Journal of Applied Mathematics and Computing (2023) 69:2239–2261
https://doi.org/10.1007/s12190-022-01832-w

ORIGINAL RESEARCH

Parameter-uniform convergence analysis of a domain decomposition method for singularly perturbed parabolic problems with Robin boundary conditions

Sunil Kumar[1] · **Aakansha**[1] · **Joginder Singh**[2] · **Higinio Ramos**[3,4]

Received: 28 March 2022 / Revised: 30 November 2022 / Accepted: 21 December 2022 /
Published online: 29 December 2022

Abstract

We construct and analyze a domain decomposition method to solve a class of singularly perturbed parabolic problems of reaction-diffusion type having Robin boundary conditions. The method considers three subdomains, of which two are finely meshed, and the other is coarsely meshed. The partial differential equation associated with the problem is discretized using the finite difference scheme on each subdomain, while the Robin boundary conditions associated with the problem are approximated using a special finite difference scheme to maintain the accuracy. Then, an iterative algorithm is introduced, where the transmission of information to the neighbours is done using a piecewise linear interpolation. It is proved that the resulting numerical approximations are parameter-uniform and, more interestingly, that the convergence of the iterates is optimal for small values of the perturbation parameters. The numerical results support the theoretical results about convergence.

✉ Higinio Ramos
higra@usal.es

Sunil Kumar
skumar.mat@iitbhu.ac.in

Aakansha
aakansha.chauhan03@gmail.com

Joginder Singh
virk1516@gmail.com

1 Department of Mathematical Sciences, Indian Institute of Technology (BHU) Varanasi, Varanasi, India

2 Department of Mathematics, Guru Jambheshwar University of Science and Technology, Hisar, Haryana, India

3 Scientific Computing Group, Universidad de Salamanca, Plaza de la Merced, 37008 Salamanca, Spain

4 Escuela Politécnica Superior de Zamora, Campus Viriato, 49022 Zamora, Spain

Keywords Domain decomposition · Waveform relaxation · Singularly perturbed problems · Robin boundary conditions · Schwarz methods

Mathematics Subject Classification 65M06 · 65M55 · 65M15 · 65M12

1 Introduction

Singularly perturbed problems (SPPs) arise in the mathematical modeling of several practical problems in engineering and applied mathematics, for example, in describing the theory of gyroscopes [25], in studying linear spring-mass system without damping but with forcing and a small spring constant [31], in variational problems in control theory [10], etc., where the higher order derivative appears multiplied by a small parameter. The fascinating aspect of SPPs is that their solutions have boundary and/or interior layers (the regions in which there are steep gradients). Because of this reason, classical numerical schemes are not appropriate to accurately and efficiently solve these problems. This has led to the development of special numerical approaches for SPPs. The developed numerical methods for SPPs are called as parameter-robust or parameter-uniform or uniformly convergent, meaning that the convergence of the method is independent of the involved parameter. They are developed considering fitted operators, fitted meshes, and domain decomposition approaches [1, 2, 9, 11, 14–16, 26, 28–31, 35]. Further, some recent advancements in finite difference methods can be seen in [3–6].

Consider the following SPP

$$Lu := u_t(x,t) - \varepsilon u_{xx}(x,t) + a(x,t)u(x,t) = f(x,t), \quad (x,t) \in \Omega := \mathsf{D} \times (0,T], \tag{1}$$

with the Robin boundary conditions

$$u(0,t) - \sqrt{\varepsilon} u_x(0,t) = g_\ell(t), \;\; u(1,t) + \sqrt{\varepsilon} u_x(1,t) = g_r(t), \;\; t \in (0,T], \tag{2}$$

$$u(x,0) = g_b(x), \;\; x \in \overline{\mathsf{D}}, \tag{3}$$

where $\mathsf{D} = (0,1)$, $a(x,t) \geq \alpha > 0$ on $\overline{\Omega}$, and $0 < \varepsilon \leq 1$ is known as the perturbation parameter. It is known that problem (1)–(3) has a unique solution exhibiting boundary layers near $x = 0$ and $x = 1$ [12, 22, 37]. Such problems arise in the modelling of certain types of bioswitches [41].

Singularly perturbed problems similar to (1)–(3) with Dirichlet-type boundary conditions have been studied extensively in the literature (see [7, 18, 20, 24, 27, 34, 38] and the references therein). However, there are only a very few studies of such problems with Robin boundary conditions (RBCs) [12, 13, 22, 37]. Note that all these studies are based on the fitted mesh approach.

Domain decomposition is shown to be a very versatile and effective approach for solving problems in ordinary and partial differential equations [8, 17, 32]. The original approach dates back to 19th century [36]. However, the approach flourished more in the last three decades. Schwarz waveform relaxation (SWR) is a special approach that is very popular for solving time dependent problems [32]. In this technique, the space-time domain is split into several subdomains. Then, on each subdomain across the whole time interval the solution is computed and the exchange of time-space boundary values takes place. We can treat the subdomains differently using this approach. Also, we have the flexibility to locally address any singularity in the solution. Further, a non uniform mesh can be avoided with this approach. Lastly, one can speed-up the computations by implementing it on parallel computers. Domain decomposition methods for SPPs have been developed in [19–21, 33, 34, 39, 40] and the references therein. Note that in all these works model problems with Dirichlet boundary conditions are considered. Moreover, we are not aware of any studies involving the domain decomposition approach for SPPs with Robin boundary conditions.

Hence, the objective of this paper is two-fold: first, to introduce a domain decomposition method of SWR type to numerically solve problem (1)–(3), and second, to present a parameter-uniform convergence analysis of the introduced method. We consider a space-time partitioning of the original domain using the Shishkin transition point. The PDE associated with problem (1)–(3) in each subdomain is discretized by the finite difference scheme, while the Robin boundary conditions of problem (1)–(3) are approximated by a special finite difference scheme to maintain the accuracy. Then an iterative algorithm is introduced, where transmission of the information to the neighbours is done using the piecewise-linear interpolation. We discuss parameter-uniform convergence of the constructed method using auxiliary problems that separates the discretization and iteration errors. Note that the convergence analysis of domain decomposition methods for problems with Dirichlet boundary conditions cannot be straightforwardly applied to the present method due to the presence of the Robin boundary conditions. Firstly, we require a different definition of auxiliary problems to handle the Robin boundary conditions. The boundary and initial conditions of the auxiliary problems in earlier works are simply defined using the exact solution of the problem, but that is not entirely true for the present problem. Secondly, we require a more complex constant coefficient problem (in Theorem 3.2) and altogether different barrier functions (in Theorems 3.2 and 3.3) for bounding the error between the auxiliary solution and the Schwarz iterates. It is proved that the resulting numerical approximations are parameter-uniform and, more interestingly, that the iteration convergence is optimal for small ε. Finally, numerical results are provided to support the convergence theory.

The rest of the article is organized in different sections as follows. Section 2 provides a priori bounds for the solution derivatives, and Sect. 3 includes the development of the SWR method for problem (1)–(3). Section 4 includes the convergence analysis of the developed method. Finally, in Sect. 5, we present the numerical results for two test examples confirming our convergence theory.

Notation: C denotes a generic positive constant which is independent of the parameters N, M, ε, and k. For $v \in C(\overline{\Omega})$, let us define $v_{i,j} = v(x_i, t_j)$. Here, $\|.\|_Q$ is used to define the maximum norm on a closed and bounded set Q, and $\|.\|_{Q^{N,M}}$ is used to define the corresponding discrete maximum norm.

2 Asymptotic behavior

We discuss a priori bounds on the solution derivatives of (1)–(3). The existence of a unique solution of problem (1)–(3) is established in the following lemma.

Lemma 1 *Suppose* $a, f \in C^{(\beta,\beta/2)}(\overline{\Omega})$, $g_\ell,\ g_r \in C^{\frac{1+\beta}{2}}([0,T])$, $g_b \in C^{2+\beta}(\overline{D})$, $\beta \in (0,1)$, *and the compatibility conditions up to the first order are true. Then, problem (1)–(3) has a unique solution* $u \in C^{(2+\beta,1+\beta/2)}(\overline{\Omega})$.

Proof See [23, Chapter IV, Section 5]). □

In the following lemma, crude bounds for the derivatives of u are given.

Lemma 2 *Suppose* $a, f \in C^{(2+\beta,1+\beta/2)}(\overline{\Omega})$, $g_\ell,\ g_r \in C^{\frac{3+\beta}{2}}([0,T])$, $g_b \in C^{4+\beta}(\overline{D})$, $\beta \in (0,1)$, *and the compatibility conditions up to the second order are true. Then, problem (1)–(3) has a unique solution* $u \in C^{(4+\beta,2+\beta/2)}(\overline{\Omega})$. *Moreover, it holds*

$$\left\| \frac{\partial^{q_1+q_2} u}{\partial x^{q_1} \partial t^{q_2}} \right\|_{\overline{\Omega}} \le C\varepsilon^{-q_1/2} \quad for \ \ 0 \le q_1 + 2q_2 \le 4, \tag{4}$$

Proof The proof follows from the arguments in [37, Theorem 5]. □

The above bounds are not enough for the convergence analysis in Sect. 4. We require to split u into regular and singular parts, and bounds for their derivatives as given in the following lemma.

