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Richards's curve induced Banach space valued multivariate neural network approximation

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Abstract Here, we present multivariate quantitative approximations of Banach space valued continuous multivariate functions on a box or $\mathbb{R}^N$, $N \in \mathbb{N}$, by the multivariate normalized, quasi-interpolation, Kantorovich-type and quadrature-type neural network operators. We examine also the case of approximation by iterated operators of the last four types. These approximations are achieved by establishing multidimensional Jackson type inequalities involving the multivariate modulus of continuity of the engaged function or its high-order Fréchet derivatives. Our multivariate operators are defined using a multidimensional density function induced by the Richards's curve, which is a generalized logistic function. The approximations are pointwise, uniform and L_p. The related feed-forward neural network is with one hidden layer.

Mathematics Subject Classification 41A17 · 41A25 · 41A30 · 41A36

1 Introduction

Anastassiou in [2,3], see chapters 2–5, was the first to establish neural network approximations to continuous functions with rates by very specifically defined neural network operators of Cardaliaguet–Euvrard and "Squashing" types, by employing the modulus of continuity of the engaged function or its high-order derivative, and producing very tight Jackson type inequalities. He treats there both the univariate and multivariate cases. The defining these operators "bell-shaped" and "squashing" functions are assumed to be of compact support. Also, in [3], he gives the Nth-order asymptotic expansion for the error of weak approximation of these two operators to a special natural class of smooth functions; see chapters 4–5 there.

Motivations for this work are the article [22] of Chen and Cao, and [4–20,23,24].

Here, we perform multivariate sigmoid function by Richards's curve [30] based neural network approximations to continuous functions over boxes or over the whole $\mathbb{R}^N$, $N \in \mathbb{N}$, and also iterated and L_p approximations. All convergences here are with rates expressed via the multivariate modulus of continuity of the involved function or its high-order Fréchet derivative and given by very tight multidimensional Jackson type inequalities.

We come up with the "right" precisely defined multivariate normalized, quasi-interpolation neural network operators related to boxes or $\mathbb{R}^N$, as well as Kantorovich-type and quadrature-type-related operators on $\mathbb{R}^N$. Our boxes are not necessarily symmetric to the origin. In preparation to prove our results, we establish important

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properties of the basic multivariate density function induced by the sigmoid function related to Richards's curve and defining our operators. Richards's curve among others has been used for modeling COVID-19 infection trajectory [26].

Feed-forward neural networks (FNNs) with one hidden layer, the only type of networks we deal with in this article, are mathematically expressed as

$$N_n(x) = \sum_{j=0}^{n} c_j \sigma\left(\left\langle a_j \cdot x\right\rangle + b_j\right), \quad x \in \mathbb{R}^s, \ s \in \mathbb{N},$$

where for $0 \leq j \leq n$, $b_j \in \mathbb{R}$ are the thresholds, $a_j \in \mathbb{R}^s$ are the connection weights, $c_j \in \mathbb{R}$ are the coefficients, $\left\langle a_j \cdot x\right\rangle$ is the inner product of a_j and x, and σ is the activation function of the network. In many fundamental network models, the activation function is based on the Richards's curve sigmoid function. About neural networks, see [25,27,28].

2 Background

A Richards's curve is [30]

$$\varphi(x) = \frac{1}{1+\mathrm{e}^{-\mu x}}; \quad x \in \mathbb{R}, \mu > 0, \tag{1}$$

which is strictly increasing on $\mathbb{R}$, and it is a sigmoid function. For small $0 < \mu < 1$, our Richards's curve, which is a smooth function, is expected to behave better than the ReLu activation function. We have that $\varphi(+\infty) = 1$ and $\varphi(-\infty) = 0$.

We consider the following activation function:

$$G(x) = \frac{1}{2}\left(\varphi(x+1) - \varphi(x-1)\right), \quad x \in \mathbb{R}, \tag{2}$$

which is $G(x) > 0$, all $x \in \mathbb{R}$.

