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RESEARCH ARTICLE

Eigenvalues and Eigenfunctions of One-Dimensional Fractal Laplacians

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Abstract
We study the eigenvalues and eigenfunctions of one-dimensional weighted fractal Laplacians. These Laplacians are defined by self-similar measures with overlaps. We first prove the existence of eigenvalues and eigenfunctions. We then set up a framework for one-dimensional measures to discretize the equation defining the eigenvalues and eigenfunctions, and obtain numerical approximations of the eigenvalue and eigenfunction by using the finite element method. Finally, we show that the numerical eigenvalues and eigenfunctions converge to the actual ones and obtain the rate of convergence.

Keywords Fractal · Eigenvalue · Eigenfunction · Laplacian · Self-similar measure with overlaps

Mathematics Subject Classification Primary: 28A80 · 34L16 · Secondary: 65L60

1 Introduction

Let μ be a continuous, positive, finite Borel measure on $\mathbb{R}$ with support $\operatorname{supp}(\mu) = [a, b]$. Consider the non-negative quadratic form $\mathcal{E}(\cdot, \cdot)$ in $L^2((a, b), \mu)$ defined by

$$\mathcal{E}(u, v) := \int_a^b u'(x)v'(x)dx,$$

with domain dom $\mathcal{E}$ equal to the Sobolev space

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$$H_0^1(a,b) := \{u \in L^2((a,b),dx) : u' \in L^2((a,b),dx), u(a) = u(b) = 0\}$$

(Dirichlet boundary condition). It is well known that $\operatorname{dom}\mathcal{E}$ is dense in $L^2((a,b),\mu)$, and the quadratic form $(\mathcal{E}, \operatorname{dom}\mathcal{E})$ is closed. Hence there exists a non-negative self-adjoint operator A_D such that

$$\operatorname{dom}\mathcal{E} = \operatorname{dom}(A_D^{1/2}) \quad \text{and} \quad \mathcal{E}(u,v) = (A_D^{1/2}u, A_D^{1/2}v)_\mu$$

for all $u, v \in \operatorname{dom}\mathcal{E}$. We write $-\Delta_\mu^D := A_D$ and call it the *(Dirichlet) Laplacian with respect to* μ. If no confusion is possible, we denote Δ_μ^D simply by Δ_μ. Throughout this paper, we let

$$\|\cdot\|_{\operatorname{dom}\mathcal{E}} = \sqrt{\mathcal{E}(\cdot,\cdot)}.$$

Recently, the Dirichlet Laplacian Δ_μ has been studied extensively in connection with fractal measures (see [1–4, 6–12, 14–17, 19, 20, 22, 25–28] and the references therein). These papers mainly study the spectral asymptotics of Δ_μ and the associated Schrödinger operators, and wave, heat, and Schrödinger equations defined by Δ_μ.

In this paper, we consider the following eigenvalue problem

$$-\Delta_\mu u = \lambda V(x)u \qquad x \in (a,b), \tag{1.1}$$

where $V : [a,b] \to [0,+\infty)$. We say λ is an *eigenvalue* of Eq. (1.1) corresponding to a non-zero eigenfunction $\varphi(x) \in \operatorname{dom}\mathcal{E}$ if

$$\mathcal{E}(\varphi, v) = \lambda(V\varphi, v)_\mu$$

holds for all $v \in \operatorname{dom}\mathcal{E}$. We remark that for the case $V \equiv 1$, the finite element method is used to obtain numerical solutions to the eigenproblem (1.1) for self-similar measures satisfying a family of second-order self-similar identities in [3]. These identities were first introduced by Strichartz and are used in [24] to approximate the density of the infinite Bernoulli convolution associated with the golden ratio. To the best of the authors' knowledge, in the absence of second-order identities, the eigenvalue problem defined by IFSs with overlaps has not been obtained before, and this is a main motivation of this paper.

The first objective of this paper is to obtain the existence result of eigenvalues and eigenfunctions of (1.1).

Theorem 1.1 *Let $V(x) \geq 0$ on $[a, b]$ with $c_V := \|V\|_{L^1((a,b),\mu)} > 0$. Then there exists a complete orthonormal basis $\{\varphi_n\}_{n=1}^\infty$ of $L^2((a,b),\mu)$ such that each φ_n is an eigenfunction of Eq. (1.1) corresponding to an eigenvalue λ_n satisfying*

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \leq \cdots$$

and $\lim_{n\to\infty} \lambda_n = \infty$. *Moreover, $\{\varphi_n(x)\}_{n=1}^\infty$ also forms an orthogonal basis of* $\operatorname{dom}\mathcal{E}$.

To prove Theorem 1.1, we modify the classical argument (see [21]) and replace Lebesgue measure by a more general measure μ.

The second objective of this paper is to study Eq. (1.1) from a numerical point of view. Two closed sub-intervals I, J of $[a, b]$ are *measure disjoint* with respect to μ if $\mu(I \cap J) = 0$. Let $I \subseteq [a, b]$ be a closed interval. We call a finite family $\mathbf{P}$ of measure disjoint cells a *μ-partition* of I if $J \subseteq I$ for all $J \in \mathbf{P}$, and $\mu(I) = \sum_{J\in\mathbf{P}} \mu(J)$. A sequence of μ-partitions $(\mathbf{P}_m)_{m\geq 1}$ of $[a, b]$ is *compatible* if (1) for any $m \geq 1$, each member of $\mathbf{P}_{m+1}$ is a proper subset of some member of $\mathbf{P}_m$; (2) for any $m \geq 2$ and any $J \in \mathbf{P}_m$, there exist similitudes $(\tau_{IJ})_{I\in\mathbf{P}_1}$ of the form $\tau_{IJ}(x) = r_{IJ}x + b_{IJ}$ $(r_{IJ} \in (0, 1), b_{IJ} \in \mathbb{R})$ and positive constants $(c_{IJ})_{I\in\mathbf{P}_1}$ such that $\tau_{IJ}(I) \subseteq J$ and

$$\mu|_J = \sum_{I\in\mathbf{P}_1} c_{IJ} \cdot \mu|_I \circ \tau_{IJ}^{-1}. \tag{1.2}$$

Intuitively, (1.2) means that the μ measure of each closed interval in $\mathbf{P}_m$ for $m \geq 2$ can be expressed as a linear combination of $\{\mu(I) : I \in \mathbf{P}_1\}$. By making use of (1.2), some results concerning Δ_μ have been obtained (see, e.g., [17, 25–27]).

