_Q |\phi|^p dx}.$$

We notice that $v_p(Q) = d\pi_p^p$, with a minimizer given by

$$\phi(x_1, \dots, x_d) = u(x_1) \dots u(x_d), \tag{3.11}$$

where $u \in W^{1,p}(0, 1)$ is a non negative function, optimal for (2.1), with unitary L^p norm. Now, the case when $p > 2$ follows by strict convexity of the map $t \rightarrow t^{p/2}$: indeed, being $d \geq 2$, integrating over Q the inequality

$$|\nabla \phi(x)|^p < d^{p/2-1} \sum_i \left| \frac{\partial \phi(x)}{\partial x_i} \right|^p,$$

we obtain

$$\lambda_p(Q) < d^{p/2-1} v_p(Q) = d^{p/2} \pi_p^p.$$

Similarly, when $p < 2$, we can consider $\tilde{\phi}$ to be the optimal positive function for $\lambda_p(Q)$, with unitary L^p norm. Then, being $d \geq 2$, the strict concavity of the map $t \rightarrow t^{p/2}$ gives

$$|\nabla \tilde{\phi}(x)|^p > d^{p/2-1} \sum_i \left| \frac{\partial \tilde{\phi}(x)}{\partial x_i} \right|^p,$$

which, integrated over Q , implies that

$$\lambda_p(Q) > d^{p/2-1} v_p(Q) = d^{p/2} \pi_p^p.$$

This concludes the proof of the lemma. □

We are now in a position to prove the following existence result in the case $d = 2$.

Theorem 3.8 *Let $1 \leq q < p \leq +\infty$. Suppose $q \leq 2 \leq p$, then there exists $\Omega^* \in \mathcal{A}_c^2$ such that*

$$\mathcal{F}_{p,q}(\Omega^*) = \min\{\mathcal{F}_{p,q}(\Omega) : \Omega \in \mathcal{A}_c^2\}.$$

Proof By Theorem 3.6 it is sufficient to show that

$$\bar{m}_2(p, q) < \bar{m}_1(p, q) = \pi_p/\pi_q.$$

The cases when $q = 1$ or $p = +\infty$ follow at once by inequalities (3.1) and (3.2). The remaining cases follow by combining the upper estimate for $\lambda_p(Q)$ and the lower estimate for $\lambda_q(Q)$ given by Lemma 3.7; notice that since $p \neq q$, at least one of these two inequalities is strict. □

Remark 3.9 We notice that, by Proposition 3.5, one readily concludes that if there exists a maximizing sequence $(\Omega_n) \subset \mathcal{A}_c^d$ such that $|\Omega_n| = 1$ and satisfying $\text{diam}(\Omega_n) \rightarrow +\infty$, then

$$\bar{M}_d(p, q) = \bar{M}_{d-1}(p, q).$$

In particular when $d = 2$, this argument would prove Conjecture 3.1. On the other hand, if any maximizing sequence $(\Omega_n) \subset \mathcal{A}_c^d$ with $|\Omega_n| = 1$ is contained (up to translation) in a compact set, arguing as in Theorem 3.6 it is easy to conclude that a convex maximizer exists. This suggests a dichotomy between the equality $\bar{M}_d(p, q) = \pi_p/\pi_q$, stated in Conjecture 3.1, and the existence of a convex maximizer.

4 Further remarks and open problems

Several interesting problems and questions about the shape functionals $\mathcal{F}_{p,q}$ are still open; in this section we list some of them.

Problem 1 In Theorem 2.9 we have shown that $M_d(p, q) < +\infty$ when $q > d$; it would be interesting to give a characterization of the quantity $M_d(p, q)$ in these cases. In addition, even if we believe that the value $M_d(p, q)$ is not a maximum, it would be interesting to describe the behavior of maximizing sequences (Ω_n) . It is reasonable to expect that Ω_n is made by a domain Ω where n points are removed; the locations of these points in Ω is an interesting issue: is it true that in the two-dimensional case they are the centers of an hexagonal tiling?

Problem 2 Proving or disproving the existence of a domain Ω minimizing the shape functional $\mathcal{F}_{p,q}$ in the class \mathcal{A}^d is another very interesting issue. The presence of small holes in a domain Ω does not seem to decrease the value of $\mathcal{F}_{p,q}(\Omega)$, which could be a point in favor of the existence of an optimal domain $\Omega_{p,q}$.