Lemma 3 *Suppose* $a, f \in C^{(4+\beta,2+\beta/2)}(\overline{\Omega})$, $g_\ell,\ g_r \in C^{\frac{5+\beta}{2}}([0,T])$, $g_b \in C^{6+\beta}(\overline{D})$, $\beta \in (0,1)$, *and sufficient compatibility conditions hold. Then, one can split u as* $u = u_1 + u_2$, *where* u_1 *and* u_2 *are the regular and singular parts, respectively, satisfying*

$$|\partial_x^{q_1} u_1(x,t)| \le C(1+\varepsilon^{(2-q_1)/2}), \tag{5}$$

$$|\partial_x^{q_1} u_2(x,t)| \le C\varepsilon^{-q_1/2}\left(e^{(-x\sqrt{\alpha/\varepsilon})} + e^{(-(1-x)\sqrt{\alpha/\varepsilon})}\right), \tag{6}$$

for $(x,t) \in \overline{\Omega}$, $0 \le q_1 \le 4$.

Proof The lemma can be proved following the arguments in [37, Theorem 6]. □

3 The domain decomposition method

We partition Ω into $\Omega_p = \mathsf{D}_p \times (0,T]$, $p = \ell, m, r$, where $\mathsf{D}_\ell = (0, 2\rho)$, $\mathsf{D}_m = (\rho, 1-\rho)$, $\mathsf{D}_r = (1-2\rho, 1)$ with the parameter ρ defined by

$$\rho = \min\left\{\frac{1}{4},\ 2\sqrt{\frac{\varepsilon}{\alpha}}\ln N\right\}. \tag{7}$$

Here, ρ is chosen as the Shishkin transition parameter [9, 26], which enables us to have fine meshes (for small ε) on Ω_ℓ and Ω_r (the subdomains corresponding to the boundary layers). Further, we discretize each subdomain Ω_p with a uniform mesh in both the spatial and temporal directions. On each spatial subdomain $\overline{\mathsf{D}}_p = [a, b]$, we define a uniform mesh $\overline{\mathsf{D}}_p^N = \{x_{p;i} = a + ih_p, i = 0, \ldots, N,\ h_p = (b-a)/N\}$. When the domain is obvious, for convenience, the term p will be dropped from the subscript in $x_{p;i}$. On the interval $\overline{\Omega}_t = [0, T]$, we define a uniform mesh $\overline{\Omega}_t^M = \{t_j = j\Delta t,\ j = 0, \ldots, M,\ \text{with } \Delta t = T/M\}$. Here, M and N are discretization parameters in time and space directions respectively. Defining $\mathsf{D}_p^N = \overline{\mathsf{D}}_p^N \cap \mathsf{D}_p$, and $\Omega_t^M = \overline{\Omega}_t^M \cap (0, T]$, the mesh $\Omega_p^{N,M}$ corresponding to Ω_p is defined as $\Omega_p^{N,M} = \mathsf{D}_p^N \times \Omega_t^M$. Further, we define $\bar{\Omega}_p^{N,M} = \bar{\mathsf{D}}_p^N \times \bar{\Omega}_t^M$ and $\gamma_{p,\ell}^{N,M} = \{x_{p;0}\} \times \Omega_t^M$, $\gamma_{p,r}^{N,M} = \{x_{p;N}\} \times \Omega_t^M$, $\gamma_{p,b}^{N,M} = \overline{\mathsf{D}}_p^N \times \{t_0\}$. Then, on each $\Omega_p^{N,M}$, $p = \ell, m, r$, the discretization of (1) is defined as follows

$$L_p^{N,M} U_{p;i,j} := B_t^- U_{p;i,j} - \varepsilon\delta_x^2 U_{p;i,j} + a_{i,j} U_{p;i,j} = f_{i,j}, \quad (x_{p;,i}, t_j) \in \Omega_p^{N,M},$$

where

$$\begin{aligned}
&\delta_x^2 U_{p;i,j} = \frac{[F_x^+ U_p - F_x^- U_p]_{i,j}}{h_p},\\
&F_x^+ U_{p;i,j} = (U_{p;i+1,j} - U_{p;i,j})/h_p, \quad F_x^- U_{p;i,j} = (U_{p;i,j} - U_{p;i-1,j})/h_p,\\
&\text{and } \ B_t^- U_{p;i,j} = (U_{p;i,j} - U_{p;i,j-1})/\Delta t.
\end{aligned}$$

We discretize (2) as follows

$$\begin{cases}
\Gamma_\ell^{N,M} U_\ell(0, t_j) := U_\ell(0, t_j) - \sqrt{\varepsilon} F_x^+ U_\ell(0, t_j) + \frac{h_\ell}{2\sqrt{\varepsilon}}(aU_\ell + B_t^- U_\ell)(0, t_j)\\
\qquad\qquad = g_\ell(t_j) + \frac{h_\ell}{2\sqrt{\varepsilon}} f(0, t_j), \quad t_j \in \Omega_t^M,\\
\Gamma_r^{N,M} U_r(1, t_j) := U_r(1, t_j) + \sqrt{\varepsilon} F_x^- U_r(1, t_j) + \frac{h_r}{2\sqrt{\varepsilon}}(aU_r + B_t^- U_r)(1, t_j)\\
\qquad\qquad = g_r(t_j) + \frac{h_r}{2\sqrt{\varepsilon}} f(1, t_j), \quad t_j \in \Omega_t^M.
\end{cases} \tag{8}$$

Note that we have approximated the boundary conditions using a special discretization scheme. If we had used the standard upwind scheme for the discretization of the boundary conditions we would have obtained only first order accuracy. Therefore, we use a special discretization scheme for the boundary conditions, which is based on improving the truncation error to maintain second order accuracy.

Consider $\overline{\Omega}^{N,M} = (\overline{\Omega}_\ell^{N,M} \backslash \overline{\Omega}_m) \bigcup \overline{\Omega}_m^{N,M} \bigcup (\overline{\Omega}_r^{N,M} \backslash \overline{\Omega}_m)$. To compute as the approximate solution of problem (1)–(3) the algorithm is defined as follows. We start the iterative process with $U^{[0]}(x_i, t_j)$ as the initial approximation, defined as follows: $U^{[0]}(x_i, 0) = g_b(x_i), 0 \le x_i \le 1$; $U^{[0]}(x_i, t_j) = 0,\ 0 < x_i < 1,\ 0 < t_j \le T$. Suppose $\mathcal{I}_j Z$ is used to denote the piecewise-linear interpolant function of Z at t_j.

For each $k \geq 1$, we solve

$$\begin{cases} L_\ell^{N,M} U_\ell^{[k]} = f & \text{in } \Omega_\ell^{N,M}, \\ \Gamma_\ell^{N,M} U_\ell^{[k]}(0, t_j) = g_\ell(t_j) + \frac{h_\ell}{2\sqrt{\varepsilon}} f(0, t_j) & \text{if } t_j \in \Omega_t^M, \\ U_\ell^{[k]}(2\rho, t_j) = \mathcal{I}_j U^{[k-1]}(2\rho, t_j) & \text{if } t_j \in \Omega_t^M, \\ U_\ell^{[k]}(x_i, 0) = g_b(x_i) & \text{if } x_i \in \overline{\mathsf{D}}_\ell^N, \end{cases}$$

$$\begin{cases} L_r^{N,M} U_r^{[k]} = f & \text{in } \Omega_r^{N,M}, \\ U_r^{[k]}(1-2\rho, t_j) = \mathcal{I}_j U^{[k-1]}(1-2\rho, t_j) & \text{if } t_j \in \Omega_t^M, \\ \Gamma_r^{N,M} U_r^{[k]}(1, t_j) = g_r(t_j) + \frac{h_r}{2\sqrt{\varepsilon}} f(1, t_j) & \text{if } t_j \in \Omega_t^M, \\ U_r^{[k]}(x_i, 0) = g_b(x_i) & \text{if } x_i \in \overline{\mathsf{D}}_r^N, \end{cases}$$

$$\begin{cases} L_m^{N,M} U_m^{[k]} = f & \text{in } \Omega_m^{N,M}, \\ U_m^{[k]}(\rho, t_j) = \mathcal{I}_j U_\ell^{[k]}(\rho, t_j), & \text{for } t_j \in \Omega_t^M, \\ U_m^{[k]}(1-\rho, t_j) = \mathcal{I}_j U_r^{[k]}(1-\rho, t_j) & \text{for } t_j \in \Omega_t^M, \\ U_m^{[k]}(x_i, 0) = g_b(x_i) & \text{for } x_i \in \overline{\mathsf{D}}_m^N. \end{cases}$$

We then compute $U^{[k]}$ by

$$U^{[k]} = \begin{cases} U_\ell^{[k]} & \text{in } \overline{\Omega}_\ell^{N,M} \setminus \overline{\Omega}_m, \\ U_m^{[k]} & \text{in } \overline{\Omega}_m^{N,M}, \\ U_r^{[k]} & \text{in } \overline{\Omega}_r^{N,M} \setminus \overline{\Omega}_m. \end{cases} \tag{9}$$

The iterations are performed until

$$||U^{[k+1]} - U^{[k]}||_{\overline{\Omega}^{N,M}} \leq tol$$

is reached, where tol is a specified user tolerance.

For $p = \ell,\ r,\ m$, suppose the operator $\mathcal{T}_p^{N,M}$ is defined as $\mathcal{T}_p^{N,M} Z(y, t_j) = Z(y, t_j),\ y = \rho,\ 2\rho,\ 1-\rho,\ 1-2\rho, t_j \in \Omega_t^M$ and $\mathcal{T}_\ell^{N,M} Z(0, t_j) = \Gamma_\ell^{N,M} Z(0, t_j),\ \mathcal{T}_r^{N,M} Z(1, t_j) = \Gamma_r^{N,M} Z(1, t_j),\ t_j \in \Omega_t^M$. Now using the arguments in [37, Theorem 7] we can prove the following maximum principle for $L_p^{N,M}$.