In order to discretize (1.1) and obtain numerical approximations of the eigenvalue and eigenfunction, we will assume that there exists a sequence of compatible μ-partitions $(\mathbf{P}_m)_{m\geq 1}$. Thus the μ measure of each closed interval in the partition can be computed by using (1.2), making it possible to discretize the Eq. (1.1). We remark that the assumption $\mathrm{supp}(\mu) = [a, b]$ guarantees that the mass matrix that arises in the finite element method is positive definite (see [2, Proposition 3.1]),

Let $V \equiv 1$ in (1.1). If λ is an eigenvalue of Eq. (1.1) corresponding to a non-zero eigenfunction $u \in \mathrm{dom}\,\mathcal{E}$, then

$$\int_a^b u'v'\,dx = \lambda \int_a^b uv\,d\mu \qquad \text{for all } v \in \mathrm{dom}\,\mathcal{E}. \tag{1.3}$$

Theorem 1.2 *Let μ be a continuous positive finite Borel measure on $\mathbb{R}$ with* $\mathrm{supp}(\mu) = [a, b]$. *Assume that there exists a sequence of compatible μ-partitions* $(\mathbf{P}_m)_{m\geq 1}$ *of* $[a, b]$ *and the integrals $\int_I x^k\,d\mu$, $I \in \mathbf{P}_1$, $k = 0, 1, 2$, can be evaluated explicitly. Then Eq.* (1.3) *can be discretized into a matrix Eq.* (3.5) *by finite element method. Moreover, Eq.* (3.5) *can be solved numerically.*

We are mainly interested in fractal measures with overlaps. Theorem 1.2 provides a framework under which discretization can be performed. We remark that if self-similar measures μ satisfies a family of second-order self-similar identities, then μ satisfies (1.2) (see [26, Proposition 5.1]). Hence, Theorem 1.1 cannot be deduced from [3, Theorem 1.2].

The following theorem shows that the approximate eigenvalues and eigenfunctions obtained in Theorem 1.2 converge to the actual ones, and we also obtain a rate of convergence.

Theorem 1.3 *Assume the hypotheses of Theorem 1.2. If there exist constants $r \in (0,1)$ and $c > 0$ such that $\max\{|J| : J \in \mathbf{P}_k\} \le c\,r^k$ for all $k \ge 1$, then the numerical eigenvalues $\hat{\lambda}_n^{(m)}$ and normalized eigenfunctions $\hat{\varphi}_n^{(m)}$, obtained in Theorem 1.2, converge to the corresponding theoretical λ_n and φ_n, respectively. Moreover, for each $n \ge 1$, there exists a constant $C := C(n) > 0$ (depending only on n) such that for all $m \ge 1$,*

$$\lambda_n \le \hat{\lambda}_n^{(m)} \le \lambda_n + Cr^{m/2}\lambda_n \quad \text{and} \quad \|\varphi_n - \hat{\varphi}_n^{(m)}\| \le Cr^{m/2}. \tag{4}$$

We illustrate Theorems 1.2 and 1.3 by a class of self-similar measures with overlaps that we call *essentially of finite type (EFT)* (see [19]). This class is used in [19] to illustrate self-similar measures satisfying EFT. These measures have been studied extensively (see, e.g., [16, 25, 27]). However, it is not clear whether they satisfy a family of second-order identities.

The rest of this paper is organized as follows. In Sect. 2, we prove Theorem 1.1. In Sect. 3, we give the proofs of Theorems 1.2 and 1.3, and apply them to a class of self-similar measures with overlaps.

2 Existence of Eigenvalues and Eigenfunctions

In this section, we mainly prove Theorem 1.1. The technique used here is the same in spirit as in classical setting.

Proof of Theorem 1.1 **Step 1.** We show the existence of the smallest eigenvalue λ_1. We remark that for each $u \in \operatorname{dom}\mathcal{E}$,

$$|u(x)| = \left|\int_a^x u'(x)\,dx\right| \le (b-a)^{1/2}\mathcal{E}^{1/2}(u,u)$$

for all $x \in [a,b]$. Then $\sup_{x\in[a,b]}|u(x)| \le (b-a)^{1/2}\mathcal{E}^{1/2}(u,u)$ for all $u \in \operatorname{dom}\mathcal{E}$. It follows that

$$(Vu,u)_\mu = \int_a^b V(x)u^2(x)\,d\mu(x) \le (b-a)\mathcal{E}(u,u)\int_a^b V(x)\,d\mu = \beta\mathcal{E}(u,u) \tag{1.4}$$

for all $u \in \operatorname{dom}\mathcal{E}$, where $\beta := (b-a)c_V$. Define the Rayleigh quotient $R(u)$ on $\operatorname{dom}\mathcal{E}$ by

$$R(u) := \frac{\mathcal{E}(u,u)}{(Vu,u)_\mu}, \quad u \ne 0.$$

Together with (2.1), it yields

$$R(u) \ge \frac{\mathcal{E}(u,u)}{\beta\mathcal{E}(u,u)} = \frac{1}{\beta} \quad \text{for all } u \in \operatorname{dom}\mathcal{E},\, u \ne 0.$$

