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Global existence and asymptotic behavior for a Timoshenko system with internal damping and logarithmic source terms

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Abstract This manuscript deals with a Timoshenko system with damping and source. The existence and stability of the solution are analyzed taking into account the competition of the internal damping versus the logarithmic source. We use the potential well theory. For initial data in the stability set created by the Nehari surface, the existence of global solutions is proved using Faedo–Galerkin's approximation. The exponential decay is given by the Nakao theorem. A numerical approach is presented to illustrate the results obtained.

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1 Introduction

The Timoshenko system is a model widely studied in the scientific community for vibrations of elastic beams. Its mathematical formulation is given by a system of two partial differential equations

$$\begin{aligned} \rho A\varphi_{tt} &= [S(x,t)]_x + N_1(x,t), \\ \rho I\psi_{tt} &= [M(x,t)]_x - S(x,t) + N_2(x,t), \end{aligned}$$

where the functions φ and ψ depending upon $(x,t) \in (0,L)\times(0,T)$ model the transverse displacement of a beam with reference configuration $(0,L) \subset \mathbb{R}$ are transverse displacement and the rotations in the transverse sections, respectively. The functions M and S represent, respectively, the bending moment and the shear stress and satisfies

$$\begin{aligned} S(x,t) &= kAG\left(\varphi_x + \psi\right), \\ M(x,t) &= EI\psi_x, \end{aligned}$$

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$$N_1(x,t) = \mu_1\varphi \ \ln|\varphi|^2_{\mathbb{R}},$$
$$N_2(x,t) = \mu_2\psi \ \ln|\psi|^2_{\mathbb{R}},$$

where $\mu_j > 0$, $j = 1, 2$ and $|\cdot|_{\mathbb{R}}$ denote the absolute value of a real number.

The constants, ρ is the mass density, A the cross-sectional area, and I the moment of inertia. To simplify the notation, let us denote by $\rho_1 = \rho A$, $\rho_2 = \rho I$, $\kappa = kAG$ and $b = EI$. Under these conditions, we consider the initial-boundary problem for the following logarithmic Timoshenko System:

$$\rho_1\varphi_{tt} - \kappa\,(\varphi_x + \psi)_x + \gamma_1\varphi_t = \mu_1\varphi \ \ln|\varphi|^2 \quad \text{in } (0, L) \times (0, \infty), \tag{1}$$
$$\rho_2\psi_{tt} - b\psi_{xx} + \kappa\,(\varphi_x + \psi) + \gamma_2\psi_t = \mu_2\psi \ \ln|\psi|^2 \quad \text{in } (0, L) \times (0, \infty), \tag{2}$$
$$\varphi(x, 0) = \varphi_0(x), \quad \varphi_t(x, 0) = \varphi_1(x), \quad x \in (0, L), \tag{3}$$
$$\psi(x, 0) = \psi_0(x), \quad \psi_t(x, 0) = \psi_1(x), \quad x \in (0, L), \tag{4}$$
$$\varphi(0, t) = \varphi(L, t) = \psi(0, t) = \psi(L, t) = 0, \quad t \geq 0, \tag{5}$$

where $\gamma_i > 0$, $i = 1, 2$. Shear deformation effects were first introduced by Rankine [24] in 1858. Rotary inertia effects were apparently discovered independently by Bresse [6] in 1859 and Rayleigh [25] in 1945. One contributor to developing the theory that takes into account both effects was Paul Ehrenfest, who was cited by Timoshenko [27] in the footnote of his book, in Russian, Course in Elasticity (second volume) in 1916. Nowadays, this celebrated theory is often knowledge by Timoshenko's paper [28] of 1921. For more detailed historic context, see [10–12] with references therein.

The internal damping is associated with an oscillating system and produces a loss of energy to overcome external sources that act in the mechanical resistance of the material. Logarithmic non-linearity is a class of nonlinearities distinguished by several interesting physical properties, see [7]. It appears, for instance, in dynamics of Q-ball in theoretical physics [14], theories of quantum gravity [31], inflationary models [4], and quantum mechanics [5].

There are several studies on this competition, that is, stability analysis of the global solution taking account the effect provoked by the presence of both, stabilizing mechanism and source term. Below, we cite a few. [9] studied the existence and exponential stability of the global solution to a Klein–Gordon equation of Kirchhoff–Carrier type with strong damping and logarithmic source term. An extensible beam equation of Kirchhoff type with internal damping and source term was investigated in [22]. Kirchhoff plate equations with internal damping and logarithmic non-linearity were considered in [23]. General decay result for a plate equation with non-linear damping and a logarithmic source term was established in [2]. For global solution and blow-up of logarithmic Klein–Gordon equation, see [29].

Motivated by the above studies, in this paper, we prove the global existence for the problem (1)–(5) by applying the potential well theory introduced by Payne and Sattinger [20] and Sattinger [26]. Furthermore, we obtain the exponential decay of solution for this problem.

This paper is organized as follows: In the next section, we are going to give some preliminaries. Section 3 deals with potential well theory. We introduce the stability set. In Sect. 4, we prove the existence of global solution. In Sect. 5, we study the exponential decay. Finally, Sect. 6 is devoted to the numerical approach.

2 Preliminaries

We denote $L^2(0, L)$ the Hilbert's space of square-integrable function on the interval $(0, L)$, with the inner product

$$(u, v) = \int_0^L uv \, \mathrm{d}x, \quad \forall u, v \in L^2(0, L)$$

and norm

$$|u|^2 = (u, u) \quad \forall u \in L^2(0, L).$$

We use Sobolev space notation and properties as in [1]. We denote

$$H^1(0, L) = \{u \mid u \in L^2(0, L), \ u_x \in L^2(0, L)\}$$

and

$$H_0^1(0, L) = \left\{u \in H^1(0, L) \mid \; u(0) = u(L) = 0\right\}.$$

In this section, we present some results needed for the proof of our results. We start defining the energy functional associated with the problem (1)–(5)

$$E(t) = \frac{1}{2}\Big(\rho_1|\varphi_t(t)|^2 + \rho_2|\psi_t(t)|^2 + \kappa|\varphi_x(t) + \psi(t)|^2 + b|\psi_x(t)|^2 + \mu_1|\varphi(t)|^2 + \mu_2|\psi(t)|^2 \\ -\mu_1\int_0^L \varphi^2(t)\;\ln|\varphi(t)|_{\mathbb{R}}^2\,\mathrm{d}x - \mu_2\int_0^L \psi^2(t)\;\ln|\psi(t)|_{\mathbb{R}}^2\,\mathrm{d}x\Big). \tag{6}$$

Direct differentiation of (6) gives us

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) = -\gamma_1|\varphi_t(t)|^2 - \gamma_2\,|\psi_t(t)|^2. \tag{7}$$

Now, consider the following lemmas:

Lemma 2.1 (Sobolev–Poincaré Inequality) *Let p be a number with in $2 < p < \infty$ if $n = 1, 2$ or $2 \le p \le \dfrac{2n}{n-2}$ if $n \ge 3$, then there exist a constant $C > 0$, such that*

$$||u||_p \le C|u_x|, \quad \forall\; u \in H_0^1(0, L). \tag{8}$$

Lemma 2.2 (Aubin–Lions compactness Theorem [16], Theorem 5.1) *Let $T > 0$, $1 < p_0, p_1 < \infty$. Consider $B_0 \subset B \subset B_1$ Banach spaces, B_0, B_1 reflexives, B_0 with compact embedding in B. Define $W = \{u \mid u \in L^{p_0}(0, T; B_0),\; u_t \in L^{p_1}(0, T; B_1)\}$ equipped with the norm $||u||_W = ||u||_{L^{p_0}(0,T;B_0)} + ||u_t||_{L^{p_1}(0,T;B_1)}$. Then, W has compact embedding in $L^{p_0}(0, T; B)$.*

Lemma 2.3 (Lions [16], Lemma 1.3) *Let $Q = \Omega \times (0, T)$, $T > 0$ a bounded open set of $\mathbb{R}^n \times \mathbb{R}$ and $g_m, g : Q \to \mathbb{R}$ functions of $L^p(0, T; L^p(\Omega)) = L^p(Q)$, $1 < p < \infty$, such that $||g_m||_{L^p(Q)} \le C$, $g_m \to g$ a.e. in Q. Then, $g_m \rightharpoonup g$ in $L^p(Q)$ as $m \to \infty$.*

Lemma 2.4 (Nakao's Lemma) [18] *Suppose that $\phi(t)$ is a bounded nonnegative function on $\mathbb{R}^+$, satisfying*

$$\operatorname*{sup\,ess}_{t\le s\le t+1}\phi(s) \le C_0\left[\phi(t) - \phi(t+1)\right],$$

for any $t \ge 0$, where C_0 is a positive constant. Then

$$\phi(t) \le Ce^{-\alpha t}, \forall\, t \ge 0,$$

where C and α are positive constants.

3 The potential well

In this section, we present the potential well corresponding to the Eqs. (1)–(2). We define the operator $J : \left(H_0^1(0, L)\right)^2 \to \mathbb{R}$ by

$$J(\varphi, \psi) \overset{\text{def}}{=} \frac{1}{2}\left[\kappa|\varphi_x + \psi|^2 + b|\psi_x|^2 + \mu_1|\varphi|^2 + \mu_2|\psi|^2 - \mu_1\int_0^L \varphi^2\;\ln\;|\varphi|_{\mathbb{R}}^2\mathrm{d}x - \mu_2\int_0^L \psi^2\;\ln\;|\psi|_{\mathbb{R}}^2\mathrm{d}x\right].$$

For $(\varphi,\; \psi) \in \left(H_0^1(0, L)\right)^2$ and $\lambda > 0$, we have

$$J(\lambda\varphi, \lambda\psi) \overset{\text{def}}{=} \frac{\lambda^2}{2}\Big[\kappa|\varphi_x + \psi|^2 + b|\psi_x|^2 + \mu_1|\varphi|^2 + \mu_2|\psi|^2 - 2\mu_1\ln\,\lambda\int_0^L \varphi^2\,\mathrm{d}x \\ -\mu_1\int_0^L \varphi^2\ln\;|\varphi|_{\mathbb{R}}^2\,dx - 2\mu_1\ln\,\lambda\int_0^L \psi^2\,\mathrm{d}x - \mu_2\int_0^L \psi^2\;\ln\;|\psi|_{\mathbb{R}}^2\,\mathrm{d}x\Big].$$

Associated with J, we have the well-known Nehari Manifold

$$\mathcal{N} \overset{def}{=} \left\{ \left(\varphi, \psi\right) \in \left(H_0^1(0, L)\right)^2 / \{0\}; \quad \left[\frac{\mathrm{d}}{\mathrm{d}\lambda} J(\lambda\varphi, \lambda\psi)\right]_{\lambda=1} = 0 \right\}.$$

Equivalently

$$\mathcal{N} = \left\{ \left(\varphi, \psi\right) \in \left(H_0^1(0, L)\right)^2;\ \kappa|\varphi_x + \psi|^2 + b|\psi_x|^2 = \mu_1 \int_0^L \varphi^2 \ \ln\ |\varphi|_{\mathbb{R}}^2 \,\mathrm{d}x + \mu_2 \int_0^L \psi^2 \ \ln\ |\psi|_{\mathbb{R}}^2 \,\mathrm{d}x \right\}.$$

We define as in the Mountain Pass theorem due to Ambrosetti and Rabinowitz [3]

$$d \overset{\text{def}}{=} \inf_{\left(\varphi,\psi\right) \in \left(H_0^1(0,L)\right)^2 / \{0\}} \sup_{\lambda > 0} J(\lambda u).$$

According to Willem [30], Theorem 4.2, the depth of the well d is a strictly positive constant given by

$$0 < d = \inf_{\varphi, \psi\ \in \mathcal{N}} J(\lambda u).$$

Now, we introduce

$$W = \left\{ \left(\varphi, \psi\right) \in \left(H_0^1(0, L\right)^2;\ \ J(\varphi, \psi) < d \right\} \cup \{0\},$$

and partition it into two sets as follows:

$$W_1 = \left\{ \left(\varphi, \psi\right) \in W;\ \kappa|\varphi_x + \psi|^2 + b|\psi_x|^2 > \mu_1 \int_0^l \varphi^2 \ \ln\ |\varphi|_{\mathbb{R}}^2 + \mu_2 \int_0^l \psi^2 \ \ln\ |\psi|_{\mathbb{R}}^2 \right\} \cup \{0\}$$

and

$$W_2 = \left\{ \left(\varphi, \psi\right) \in W;\ \kappa|\varphi_x + \psi|^2 + b|\psi_x|^2 < \mu_1 \int_0^l \varphi^2 \ \ln\ |\varphi|_{\mathbb{R}}^2 + \mu_2 \int_0^l \psi^2 \ \ln\ |\psi|_{\mathbb{R}}^2 \right\}.$$

Therefore, we define by W_1 the set of stability for the problem (1)–(5).

Proposition 3.1 *Let* $\left(\varphi_0, \psi_0\right) \in W_1$ *and* $\left(\varphi_1, \psi_1\right) \in \left(L^2(0, L)\right)^2$. *If* $E(0) < d$, *then* $(\varphi, \psi) \in W_1$.