Lemma 4 *Suppose $Z_p,\ p = m, r, \ell,$ satisfies $Z_p(x_i, 0) \geq 0$ for $x_i \in \overline{D}_p^N$ and $\mathcal{T}_p^{N,M} Z_p(x_0, t_j) \geq 0,\ \mathcal{T}_p^{N,M} Z_p(x_N, t_j) \geq 0,\ t_j \in \Omega_t^M$.*

Then, if $L_p^{N,M} Z_{p;i,j} \geq 0$ in $\Omega_p^{N,M}$, it holds that $Z_{p;i,j} \geq 0$ in $\overline{\Omega}_p^{N,M}$.

The stability of the numerical scheme is given by the following lemma.

Lemma 5 *For any mesh function Z_p, we have*

$$||Z_p||_{\overline{\Omega}_p^{N,M}} \leq \max\left\{ ||\mathcal{T}_p^{N,M} Z_p||_{\gamma_{p,\ell}^{N,M}}, ||\mathcal{T}_p^{N,M} Z_p||_{\gamma_{p,r}^{N,M}}, ||Z_p||_{\gamma_{p,b}^{N,M}}, \frac{1}{\alpha} ||L_p^{N,M} Z_p||_{\Omega_p^{N,M}} \right\}.$$

Proof Suppose $\Phi^{\pm}(x_i, t_j) = \Theta \pm Z(x_i, t_j)$ is a mesh function with

$$\Theta = \max\left\{ ||\mathcal{T}_p^{N,M} Z_p||_{\gamma_{p,\ell}^{N,M}}, ||\mathcal{T}_p^{N,M} Z_p||_{\gamma_{p,r}^{N,M}}, ||Z_p||_{\gamma_{p,b}^{N,M}}, \frac{1}{\alpha}||L_p^{N,M} Z_p||_{\Omega_p^{N,M}} \right\}.$$

Then, it is easy to verify that $\mathcal{T}_\ell^{N,M}\Phi^{\pm}(x_0, t_j) \geq 0$, $\mathcal{T}_r^{N,M}\Phi^{\pm}(x_N, t_j) \geq 0$, $t_j \in \Omega_t^M$, $\Phi^{\pm}(x_i, 0) \geq 0$ for $x_i \in \overline{\mathsf{D}}_p^N$, and $L_p^{N,M}\Phi^{\pm}(x_i, t_j) \geq 0$, $(x_i, t_j) \in \Omega_p^{N,M}$. Hence, from Lemma 4, the proof follows. □

4 Convergence analysis

We now establish that the method gives parameter-uniform approximations to the solution of problem (1)–(3). Further, we prove that the iterative process converges optimally for small ε. We consider the following auxiliary mesh function

$$\overline{U} = \begin{cases} \overline{U}_\ell & \text{in } \overline{\Omega}_\ell^{N,M} \setminus \overline{\Omega}_m, \\ \overline{U}_m & \text{in } \overline{\Omega}_m^{N,M}, \\ \overline{U}_r & \text{in } \overline{\Omega}_r^{N,M} \setminus \overline{\Omega}_m, \end{cases} \tag{10}$$

where $\overline{U}_p$, $p = \ell, m, r$, are such that

$$\begin{cases} L_\ell^{N,M}\overline{U}_\ell = f & \text{in } \Omega_\ell^{N,M}, \\ \Gamma_\ell^{N,M}\overline{U}_\ell(0, t_j) = \Gamma_\ell^{N,M}U^{[k]}(0, t_j) & \text{for } t_j \in \Omega_t^M, \\ \overline{U}_\ell(2\rho, t_j) = u(2\rho, t_j) & \text{for } t_j \in \Omega_t^M, \\ \overline{U}_\ell(x_i, 0) = u(x_i, 0) & \text{if } x_i \in \overline{\mathsf{D}}_\ell^N, \end{cases}$$

$$\begin{cases} L_m^{N,M}\overline{U}_m = f & \text{in } \Omega_m^{N,M}, \\ \overline{U}_m(\rho, t_j) = u(\rho, t_j) & \text{for } t_j \in \Omega_t^M, \\ \overline{U}_m(1-\rho, t_j) = u(1-\rho, t_j) & \text{for } t_j \in \Omega_t^M, \\ \overline{U}_m(x_i, 0) = u(x_i, 0) & \text{if } x_i \in \overline{\mathsf{D}}_m^N, \end{cases}$$

$$\begin{cases} L_r^{N,M}\overline{U}_r = f & \text{in } \Omega_r^{N,M}, \\ \overline{U}_r(1-2\rho, t_j) = u(1-2\rho, t_j) & \text{for } t_j \in \Omega_t^M, \\ \Gamma_r^{N,M}\overline{U}_r(1, t_j) = \Gamma_r^{N,M}U^{[k]}(1, t_j) & \text{for } t_j \in \Omega_t^M, \\ \overline{U}_r(x_i, 0) = u(x_i, 0) & \text{if } x_i \in \overline{\mathsf{D}}_r^N. \end{cases}$$

Here, the discrete operators $L_\ell^{N,M}$, $L_m^{N,M}$, $L_r^{N,M}$, $\Gamma_r^{N,M}$, and $\Gamma_\ell^{N,M}$ are the ones that were defined in the previous section. The main difference between these problems and the problems in the previous section is that we used the solution u of problem (1)–(3) in the boundary conditions for $\overline{U}_\ell$ at $(2\rho, t_j)$, $\overline{U}_r$ at $(1-2\rho, t_j)$, and $\overline{U}_m$ at (ρ, t_j)

and $(1-\rho, t_j)$. By the triangle inequality, we get

$$||u - U^{[k]}||_{\overline{\Omega}^{N,M}} \leq ||u - \overline{U}||_{\overline{\Omega}^{N,M}} + ||\overline{U} - U^{[k]}||_{\overline{\Omega}^{N,M}}. \tag{11}$$

The following lemma gives the bound of the first term on the LHS of the inequality in (11).

Lemma 6 *Suppose u is the solution of (1)–(3) and $\overline{U}$ is the auxiliary mesh function defined in (10). Then*

$$||u - \overline{U}||_{\overline{\Omega}^{N,M}} \leq C(\Delta t + (N^{-1}\ln N)^2). \tag{12}$$

Proof Note that $(u - \overline{U_\ell})(2\rho, t_j) = 0$ for $t_j \in \Omega_t^M$. On the other hand, from (1)–(2) and (8) it follows that

$$\begin{aligned}
\Gamma_\ell^{N,M}(u - \overline{U}_\ell)(0,t_j) &= \left(u - \sqrt{\varepsilon}F_x^+ u + \frac{h_\ell}{2\sqrt{\varepsilon}}(au + B_t^- u) - (u - \sqrt{\varepsilon}u_x) - \frac{h_\ell}{2\sqrt{\varepsilon}}f\right)(0,t_j)\\
&= \left(\sqrt{\varepsilon}(u_x - F_x^+ u) + \frac{h_\ell}{2\sqrt{\varepsilon}}(au - f + u_t)\right)(0,t_j) + \frac{h_\ell}{2\sqrt{\varepsilon}}(B_t^- u - u_t)(0,t_j)\\
&= \sqrt{\varepsilon}\left(u_x - F_x^+ u + \frac{h_\ell}{2}u_{xx}\right)(0,t_j) + \frac{h_\ell}{2\sqrt{\varepsilon}}(B_t^- u - u_t)(0,t_j), \quad t_j \in \Omega_t^M.
\end{aligned}$$

Using Taylor expansions and (4) we get

$$\begin{aligned}
\left|\Gamma_\ell^{N,M}(u - \overline{U_\ell})(0,t_j)\right| &\leq \frac{\sqrt{\varepsilon}}{6}h_\ell^2 \left\|u_{xxx}(.,t_j)\right\|_{[x_0,x_1]} + \frac{h_\ell}{4\sqrt{\varepsilon}}(t_j - t_{j-1}) \left\|u_{tt}(x_i,.)\right\|_{[t_{j-1},t_j]}\\
&\leq C(\Delta t + (N^{-1}\ln N)^2).
\end{aligned}$$

Now

$$L_p^{N,M}(u - \overline{U}_p) = (L_p^{N,M} - L)u = \left(B_t^- u - u_t\right) - \varepsilon\left(\delta_x^2 u - u_{xx}\right) \text{ in } \Omega_p^{N,M}, \ p = \ell, m, r, \tag{13}$$

and

$$\left|L_p^{N,M}(u - \overline{U}_p)_{i,j}\right| \leq \frac{(t_j - t_{j-1})}{2}\left\|u_{tt}(x_i,.)\right\|_{[t_{j-1},t_j]} + \frac{\varepsilon}{12}h_p^2 \left\|u_{xxxx}(.,t_j)\right\|_{[x_{i-1},x_{i+1}]}, \tag{14}$$

where we have used Taylor expansions to get (14). Since $h_\ell \leq C\sqrt{\varepsilon}N^{-1}\ln N$, using (4) it follows from (14) that

$$\left|L_\ell^{N,M}(u - \overline{U}_\ell)_{i,j}\right| \leq C(\Delta t + (N^{-1}\ln N)^2) \quad \text{in } \Omega_\ell^{N,M}.$$