Hence, $R(u)$ has a positive infimum. Let $\mathcal{A} = \{u \in \operatorname{dom}\mathcal{E} : (Vu,u)_\mu = 1\}$. We define

$$\lambda_1 := \inf_{u\in \mathrm{dom}\,\mathcal{E},\, u\neq 0} R(u) = \inf_{u\in\mathcal{A}} \mathcal{E}(u,u). \tag{2.1}$$

Then $\lambda_1 \geq 1/\beta > 0$. We claim that λ_1 is the smallest eigenvalue of Eq. (1.1). We choose a minimising sequence $\{u_m\} \subset \mathcal{A}$ such that

$$R(u_m) = \mathcal{E}(u_m, u_m) \to \lambda_1. \tag{2.2}$$

By the definition of $\mathcal{A}$, we have the sequence $\{u_m\}$ is bounded in $\mathrm{dom}\,\mathcal{E}$. Combining it with the fact $\mathrm{dom}\,\mathcal{E}$ is compactly embedded into the space $C_0(a,b) := \{u \in C(a,b) : u(a) = u(b) = 0\}$, there exists a subsequence (also denoted by $\{u_m\}$, for convenience) and a function $\varphi_1 \in C_0(a,b)$ such that

$$\lim_{m\to\infty} \sup_{x\in[a,b]} |u_m(x) - \varphi_1(x)| = 0.$$

Thus we can deduce from the Lebesgue dominated convergence theorem that

$$(V\varphi_1, \varphi_1)_\mu = 1. \tag{2.3}$$

Combining Eqs. (2.2)–(2.4) and the fact that

$$\mathcal{E}(u_m - u_k, u_m - u_k) + \mathcal{E}(u_m + u_k, u_m + u_k) = 2\big(\mathcal{E}(u_m, u_m) + \mathcal{E}(u_k, u_k)\big),$$

we can see that

$$\begin{aligned}\mathcal{E}(u_m - u_k, u_m - u_k) &\leq 2\big(\mathcal{E}(u_m, u_m) + \mathcal{E}(u_k, u_k)\big) - \lambda_1\big(V(u_m + u_k), u_m + u_k\big)_\mu \\ &\to 4\lambda_1 - 4\lambda_1 (V\varphi_1, \varphi_1)_\mu = 0 \qquad \text{as } m, k \to \infty,\end{aligned}$$

which implies that $\{u_m\}$ is a Cauchy sequence in $\mathrm{dom}\,\mathcal{E}$. Hence

$$u_m \to \varphi_1 \qquad \text{in } \mathrm{dom}\,\mathcal{E} \quad \text{and} \quad \lambda_1 = \mathcal{E}(\varphi_1, \varphi_1). \tag{2.4}$$

For $w \in \mathrm{dom}\,\mathcal{E}$, we define $g(t) := R(\varphi_1 + tw)$. Then

$$g(t) = \frac{\mathcal{E}(\varphi_1, \varphi_1) + 2t\mathcal{E}(\varphi_1, w) + t^2\mathcal{E}(w,w)}{(V\varphi_1, \varphi_1)_\mu + 2t(V\varphi_1, w)_\mu + t^2(Vw, w)_\mu}.$$

It follows from (2.2) and (2.5) that $g(t)$ reaches its minimum at $t = 0$. Then $g'(0) = 0$. By virtue of (2.4) and (2.5), we have

$$\begin{aligned}g'(0) &= \frac{2\mathcal{E}(\varphi_1, w)(V\varphi_1, \varphi_1)_\mu - 2\mathcal{E}(\varphi_1, \varphi_1)(V\varphi_1, w)_\mu}{(V\varphi_1, \varphi_1)_\mu^2} \\ &= \frac{2\Big(\mathcal{E}(\varphi_1, w) - \lambda_1 (V\varphi_1, w)_\mu\Big)}{(V\varphi_1, \varphi_1)_\mu} = 0\end{aligned}$$

for all $w \in \mathrm{dom}\,\mathcal{E}$, which implies that λ_1 is an eigenvalue of Eq. (1.1). We remark that λ_1 is the smallest eigenvalue following from (2.2).

Step 2. We prove the existence of the smallest eigenvalues $\lambda_2 \leq \lambda_3 \leq \cdots$. Since the existence of λ_1 has been obtained, we can assume that there exists $m-1$ eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_{m-1}$ with corresponding to the eigenfunctions $\varphi_1, \varphi_2, \cdots, \varphi_{m-1}$, and $(V\varphi_k, \varphi_k)_\mu = 1$ for all $k = 1, 2, \cdots, m-1$, where $m \geq 2$. We define

$$\begin{aligned} E_{m-1} &:= \operatorname{span}\{\varphi_1, \cdots, \varphi_{m-1}\}, \\ E_{m-1}^{\perp} &:= \{\varphi : \varphi \neq 0 \text{ and } (V\varphi, w)_\mu = 0 \text{ for all } w \in E_{m-1}\}. \end{aligned}$$

Similarly, we can prove that there exist some positive constant λ_m and $\varphi_m \in \operatorname{dom}\mathcal{E} \cap E_{m-1}^{\perp}$ satisfying $(V\varphi_m, \varphi_m)_\mu = 1$, and $\mathcal{E}(\varphi_m, w) = \lambda_m (V\varphi_m, w)_\mu$ for all $w \in \operatorname{dom}\mathcal{E} \cap E_{m-1}^{\perp}$. Furthermore,

$$\lambda_m = R(\varphi_m) = \inf_{u \in \operatorname{dom}\mathcal{E} \cap E_{m-1}^{\perp}, u \neq 0} R(u).$$

For each $w \in \operatorname{dom}\mathcal{E}$, we can write w as

$$w = w_1 + w_2, \quad w_1 = \sum_{k=1}^{m-1} \alpha_k \varphi_k \in E_{m-1}, \quad w_2 \in E_{m-1}^{\perp}.$$

It follows that

$$\begin{aligned} \mathcal{E}(\varphi_m, w) = \mathcal{E}(\varphi_m, w_1) + \mathcal{E}(\varphi_m, w_2) &= \sum_{k=1}^{m-1} \alpha_k \mathcal{E}(\varphi_m, \varphi_k) + \mathcal{E}(\varphi_m, w_2) \\ &= \mathcal{E}(\varphi_m, w_2) = \lambda_m (V\varphi_m, w_2)_\mu = \lambda_m (V\varphi_m, w_1 + w_2)_\mu = \lambda_m (V\varphi_m, w)_\mu \end{aligned}$$

for all $w \in \operatorname{dom}\mathcal{E}$. Thus λ_m is an eigenvalue of Eq. (1.1). It is easy to see that $\lambda_m \geq \lambda_{m-1}$. Therefore, Eq. (1.1) has a sequence of positive eigenvalues such that $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m \leq \cdots$.