Proof We introduce the functional $I(\varphi, \psi)$ given by

$$I(\varphi, \psi) \overset{\text{def}}{=} \frac{1}{2} \left[\kappa|\varphi_x + \psi|^2 + b|\psi_x|^2 - \mu_1 \int_\Omega \varphi^2 \ln |\varphi|_{\mathbb{R}}^2 \mathrm{d}x - \mu_2 \int_\Omega \psi^2 \ln |\psi|_{\mathbb{R}}^2 \mathrm{d}x \right].$$

Let $T > 0$. From (7), we get

$$E(t) \le E(0) < d, \ \text{ for all } t \in [0, T),$$

and then

$$\frac{1}{2} \left[|\varphi_t(t)|^2 + |\psi_t(t)|^2 \right] + J(\varphi(t), \psi(t)) < d, \ \text{ for all } t \in [0, T). \tag{9}$$

Note that in W_1, we have $I(\varphi(t), \psi(t)) > 0$ for all $t \in (0, T)$. Arguing by contradiction, we suppose that there exists a first $t_0 \in (0, T)$, such that $I(\varphi(t_0), \psi(t_0)) = 0$ and $I(\varphi(t), \psi(t)) > 0$ for all $0 \le t < t_0$, that is

$$\frac{1}{2} \left[|\varphi_t(t_0)|^2 + |\psi_t(t_0)|^2 \right] + J(u(t_0), v(t_0)) = 0.$$

From the definition of $\mathcal{N}$, we have that $(\varphi(t_0), \psi(t_0)) \in \mathcal{N}$, which leads to

$$J(\varphi(t_0), \psi(t_0)) \ge \inf_{(\varphi(t), \psi(t)) \in \mathcal{N}} J(u(t), v(t)) = d.$$

We deduce

$$\frac{1}{2} \left[|\varphi_t(t_0)|^2 + |\psi_t(t_0)|^2 \right] + J(\varphi(t_0), \psi(t_0)) \ge d,$$

which contradicts with (9). Then, $(\varphi(t), \psi(t)) \in W_1$ for all $t \in [0, T)$. □

4 Existence of global weak solution

In this section, we prove the existence of global weak solutions.

Theorem 4.1 *Let $(\varphi_0, \psi_0) \in W_1$, $E(0) < d$ and $(\varphi_1, \psi_1) \in (L^2(0, L))^2$. Then, the problem (1)–(5) admits a weak solution (φ, ψ) in the class*

$$(\varphi, \psi) \in \left(L^\infty_{\text{loc}}\left(0, \infty; H^1_0(0, L)\right)\right)^2 \tag{10}$$

$$(\varphi_t, \psi_t) \in \left(L^\infty_{\text{loc}}\left(0, \infty; L^2(0, L)\right)\right)^2 \tag{11}$$

satisfying $w, z \in H^1_0(0, L)$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_1\varphi_t(t), w\right) + \left(\kappa(\varphi_x + \psi)(t), w_x\right) + \left(\gamma_1\varphi_t(t), w\right) - \left(\mu_1\varphi(t)\ \ln|\varphi(t)|^2_{\mathbb{R}}, w\right) = 0, \tag{12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_2\psi_t(t), z\right) + \left(b\psi_x(t), z_x\right) + \left(\kappa(\varphi_x + \psi)(t), z\right) + \left(\gamma_2\psi_t(t), z\right) - \left(\mu_2\psi(t)\ \ln|\psi(t)|^2_{\mathbb{R}}, z\right) = 0, \tag{13}$$

$$(\varphi, \psi)(x, 0) = (\varphi_0, \psi_0), \tag{14}$$

$$(\varphi_t, \psi_t)(x, 0) = (\varphi_1, \psi_1), \tag{15}$$

in $\mathcal{D}'(0, T)$.

Proof We use the Faedo–Galerkin's method. The proof of the global existence of solutions will be made in three steps: approximated problem, a priori estimates, and passage to the limit. □

4.1 Approximated problem

Let $(w_\nu)_{\nu\in\mathbb{N}}$ be a basis of $H^1_0(0, L)$ from the eigenvectors of the operator $-\Delta$, and

$$V_m = \operatorname{span}\{w_1,\ w_2,\ \dots, w_m\}.$$

Consider

$$\varphi^m(t) = \sum_{j=1}^{m} g_{jm}(t)w_j \quad \text{and} \quad \psi^m(t) = \sum_{j=1}^{m} h_{jm}(t)w_j$$

a solution of the approximated problem

$$\left(\rho_1\varphi^m_{tt}(t), w\right) + \left(\kappa(\varphi^m_x(t) + \psi^m(t)), w_x\right) + (\gamma_1\varphi^m_t(t), w) - \left(\mu_1\varphi^m(t)\ \ln|\varphi^m(t)|^2, w\right) = 0, \tag{16}$$

$$(\rho_2\psi^m_{tt}(t), z) + \left(b\psi^m_x(t), z_x\right) + \left(\kappa(\varphi^m_x(t) + \psi^m(t)), z\right) + (\gamma_2\psi^m_t(t), z) - \left(\mu_2\psi^m(t)\ \ln|\psi^m(t)|^2, z\right) = 0, \tag{17}$$

$$\left(\varphi^m(0), \psi^m(0)\right) = (\varphi_{0m}, \psi_{0m}) \longrightarrow (\varphi_0, \psi_0) \text{ strongly in } \left(H^1_0(0, l)\right)^2, \tag{18}$$

$$\left(\varphi^m_t(0), \psi^m_t(0)\right) = (\varphi_{1m}, \psi_{1m}) \longrightarrow (\varphi_1, \psi_1) \text{ strongly in } \left(L^2(0, l)\right)^2, \tag{19}$$

$\forall\ w, z \in V_m$. By virtue of Carathéodory's theorem, see [8], the system (16) has a local solution in $[0, t_m)$, $0 < t_m \leq T$. The extension of the solution to the whole interval $[0, T]$ is a consequence of the following a priori estimates.

4.2 A priori estimates

Let $w = \varphi_t^m(t)$ and $z = \psi_t^m(t)$ in (16) and (17), respectively. Then, we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\rho_1\left|\varphi_t^m(t)\right|^2 + \rho_2\left|\psi_t^m(t)\right|^2 + \kappa\left|\varphi_x^m(t) + \psi^m(t)\right|^2 + b\left|\psi_x^m(t)\right|^2 + \mu_1\left|\varphi^m(t)\right|^2 + \mu_2\left|\psi^m(t)\right|^2$$
$$-\mu_1\int_0^L \varphi^m(t)^2 \ \ln\ \left|\varphi^m(t)\right|_{\mathbb{R}}^2 - \mu_2\int_0^L \psi^m(t)^2 \ \ln\ \left|\psi^m(t)\right|_{\mathbb{R}}^2 \ \mathrm{d}x\Big] + \gamma_1\left|\varphi_t^m(t)\right|^2 + \gamma_2\left|\psi_t^m(t)\right|^2 = 0.$$

From (6), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}E_m(t) + \gamma_1\left|\varphi_t^m(t)\right|^2 + \gamma_2\left|\psi_t^m(t)\right|^2 = 0, \tag{20}$$

where $E_m(t)$ is the approximated energy of the problem (16). Now, integrating (20) from 0 to t, $0 \le t \le t_m$, we obtain

$$E_m(t) + \gamma_1\int_0^t \left|\varphi_t^m(t)\right|^2 \mathrm{d}s + \gamma_2\int_0^t \left|\psi_t^m(s)\right|^2 \mathrm{d}s\,\mathrm{d}s = E_m(0). \tag{21}$$