Hence, by Lemma 4 with $C(\Delta t + (N^{-1}\ln N)^2) \pm (u - \overline{U}_\ell)_{i,j}$, one gets

$$||u - \overline{U}_\ell||_{\overline{\Omega}_\ell^{N,M}} \leq C(\Delta t + (N^{-1}\ln N)^2).$$

A similar analysis gives

$$||u-\overline{U}_r||_{\overline{\Omega}_r^{N,M}} \le C(\Delta t + (N^{-1}\ln N)^2).$$

Suppose $\rho = 1/4$. Then, we have $h_m = 0.5N^{-1}$ and $\varepsilon^{-1} \le C\ln^2 N$. Again, using (4) it follows from (14) that

$$\left|L_m^{N,M}(u-\overline{U}_m)_{i,j}\right| \le C(\Delta t + (N^{-1}\ln N)^2) \quad \text{in } \Omega_m^{N,M}.$$

If $\rho = 2\sqrt{\frac{\varepsilon}{\alpha}}\ln N$, then $h_m \le CN^{-1}$. For $(B_t^- u - u_t)$ use Taylor expansion and (4) to get

$$\left|\left(B_t^- u - u_t\right)_{i,j}\right| \le C\Delta t \quad \text{in } \Omega_m^{N,M}.$$

For $\varepsilon(\delta_x^2 u - u_{xx})$ use the decomposition $u = u_1 + u_2$, Taylor expansions, and (5)-(6) to get

$$\begin{aligned}
\varepsilon\left|\left(\delta_x^2 u - u_{xx}\right)_{i,j}\right| &\le \varepsilon\left(\left|\left(\delta_x^2 u_1 - \frac{\partial^2 u_1}{\partial x^2}\right)_{i,j}\right| + \left|\left(\delta_x^2 u_2 - \frac{\partial^2 u_2}{\partial x^2}\right)_{i,j}\right|\right)\\
&\le C\varepsilon h_m^2\left\|\frac{\partial^4 u_1(.,t_j)}{\partial x^4}\right\|_{[x_{i-1},x_{i+1}]} + C\varepsilon\left\|\frac{\partial^2 u_2(.,t_j)}{\partial x^2}\right\|_{[x_{i-1},x_{i+1}]}\\
&\le CN^{-2} + 2Ce^{-\rho\sqrt{\alpha/\varepsilon}} \le CN^{-2}.
\end{aligned}$$

Thus, $\left|L_m^{N,M}(u-\overline{U}_m)_{i,j}\right| \le C(\Delta t + (N^{-1}\ln N)^2)$ in $\Omega_m^{N,M}$. So, applying Lemma 4 to $C(\Delta t + (N^{-1}\ln N)^2) \pm (u-\overline{U}_m)$, one gets

$$||u-\overline{U}_m||_{\overline{\Omega}_m^{N,M}} \le C(\Delta t + (N^{-1}\ln N)^2).$$

This completes the proof. □

The following notation will be used in the next two theorems.

$$\mu^{[k]} = \max\left\{\max_{t_j\in\Omega_t^M}|(\overline{U}_\ell - \mathcal{I}_j U^{[k-1]})(2\rho,t_j)|,\ \max_{t_j\in\Omega_t^M}|(\overline{U}_r - \mathcal{I}_j U^{[k-1]})(1-2\rho,t_j)|\right\},$$

$$\mu_\rho = \max\left\{\max_{t_j\in\Omega_t^M}|(\overline{U}_m - \overline{U}_\ell)(\rho,t_j)|,\ \max_{t_j\in\Omega_t^M}|(\overline{U}_m - \overline{U}_r)(1-\rho,t_j)|\right\}.$$

We will show the parameter-uniform convergence of the method in two cases: when $\rho = 2\sqrt{\frac{\varepsilon}{\alpha}}\ln N$ and $\rho = 1/4$. In the next theorem, for $\rho = 2\sqrt{\frac{\varepsilon}{\alpha}}\ln N$, we first obtain a bound of $||\overline{U} - U^{[1]}||_{\overline{\Omega}^{N,M}}$ and then combine it with Lemma 6 to get a bound of

$||u - U^{[1]}||_{\overline{\Omega}^{N,M}}$. So, this result demonstrates that only one iteration is enough to achieve the required accuracy.

Theorem 1 *Suppose u is the exact solution of (1)–(3) and $U^{[1]}$ is its approximation obtained after the first iterate of the proposed method. Then, for $\rho = 2\sqrt{\frac{\varepsilon}{\alpha}} \ln N$, it holds*

$$||u - U^{[1]}||_{\overline{\Omega}^{N,M}} \le C(\Delta t + (N^{-1} \ln N)^2).$$

Proof Observe that

$$\begin{cases} L_\ell^{N,M}(\overline{U}_\ell - U_\ell^{[1]}) = 0 & \text{in } \Omega_\ell^{N,M}, \\ (\overline{U}_\ell - U_\ell^{[1]})(x_i, 0) = 0 & \text{for } x_i \in \overline{\mathsf{D}}_\ell^N, \\ \Gamma_\ell^{N,M}(\overline{U}_\ell - U_\ell^{[1]})(0, t_j) = 0 & \text{for } t_j \in \Omega_t^M, \\ |(\overline{U}_\ell - U_\ell^{[1]})(2\rho, t_j)| \le \mu^{[1]} & \text{for } t_j \in \Omega_t^M. \end{cases}$$

For an arbitrary mesh function Y, let us define

$$\widetilde{\Gamma}_\ell^{N,M} Y(0, t_j) := Y(0, t_j) - \sqrt{\varepsilon} F_x^+ Y(0, t_j) + \frac{h_\ell}{2\sqrt{\varepsilon}}(\alpha Y + B_t^- Y)(0, t_j).$$

For $(x_i, t_j) \in \overline{\Omega}_\ell^{N,M}$, consider

$$\chi^\pm(x_i, t_j) = \left[\chi_\ell \pm (\overline{U}_\ell - U_\ell^{[1]})\right](x_i, t_j),$$

where χ_ℓ solves

$$\begin{cases} B_t^- \chi_\ell - \varepsilon \delta_x^2 \chi_\ell + \alpha \chi_\ell = 0 \text{ in } \Omega_\ell^{N,M}, \\ \chi_\ell(x_i, 0) = \mu^{[1]} \frac{B\xi_1^i - A\xi_2^i}{B\xi_1^N - A\xi_2^N} & \text{for } x_i \in \overline{\mathsf{D}}_\ell^N, \\ \widetilde{\Gamma}_\ell^{N,M} \chi_\ell(0, t_j) = 0 & \text{for } t_j \in \Omega_t^M, \\ \chi_\ell(2\rho, t_j) = \mu^{[1]} & \text{for } t_j \in \Omega_t^M \end{cases} \tag{15}$$

with

$$A = 2\sqrt{\varepsilon} h_\ell - 2\varepsilon(\xi_1 - 1) + h_\ell^2 a\ , \ B = 2\sqrt{\varepsilon} h_\ell - 2\varepsilon(\xi_2 - 1) + h_\ell^2 a,$$

and ξ_i, $i = 1, 2$, are as follows

$$\xi_1 = \lambda_1 + \lambda_2 \quad \text{and} \quad \xi_2 = \lambda_1 - \lambda_2$$

with

$$\lambda_1 = 1 + \left(\frac{\rho}{N}\sqrt{\frac{\alpha}{\varepsilon}}\right)^2, \ \lambda_2 = 2\left(\frac{\rho}{N}\sqrt{\frac{\alpha}{\varepsilon}}\right)\sqrt{1 + \left(\frac{\rho}{N}\sqrt{\frac{\alpha}{\varepsilon}}\right)^2}.$$

The solution of (15) is given by

$$\chi_\ell(x_i, t_j) = \mu^{[1]} \frac{\varphi \xi_1^i - \xi_2^i}{\varphi \xi_1^N - \xi_2^N}, \quad \varphi = B/A, \tag{16}$$

which is monotonically increasing and satisfies $\chi_\ell \geq 0$ in $\overline{\Omega}_\ell^{N,M}$. Thus, one gets

$$\chi^{\pm}(x_i, 0) \geq 0 \;\text{ if }\; x_i \in \overline{\mathsf{D}}_\ell^N, \quad \Gamma_\ell^{N,M} \chi^{\pm}(0, t_j) \geq 0, \;\; \chi^{\pm}(2\rho, t_j) \geq 0 \;\text{ if } t_j \in \Omega_t^M,$$

and $L_\ell^{N,M} \chi^{\pm}(x_i, t_j) \geq 0$ if $(x_i, t_j) \in \Omega_\ell^{N,M}$. Hence, applying Lemma 4 to $\chi^{\pm}$, we obtain

$$|(\overline{U}_\ell - U_\ell^{[1]})_{i,j}| \leq \chi_\ell(x_i, t_j), \quad \text{if } \;(x_i, t_j) \in \overline{\Omega}_\ell^{N,M}.$$

Further, since $x_i \leq \rho$ for $(x_i, t_j) \in \overline{\Omega}_\ell^{N,M} \backslash \overline{\Omega}_m$, it follows from (16) that

$$\chi_\ell(x_i, t_j) \leq \mu^{[1]} \frac{\varphi \xi_1^{N/2} - \xi_2^{N/2}}{\varphi \xi_1^N - \xi_2^N} = \mu^{[1]} \frac{\varphi^2 \xi_1^N - \xi_2^N}{(\varphi \xi_1^N - \xi_2^N)(\varphi \xi_1^{N/2} + \xi_2^{N/2})}.$$