Step 3. We show that for any distinct eigenvalues λ_i and λ_j, the corresponding eigenfunctions φ_i and φ_j are orthogonal in $\operatorname{dom}\mathcal{E}$, that is

$$\mathcal{E}(\varphi_i, \varphi_j) = 0. \tag{2.5}$$

Since $0 = \mathcal{E}(\varphi_i, \varphi_j) - \mathcal{E}(\varphi_j, \varphi_i) = \lambda_i (V\varphi_i, \varphi_j)_\mu - \lambda_j (V\varphi_j, \varphi_i)_\mu$, we have

$$(\lambda_i - \lambda_j)(V\varphi_i, \varphi_j)_\mu = 0,$$

which implies (2.6).

Step 4. We claim that the dimension of the space consisting of eigenfunctions corresponding to a fixed eigenvalue is finite. Suppose, on the contrary, that there exists countably infinite sequence of eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$ corresponding to the same eigenvalue λ, which are linearly independent in $\operatorname{dom}\mathcal{E}$. So we can renormalize such that

$$(V\phi_k, \phi_k)_\mu = 1 \qquad \text{for all } k \geq 1.$$

Since $\mathcal{E}(\phi_k, \phi_k) = \lambda(V\phi_k, \phi_k)_\mu = \lambda$, there exists a convergent subsequence of $\{\phi_k\}$ in $C_0(a, b)$. However, we can see that

$$\begin{aligned}(V(\phi_l - \phi_k), \phi_l - \phi_k)_\mu &= (V\phi_l, \phi_l)_\mu + (V\phi_k, \phi_k)_\mu - 2(V\phi_l, \phi_k)_\mu \\ &= (V\phi_l, \phi_l)_\mu + (V\phi_k, \phi_k)_\mu = 2, \quad \text{if } l \neq k,\end{aligned}$$

which is a contradiction. Thus the claim holds.

Step 4. We show that

$$\lim_{n\to\infty} \lambda_n = +\infty. \tag{2.6}$$

Suppose, on the contrary, that there exists some positive constant M such that $0 < \lambda_n \le M$ for all $n \ge 1$. Without loss of generality, we assume that the sequence of corresponding eigenfunctions $\{\varphi_n\}$ is orthogonal in $L^2((a, b), Vd\mu)$. Then

$$\mathcal{E}(\varphi_n, \varphi_n) = \lambda_n(V\varphi_n, \varphi_n)_\mu \le M,$$

and hence there exists a convergent subsequence of $\{\varphi_m\}$ in the space $C_0(a, b)$, which is a contradiction, since $(V(\varphi_m - \varphi_k), \varphi_m - \varphi_k)_\mu = 2$. Thus (2.7) holds.

Step 5. We show that the eigenfunctions $\{\varphi_m\}$ corresponding to $\{\lambda_m\}$ form a basis of $\operatorname{dom}\mathcal{E}$. Suppose that there exists some non-zero $w \in \operatorname{dom}\mathcal{E}$ such that $\mathcal{E}(\varphi_m, w) = 0$ for all $m \ge 1$. Then $(V\varphi_m, w)_\mu = \mathcal{E}(\varphi_m, w)/\lambda_m = 0$ for all $m \ge 1$. Therefore, $w \in E_{m-1}^\perp$ for all $m \ge 2$, which implies

$$\lambda_m \le R(w) = \frac{\mathcal{E}(w, w)}{(Vw, w)_\mu}, \qquad m \ge 2. \tag{2.7}$$

However, $\lambda_m \to \infty$ as $m \to \infty$, and then (2.8) is impossible. Hence, the desired result holds. □

Since $\operatorname{dom}\mathcal{E}$ is dense in $C_0(a, b)$, we see that $\{\varphi_m\}_{m=1}^\infty$ is also an orthogonal basis of $L^2((a, b), Vd\mu)$. We remark that $(V\varphi_m, \varphi_m)_\mu = 1$. Thus $\{\varphi_m\}_{m=1}^\infty$is an orthonormal basis of $L^2((a, b), Vd\mu)$. It is well known that μ can also define a Neumann Laplace operator Δ_μ^N in $L^2((a, b), \mu)$. We remark that the eigenproblem (1.1) replacing Δ_μ by Δ_μ^N also has a sequence of orthogonal eigenfunctions in $L^2((a, b), \mu)$, but the first eigenvalue is zero, corresponding to the constant eigenfunction. The proof is similar to that of Theorem 1.1. We omit the details.

Corollary 2.1 *Assume the hypotheses of Theorem 1.1 and let λ_1 be the first eigenvalue of eigenproblem (1.1). Then there exists a non-negative eigenfunction corresponding to λ_1.*

Proof Let φ_1 be an eigenfunction of (1.1) corresponding to λ_1. It suffices to prove that $|\varphi_1|$ is also an eigenfunction corresponding to λ_1. We remark that $|\varphi_1| \in \operatorname{dom}\mathcal{E}$ and $\mathcal{E}(|\varphi_1|, |\varphi_1|) \le \mathcal{E}(\varphi_1, \varphi_1)$. Combining it with (2.4) and (2.5), we can see that

$$R(|\varphi_1|) \leq \lambda_1. \tag{2.8}$$

On the other hand, by the definition of λ_1, we have

$$R(|\varphi_1|) \geq \lambda_1. \tag{2.9}$$

Thus the desired result follows by combining (2.9) and (2.10). □

3 The Finite Element Method and Convergence of Numerical Solutions

In this section, we mainly prove Theorems 1.2 and 1.3. Let $V \equiv 1$ in Eq. (1.1). We first use the finite element method to solve the equation, and then prove the convergence of numerical approximations of the eigenvalue and eigenfunction.