Thus

$$E_m(t) + \gamma_1\int_0^t |\varphi_t^m(s)|^2\,\mathrm{d}s + \gamma_2\int_0^t \left|\psi_t^m(s)\right|^2\mathrm{d}s = \rho_1\,|\varphi_{1m}|^2 + \rho_2\,|\psi_{1m}|^2 + \kappa\,|\varphi_{0mx} + \psi_{0m}|^2$$
$$+b\,|\psi_{0mx}|^2 + \gamma_1\,|\varphi_{0m}|^2 + \gamma_2\,|\psi_{0m}|^2 - \mu_1\int_0^L \varphi_{0m}^2\ \ln\ |\varphi_{0m}|_{\mathbb{R}}^2 - \mu_2\int_0^L \psi_{0m}^2\ \ln\ |\psi_{0m}|_{\mathbb{R}}^2\ \mathrm{d}x,$$

which gives us the following estimate:

$$E_m(t) + \gamma_1\int_0^t \left|\varphi_t^m(s)\right|^2\mathrm{d}s + \gamma_2\int_0^t \left|\psi_t^m(s)\right|^2\mathrm{d}s \le \rho_1\,|\varphi_{1m}|^2 + \rho_2\,|\psi_{1m}|^2 + J\left(\varphi_{0m}, \psi_{0m}\right).$$

We have that $J(\varphi_{0m}, \psi_{0m}) < d$, and then, by (16), we get

$$E_m(t) + \mu_1\int_0^t \left|\varphi_t^m(s)\right|^2\mathrm{d}s + \mu_2\int_0^t \left|\psi_t^m(s)\right|^2\mathrm{d}s \le C_1, \tag{22}$$

where C_1 is a positive constant independent of m and t.

These estimates imply that the approximated solution (φ^m, ψ^m) exists globally in $[0, \infty)$. See [13]. Then, by estimate (22), we have

$$\left(\varphi^m\right), \left(\psi^m\right) \ \text{ are bounded in } \ L_{\mathrm{loc}}^\infty\left(0, T; H_0^1(0, L)\right) \tag{23}$$
$$\left(\varphi_t^m\right), \left(\psi_t^m\right) \ \text{ are bounded in } \ L_{\mathrm{loc}}^\infty\left(0, T; L^2(0, L)\right). \tag{24}$$

Now, by the logarithmic inequality

$$\left|t^2\ \ln t\right| \le C\left(1 + |t|^3\right),$$

we get

$$\mu_1\int_0^L \left|\varphi^m(t)\ \ln\left|\varphi^m(t)\right|_{\mathbb{R}}^2\right|^2\ \mathrm{d}x = 4\mu_1\int_0^L \left|\varphi^m(t)\right|_{\mathbb{R}}^2\ \ln\left|\varphi^m(t)\right|_{\mathbb{R}}^2\ \mathrm{d}x$$
$$= 4\mu_1\int_{x\in(0,L);\ |\varphi^m|<1} \left|\varphi^m(t)\right|_{\mathbb{R}}^2 \ln\left|\varphi^m(t)\right|^2\mathrm{d}x + 4\mu_1\int_{x\in(0,L);\ |\varphi^m|\ge 1} \left|\varphi^m(t)\right|_{\mathbb{R}}^2\ \ln\left|\varphi^m(t)\right|^2\mathrm{d}x$$
$$\le 4\mu_1\int_0^L \left|\varphi^m(t)\right|_{\mathbb{R}}^2\ \mathrm{d}x + 4\mu_1\int_0^L \left|\varphi^m(t)\right|_{\mathbb{R}}^4\ \ln\left|\varphi^m(t)\right|_{\mathbb{R}}^2\ \mathrm{d}x \le 4\mu_1\left|\varphi^m(t)\right|^2 + 4\mu_1 C\int_0^L \left(1 + \left|\varphi^m(t)\right|_{\mathbb{R}}^6\right)\mathrm{d}x$$
$$= 4\mu_1\left|\varphi^m(t)\right|^2 + 4\mu_1 CL + C\left|\varphi^m(t)\right|_2^6 \le \mu_1\left|\varphi^m(t)\right|^2 + CL + C\left|\varphi^m(t)\right|^6 \le \tilde{C}_1. \tag{25}$$

Analogously, we have

$$\mu_2 \int_0^L |\varphi^m(t)\ \ln \left|\varphi^m(t)\right|^2_{\mathbb{R}}|^2 \,\mathrm{d}x \leq \tilde{C}_2, \tag{26}$$

where $\tilde{C}_1$ and $\tilde{C}_2$ are constant independent of m and t. From (25) and (26), we get

$$\varphi^m \ln|\varphi|^2_{\mathbb{R}} \text{ are bounded in } L^2_{loc}\left(0, \infty; L^2(0, L)\right), \tag{27}$$
$$\psi^m \ln|\psi|^2_{\mathbb{R}} \text{ are bounded in } L^2_{loc}\left(0, \infty; L^2(0, L)\right). \tag{28}$$

4.3 Passage to the limit

From estimates (23) and (24), there exists a subsequence of (φ^m), (ψ^m) also denoted by (φ^m), (ψ^m), such that

$$(\varphi^m), (\psi^m) \overset{*}{\rightharpoonup} \varphi, \psi \text{ weakly star in } L^\infty_{loc}\left(0, \infty; H^1_0(0, L)\right), \tag{29}$$
$$(\varphi^m_t), (\psi^m_t) \overset{*}{\rightharpoonup} \varphi_t, \psi_t \text{ weakly in } L^\infty_{loc}; L^2(0, L)). \tag{30}$$

Applying the Aubin–Lions compactness Theorem (Lemma 2.2), we get from (29) and (30)

$$\left(\varphi^m\right), \left(\psi^m\right) \longrightarrow \varphi, \psi \text{ strongly in } L^2_{loc}\left(0, \infty; L^2(0, L)\right), \tag{31}$$

and for all $T > 0$

$$\left(\varphi^m\right) \longrightarrow \varphi \text{ a.e in } (0, L) \times (0, T) \tag{32}$$
$$\left(\psi^m\right) \longrightarrow \psi \text{ a.e in } (0, L) \times (0, T). \tag{33}$$

Now, since that $f(s) = s\ ln|s|^2$ is continuous, we have the convergence

$$\mu_1\varphi^m\ \ln\left|\varphi^m\right|^2_{\mathbb{R}} \longrightarrow \mu_1\varphi\ \ln|\varphi|^2_{\mathbb{R}} \text{ a.e in } (0, L) \times (0, T) \tag{34}$$

and

$$\mu_2\psi^m\ \ln\left|\psi^m\right|^2_{\mathbb{R}} \longrightarrow \mu_2\psi\ \ln|\psi|^2_{\mathbb{R}} \text{ a.e in } (0, L) \times (0, T). \tag{35}$$

From (27), (28), (34), and (35) using the Lions's Lemma (Lemma 2.3), we obtain

$$\mu_1\varphi^m\ \ln\left|\varphi^m\right|^2_{\mathbb{R}} \rightharpoonup \mu_1\varphi\ \ln|\varphi|^2_{\mathbb{R}} \text{ weakly in } L^2_{loc}\left(0, \infty; L^2(0, L)\right) \tag{36}$$

and

$$\mu_2\psi^m\ \ln\left|\psi^m\right|^2_{\mathbb{R}} \rightharpoonup \mu_2\psi\ \ln|\psi|^2_{\mathbb{R}} \text{ weakly in } L^2_{loc}\left(0, \infty; L^2(0, L)\right). \tag{37}$$

By the convergences (23), (24), (34) and (35), we can pass to the limit in the approximate system (16) and (17) and obtain for all $w, z \in H^1_0(0, L)$

$$\frac{\mathrm{d}}{\mathrm{d}t}(\rho_1\varphi_t(t), w) + (\kappa(\varphi_x + \psi)(t), w_x) + (\gamma_1\varphi_t(t), w) - \left(\mu_1\varphi(t)\ \ln|\varphi(t)|^2_{\mathbb{R}}, w\right) = 0, \tag{38}$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_2\psi_t(t), z\right) + \left(b\psi_x(t), z_x\right) + \left(\kappa(\varphi_x + \psi)(t), z\right) + \left(\gamma_2\psi_t(t), z\right) - \left(\mu_2\psi(t)\ \ln|\psi(t)|^2_{\mathbb{R}}, z\right) = 0,$$
$$\text{in} \quad \mathcal{D}'(0, T). \tag{39}$$

The verification of the initial data is obtained in a standard way.