This implies that

$$\chi_\ell(x_i, t_j) \leq \frac{\mu^{[1]}}{\varphi \xi_1^{N/2} + \xi_2^{N/2}} \left[\frac{\varphi^2 - \xi_2^N/\xi_1^N}{\varphi - \xi_2^N/\xi_1^N} \right].$$

Now, since $\varphi > 1$ and $\xi_2/\xi_1 < 1$, we have $(\varphi^2 - \xi_2^N/\xi_1^N)/(\varphi - \xi_2^N/\xi_1^N) \leq C\varphi$. Thus, one gets

$$\chi_\ell(x_i, t_j) \leq \mu^{[1]} \frac{C\varphi}{\varphi \xi_1^{N/2} + \xi_2^{N/2}} \leq \frac{C\mu^{[1]}}{\xi_1^{N/2}}.$$

For $\rho = 2\varepsilon^{1/2}\alpha^{-1/2} \ln N$, it holds

$$\xi_1^{-N/2} \leq \left(1 + \frac{\rho}{N}\sqrt{\frac{\alpha}{\varepsilon}}\right)^{-N} \leq 4N^{-2}, \;\; \text{if } N \geq 1,$$

having used $\lambda_2 \geq 2\left(\frac{\rho}{N}\sqrt{\frac{\alpha}{\varepsilon}}\right)$ and the arguments in [26, Lemma 5.1]. Further, since $\mu^{[1]} \leq C$, we have $\chi_\ell(x_i, t_j) \leq CN^{-2}$ in $\overline{\Omega}_\ell^{N,M} \backslash \overline{\Omega}_m$. Thus, we have

$$||\overline{U}_\ell - U_\ell^{[1]}||_{\overline{\Omega}_\ell^{N,M} \backslash \overline{\Omega}_m} \leq CN^{-2}. \tag{17}$$

Similarly, it is

$$||\overline{U}_r - U_r^{[1]}||_{\overline{\Omega}_r^{N,M}\setminus\overline{\Omega}_m} \leq CN^{-2}. \tag{18}$$

Next, consider $\overline{U}_m - U_m^{[1]}$ which satisfies

$$\begin{aligned}
&L_m^{N,M}(\overline{U}_m - U_m^{[1]}) = 0 \quad \text{in } \Omega_m^{N,M},\\
&(\overline{U}_m - U_m^{[1]})(x_i, 0) = 0, \quad \text{if } x_i \in \overline{\mathsf{D}}_m^N,\\
&|(\overline{U}_m - U_m^{[1]})(\rho, t_j)| = |(\overline{U}_m - \mathcal{I}_j U_\ell^{[1]})(\rho, t_j)|\\
&\leq |(\overline{U}_m - \overline{U}_\ell)(\rho, t_j)| + |(\overline{U}_\ell - U_\ell^{[1]})(\rho, t_j)|\\
&\leq \mu_\rho + CN^{-2}, \quad t_j \in \Omega_t^M,
\end{aligned}$$

and

$$\begin{aligned}
&|(\overline{U}_m - U_m^{[1]})(1-\rho, t_j)| = |(\overline{U}_m - \mathcal{I}_j U_r^{[1]})(1-\rho, t_j)| \leq |(\overline{U}_m - \overline{U}_r)(1-\rho, t_j)|\\
&\quad + |(\overline{U}_r - U_r^{[1]})(1-\rho, t_j)|\\
&\leq \mu_\rho + CN^{-2} \ \text{ for } t_j \in \Omega_t^M,
\end{aligned}$$

since the mesh points (ρ, t_j) and $(1-\rho, t_j)$ belong to $\Omega_\ell^{N,M}$ and $\Omega_r^{N,M}$, respectively. Therefore, Lemma 4 gives

$$||\overline{U}_m - U_m^{[1]}||_{\overline{\Omega}_m^{N,M}} \leq \mu_\rho + CN^{-2}. \tag{19}$$

Combining (17), (18), and (19), we obtain

$$||\overline{U} - U^{[1]}||_{\overline{\Omega}^{N,M}} \leq \mu_\rho + CN^{-2}. \tag{20}$$

Since $(\rho, t_j) \in \overline{\Omega}_\ell^{N,M}$ and $(1-\rho, t_j) \in \overline{\Omega}_r^{N,M}$, by Lemma 6 one gets $\mu_\rho \leq C(\Delta t + (N^{-1}\ln N)^2)$. Hence, using (20) and Lemma 6 in (11) the proof is complete. □

In the next theorem, for $\rho = 1/4$, we first obtain a bound on $||\overline{U} - U^{[k]}||_{\overline{\Omega}^{N,M}}$ and then combine it with Lemma 6 to get a bound on $||u - U^{[k]}||_{\overline{\Omega}^{N,M}}$.

Theorem 2 *Suppose $U^{[k]}$ is the approximation of the exact solution u of (1)–(3), generated by the k-th iterate of the method. Then, for $\rho = 1/4$, it holds*

$$||u - U^{[k]}||_{\overline{\Omega}^{N,M}} \leq C\left((5/6)^k + (\Delta t + (N^{-1}\ln N)^2)\right). \tag{21}$$

Proof The following notation is used

$$\nu^{[k]} = \max\left\{||\overline{U}_r - U^{[k]}||_{\overline{\Omega}_r^{N,M}\backslash\overline{\Omega}_m}, ||\overline{U}_\ell - U^{[k]}||_{\overline{\Omega}_\ell^{N,M}\backslash\overline{\Omega}_m}, ||\overline{U}_m - U^{[k]}||_{\overline{\Omega}_m^{N,M}}\right\},$$

$$\mu_{2\rho} = \max\left\{\max_{t_j\in\Omega_t^M}|(\overline{U}_\ell - \overline{U}_m)(2\rho, t_j)|, \max_{t_j\in\Omega_t^M}|(\overline{U}_r - \overline{U}_m)(1-2\rho, t_j)|\right\}.$$

We have

$$\begin{cases} L_\ell^{N,M}(\overline{U}_\ell - U_\ell^{[1]}) = 0 & \text{in } \Omega_\ell^{N,M}, \\ (\overline{U}_\ell - U_\ell^{[1]})(x_i, 0) = 0, & \text{if } x_i \in \overline{\mathsf{D}}_\ell^N, \\ \Gamma_\ell^{N,M}(\overline{U}_\ell - U_\ell^{[1]})(0, t_j) = 0 & \text{if } t_j \in \Omega_t^M, \\ |(\overline{U}_\ell - U_\ell^{[1]})(2\rho, t_j)| \le \mu^{[1]} & \text{if } t_j \in \Omega_t^M. \end{cases}$$

We define

$$\Psi^{\pm}(x_i, t_j) = \mu^{[1]}\frac{x_i + \sqrt{\varepsilon}}{2\rho + \sqrt{\varepsilon}} \pm (\overline{U}_\ell - U_\ell^{[1]})(x_i, t_j),$$

which verify that $\Psi^{\pm}(x_i, 0) \ge 0$, $x_i \in \overline{\mathsf{D}}_\ell^N$; $\Gamma_\ell^{N,M}\Psi^{\pm}(0, t_j) \ge 0$, $\Psi^{\pm}(2\rho, t_j) \ge 0$, $t_j \in \Omega_t^M$;

$$L_\ell^{N,M}\Psi^{\pm}(x_i, t_j) = \left(\frac{x_i + \sqrt{\varepsilon}}{2\rho + \sqrt{\varepsilon}}\right)\mu^{[1]}a(x_i, t_j) \pm 0 \ge 0, (x_i, t_j) \in \Omega_\ell^{N,M}.$$

So, Lemma 4 gives

$$|(\overline{U}_\ell - U_\ell^{[1]})(x_i, t_j)| \le \mu^{[1]}\left(\frac{x_i + \sqrt{\varepsilon}}{2\rho + \sqrt{\varepsilon}}\right). \tag{22}$$

Since $x_i \le \rho$ for $(x_i, t_j) \in \overline{\Omega}_\ell^{N,M}\backslash\overline{\Omega}_m$, it follows from (22) that

$$||\overline{U}_\ell - U_\ell^{[1]}||_{\overline{\Omega}_\ell^{N,M}\backslash\overline{\Omega}_m} \le \frac{5}{6}\mu^{[1]}. \tag{23}$$

Similarly, we have

$$||\overline{U}_r - U_r^{[1]}||_{\overline{\Omega}_r^{N,M}\backslash\overline{\Omega}_m} \le \frac{5}{6}\mu^{[1]}. \tag{24}$$

For $\overline{U}_m - U_m^{[1]}$, we have

$$L_m^{N,M}(\overline{U}_m - U_m^{[1]}) = 0 \ \text{ in } \Omega_m^{N,M}, \ \ (\overline{U}_m - U_m^{[1]})(x_i, 0) = 0, \ \text{if } x_i \in \overline{\mathsf{D}}_m^N;$$

and for $t_j \in \Omega_t^M$,

$$|(\overline{U}_m - U_m^{[1]})(\rho, t_j)| = |(\overline{U}_m - \mathcal{I}_j U_\ell^{[1]})(\rho, t_j)| \le \mu_\rho + \frac{5}{6}\mu^{[1]}$$
$$\text{(since } (\rho, t_j) \in \overline{\Omega}_\ell^{N,M}\text{)}$$
$$|(\overline{U}_m - U_m^{[1]})(1-\rho, t_j)| = |(\overline{U}_m - \mathcal{I}_j U_r^{[1]})(1-\rho, t_j)| \le \mu_\rho + \frac{5}{6}\mu^{[1]}$$
$$\text{(since } (1-\rho, t_j) \in \overline{\Omega}_r^{N,M}\text{)}.$$