Let μ be a continuous positive finite Borel measure on $\mathbb{R}$ with $\mathrm{supp}(\mu) = [a, b]$, and $(\mathbf{P}_m)_{m\geq 1} = (\{I_{m,i}\}_{i=1}^{N(m)})_{m\geq 1}$ be a sequence of μ-partitions of $[a, b]$. Thus we can write $I_{m,i} = [x_{m,i-1}, x_{m,i}]$ for $m \geq 1$ and $1 \leq i \leq N(m)$. Note that $x_{m,0} = a$ and $x_{m,N(m)} = b$ for all $m \geq 1$. Let $W_m := \{x_{m,i} : i = 0, \dots, N(m)\}$ be the set of endpoints of all level-m subintervals in $\mathbf{P}_m$, and S^m be the space of continuous piecewise linear functions on $[a, b]$ with nodes W_m, and let

$$S_D^m = \{u \in S^m : u(a) = u(b) = 0\}$$

be the subspaces of S^m consisting of functions satisfying the Dirichlet boundary condition. Then

$$\dim(S^m) = N(m) + 1 \quad \text{and} \quad \dim(S_D^m) = N(m) - 1.$$

We choose the basis of S^m consisting of the following tent functions:

$$\phi_{m,i}(x) := \begin{cases} \dfrac{x - x_{m,i-1}}{x_{m,i} - x_{m,i-1}} & \text{if } x \in I_{m,i},\ i = 1, 2, \dots, N(m), \\ \dfrac{x - x_{m,i+1}}{x_{m,i} - x_{m,i+1}} & \text{if } x \in I_{m,i+1},\ i = 0, 1, \dots, N(m) - 1, \\ 0 & \text{otherwise}, \end{cases} \tag{2.10}$$

and choose the basis $\{\phi_{m,i}\}_{i=1}^{N(m)-1}$ for S_D^m.

Proof of Theorem 1.2 Use the notation above. We use the finite element method to discretize Eq. (1.3). Let $u(x)$ be an eigenfunction of Eq. (1.1) corresponding to an eigenvalue λ, that is, $u(x)$ satisfies Eq. (1.3). We approximate $u(x)$ by

$$u_m(x) = \sum_{i=0}^{N(m)} w_{m,i}\phi_{m,i} \qquad \text{for } m \geq 1, \tag{3.1}$$

where each $w_{m,i}$ is a constant to be determined. Fix any $m \geq 1$. We require $u_m(x)$ to satisfy the integral form of the eigenvalue equation

$$\int_a^b u_m' \phi_{m,j}' \, dx = \lambda \int_a^b u_m \phi_{m,j} \, d\mu \tag{3.2}$$

for $j = 1, \ldots, N(m) - 1$, and the Dirichlet boundary condition $u_m(x) = 0$ on $\{a, b\}$. It follows that $w_{m,0} = w_{m,N(m)} = 0$. Using this and substituting (3.2) into (3.3), we can deduce that

$$\sum_{i=1}^{N(m)-1} w_{m,i} \int_a^b \phi_{m,i}'(x)\phi_{m,j}'(x) \, dx = \lambda \sum_{i=1}^{N(m)-1} w_{m,i} \int_a^b \phi_{m,i}(x)\phi_{m,j}(x) \, d\mu \tag{3.3}$$

for $1 \leq j \leq N(m) - 1$. We define the mass matrix $\mathbf{M} = \mathbf{M}^{(m)} = (M_{ij}^{(m)})$ and stiffness matrix $\mathbf{K} = \mathbf{K}^{(m)} = (K_{ij}^{(m)})$, respectively, by

$$M_{ij}^{(m)} = \int_a^b \phi_{m,i}(x)\phi_{m,j}(x) \, d\mu \quad \text{and} \quad K_{ij}^{(m)} = \int_a^b \phi_{m,i}'(x)\phi_{m,j}'(x) \, dx,$$

where $1 \leq i, j \leq N(m) - 1$. Thus $\mathbf{M}$ and $\mathbf{K}$ are tridiagonal following from the definition of $\phi_{m,j}(x)$. Hence, (3.4) can be expressed into matrix form as

$$\mathbf{K}\mathbf{w} = \lambda \mathbf{M}\mathbf{w}, \tag{3.4}$$

where

$$\mathbf{w} = \mathbf{w}_m := \begin{bmatrix} w_{m,1} \\ \vdots \\ w_{m,N(m)-1} \end{bmatrix}.$$

We remark that the matrix $\mathbf{K}$ can be computed directly. We claim that the matrix $\mathbf{M}$ also can be computed. Recall that $(\mathbf{P}_m)_{m\geq 1}$ is a sequence of compatible μ-partitions of $[a, b]$. Then (1.2) holds, which implies that the matrix $\mathbf{M}$ is completely determined by

$$\left\{ \int_I x^k \, d\mu \, : I \in \mathbf{P}_1, k = 0, 1, 2 \right\}. \tag{3.5}$$

Thus the claim holds, since the integrals in (3.6) can be evaluated explicitly for $k = 0, 1, 2$ and $I \in \mathbf{P}_1$. Since $\mathrm{supp}(\mu) = [a, b]$, $\mathbf{M}$ is invertible (see [2, Proposition 3.1]). It follows that the eigenvalues and eigenfunctions of Eq. (3.5) can be solved numerically. $\square$

We remark that numerical algorithms for general eigenproblem as (3.5) have been well developed. We can get good approximations for eigenvalues and eigenfunctions by choosing m sufficiently large (see Theorem 1.3 for details).