5 Exponential decay

In this section, we provide the exponential decay of the energy associated with the system solution (1)–(5).

Theorem 5.1 *Under the hypothesis of Theorem 4.1, the energy associated with problem (1)–(5) satisfies*

$$E(t) \leq C_0\, e^{-\alpha t}, \quad \forall\, t \geq 0,$$

where C_0 and α are positive constants.

Proof Let $w = \varphi_t(t)$ and $z = \psi_t(t)$ in (38) and (39), respectively, and summing up the result, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\Big[\rho_1|\varphi_t(t)|^2 + \rho_2|\psi(t)|^2 + \kappa|\varphi_x(t) + \psi(t)|^2) + b|\psi_x(t)|^2 + \mu_1|\varphi(t)|^2 + \mu_2|\psi(t)|^2\\ &-\mu_1\int_0^L |\varphi(t)|_{\mathbb{R}}^2\, \ln|\varphi(t)|_{\mathbb{R}}^2\,\mathrm{d}x - \mu_2\int_0^L |\psi(t)|_{\mathbb{R}}^2\,\ln|\psi(t)|_{\mathbb{R}}^2\,\mathrm{d}x\Big] + \gamma_1|\varphi_t(t)|^2 + \gamma_2|\psi_t(t)|^2 = 0, \quad (40)\end{aligned}$$

that is

$$\frac{\mathrm{d}}{\mathrm{d}t}E(t) + \gamma_1|\varphi_t(t)|^2 + \gamma_2|\psi_t(t)|^2 \leq 0, \tag{41}$$

where $E(t)$ is define in (6). Integrating (40) from t to $t+1$, we obtain

$$\int_t^{t+1} \left[\gamma_1|\varphi_t(s)|^2 + \gamma_2|\psi_t(s)|^2\right]\,\mathrm{d}s \leq E(t) - E(t+1) \overset{\text{def}}{:=} F^2(t); \tag{42}$$

therefore, there exist $t_1 \in \left[t, t+\frac{1}{4}\right]$ and $t_2 \in \left[t+\frac{3}{4}, t+1\right]$, such that

$$\gamma_1\,|\varphi_t(t_i)|^2 + \gamma_2\,|\psi_t(t_i)|^2 \leq 4F(t_i), \qquad i = 1, 2. \tag{43}$$

Let $w = \varphi(t)$ and $z = \psi(t)$ in (38) and (39), respectively. Summing the result, we get

$$\begin{aligned}&b|\psi_x(t)|^2 + \kappa|\varphi_x(t) + \psi(t)|^2 - \mu_1\int_0^L (\varphi(t))^2\,\ln|\varphi(t)|_{\mathbb{R}}^2\,\mathrm{d}x - \mu_2\int_0^L (\psi(t))^2\,\ln|\psi(t)|_{\mathbb{R}}^2\,\mathrm{d}x\\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\rho_1\,(\varphi_t(t), \varphi(t)) + \rho_1|\varphi_t(t)|^2 - \frac{\mathrm{d}}{\mathrm{d}t}\rho_2\,(\psi_t(t), \psi(t)) + \rho_2\,|\psi_t(t)|^2 - \gamma_1\,(\varphi_t(t), \varphi(t))\\ &-\gamma_2\,(\psi_t(t), \psi(t))\,. \end{aligned}\tag{44}$$

Integrating (44) from t_1 to t_2, and using (43), we obtain

$$\begin{aligned}&\int_{t_1}^{t_2}\left[b\,|\psi_x(t)|^2 + \kappa\,|\varphi_x(t) + \psi(t)|^2 - \mu_1\int_0^L (\varphi(t))^2\,\ln|\varphi(t)|_{\mathbb{R}}^2\,\mathrm{d}x - \mu_2\int_0^L (\psi(t))^2\,\ln|\psi(t)|_{\mathbb{R}}^2\,\mathrm{d}x\right]\mathrm{d}s\\ &\leq \rho_1\,|\varphi_t(t_1)|\,|\varphi(t_1)| + \rho_1\,|\varphi_t(t_2)|\,|\varphi(t_2)| + \rho_2\,|\psi_t(t_1)|\,|\psi(t_1)| + \rho_2\,|\psi_t(t_2)|\,|\psi(t_2)|\\ &+\rho_1\int_{t_1}^{t_2}|\varphi_t(s)|^2\,\mathrm{d}s + \rho_2\int_{t_1}^{t_2}|\psi_t(s)|^2\,\mathrm{d}s + \gamma_1\int_{t_1}^{t_2}|\varphi_t(s)|\;|\varphi(s)|\;\mathrm{d}s + \gamma_2\int_{t_1}^{t_2}|\psi_t(s)|\,|\psi(s)|\,\mathrm{d}s;\end{aligned}$$

therefore

$$\begin{aligned}&\int_{t_1}^{t_2}\left[b|\psi_x(t)|^2 + \kappa\,|\varphi_x(t) + \psi(t)|^2 - \mu_1\int_0^L (\varphi(t))^2\,\ln|\varphi(t)|_{\mathbb{R}}^2\,\mathrm{d}x - \mu_2\int_0^L (\psi(t))^2\,\ln|\psi(t)|_{\mathbb{R}}^2\,\mathrm{d}x\right]\mathrm{d}s\\ &\leq C_1\left[F(t)\,\underset{t\leq s\leq t+1}{\text{sup ess}}\,E^{1/2}(s) + \frac{1}{4}\,\underset{t\leq s\leq t+1}{\text{sup ess}}\,E(s) + F^2(t)\right] \overset{\text{def}}{:=} G^2(t),\end{aligned}\tag{45}$$

where $C_1 = C_1(\rho_1, \rho_2, \gamma_1, \gamma_2) > 0$ is a constant. Now, from (42) and (45), we get

$$\int_{t_1}^{t_2}\Bigg[\rho_1|\varphi_t(t)|^2+\rho_2|\psi(t)|^2+b|\psi_x(t)|^2+\kappa|\varphi_x(t)+\psi(t)|^2-\mu_1\int_0^L(\varphi(t))^2\,\ln|\varphi(t)|^2_{\mathbb{R}}\,\mathrm{d}x$$
$$-\mu_2\int_0^L(\psi(t))^2\,\ln|\psi(t)|^2_{\mathbb{R}}\,\mathrm{d}x\Bigg]\,\mathrm{d}s\le 2\left[F^2(t)+G^2(t)\right]; \tag{46}$$