Problem 3 In the smaller class \mathcal{A}_c^d we know that $\bar{M}_d(p, q)$ is always finite. It would be interesting to prove (or disprove) Conjecture 3.1 (formulated by Parini in [20]), that is,

$$\bar{M}_d(p, q) = \pi_p/\pi_q \text{ and no maximizer exists.}$$

In other words, maximizing sequences are made by thin slabs

$$\Omega_\varepsilon = A \times (0, \varepsilon) \quad \text{with } \varepsilon \rightarrow 0 \text{ and } A \text{ a smooth } d - 1 \text{ dimensional domain.}$$

At present the problem is open even in the case $d = 2$, see also Remark 3.9.

Problem 4 Concerning the minimum $\bar{m}_d(p, q)$ of $\mathcal{F}_{p,q}$ in the class \mathcal{A}_c^d , establishing if it is attained is an interesting issue. Theorem 3.8 gives an affirmative answer in the case $d = 2$ and $q \leq 2 \leq p$; in particular, this happens for $d = 2$ and $q = 1, p = 2$, which is the original Cheeger case and, according to some indications by E. Parini [20], the optimal domain could be in this case a square. This is not yet known.

We expect the existence of an optimal domain for every dimension d and every p, q and, as stated in Theorem 3.6, this would follow once the strict monotonicity of $\bar{m}_d(p, q)$ with respect to the dimension d is proved. At present however, a general proof of this strict monotonicity is missing.

Appendix

We devote this appendix to briefly describe the classical strategy of Cioranescu–Murat (see [7]) which can be used to prove Theorem 2.9, see Remark 2.10. These results are well known, but in the case $p \neq 2$ it is not easy to find precise references, hence we add them for the sake of completeness and for the readers’ convenience. We limit ourselves to prove only what we need in the paper, pointing out that the following results can be obtained in a more general framework of γ -convergence (see for instance the monographs [6, 11] and references therein).

Let $1 \leq p \leq d, \Omega$ be a bounded connected smooth open set, and $\varepsilon > 0$. We consider in \mathbb{R}^d ($d \geq 2$) the lattice of parallel cubes P_ε^i of size 2ε and we denote by x_i^ε their centers. In each cube we consider a tiny ball $B_{r_\varepsilon}(x_i^\varepsilon)$ of radius r_ε , where $r_\varepsilon < \varepsilon$. Finally we set

$$C_\varepsilon = \{x_i^\varepsilon : P_\varepsilon^i \Subset \Omega\}.$$

and

$$\Omega_\varepsilon = \Omega \setminus \bigcup_{x \in C_\varepsilon} \bar{B}_{r_\varepsilon}(x).$$

Our goal is to determine the behavior of $\lambda_p(\Omega_\varepsilon)$ as $\varepsilon \rightarrow 0$. This depends on the size of r_ε , and more precisely on the following ratio:

$$a_\varepsilon = \begin{cases} \varepsilon^{-d} r_\varepsilon^{d-p} & \text{if } p < d, \\ \varepsilon^{-d} (-\ln(r_\varepsilon))^{1-d} & \text{if } p = d. \end{cases} \tag{5.1}$$

Proposition A.1 (super-critical case) *If $a_\varepsilon \rightarrow +\infty$ as $\varepsilon \rightarrow 0$, then $\lambda_p(\Omega_\varepsilon) \rightarrow +\infty$.*

Proof Given $R > r > 0$, we denote by $\mu_{R,r}$ the least eigenvalue of $B_R \setminus \bar{B}_r$ with Dirichlet boundary condition on ∂B_r and Neumann boundary condition on ∂B_R , that is,

$$\mu_{R,r} = \inf \left\{ \frac{\int_{B_R \setminus B_r} |\nabla v|^p dx}{\int_{B_R \setminus B_r} |v|^p dx} : v \in W^{1,p}(B_R \setminus \overline{B_r}), v = 0 \text{ on } \partial B_r \right\}, \tag{5.2}$$

where the condition $v = 0$ on ∂B_r is intended in the usual trace sense. Notice that by exploiting the convexity property of the functional $u \mapsto \int |\nabla u|^{1/p}|^p$ (as done in [1, 2] and [8]), we can infer that there exists a unique positive minimizer v for (5.2) with unitary L^p norm. In particular, being the domain $B_R \setminus B_r$ radial, v is a radially symmetric function in $W^{1,p}(B_R \setminus \overline{B_r})$.