Therefore, Lemma 4 gives

$$||\overline{U}_m - U_m^{[1]}||_{\overline{\Omega}_m^{N,M}} \le \mu_\rho + \frac{5}{6}\mu^{[1]}. \tag{25}$$

We next obtain an estimate for $\mu^{[2]}$. Since $(2\rho, t_j)$ and $(1-2\rho, t_j)$ belong to $\overline{\Omega}_m^{N,M}$, it follows that

$$|(\overline{U}_\ell - \mathcal{I}_j U^{[1]})(2\rho, t_j)| \le \mu_{2\rho} + \mu_\rho + \frac{5}{6}\mu^{[1]}$$
$$\text{and} \qquad |(\overline{U}_r - \mathcal{I}_j U^{[1]})(1-2\rho, t_j)| \le \mu_{2\rho} + \mu_\rho + \frac{5}{6}\mu^{[1]}.$$

Thus, $\mu^{[2]} \le \mu_{2\rho} + \mu_\rho + \frac{5}{6}\mu^{[1]}$. Therefore, it holds that

$$\max\{\nu^{[1]}, \mu^{[2]}\} \le \lambda + \frac{5}{6}\mu^{[1]}, \quad \lambda = \mu_{2\rho} + \mu_\rho.$$

After repeating the above arguments one gets

$$\max\{\nu^{[k]}, \mu^{[k+1]}\} \le \lambda + \frac{5}{6}\mu^{[k]}.$$

We simplify this to get

$$\mu^{[k]} \le 6\lambda + \left(\frac{5}{6}\right)^{k-1}\mu^{[1]}.$$

Thus, it is

$$\nu^{[k]} \le 6\lambda + \left(\frac{5}{6}\right)^{k}\mu^{[1]}. \tag{26}$$

Further, since $(2\rho, t_j), (1-2\rho, t_j) \in \overline{\Omega}_m^{N,M}$, $(\rho, t_j) \in \overline{\Omega}_\ell^{N,M}$, and $(1-\rho, t_j) \in \overline{\Omega}_r^{N,M}$, by Lemma 6 $\lambda \le C(\Delta t + (N^{-1}\ln N)^2)$. Since $\mu^{[1]} \le C$, using (26) and Lemma 6 in (11) the proof is complete. □

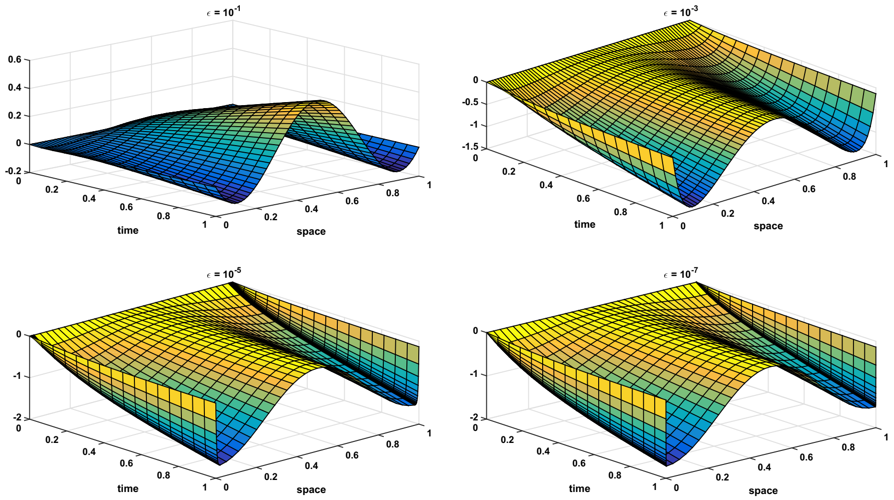

Fig. 1 Solution plots for Example 1 taking $N = 64,\ M = 16$, and various values of ε

5 Numerical results

Numerical results considering a couple of examples are given that verify the convergence theory.

Example 1 *Consider*

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + (1 + xe^{-t})u(x,t) = f(x,t) & (x,t) \in \Omega = \mathsf{D} \times (0,1], \\ u(0,t) - \sqrt{\varepsilon}\frac{\partial u}{\partial x}(0,t) = g_\ell(t) & t \in (0,1], \\ u(1,t) + \sqrt{\varepsilon}\frac{\partial u}{\partial x}(1,t) = g_r(t) & t \in (0,1], \\ u(x,0) = 0 & x \in [0,1]. \end{cases}$$

where $f,\ g_\ell$, and g_r are such that

$$u(x,t) = t\left[\frac{e^{-x/\sqrt{\varepsilon}} + e^{(x-1)/\sqrt{\varepsilon}}}{1 + e^{-1/\sqrt{\varepsilon}}} - \cos^2(\pi x)\right].$$

The solution plots for various values of ε are given in Fig. 1. Note that the boundary layers are close to $x = 0, 1$. Taking the tolerance error $tol = N^{-2}$, we compute the approximate solution and denote it by $U^{N,\Delta t}$. We then evaluate parameter uniform errors and convergence orders as follows

$$E^{N,\Delta t} = \max_{\forall \varepsilon} E_\varepsilon^{N,\Delta t} \text{ and } R^{N,\Delta t} = \log_2\left(\frac{E^{N,\Delta t}}{E^{2N,\Delta t/4}}\right),$$

where $E_\varepsilon^{N,\Delta t} = ||u - U^{N,\Delta t}||_{\overline{\Omega}^{N,M}}$ are the maximum pointwise errors. Table 1 displays the computed errors $E_\varepsilon^{N,\Delta t},\ E^{N,\Delta t}$, and parameter uniform orders $R^{N,\Delta t}$ for

Table 1 Errors and convergence orders for Example 1

$\varepsilon = 10^{-p}$	$N = 64$ $\Delta t = 0.25/4$	$N = 128$ $\Delta t = 0.25/4^2$	$N = 256$ $\Delta t = 0.25/4^3$	$N = 512$ $\Delta t = 0.25/4^4$	$N = 1024$ $\Delta t = 0.25/4^5$
$p = 1$	6.855E-05	1.712E-05	4.285E-06	1.069E-06	2.674E-07
2	3.193E-04	8.292E-05	2.072E-05	5.181E-06	1.295E-06
3	3.195E-03	8.063E-04	2.020E-04	5.054E-05	1.263E-05
4	3.530E-03	1.212E-03	3.972E-04	1.258E-04	3.885E-05
5	3.530E-03	1.212E-03	3.972E-04	1.258E-04	3.885E-05
6	3.530E-03	1.212E-03	3.972E-04	1.258E-04	3.885E-05
7	3.530E-03	1.212E-03	3.972E-04	1.258E-04	3.885E-05
8	3.530E-03	1.212E-03	3.972E-04	1.258E-04	3.885E-05
$E^{N,\Delta t}$	3.530E-03	1.212E-03	3.972E-04	1.258E-04	3.885E-05
$R^{N,\Delta t}$	1.542	1.609	1.658	1.695	

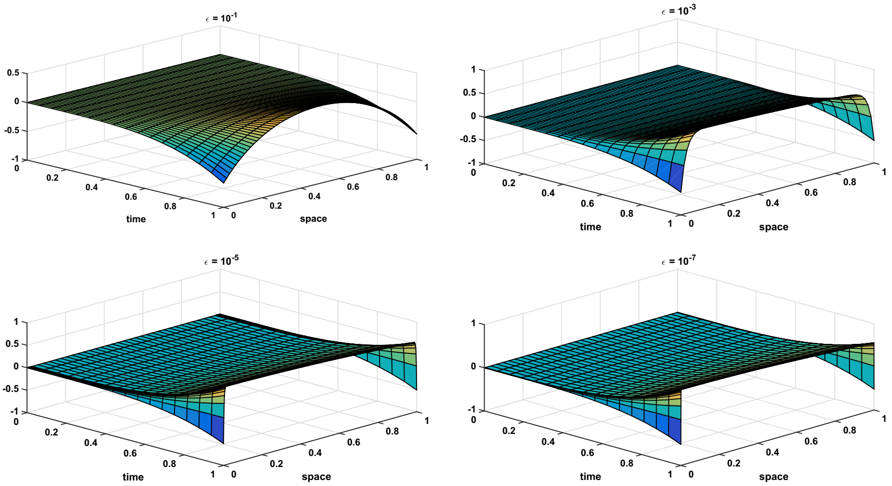

Fig. 2 Solution plots for Example 2 taking $N = 64,\ M = 16$, and various values of ε

Table 2 Iterations for Example 1

$\varepsilon = 10^{-p}$	$N = 64$ $\Delta t = 0.25/4$	$N = 128$ $\Delta t = 0.25/4^2$	$N = 256$ $\Delta t = 0.25/4^3$	$N = 512$ $\Delta t = 0.25/4^4$	$N = 1024$ $\Delta t = 0.25/4^5$
$p = 1$	4	4	4	5	5
2	1	2	2	2	2
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1
6	1	1	1	1	1
7	1	1	1	1	1
8	1	1	1	1	1

Example 1. Clearly, we can confirm the parameter uniform convergence from this table. In addition, Table 2 gives the iterations needed for convergence; from this, we observe that one iteration is enough to get the desired results for very small values of ε.