thus, there exists $t^*\in[t_1,t_2]$, such that

$$\rho_1\left|\varphi_t(t^*)\right|^2+\rho_2\left|\psi(t^*)\right|^2+b\left|\psi_x(t^*)\right|^2+\kappa\left|\varphi_x(t^*)+\psi(t^*)\right|^2-\mu_1\int_\Omega\left(\varphi(t^*)\right)^2\,\ln\left|\varphi(t^*)\right|^2_{\mathbb{R}}\,\mathrm{d}x$$
$$-\mu_2\int_0^L\left(\psi(t^*)\right)^2\,\ln|\psi(t^*)|^2_{\mathbb{R}}\,\mathrm{d}x\le C_2\left[F^2(t)\,+\,G^2(t)\right]. \tag{47}$$

We deduce

$$\left|\varphi(t^*)\right|^2+\left|\psi(t^*)\right|^2\le C_3\left[\left|\varphi_x(t^*)+\psi(t^*)\right|^2+\left|\psi_x(t^*)\right|^2\right]. \tag{48}$$

By (47) and (48), we have

$$E(t^*)\le C_4\left[F^2(t)+G^2(t)\right]. \tag{49}$$

Since that $E(t)$ is increasing, by (42), (48), and (49), we obtain

$$\begin{aligned}\sup_{t\le s\le t+1}\text{ess}\;\;E(s)&\le E(t^*)+\int_t^{t+1}\left[\gamma_1|\varphi_t(s)|^2+\gamma_2|\psi_t(s)|^2\right]\,\mathrm{d}s\\&\le C_5\left[F^2(t)\,+\,G^2(t)\right]\\&\le C_6\left[F(t)\sup_{t\le s\le t+1}\text{ess}\;\;E^{1/2}(s)+F^2(t)+\frac{1}{4}\sup_{t\le s\le t+1}\text{ess}\;\;E(s)\right]\\&\le C_7F^2(t)+\frac{1}{2}\sup_{t\le s\le t+1}\text{ess}\;\;E(s).\end{aligned}$$

Hence, by Nakao's Lemma (Lemma 42)

$$\sup_{t\le s\le t+1}\text{ess}\;\;E(s)\;\le\;C_8F^2(t)=C_9[E(t)-E(t+1)],$$

where $C_i=1,2,\ldots,9$ are positive constants. By Lemma (2.4), we conclude

$$E(t)\le C_0e^{-\alpha t},\quad\forall\,t\ge 0,$$

where C_0 and α are positive constants. □

6 Numerical approach

6.1 Variational formulation

Here, we use a representation to the functions φ, ψ and logarithmic source terms by component vectorial

$$\mathbf{u}=[\varphi,\psi]^\top\quad\text{and}\;\mathcal{F}(\mathbf{u})=\left[\mu_1\varphi\ln|\varphi|^2_{\mathbb{R}},\mu_2\psi\ln|\psi|^2_{\mathbb{R}}\right]^\top.$$

Thus, from (1) to (5), we get the following variational problem:

$$(\mathbf{u}_{tt}(t),\tilde{\mathbf{u}})+a_1(\mathbf{u}(t),\tilde{\mathbf{u}})+a_2(\mathbf{u}_t(t),\tilde{\mathbf{u}})=a_3(\mathcal{F}(\mathbf{u}),\tilde{\mathbf{u}}), \tag{50}$$

where $\mathbf{u}$ satisfies the initial conditions

$$(\mathbf{u}(0),\tilde{\mathbf{u}})=(\mathbf{u}_0,\tilde{\mathbf{u}}),\quad(\mathbf{u}_t(0),\tilde{\mathbf{u}})=(\mathbf{u}_1,\tilde{\mathbf{u}}). \tag{51}$$

Here

$$\begin{aligned}
(\mathbf{u}_{tt}(t), \tilde{\mathbf{u}}) &= \rho_1 (\varphi_t, u_1) + \rho_2 (\psi_t, u_2),\\
a(\mathbf{u}(t), \tilde{\mathbf{u}}) &= \kappa \left(\varphi_x + \psi, u_{1,x} + u_2\right) + b\left(\psi_x, u_{2,x}\right),\\
(\mathbf{u}_t(t), \tilde{\mathbf{u}}) &= \gamma_1 (\varphi_t, u_1) + \gamma_2 (\psi_t, u_2),\\
(\mathcal{F}(\mathbf{u}), \tilde{\mathbf{u}}) &= \mu_1 \left(\varphi \ln |\varphi|^2_{\mathbb{R}}, u_1\right) + \mu_2 \left(\psi \ln |\psi|^2_{\mathbb{R}}, u_2\right).
\end{aligned}$$

Here

$$a : \mathbb{U} \times \mathbb{U} \mapsto \mathbb{R},$$

where

$$\mathbb{U} = H^1_0(0, L) \times H^1_0(0, L).$$

6.2 Algorithms

Here, we developed an algorithms to obtain the numerical solutions and verified the properties of theoretical results to the Timoshenko beam's with Logarithms source. We adopt our approximated solution by Finite-Element Method (FEM), in spatial variable and a finite difference method in the temporal variable with iterative methods. First, we consider a partition X_h over the interval $\Omega = (0, L)$, that is, $X_h = \{0 = x_0 < x_1 < \cdots < x_N = L\}$, $\Omega_{j+1} = (x_j, x_{j+1})$, and, $\Omega_i \bigcap \Omega_j = \emptyset$, $i \neq j$ and $\Omega = \bigcup_{e=1}^{Ne} \overline{\Omega_e}$ where N_e is the number of the elements obtained of partition. We consider the following finite-dimensional subspaces:

$$\begin{aligned}
S^h_1 &= \left\{u \in C(0, L); u\Big|_{\Omega_e} \in P_1(\Omega_e)\right\},\\
U^h &= \left\{u^h \in S^h_1; u^h(0) = u^h(L) = 0\right\},
\end{aligned}$$

where P_1 is the set linear polynomials defined over the element Ω_e. We use a representation of the numerical solution $\mathbf{u}^h = [\varphi^h, \psi^h]^\top$ analogous like in [17], and then, we have $\mathbf{u}^h(t, x) = \sum_{i=1}^{2N} d_i(t)\phi_i(x)$ where $2N$ is the number total of degrees of freedom of the finite-element approximation, and $\phi_i(x)$, $i = 1, \cdots, 2N$, are the global vector interpolation functions. Therefore, we obtain the following dynamical problem in $\mathbb{R}^{2N}$:

$$\begin{aligned}
\mathbf{M}\ddot{\mathbf{d}}(t) + \mathbf{C}\dot{\mathbf{d}}(t) + \mathbf{K}\mathbf{d}(t) &= \mathbf{F}(\mathbf{d}(t)),\\
\mathbf{d}(0) &= \mathbf{d}_0,\\
\dot{\mathbf{d}}(0) &= \dot{\mathbf{d}}_1,
\end{aligned}$$

where $\mathbf{M}$: the consistent mass matrix, $\mathbf{C}$: the damping matrix, $\mathbf{K}$: the vector of consistent nodal elastic stiffness at time t, and $\mathbf{F}(\mathbf{d}(t))$: the vector of consistent nodal to logarithmic source at time t, and $\mathbf{d}(t)$: the vector of displacement nodal generalized at time t. Furthermore, $\mathbf{d}_0$ and $\dot{\mathbf{d}}_1$ are displacement and velocities, nodal initial, respectively.