We have that

$$\varphi(0) = \frac{1}{2} \quad \text{and} \quad \varphi(x) = 1 - \varphi(-x). \tag{3}$$

Clearly, $G(-x) = G(x)$, and

$$G(0) = \frac{\mathrm{e}^{\mu}-1}{2\left(\mathrm{e}^{\mu}+1\right)}. \tag{4}$$

In [20], we prove that
$G'(x) < 0$ for $x > 0$, so that
$G(x)$ is strictly decreasing on $(0, +\infty)$.
Clearly, then $G(x)$ is strictly increasing on $(-\infty, 0)$, along with $G'(0) = 0$.
Also, it holds $G(\infty) = G(-\infty) = 0$.
Conclusion, G is a bell symmetric function with maximum as in (4)

$$G(0) = \frac{\mathrm{e}^{\mu}-1}{2\left(\mathrm{e}^{\mu}+1\right)}, \quad \mu > 0.$$

We mention

Theorem 2.1 [20] *We have*

$$\sum_{i=-\infty}^{\infty} G(x-i) = 1, \quad \forall\, x \in \mathbb{R}. \tag{5}$$

Theorem 2.2 [20] *It holds*

$$\int_{-\infty}^{\infty} G(x)\,\mathrm{d}x = 1, \tag{6}$$

so that G is a density function.

Theorem 2.3 [20] *Let* $0 < \alpha < 1$, $\mu > 0$ *and* $n \in \mathbb{N}$ *with* $n^{1-\alpha} > 2$. *It holds*

$$\sum_{\substack{k=-\infty \\ : |nx-k| \geq n^{1-\alpha}}}^{\infty} G(nx-k) < \frac{1}{\mathrm{e}^{\mu(n^{1-\alpha}-2)}}, \quad \mu > 0. \tag{7}$$

Denote by $\lfloor \cdot \rfloor$ the integral part of the number and by $\lceil \cdot \rceil$ the ceiling of the number.

Theorem 2.4 [20] *Let* $[a, b] \subset \mathbb{R}$ *and* $n \in \mathbb{N}$, *so that* $\lceil na \rceil \leq \lfloor nb \rfloor$. *It holds*

$$\frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G(nx-k)} < \frac{4\left(1+\mathrm{e}^{-2\mu}\right)}{1-\mathrm{e}^{-2\mu}}, \quad \mu > 0 \tag{8}$$

$\forall\, x \in [a, b]$.

We make

Remark 2.5 [20]

(i) We have that

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G(nx-k) \neq 1, \tag{9}$$

for at least some $x \in [a, b]$.

(ii) Let $[a, b] \subset \mathbb{R}$. For large n, we always have $\lceil na \rceil \leq \lfloor nb \rfloor$. Also, $a \leq \frac{k}{n} \leq b$, iff $\lceil na \rceil \leq k \leq \lfloor nb \rfloor$.

In general, it holds

$$\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} G(nx-k) \leq 1. \tag{10}$$

We introduce

$$Z(x_1, \ldots, x_N) := Z(x) := \prod_{i=1}^{N} G(x_i), \quad x = (x_1, \ldots, x_N) \in \mathbb{R}^N, \; N \in \mathbb{N}. \tag{11}$$

It has the properties

(i) $Z(x) > 0$, $\forall\, x \in \mathbb{R}^N$,

(ii)

$$\sum_{k=-\infty}^{\infty} Z(x-k) := \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} Z(x_1-k_1, \ldots, x_N-k_N) = 1, \tag{12}$$

where $k := (k_1, \ldots, k_n) \in \mathbb{Z}^N$, $\forall\, x \in \mathbb{R}^N$;
hence

(iii)

$$\sum_{k=-\infty}^{\infty} Z(nx-k) = 1 \tag{13}$$

$\forall\, x \in \mathbb{R}^N$; $n \in \mathbb{N}$,
and
(iv)

$$\int_{\mathbb{R}^N} Z(x)\,\mathrm{d}x = 1, \tag{14}$$

that is, Z is a multivariate density function.

Here, denote $\|x\|_\infty := \max\{|x_1|,\ldots,|x_N|\}$, $x \in \mathbb{R}^N$, also set $\infty := (\infty,\ldots,\infty)$, $-\infty := (-\infty,\ldots,-\infty)$ upon the multivariate context, and

$$\begin{aligned} \lceil na \rceil &:= (\lceil na_1 \rceil,\ldots,\lceil na_N \rceil), \\ \lfloor nb \rfloor &:= (\lfloor nb_1 \rfloor,\ldots,\lfloor nb_N \rfloor), \end{aligned} \tag{15}$$

where $a := (a_1,\ldots,a_N)$, $b := (b_1,\ldots,b_N)$.