Example 2 *Consider*

$$\begin{cases} \frac{\partial u(x,t)}{\partial t} - \varepsilon \frac{\partial^2 u(x,t)}{\partial x^2} + \frac{1+x^2}{2} u(x,t) = t^3 & (x,t) \in (0,1) \times (0,1], \\ u(0,t) - \sqrt{\varepsilon}\frac{\partial u}{\partial x}(0,t) = -(128/35)\pi^{-1/2} t^{7/2} & t \in (0,1], \\ u(1,t) + \sqrt{\varepsilon}\frac{\partial u}{\partial x}(1,t) = -(128/35)\pi^{-1/2} t^{7/2} & t \in (0,1], \\ u(x,0) = 0 & x \in [0,1]. \end{cases}$$

The solution plots for various values of ε are given in Fig. 2. Note that the boundary layers are close to $x = 0, 1$. Since the solution is unknown, we compute another

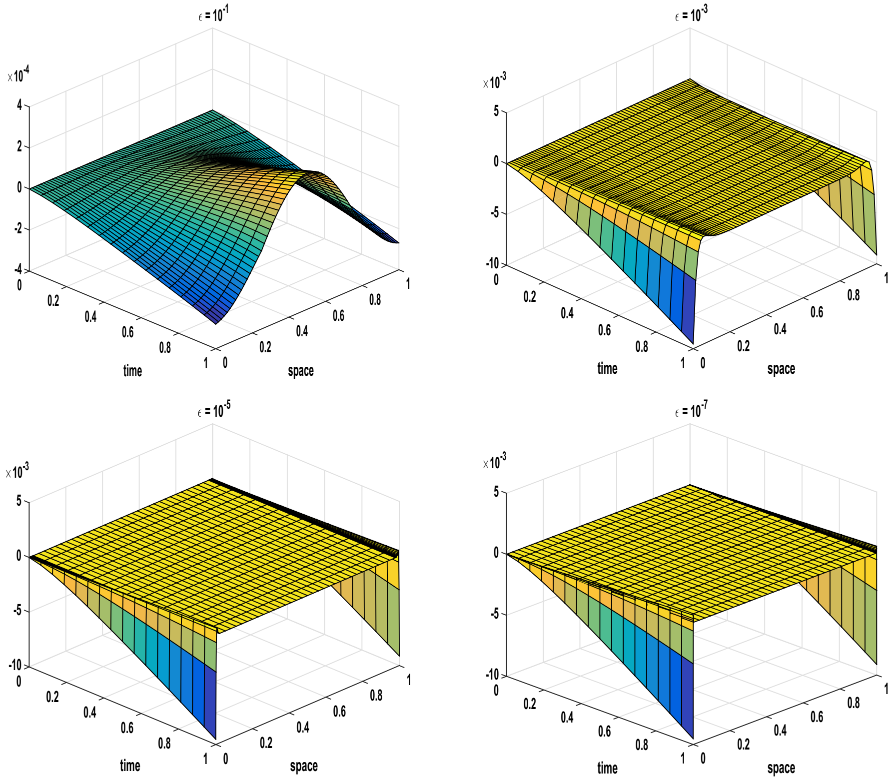

Fig. 3 Error plots for Example 1 taking $N = 64,\ M = 16$, and various values of ε

approximate solution to calculate the uniform errors and uniform convergence orders as follows

$$E_\varepsilon^{N,\Delta t} = \max_{(x_i,t_j)\in\overline{\Omega}^{N,M}} |U^{2N,\Delta t/4}(x_i,t_j) - U^{N,\Delta t}(x_i,t_j)|, \quad E^{N,\Delta t} = \max_{\varepsilon} E_\varepsilon^{N,\Delta t},$$

$$R^{N,\Delta t} = \log_2\left(\frac{E^{N,\Delta t}}{E^{2N,\Delta t/4}}\right),$$

where $U^{2N,\Delta t/4}$ is obtained taking in each subdomain $2N + 1$ points in space and $\Delta t/4$ mesh size in time, but using the same subdomain parameter ρ as is considered for $U^{N,\Delta t}$. The error plots for various values of ε are given in Figs. 3 and 4 for Examples 1 and 2 respectively.

The computed errors $E_\varepsilon^{N,\Delta t}$, $E^{N,\Delta t}$, and uniform convergence rates $R^{N,\Delta t}$ for Example 2 are tabulated in Table 3, showing the uniform convergence of the method. Table 4 gives iterations that are performed to achieve the convergence for Example 2. One can see that only one iteration is needed to get the solution up to the desired accuracy for very small values ε.

Table 3 Errors and convergence orders for Example 2

$\varepsilon = 10^{-p}$	$N = 64$ $\Delta t = 0.25/4$	$N = 128$ $\Delta t = 0.25/4^2$	$N = 256$ $\Delta t = 0.25/4^3$	$N = 512$ $\Delta t = 0.25/4^4$	$N = 1024$ $\Delta t = 0.25/4^5$
$p = 1$	7.647E-03	1.970E-03	4.965E-04	1.243E-04	3.11E-05
2	1.796E-02	4.449E-03	1.109E-03	2.772E-04	6.929E-05
3	1.858E-02	4.588E-03	1.143E-03	2.856E-04	7.138E-05
4	1.888E-02	7.929E-03	2.821E-03	9.318E-04	2.949E-04
5	1.888E-02	7.929E-03	2.821E-03	9.318E-04	2.949E-04
6	1.888E-02	7.929E-03	2.821E-03	9.318E-04	2.949E-04
7	1.888E-02	7.929E-03	2.821E-03	9.318E-04	2.949E-04
8	1.888E-02	7.929E-03	2.821E-03	9.318E-04	2.949E-04
$E^{N,\Delta t}$	1.888E-02	7.929E-03	2.821E-03	9.318E-03	2.949E-04
$R^{N,\Delta t}$	1.252	1.490	1.598	1.659	

Table 4 Iterations for Example 2

$\varepsilon = 10^{-p}$	$N = 64$ $\Delta t = 0.25/4$	$N = 128$ $\Delta t = 0.25/4^2$	$N = 256$ $\Delta t = 0.25/4^3$	$N = 512$ $\Delta t = 0.25/4^4$	$N = 1024$ $\Delta t = 0.25/4^5$
$p = 1$	3	4	4	4	4
2	2	2	2	2	2
3	1	1	1	1	1
4	1	1	1	1	1
5	1	1	1	1	1
6	1	1	1	1	1
7	1	1	1	1	1
8	1	1	1	1	1

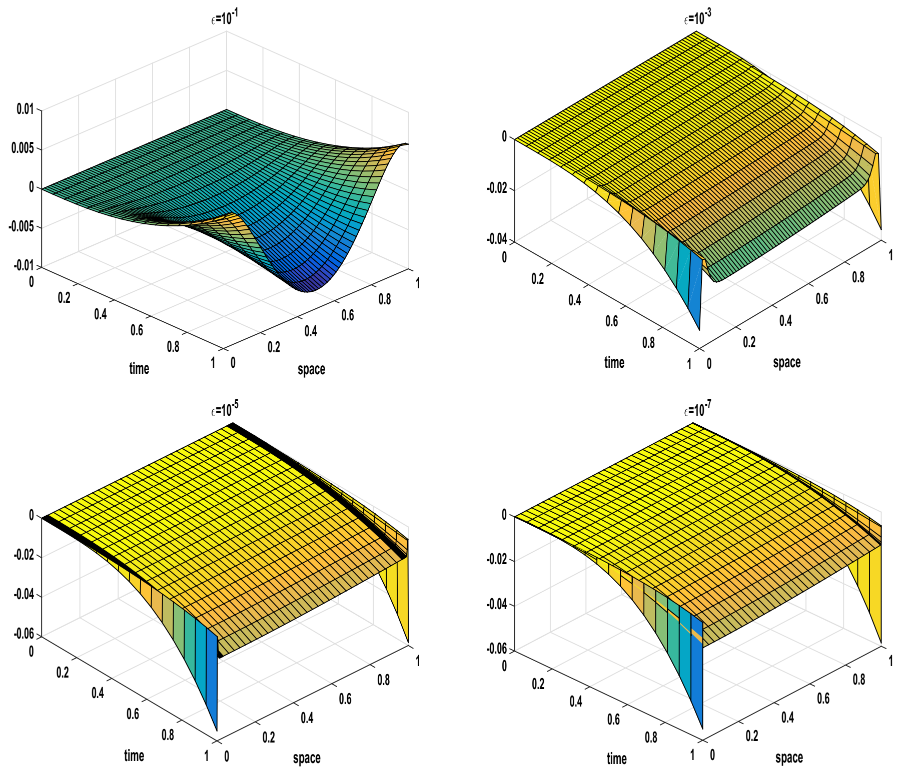

Fig. 4 Error plots for Example 2 taking $N = 64$, $M = 16$, and various values of ε

6 Conclusions

An SWR technique to solve singularly perturbed parabolic reaction-diffusion equations with Robin boundary conditions is developed and analyzed in this paper. The original domain is initially divided into three overlapping subdomains. The problem is discretized using the backward difference and central difference schemes for the time and space derivatives, respectively. It is shown that the proposed scheme is unconditionally stable. Error analysis is also discussed in this work, and it is demonstrated that the proposed approach is uniformly convergent with order almost two in space and one in time. Furthermore, it is shown that for small values of ε, one iteration is sufficient to achieve the specified accuracy. The idea discussed in this paper can also be extended to singularly perturbed semilinear differential equations having boundary conditions of Robin type. Further, it is our intention in the future to extend the SWR technique to higher dimensional singularly perturbed problems.