To solve this system above, we introduce a partition P of the time domain $[0, T]$ into M intervals of length Δt, such that $0 = t_0 < t_1 < \cdots < t_M = T$, with $t_{n+1} - t_n = \Delta t$ and we use the well-known Newmark's methods [19]. Since, in our work, we have a non-linear system we need to modify our scheme

$$\begin{aligned}
\mathbf{M}\ddot{\mathbf{d}}_{n+1} + \mathbf{C}\dot{\mathbf{d}}_{n+1} + \mathbf{K}\mathbf{d}_{n+1} &= \mathbf{F}(\mathbf{d}_{n+1})\\
\mathbf{d}_{n+1} &= \mathbf{d}_n + \Delta t\dot{\mathbf{d}}_n + \frac{\Delta t^2}{2}\left[(1 - 2\beta)\ddot{\mathbf{d}}_n + 2\beta\ddot{\mathbf{d}}_{n+1}\right]\\
\dot{\mathbf{d}}_{n+1} &= \dot{\mathbf{d}}_n + \Delta t\left[(1 - \gamma)\ddot{\mathbf{d}}_n + \gamma\ddot{\mathbf{d}}_{n+1}\right],
\end{aligned}$$

where β, γ and α are two parameters that govern the stability and accuracy of the methods. In this case

$$\mathbf{M} = \bigcup_{e=1}^{N} \mathbf{m}^e, \ \mathbf{C} = \bigcup_{e=1}^{N} \mathbf{c}^e \text{ and } \mathbf{K} = \bigcup_{e=1}^{N} (\mathbf{k}^e_b + \mathbf{k}^e_s);$$

for instance, considering linear functions, we have

$$\mathbf{m^e} = \begin{bmatrix} \rho_1 h/3 & 0 & \rho_1 h/6 & 0 \\ 0 & \rho_2 h/3 & 0 & \rho_2 h/6 \\ \rho_1 h/6 & 0 & \rho_1 h/3 & 0 \\ 0 & \rho_2 h/6 & 0 & \rho_2 h/3 \end{bmatrix}, \quad c^e = \begin{bmatrix} \gamma_1 h/3 & 0 & \gamma_1 h/6 & 0 \\ 0 & \gamma_2 h/3 & 0 & \gamma_2 h/6 \\ \gamma_1 h/6 & 0 & \gamma_1 h/3 & 0 \\ 0 & \gamma_2 h/6 & 0 & \gamma_2 h/3 \end{bmatrix}$$

$$\mathbf{k_b^e} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & b/h & 0 & -b/h \\ 0 & 0 & 0 & 0 \\ 0 & -b/h & 0 & b/h \end{bmatrix}, \quad k_s^e = \begin{bmatrix} \kappa/h & -\kappa/2 & -\kappa/h & -\kappa/2 \\ -\kappa/2 & \kappa h/3 & \kappa/2 & \kappa h/6 \\ -\kappa/h & \kappa/2 & \kappa/h & \kappa/2 \\ -\kappa/2 & \kappa h/6 & \kappa/2 & \kappa h/3 \end{bmatrix}.$$

Due to its non-linearity, we have a vector $\mathbf{F}(\mathbf{d}(t))$ with entries for each element of

$$\mathbf{F^e} = \left[\int_{\Omega_e} \mu_1(\mathbf{u}^h(t)) \ln |\mathbf{u}^h(t)|^2 \phi_i^e \, \mathrm{d}x, \int_{\Omega_e} \mu_2(\mathbf{u}^h(t)) \ln |\mathbf{u}^h(t)|^2 \phi_i^e \, \mathrm{d}x \right]^\top.$$

These vectorial components are obtained by Gaussian Quadrature using two points.

Remark 6.1 We point out to numerical pathology which occurs in penalized systems the locking problem, in particular, to Timoshenko system, it is the shear locking. It is characterized by the following over-estimation about the coefficient b, given by:

$$b = EI\left(1 + \frac{\kappa G A h^2}{12 EI}\right).$$

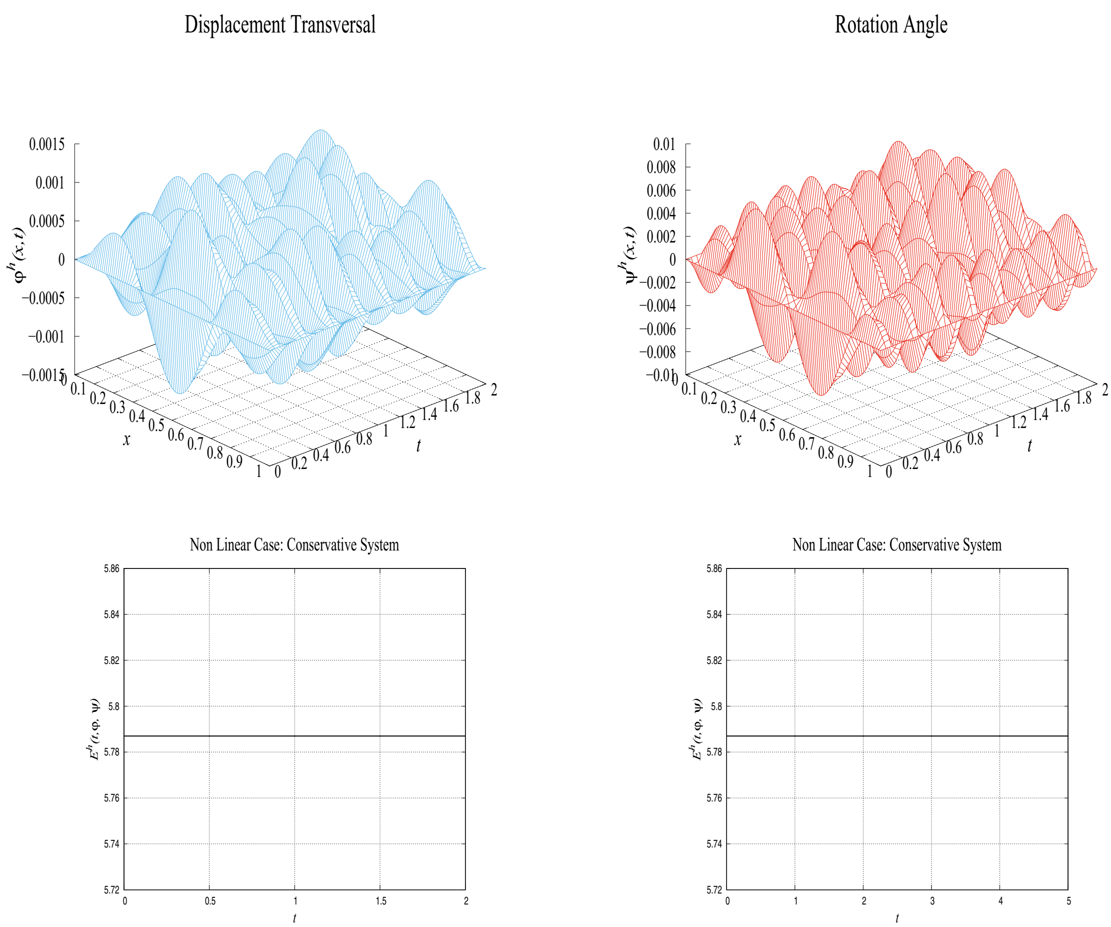

Fig. 1 Evolution of solutions: $\varphi^h(x, t)$, $\psi^h(x, t)$, respectively. Numerical energy at time 2.0 s and 5.0 s, respectively