We obviously see that

$$\begin{aligned} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) &= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left(\prod_{i=1}^{N} G(nx_i - k_i) \right) \\ &= \sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} \left(\prod_{i=1}^{N} G(nx_i - k_i) \right) = \prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} G(nx_i - k_i) \right). \end{aligned} \tag{16}$$

For $0 < \beta < 1$ and $n \in \mathbb{N}$, a fixed $x \in \mathbb{R}^N$, we have that

$$\begin{aligned} &\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \\ &= \sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty \le \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z(nx-k) + \sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z(nx-k). \end{aligned} \tag{17}$$

In the last two sums, the counting is over disjoint vector sets of k's, because the condition $\left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta}$ implies that there exists at least one $\left| \frac{k_r}{n} - x_r \right| > \frac{1}{n^\beta}$, where $r \in \{1,\ldots,N\}$.

(v) As in, Theorem 2.3, we derive that

$$\sum_{\begin{cases} k = \lceil na \rceil \\ \left\| \frac{k}{n} - x \right\|_\infty > \frac{1}{n^\beta} \end{cases}}^{\lfloor nb \rfloor} Z(nx-k) \overset{(7)}{<} \frac{1}{\mathrm{e}^{\mu\left(n^{1-\beta}-2\right)}},\ 0 < \beta < 1, \mu > 0, \tag{18}$$

with $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \prod_{i=1}^{N} [a_i, b_i]$.

(vi) By Theorem 2.4, we get that

$$0 < \frac{1}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} < \left(\frac{4\left(1+\mathrm{e}^{-2\mu}\right)}{1-\mathrm{e}^{-2\mu}} \right)^N, \tag{19}$$

$\mu > 0$, $\forall\, x \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$, $n \in \mathbb{N}$.

It is also clear that

(vii)

$$\sum_{\substack{k=-\infty \\ \left\{ \left\| \frac{k}{n} - x \right\|_{\infty} > \frac{1}{n^{\beta}} \right.}}^{\infty} Z(nx - k) < \frac{1}{e^{\mu\left(n^{1-\beta} - 2\right)}}, \tag{20}$$

$\mu > 0$, $0 < \beta < 1$, $n \in \mathbb{N} : n^{1-\beta} > 2$, $x \in \mathbb{R}^N$.

Furthermore, it holds

$$\lim_{n\to\infty} \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k) \neq 1, \tag{21}$$

for at least some $x \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$.

Here, $\left(X, \|\cdot\|_{\gamma}\right)$ is a Banach space.

Let $f \in C\left(\prod_{i=1}^{N}[a_i, b_i], X\right)$, $x = (x_1, \ldots, x_N) \in \prod_{i=1}^{N}[a_i, b_i]$, $n \in \mathbb{N}$, such that $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, \ldots, N$.

We introduce and define the following multivariate linear normalized neural network operator ($x := (x_1, \ldots, x_N) \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$):

$$\begin{aligned} L_n(f, x_1, \ldots, x_N) := L_n(f, x) &:= \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} \\ &= \frac{\sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \ldots, \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} G(nx_i - k_i)\right)}{\prod_{i=1}^{N} \left(\sum_{k_i=\lceil na_i \rceil}^{\lfloor nb_i \rfloor} G(nx_i - k_i)\right)}. \end{aligned} \tag{22}$$

For large enough $n \in \mathbb{N}$, we always obtain $\lceil na_i \rceil \leq \lfloor nb_i \rfloor$, $i = 1, \ldots, N$. Also, $a_i \leq \frac{k_i}{n} \leq b_i$, iff $\lceil na_i \rceil \leq k_i \leq \lfloor nb_i \rfloor$, $i = 1, \ldots, N$.

When $g \in C\left(\prod_{i=1}^{N}[a_i, b_i]\right)$, we define the companion operator

$$\widetilde{L}_n(g, x) := \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} g\left(\frac{k}{n}\right) Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)}. \tag{23}$$

Clearly, $\widetilde{L}_n$ is a positive linear operator. We have that

$$\widetilde{L}_n(1, x) = 1, \quad \forall\, x \in \left(\prod_{i=1}^{N}[a_i, b_i]\right).$$

Notice that $L_n(f) \in C\left(\prod_{i=1}^{N}[a_i, b_i], X\right)$ and $\widetilde{L}_n(g) \in C\left(\prod_{i=1}^{N}[a_i, b_i]\right)$.