Acknowledgements Sunil Kumar is thankful to the Science and Engineering Research Board (SERB) for providing support with Project No. MTR/2017/001036. Aakansha is thankful to the Council of Scientific & Industrial Research(CSIR), New Delhi, India, for the research fellowship with Sanction No.: 09/1217(0035)/2018-EMR-I. The authors would like to express great appreciation to anonymous reviewers for their valuable comments and suggestions, which have helped to improve the quality and presentation of this paper.

Funding Open Access funding provided thanks to the CRUE-CSIC agreement with Springer Nature.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. Amiraliyev, G.M., Duru, H.: A note on a parameterized singular perturbation problem. J. Comput. Appl. Math. **182**(1), 233–242 (2005)
2. Amiraliyev, G.M., Kudu, M., Duru, H.: Uniform difference method for a parameterized singular perturbation problem. Appl. Math. Comput. **175**(1), 89–100 (2006)
3. Buranay, S.C., Arshad, N., Matan, A.H.: Hexagonal grid computation of the derivatives of the solution to the heat equation by using fourth-order accurate two-stage implicit methods. Fractal Fract. **5**(4), 203 (2021)
4. Buranay, S.C., Matan, A.H., Arshad, N.: Two stage implicit method on hexagonal grids for approximating the first derivatives of the solution to the heat equation. Fractal Fract. **5**(1), 19 (2021)
5. Buranay, S.C., Arshad, N.: Hexagonal grid approximation of the solution of heat equation on special polygons. Adv. Differ. Equ. **2020**, 309 (2020)
6. Buranay, S.C., Farinola, L.A.: Implicit methods for the first derivative of the solution to heat equation. Adv. Differ. Equ. **2018**, 430 (2018)
7. Clavero, C., Gracia, J.L.: On the uniform convergence of a finite difference scheme for time dependent singularly perturbed reaction-diffusion problems. Appl. Math. Comput. **216**, 1478–1488 (2010)
8. Dehghan, M.: Numerical solution of the three-dimensional advection-diffusion equation. Appl. Math. Comput. **150**, 5–19 (2004)

9. Farrell, P., Hegarty, A., Miller, J.M., O'Riordan, E., Shishkin, G.I.: Robust computational techniques for boundary layers. Chapman and Hall/CRC, London (2000)
10. Glizer, V.Y.: Asymptotic solution of a singularly perturbed set of functional-differential equations of riccati type encountered in the optimal control theory. NoDEA Nonlinear Differ. Equ. Appl. **5**, 491–515 (1998)
11. Gupta, V., Sahoo, S.K., Dubey, R.K.: Robust higher order finite difference scheme for singularly perturbed turning point problem with two outflow boundary layers. Comput. Appl. Math. **40**, 179 (2021)
12. Hemker, P., Shishkin, G., Shishkina, L.: The numerical solution of a Neumann problem for parabolic singularly perturbed equations with high-order time accuracy. In: Recent Advances in Numerical Methods and Applications II, pp. 27–39. World Scientific (1999)
13. Hemker, P.W., Shishkin, G.I., Shishkina, L.P.: High-order time-accurate schemes for singularly perturbed parabolic convection-diffusion problems with Robin boundary conditions. Comput. Methods Appl. Math. **2**(1), 3–25 (2002)
14. Iragi, B.C., Munyakazi, J.B.: A uniformly convergent numerical method for a singularly perturbed Volterra integro-differential equation. Int. J. Comput. Math. **97**(4), 759–771 (2020)
15. Kadalbajoo, M.K., Gupta, V.: A brief survey on numerical methods for solving singularly perturbed problems. Appl. Math. Comput. **217**(8), 3641–3716 (2010)
16. Kadalbajoo, M.K., Gupta, V., Awasthi, A.: A uniformly convergent B-spline collocation method on a nonuniform mesh for singularly perturbed one-dimensional time-dependent linear convection-diffusion problem. J. Comput. Appl. Math. **220**(1), 271–289 (2008)
17. Kamranian, M., Dehghan, M., Tatari, M.: An image denoising approach based on a meshfree method and the domain decomposition technique. Eng. Anal. Bound. Elem. **39**, 101–110 (2014)
18. Kumar, M., Rao, S.C.S.: High order parameter-robust numerical method for time dependent singularly perturbed reaction-diffusion problems. Computing **90**, 15–38 (2010)
19. Kumar, S., Kumar, M.: An analysis of overlapping domain decomposition methods for singularly perturbed reaction-diffusion problems. J. Comput. Appl. Math. **281**, 250–262 (2015)
20. Kumar, S., Rao, S.C.S.: A robust overlapping Schwarz domain decomposition algorithm for time-dependent singularly perturbed reaction-diffusion problems. J. Comput. Appl. Math. **261**, 127–138 (2014)
21. Kumar, S., Singh, J., Kumar, M.: A robust domain decomposition method for singularly perturbed parabolic reaction-diffusion systems. J. Math. Chem. **57**, 1557–1578 (2019)
22. Kumar, S., Sumit, Ramos, Ramos, H.: Parameter-uniform approximation on equidistributed meshes for singularly perturbed parabolic reaction-diffusion problems with Robin boundary conditions. Appl. Math. Comput. **392**, 125–677 (2021)
23. Ladyzhenskaya, O.A., Solonnikov, V.A., Ural'tseva, N.N.: Linear and quasilinear equations of parabolic type. Translations of Mathematical Monographs, Vol. 23. American Mathematical Society, Providence, R.I. (1968)
24. Linß, T., Madden, N.: Parameter uniform approximations for time-dependent reaction-diffusion problems. Numer. Methods Partial Differ. Equ. **23**(6), 1290–1300 (2007)
25. Lomov, S.A.: Introduction to the general theory of singular perturbations, vol. 112. American Mathematical Soc. (1992)
26. Miller, J.J., O'Riordan, E., Shishkin, G.I.: Fitted numerical methods for singular perturbation problems: error estimates in the maximum norm for linear problems in one and two dimensions. World Scientific (2012)
27. Miller, J.J.H., O'Riordan, E., Shishkin, G.I., Shishkina, L.P.: Fitted mesh methods for problems with parabolic boundary layers. Math. Proc. Royal Irish Acad. **98A**, 173–190 (1998)
28. Munyakazi, J.B.: A uniformly convergent nonstandard finite difference scheme for a system of convection-diffusion equations. Comput. Appl. Math. **34**, 1153–1165 (2015)
29. Munyakazi, J.B., Patidar, K.C.: A new fitted operator finite difference method to solve systems of evolutionary reaction-diffusion equations. Quaest. Math. **38**, 121–138 (2015)
30. Munyakazi, J.B., Patidar, K.C., Sayi, M.T.: A fitted numerical method for parabolic turning point singularly perturbed problems with an interior layer. Numer. Methods Partial Differ. Equ. **35**, 2407–2422 (2019)
31. O'Malley, R.E.: Singular perturbation methods for ordinary differential equations, vol. 89. Springer, Berlin (1991)

32. Quarteroni, A., Valli, A.: Domain decomposition methods for partial differential equations. Oxford University Press, Oxford (1999)
33. Rao, S.C.S., Kumar, S.: An almost fourth order uniformly convergent domain decomposition method for a coupled system of singularly perturbed reaction-diffusion equations. J. Comput. Appl. Math. **235**, 3342–3354 (2011)
34. Rao, S.C.S., Kumar, S., Singh, J.: A discrete Schwarz waveform relaxation method of higher order for singularly perturbed parabolic reaction-diffusion problems. J. Math. Chem. **58**(3), 574–594 (2019)
35. Roos, H.G., Stynes, M., Tobiska, L.: Robust numerical methods for singularly perturbed differential equations: convection-diffusion-reaction and flow problems, vol. 24. Springer Science & Business Media (2008)
36. Schwarz, H.A.: Gesammelte mathematische abhandlungen. Vierteljahrsschrift der naturforschenden gesselschaft in Zurich **15**, 272–286 (1870)
37. Selvi, P.A., Ramanujam, N.: A parameter uniform difference scheme for singularly perturbed parabolic delay differential equation with Robin type boundary condition. Appl. Math. Comput. **296**, 101–115 (2017)
38. Shishkin, G.I.: Approximation of the solutions of singularly perturbed boundary-value problems with a parabolic boundary layer. USSR Comput. Math. Math. Phys. **29**(4), 1–10 (1989)
39. Singh, J., Kumar, S.: A domain decomposition method of Schwarz waveform relaxation type for singularly perturbed nonlinear parabolic problems. Int. J. Comput. Math. (2022). https://doi.org/10.1080/00207160.2022.2106786
40. Singh, J., Kumar, S., Kumar, M.: A domain decomposition method for solving singularly perturbed parabolic reaction-diffusion problems with time delay. Numer. Methods Partial Differ. Equ. **34**(5), 1849–1866 (2018)
41. Vasil'eva, A.B., Kalachev, L.: Alternating boundary layer type solutions of some singularly perturbed periodic parabolic equations with Dirichlet and Robin boundary conditions. Comput. Math. Math. Phys. **47**(2), 215–226 (2007)