We claim that there exist constants $c \geq 1$ and $\xi > 0$ (which do not depend on ε) such that

$$\lambda_p(\Omega_\varepsilon) \geq \xi \mu_{c\varepsilon, r_\varepsilon}. \tag{5.3}$$

Assume (5.3) to be true, we obtain the thesis by proving that $\mu_{c\varepsilon, r_\varepsilon} \rightarrow +\infty$ as $\varepsilon \rightarrow 0$.

Indeed, taking for simplicity $c = 1$ and using coarea formula, we have that

$$\mu_{\varepsilon, r_\varepsilon} = \inf \left\{ \int_{r_\varepsilon}^\varepsilon |u'(t)|^p t^{d-1} dt \right\},$$

where the infimum is computed among non negative functions $u \in C^\infty(r_\varepsilon, \varepsilon)$ vanishing on r_ε and satisfying

$$\int_{r_\varepsilon}^\varepsilon |u(t)|^p t^{d-1} dt = 1. \tag{5.4}$$

If u is admissible, by Hölder inequality,

$$\int_{r_\varepsilon}^\varepsilon |u'(t)|^p t^{d-1} dt \geq \left(\int_{r_\varepsilon}^\varepsilon |u'(t)| dt \right)^p \left(\int_{r_\varepsilon}^\varepsilon \frac{1}{t^{(d-1)/(p-1)}} dt \right)^{1-p}; \tag{5.5}$$

moreover there exists $t_0 \in (r_\varepsilon, \varepsilon)$ such that

$$|u(t_0)| = \left(\int_{r_\varepsilon}^\varepsilon t^{d-1} dt \right)^{-1/p},$$

since otherwise

$$0 \leq u(t) < \left(\int_{r_\varepsilon}^\varepsilon t^{d-1} dt \right)^{-1/p} \text{ for every } t \in (r_\varepsilon, \varepsilon),$$

would imply $\int_{r_\varepsilon}^\varepsilon |u(t)|^p t^{d-1} dt < 1$, in contradiction with (5.4). Hence, using the fact that $u(r_\varepsilon) = 0$, we have

$$\int_{r_\varepsilon}^\varepsilon |u'(t)| dt \geq \left| \int_{r_\varepsilon}^{t_0} u'(t) dt \right| \geq \left(\int_{r_\varepsilon}^\varepsilon t^{d-1} dt \right)^{-1/p}.$$

Finally, the latter inequality combined with (5.5) implies

$$\int_{r_\varepsilon}^\varepsilon |u'(t)|^p t^{d-1} dt \geq \left(\int_{r_\varepsilon}^\varepsilon t^{d-1} dt \right)^{-1} \left(\int_{r_\varepsilon}^\varepsilon \frac{1}{t^{(d-1)/(p-1)}} dt \right)^{1-p}.$$

In the case $p < d$ (the case $p = d$ being similar), computing the right-hand side in the previous inequality we obtain

$$\mu_{\varepsilon, r_\varepsilon} \geq d \left(\frac{1}{\varepsilon^d - r_\varepsilon^d} \right) \left(\frac{d-p}{p-1} \right)^{p-1} \left(\frac{1}{r_\varepsilon^{(p-d)/(p-1)} - \varepsilon^{(p-d)/(p-1)}} \right)^{p-1}.$$

Taking (5.1) into account, it is easy to verify that the right hand side of the previous inequality tends to $+\infty$ as $\varepsilon \rightarrow 0$.

To conclude let us prove (5.3). We notice that there exists $c > 1$, which does not depend on ε , such that for every ε small enough the family of balls

$$\mathcal{G}^\varepsilon = \{ B_{c\varepsilon}(x) : x \in C_\varepsilon \},$$

covers Ω . Moreover there exists $N \in \mathbb{N}$, which again does not depend on ε , such that we can split \mathcal{G}^ε into N sub-families $\mathcal{G}_1^\varepsilon, \dots, \mathcal{G}_N^\varepsilon$ made up of disjoint balls. This latter assertion can be easily proved once noticed that any ball in \mathcal{G}^ε can intersect only a bounded number of different balls in \mathcal{G}^ε , and such a bound does not depend on ε . Indeed suppose that $B_{c\varepsilon}(\bar{x}) \in \mathcal{G}^\varepsilon$ intersects $B_{c\varepsilon}(x_1), \dots, B_{c\varepsilon}(x_m) \in \mathcal{G}^\varepsilon$, then we have also

$$\bigcup_{i=1}^m B_\varepsilon(x_i) \subseteq B_{3c\varepsilon}(\bar{x}),$$

in particular, taking the measures of both sets, we get $m \leq (3c)^d$. Therefore, it is sufficient to take $N = [(3c)^d] + 1$.