Furthermore, it holds

$$\|L_n(f, x)\|_{\gamma} \leq \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| f\left(\frac{k}{n}\right)\right\|_{\gamma} Z(nx - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx - k)} = \widetilde{L}_n\left(\|f\|_{\gamma}, x\right) \tag{24}$$

$\forall\, x \in \prod_{i=1}^{N}[a_i, b_i]$.

Clearly, $\|f\|_{\gamma} \in C\left(\prod_{i=1}^{N}[a_i, b_i]\right)$.

Therefore, we have that

$$\|L_n(f,x)\|_\gamma \le \widetilde{L}_n\left(\|f\|_\gamma, x\right), \tag{25}$$

$\forall\, x \in \prod_{i=1}^N [a_i, b_i]$, $\forall\, n \in \mathbb{N}$, $\forall\, f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$.

Let $c \in X$ and $g \in C\left(\prod_{i=1}^N [a_i, b_i]\right)$, then $cg \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$.

Furthermore, it holds

$$L_n(cg, x) = c\widetilde{L}_n(g, x), \quad \forall\, x \in \prod_{i=1}^N [a_i, b_i]. \tag{26}$$

Since $\widetilde{L}_n(1) = 1$, we get that

$$L_n(c) = c, \quad \forall\, c \in X. \tag{27}$$

We call $\widetilde{L}_n$ the companion operator of L_n.

For convenience, we call

$$\begin{aligned} L_n^*(f,x) &:= \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f\left(\frac{k}{n}\right) Z(nx-k) \\ &= \sum_{k_1=\lceil na_1 \rceil}^{\lfloor nb_1 \rfloor} \sum_{k_2=\lceil na_2 \rceil}^{\lfloor nb_2 \rfloor} \cdots \sum_{k_N=\lceil na_N \rceil}^{\lfloor nb_N \rfloor} f\left(\frac{k_1}{n}, \ldots, \frac{k_N}{n}\right) \left(\prod_{i=1}^N G(nx_i - k_i)\right) \end{aligned} \tag{28}$$

$\forall\, x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$.

That is

$$L_n(f,x) := \frac{L_n^*(f,x)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)} \tag{29}$$

$\forall\, x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$, $n \in \mathbb{N}$.

Hence

$$L_n(f,x) - f(x) = \frac{L_n^*(f,x) - f(x)\left(\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)\right)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k)}. \tag{30}$$

Consequently, we derive

$$\|L_n(f,x) - f(x)\|_\gamma \overset{(19)}{\le} \left(\frac{4\left(1+e^{-2\mu}\right)}{1-e^{-2\mu}}\right)^N \left\| L_n^*(f,x) - f(x) \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx-k) \right\|_\gamma \tag{31}$$

$\forall\, x \in \left(\prod_{i=1}^N [a_i, b_i]\right)$.

We will estimate the right-hand side of (31).

For the last and others, we need the following.

Definition 2.6 [15, p. 274] Let M be a convex and compact subset of $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1, \infty]$, and $\left(X, \|\cdot\|_\gamma\right)$ be a Banach space. Let $f \in C(M, X)$. We define the first modulus of continuity of f as

$$\omega_1(f, \delta) := \sup_{\substack{x, y \in M: \\ \|x-y\|_p \le \delta}} \|f(x) - f(y)\|_\gamma, \ 0 < \delta \le \operatorname{diam}(M). \tag{32}$$

If $\delta > \operatorname{diam}(M)$, then

$$\omega_1(f, \delta) = \omega_1(f, \operatorname{diam}(M)). \tag{33}$$

Notice $\omega_1(f, \delta)$ is increasing in $\delta > 0$. For $f \in C_B(M, X)$ (continuous and bounded functions), $\omega_1(f, \delta)$ is defined similarly.

Lemma 2.7 [15, p. 274] *We have* $\omega_1(f, \delta) \to 0$ *as* $\delta \downarrow 0$, *iff* $f \in C(M, X)$, *where* M *is a convex compact subset of* $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $p \in [1, \infty]$.