We now prove Theorem 1.3. Define the linear map $\mathcal{F}_m : \operatorname{dom}\mathcal{E} \to S_D^m$ by

$$\mathcal{F}_m u(x) = \sum_{i=1}^{N(m)-1} u(x_{m,i})\phi_{m,i}(x) \qquad u \in \operatorname{dom}\mathcal{E} \text{ and } m \geq 1.$$

We call $\mathcal{F}_m$ the *Rayleigh-Ritz projection* with respect to W_m. Using [23, Theorem 1.1], we have

$$\mathcal{E}(u - \mathcal{F}_m u, v) = 0 \tag{3.6}$$

for all $u \in \operatorname{dom}\mathcal{E}$ and $v \in S_D^m$. We remark that

$$\|u - \mathcal{F}_m u\|_{L^2((a,b),\mu)} \leq 2\|W_m\|^{1/2}\|u\|_{\operatorname{dom}\mathcal{E}} \tag{3.7}$$

for all $u \in \operatorname{dom}\mathcal{E}$ and $m \geq 1$ (see [2, Lemma 5.3]), where

$$\|W_m\| := \max\{|x_{m,i} - x_{m,i-1}| : 1 \leq i \leq N(m)\}$$

is the *norm* of W_m for $m \geq 1$. Recall that E_n is the subspace spanned by eigenfunctions $\varphi_1, \cdots, \varphi_n$ of Eq. (1.1) with $V \equiv 1$. Let e_n be the set of unit vectors in E_n and let

$$\sigma_n^{(m)} := \max_{u \in e_n} \left|2(u, u - \mathcal{F}_m u)_\mu - (u - \mathcal{F}_m u, u - \mathcal{F}_m u)_\mu\right|.$$

Then provided that $\sigma_n^{(m)} < 1$, the approximated eigenvalues are bounded above by

$$\hat{\lambda}_n^{(m)} \leq \frac{\lambda_n}{1 - \sigma_n^{(m)}} \tag{3.8}$$

(see [3, Lemma 5.4]).

Proof of Theorem 1.3 We want to estimate $\sigma_n^{(m)}$. Fix any $n \geq 1$ and let $u \in e_n$. Then $\|u\|_{L^2((a,b),\mu)} = 1$ and u can be expressed as $u = \sum_{i=1}^n a_i\varphi_i$, where each φ_i is an eigenfunction corresponding to eigenvalue λ_i. It follows that

$$\|u\|^2_{\operatorname{dom}\mathcal{E}} = \sum_{i=1}^n a_i^2\lambda_i \leq \lambda_n \sum_{i=1}^n a_i^2 = \lambda_n\|u\|^2_{L^2((a,b),\mu)} = \lambda_n.$$

Combining it with Hölder inequality and (3.8), we have

$$\begin{aligned} 2(u, u - \mathcal{F}_m u)_\mu &\leq 2\int_a^b |u(u - \mathcal{F}_m u)|\, d\mu \leq 2\|u\|_{L^2((a,b),\mu)}\|u - \mathcal{F}_m u\|_{L^2((a,b),\mu)} \\ &\leq 4\sqrt{\lambda_n}\|W_m\|^{1/2} \end{aligned} \tag{3.9}$$

for all $m \geq 1$. According to (3.8), (3.9), and (3.10), we have

$$\sigma_n^{(m)} \le 4\sqrt{\lambda_n}\|W_m\|^{1/2} + 4\lambda_n\|W_m\| \le M_0 r^{m/2},$$

where the last inequality used the assumption $\max\{|J| : J \in \mathbf{P}_k\} \le c\, r^k$ for all $k \ge 1$, and M_0 is a constant (depending only on n). We first assume that m is sufficiently large so that $\sigma_n^{(m)} < 1/2$. Then by (3.9), we get

$$\hat{\lambda}_n^{(m)} \le \frac{\lambda_n}{1-\sigma_n^{(m)}} \le \lambda_n(1+\sigma_n^{(m)}).$$

Thus there exists a constant $C > 0$ (depending only on n) such that for all $m \ge 1$,

$$\hat{\lambda}_n^{(m)} \le \lambda_n + Cr^{m/2}\lambda_n. \tag{3.10}$$

On the other hand, every eigenvalue is approximated from above(see [3]):

$$\hat{\lambda}_n^{(m)} \ge \lambda_n \qquad \text{for all } m.$$

Together with (3.11), it yields the convergence of the eigenvalues as the first inequality in (1.4). It suffices to prove the convergence of the numerical eigenfunctions.

Let $\mathcal{B} := \{\hat{\varphi}_1^{(m)}, \cdots, \hat{\varphi}_{N(m)}^{(m)}\}$ be the set of normalized approximate eigenfunctions. Then $\mathcal{B}$ forms an orthonormal basis for S_D^m, which implies

$$\mathcal{F}_m\varphi_n = \sum_{k=1}^{N(m)} (\mathcal{F}_m\varphi_n, \hat{\varphi}_k^{(m)})_\mu \hat{\varphi}_k^{(m)}. \tag{3.11}$$

We claim that $(\hat{\lambda}_k^{(m)} - \lambda_n)(\mathcal{F}_m\varphi_n, \hat{\varphi}_k^{(m)})_\mu = \lambda_n(\varphi_n - \mathcal{F}_m\varphi_n, \hat{\varphi}_k^{(m)})_\mu$. It suffices to show that

$$\hat{\lambda}_k^{(m)}(\mathcal{F}_m\varphi_n, \hat{\varphi}_k^{(m)})_\mu = \lambda_n(\varphi_n, \hat{\varphi}_k^{(m)})_\mu.$$