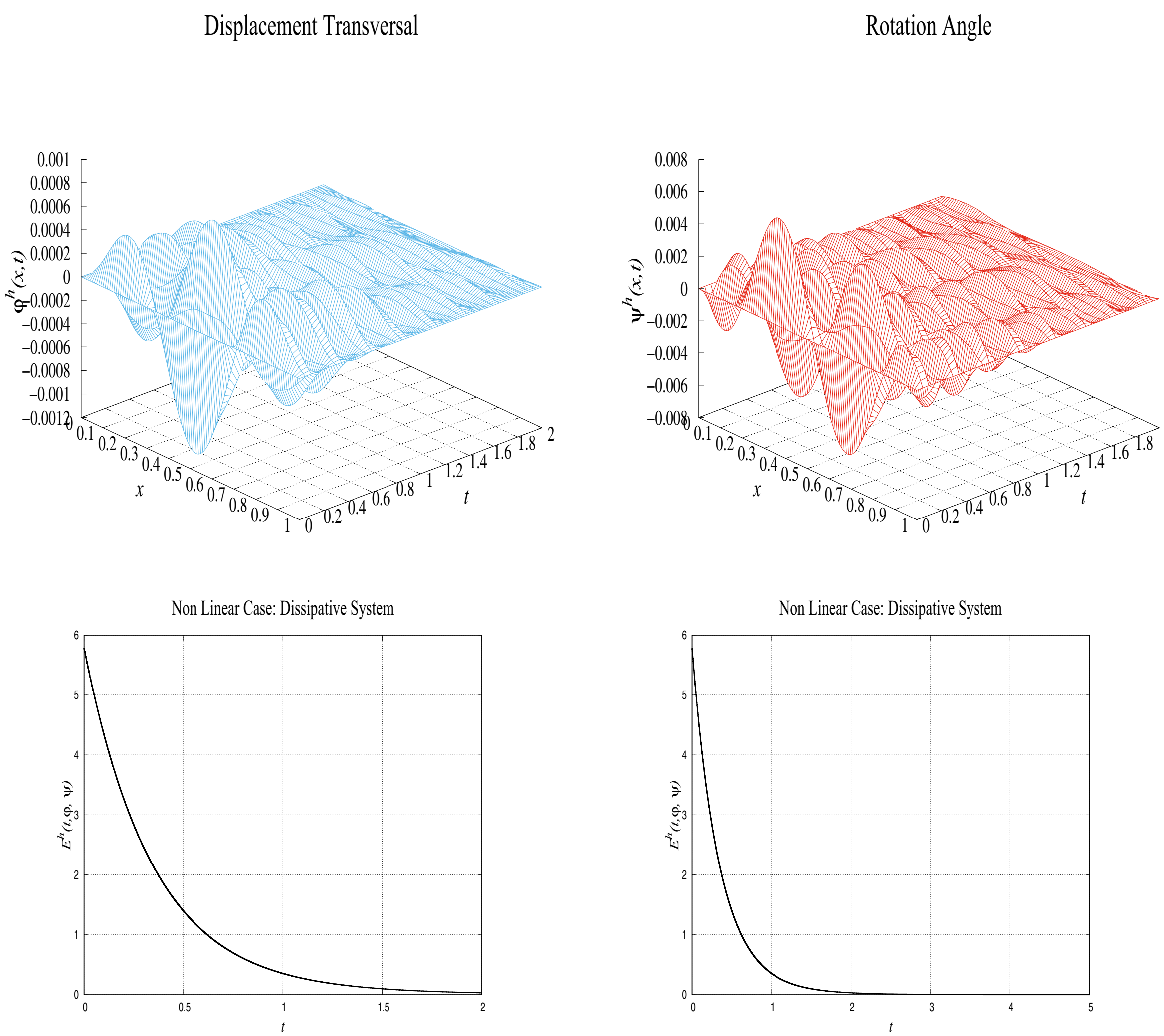

Fig. 2 Evolution of solutions: $\varphi^h(x,t)$, $\psi^h(x,t)$, respectively. Numerical Energy at time 2.0 s and 5.0 s, respectively

It is clear that the numerical alternatives to this problem were performed in the literature, and to more details, we indicate the classical reference by Hughes et al. [15] and Prathap and Bhashyam [21].

Remark 6.2 To get computational results, we use the implemented code in Language C. The graphics were developed using GNUplot.

In the sequel, we realize some numerical experiments to highlight our theoretical results.

6.3 Numerical experiments

In our performed numerical experiments to view the asymptotic properties, we consider an uniform mesh $h = 0.01$ m, $\Delta t = 10^{-5}$ s. The parameters Newmark's rules algorithms are $\gamma = \frac{1}{2}$, $\beta = \frac{1}{4}$.

Experimento 1: (*Conservative case:* $\gamma_1 = \gamma_2 = 0$)

We consider a rectangular beam with $L = 1.0$ m, thickness 0.09 m, width 0.09 m, $E = 69 \cdot 10^7 \text{N/m}^2$ $\rho = 2700 \text{ Kg/m}^3$, $\kappa = 5/6$, $r = 0.33$ (Poisson ratio). Furthermore, we have $\mu_1 = \mu_2 = 1$ and the following initial conditions:

$$\varphi(x,0) = 0,\ \varphi_t(x,0) = \sin 3\pi x,\ \psi(x,0) = 0,\ \text{and}\ \psi_t(x,0) = \sin 5\pi x.$$

Experimento 2: (*Dissipative case:* $\gamma_1 = 23, \gamma_2 = 0.015$)

We consider a rectangular beam with $L = 1.0$ m, thickness 0.09 m, width 0.09 m $E = 69 \cdot 10^7 \text{N/m}^2$ $\rho = 2700$ Kg/m^3, $\kappa = 5/6$, $r = 0.33$(Poisson ratio) and $\mu_1 = 1.0$, $\mu_2 = 1.0$. and the following initial conditions:

$$\varphi(x,0) = 0,\ \varphi_t(x,0) = \sin 3\pi x,\ \psi(x,0) = 0,\ \text{and}\ \psi_t(x,0) = \sin 5\pi x.$$

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Informed consent statement Not applicable.

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