Now, let $u \in C_c^\infty(\Omega_\varepsilon)$ and extend u by zero outside Ω_ε . We have

$$N \int_\Omega |\nabla u|^p dx \geq \sum_{i=1}^N \sum_{B \in \mathcal{G}_i^\varepsilon} \int_B |\nabla u|^p dx \geq \mu_{c\varepsilon, r_\varepsilon} \sum_{i=1}^N \sum_{B \in \mathcal{G}_i^\varepsilon} \int_B |u|^p dx \geq \mu_{c\varepsilon, r_\varepsilon} \int_\Omega |u|^p dx.$$

Thus, by the arbitrariness of u we obtain (5.3) with $\xi = N^{-1}$. □

Proposition A.2 (sub-critical case) *If $a_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, then $\lambda_p(\Omega_\varepsilon) \rightarrow \lambda_p(\Omega)$.*

Proof First we notice that by monotonicity we have

$$\lambda_p(\Omega_\varepsilon) \geq \lambda_p(\Omega).$$

Hence it is enough to prove that

$$\limsup_{\varepsilon \rightarrow 0} \lambda_p(\Omega_\varepsilon) \leq \lambda_p(\Omega). \tag{5.6}$$

Let v_ε be a competitor for $\text{cap}_p(\bar{B}_{r_\varepsilon}; B_\varepsilon)$ chosen in such a way that:

$$v_\varepsilon \in C_c^\infty(B_\varepsilon), \quad 0 \leq v_\varepsilon \leq 1, \quad v_\varepsilon = 1 \text{ on } B_{r_\varepsilon}, \quad \|\nabla v_\varepsilon\|_p \leq \text{cap}_p(\bar{B}_{r_\varepsilon}; B_\varepsilon) + o(\varepsilon^d).$$

We define V_ε in Ω to be

$$V_\varepsilon(x) = \begin{cases} 1 - v_\varepsilon(x - x_i), & \text{in } B_\varepsilon^i(x_i) \text{ if } x_i \in C_\varepsilon \\ 1, & \text{in } \Omega \setminus \bigcup_{x \in C_\varepsilon} B_\varepsilon^i(x), \end{cases}$$

and we denote by $n(\epsilon) \in \mathbb{N}$ be the number of cubes P_ϵ^i such that $P_\epsilon^i \Subset \Omega$. We have

$$\|\nabla V_\epsilon\|_{L^p(\Omega)} \leq n(\epsilon)\|\nabla v_\epsilon\|_{L^p(B_\epsilon)} \leq (2\epsilon)^{-d}|\Omega|\|\nabla v_\epsilon\|_{L^p(B_\epsilon)} \approx (2\epsilon)^{-d}|\Omega|\text{cap}_p(\bar{B}_{r_\epsilon}; B_\epsilon).$$

Since $r_\epsilon \rightarrow 0$, the latter implies

$$\|\nabla V_\epsilon\|_{L^p(\Omega)} \rightarrow 0$$

(see Section 2.2.4 of [18] for the precise value of $\text{cap}_p(\bar{B}_{r_\epsilon}; B_\epsilon)$). This means that V_ϵ weakly converge in $W^{1,p}(\Omega)$ to some constant $c \in \mathbb{R}$. Moreover, since $V_\epsilon = 1$ on $\partial\Omega$, we can infer that $c = 1$.

Now, let $u \in C^\infty(\Omega)$, and consider $u_\epsilon = V_\epsilon u$. We have $u_\epsilon \in W_0^{1,p}(\Omega_\epsilon)$ and $u_\epsilon \rightarrow u$ strongly in $W_0^{1,p}(\Omega)$. In particular

$$\frac{\int_\Omega |\nabla u|^p dx}{\int_\Omega |u|^p dx} = \lim_{\epsilon \rightarrow 0} \frac{\int_{\Omega_\epsilon} |\nabla u_\epsilon|^p dx}{\int_{\Omega_\epsilon} |u_\epsilon|^p dx} \geq \limsup_{\epsilon \rightarrow 0} \lambda_p(\Omega_\epsilon).$$

Since u is arbitrary we get (5.6). □

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