Since $\hat{\varphi}_k^{(m)}$ and φ_n are eigenfunctions, the two sides of this equation can be rewritten as $\mathcal{E}(\mathcal{F}_m\varphi_n, \hat{\varphi}_k^{(m)})$ and $\mathcal{E}(\varphi_n, \hat{\varphi}_k^{(m)})$, respectively. It follows from (3.7) that

$$\mathcal{E}(\mathcal{F}_m\varphi_n, \hat{\varphi}_k^{(m)}) - \mathcal{E}(\varphi_n, \hat{\varphi}_k^{(m)}) = \mathcal{E}(\mathcal{F}_m\varphi_n - \varphi_n, \hat{\varphi}_k^{(m)}) = 0.$$

We remark that for any fixed λ_n, there exists a constant $\gamma > 0$ such that for all m sufficiently large,

$$\frac{\lambda_n}{|\hat{\lambda}_k^{(m)} - \lambda_n|} \le \gamma \qquad \text{for all } \lambda_k \ne \lambda_n.$$

Then the size of the remaining sum is given by

$$\begin{aligned} \|\mathcal{F}_m\varphi_n - \alpha\hat{\varphi}_n^{(m)}\|^2_{L^2((a,b),\mu)} &= \sum_{k\ne n}(\mathcal{F}_m\varphi_n, \hat{\varphi}_k^{(m)})^2_\mu = \sum_{k\ne n}\left(\frac{\lambda_n}{\hat{\lambda}_k^{(m)} - \lambda_n}\right)^2(\varphi_n - \mathcal{F}_m\varphi_n, \hat{\varphi}_k^{(m)})^2_\mu \\ &\le \gamma^2\sum_{k\ne n}(\varphi_n - \mathcal{F}_m\varphi_n, \hat{\varphi}_k^{(m)})^2_\mu \le \gamma^2\|\varphi_n - \mathcal{F}_m\varphi_n\|^2_{L^2((a,b),\mu)}, \end{aligned}$$

where the claim has been used in the second equality and $\alpha := (\mathcal{F}_m\varphi_n, \hat{\varphi}_k^{(m)})_\mu$. It follows that

$$\begin{aligned}\|\varphi_n - \alpha\hat{\varphi}_n^{(m)}\|_{L^2((a,b),\mu)} &\le \|\varphi_n - \mathcal{F}_m\varphi_n\|_{L^2((a,b),\mu)} + \|\mathcal{F}_m\varphi_n - \alpha\hat{\varphi}_n^{(m)}\|_{L^2((a,b),\mu)}\\ &\le (1+\gamma)\|\varphi_n - \mathcal{F}_m\varphi_n\|_{L^2((a,b),\mu)}.\end{aligned} \tag{3.12}$$

Since φ_n and $\hat{\varphi}_n$ are normalized eigenfunctions, we have

$$|\alpha - 1| = \left|\|\alpha\hat{\varphi}_n^{(m)}\|_{L^2((a,b),\mu)} - \|\varphi_n\|_{L^2((a,b),\mu)}\right| \le \|\varphi_n - \alpha\hat{\varphi}_n^{(m)}\|_{L^2((a,b),\mu)}. \tag{3.13}$$

Combining (3.8), (3.13) and (3.14), we see that

$$\begin{aligned}\|\varphi_n - \hat{\varphi}_n^{(m)}\|_{L^2((a,b),\mu)} &\le \|\varphi_n - \alpha\hat{\varphi}_n^{(m)}\|_{L^2((a,b),\mu)} + \|\alpha\hat{\varphi}_n^{(m)} - \hat{\varphi}_n^{(m)}\|_{L^2((a,b),\mu)}\\ &= \|\varphi_n - \alpha\hat{\varphi}_n^{(m)}\|_{L^2((a,b),\mu)} + |\alpha - 1|\cdot\|\hat{\varphi}_n^{(m)}\|_{L^2((a,b),\mu)}\\ &\le 2\|\varphi_n - \alpha\hat{\varphi}_n^{(m)}\|_{L^2((a,b),\mu)}\\ &\le 2(1+\gamma)\|\varphi_n - \mathcal{F}_m\varphi_n\|_{L^2((a,b),\mu)} \le Cr^{m/2},\end{aligned}$$

which completes the proof of Theorem 1.3. □

Based on Theorems 1.2 and 1.3, we solve the eigenproblem (1.1) for a class of self-similar measures satisfying (EFT), which is defined by the following family of IFSs:

$$S_1(x) = r_1x, \quad S_2(x) = r_2x + r_1(1-r_2), \quad S_3(x) = r_2x + 1 - r_2, \tag{3.14}$$

where $r_1, r_2 \in (0,1)$ satisfy $r_1 + 2r_2 - r_1r_2 \le 1$, i.e., $S_2(1) \le S_3(0)$. Recently, the self-similar measures defined by IFSs in (3.15) have been studied extensively (see, e.g., [5, 13, 18, 19]). These papers mainly study the multifractal properties and spectral dimension of the corresponding self-similar measures.

Let μ be a self-similar measure defined by an IFS in (3.15) and a probability vector $(p_i)_{i=1}^3$. In order to define a sequence of compatible μ-partitions of [0, 1], we introduce the definition of an island, which is adopted from [19]. Let $\mathcal{M}_k := \{1,2,3\}^k$ for $k \ge 1$. A closed interval $I \subseteq [0,1]$ is called a *level-k island* with respect to $\{\mathcal{M}_k\}$ if the following conditions hold:

(1) there exist some words $\boldsymbol{i}_0, \boldsymbol{i}_1, \dots, \boldsymbol{i}_n$ in $\mathcal{M}_k$ such that $S_{\boldsymbol{i}_k}(0,1) \cap S_{\boldsymbol{i}_{k+1}}(0,1) \ne \emptyset$ for all $k = 0, \dots, n-1$, and $I = \bigcup_{k=0}^n S_{\boldsymbol{i}_k}([0,1])$;
(2) for any $\boldsymbol{j} \in \mathcal{M}_k \setminus \{\boldsymbol{i}_0, \dots, \boldsymbol{i}_n\}$ and any $k \in \{0, \dots, n\}$, $S_{\boldsymbol{j}}(0,1) \cap S_{\boldsymbol{i}_k}(0,1) = \emptyset$.

We remark that, I° is a connected component of $S_{\mathcal{M}_k}(0,1) := \bigcup_{\boldsymbol{i}\in\mathcal{M}_k} S_{\boldsymbol{i}}(0,1)$ for all level-k islands I, (see Fig. 1). For $k \ge 1$, define

$$\mathbf{P}_k := \left\{I : I \text{ is a level-k island with respect to } \{\mathcal{M}_k\}\right\}. \tag{3.15}$$

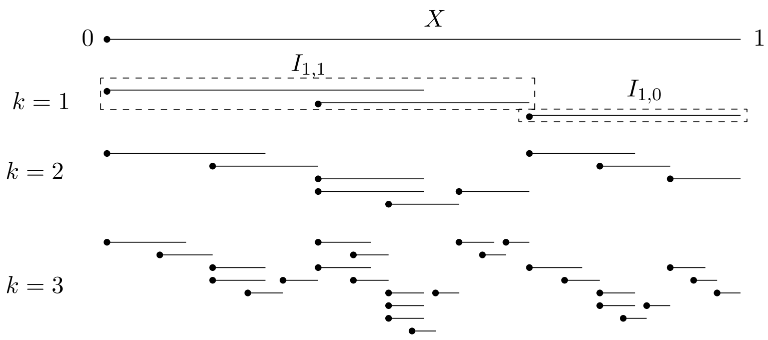

Fig. 1 μ-partitions $\mathbf{P}_k$ for $k = 1, 2, 3$, where $\mathbf{P}_k$ is defined as in (3.16). Cells that are labeled consist of line segments enclosed by a box. The figure is drawn with $r_1 = 1/2$ and $r_2 = 1/3$

Let $I_{1,1} := S_1([0,1]) \bigcup S_2([0,1])$ and $I_{1,0} := S_3([0,1])$. Then $\mathbf{P}_1 = \{I_{1,1}, I_{1,0}\}$ (see Fig. 1). It is easy to check that $\max\{|I| : I \in \mathbf{P}_k\} \le |I_{1,1}| \max\{r_1, r_2\}^{k-1} \le (r_1 + r_2)^k$ for all $k \ge 1$. Moreover, Ngai and the first author proved that $(\mathbf{P}_k)_{k\ge 1}$ satisfies (1.2) in [26]. Therefore, $(\mathbf{P}_k)_{k\ge 1}$ is a sequence of compatible μ-partitions of [0, 1].

If $S_2(1) = S_3(0)$, then $\operatorname{supp}(\mu) = [0, 1]$. In particular, we let $r_1 = 1/2$, $r_2 = 1/3$, and $p_1 = p_2 = p_3 = 1/3$. Then we obtain that

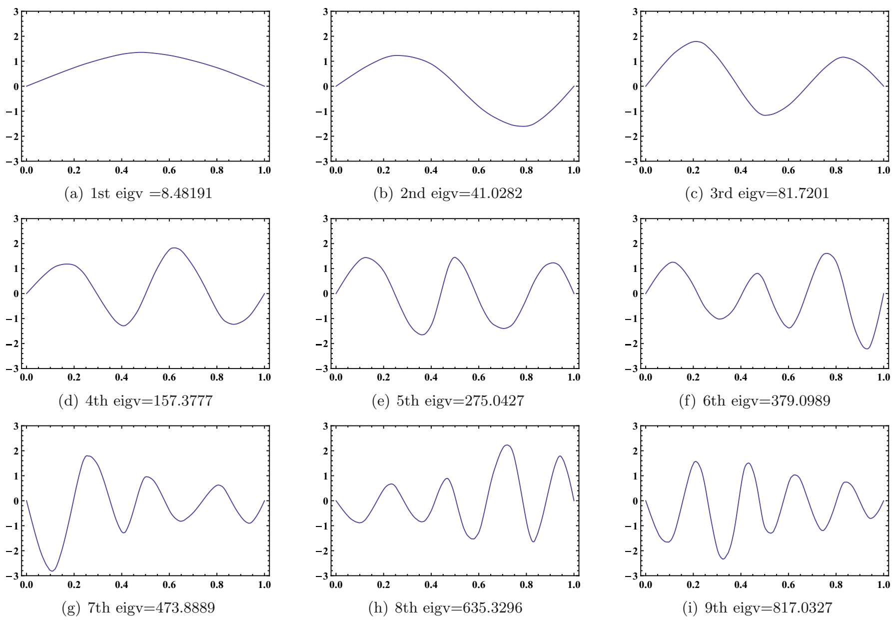

Fig. 2 Normalized eigenfunctions for the self-similar measure defined by the IFS in (3.15) with $r_1 = 1/2$, $r_2 = 1/3$ and probability weights $p_1 = p_2 = p_3 = 1/3$

$$\int_{I_{1,0}} d\mu = \frac{1}{3}, \qquad \int_{I_{1,0}} x\,d\mu = \frac{28}{99}, \qquad \int_{I_{1,0}} x^2\,d\mu = \frac{8}{33},$$
$$\int_{I_{1,1}} d\mu = \frac{2}{3}, \qquad \int_{I_{1,1}} x\,d\mu = \frac{26}{99}, \qquad \int_{I_{1,1}} x^2\,d\mu = \frac{4}{33},$$

which implies that the mass matrix **M** can be calculated. Thus Eq. (3.5) can be solved numerically. The numerical result is shown in Fig. 2.

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Declarations

Conflict of interest The authors declare that there is no competing interests.

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