Clearly, we have also: $f \in C_U\left(\mathbb{R}^N, X\right)$ (uniformly continuous functions), iff $\omega_1(f, \delta) \to 0$ as $\delta \downarrow 0$, where ω_1 is defined similarly to (32). The space $C_B\left(\mathbb{R}^N, X\right)$ denotes the continuous and bounded functions on $\mathbb{R}^N$.

When $f \in C_B\left(\mathbb{R}^N, X\right)$, we define

$$B_n(f, x) := B_n(f, x_1, \ldots, x_N) := \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx - k)$$
$$:= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} f\left(\frac{k_1}{n}, \frac{k_2}{n}, \ldots, \frac{k_N}{n}\right) \left(\prod_{i=1}^{N} G(nx_i - k_i)\right), \tag{34}$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$, $N \in \mathbb{N}$, the multivariate quasi-interpolation neural network operator.

Also, for $f \in C_B\left(\mathbb{R}^N, X\right)$, we define the multivariate Kantorovich-type neural network operator

$$C_n(f, x) := C_n(f, x_1, \ldots, x_N) := \sum_{k=-\infty}^{\infty} \left(n^N \int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\, dt\right) Z(nx - k)$$
$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} \left(n^N \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}} \int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}} \cdots \int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1, \ldots, t_N)\, dt_1 \ldots dt_N\right)$$
$$\cdot \left(\prod_{i=1}^{N} G(nx_i - k_i)\right), \tag{35}$$

$n \in \mathbb{N}$, $\forall\, x \in \mathbb{R}^N$.

Again, for $f \in C_B\left(\mathbb{R}^N, X\right)$, $N \in \mathbb{N}$, we define the multivariate neural network operator of quadrature-type $D_n(f, x)$, $n \in \mathbb{N}$, as follows.

Let $\theta = (\theta_1, \ldots, \theta_N) \in \mathbb{N}^N$, $r = (r_1, \ldots, r_N) \in \mathbb{Z}_+^N$, $w_r = w_{r_1, r_2, \ldots r_N} \geq 0$, such that $\sum_{r=0}^{\theta} w_r = \sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} \cdots \sum_{r_N=0}^{\theta_N} w_{r_1, r_2, \ldots r_N} = 1$; $k \in \mathbb{Z}^N$ and

$$\delta_{nk}(f) := \delta_{n, k_1, k_2, \ldots, k_N}(f) := \sum_{r=0}^{\theta} w_r f\left(\frac{k}{n} + \frac{r}{n\theta}\right)$$
$$= \sum_{r_1=0}^{\theta_1} \sum_{r_2=0}^{\theta_2} \cdots \sum_{r_N=0}^{\theta_N} w_{r_1, r_2, \ldots r_N} f\left(\frac{k_1}{n} + \frac{r_1}{n\theta_1}, \frac{k_2}{n} + \frac{r_2}{n\theta_2}, \ldots, \frac{k_N}{n} + \frac{r_N}{n\theta_N}\right), \tag{36}$$

where $\frac{r}{\theta} := \left(\frac{r_1}{\theta_1}, \frac{r_2}{\theta_2}, \ldots, \frac{r_N}{\theta_N}\right)$.

We set

$$D_n(f, x) := D_n(f, x_1, \ldots, x_N) := \sum_{k=-\infty}^{\infty} \delta_{nk}(f) Z(nx - k)$$
$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_N=-\infty}^{\infty} \delta_{n, k_1, k_2, \ldots, k_N}(f) \left(\prod_{i=1}^{N} G(nx_i - k_i)\right) \tag{37}$$

$\forall\, x \in \mathbb{R}^N$.

In this article, we study the approximation properties of L_n, B_n, C_n, D_n neural network operators and as well of their iterates. That is, the quantitative pointwise and uniform convergence of these operators to the unit operator I.

3 Multivariate Richards's curve neural network approximations

Here, we present several vectorial neural network approximations to Banach space valued functions given with rates.

We give

Theorem 3.1 *Let* $f \in C\left(\prod_{i=1}^{N}[a_i, b_i], X\right)$, $0<\beta<1$, $\mu>0$, $x \in \left(\prod_{i=1}^{N}[a_i, b_i]\right)$, $N, n \in \mathbb{N}$ *with* $n^{1-\beta}>2$. *Then*

(1)

$$\|L_n(f,x)-f(x)\|_\gamma \leq \left(\frac{4\left(1+e^{-2\mu}\right)}{1-e^{-2\mu}}\right)^N \left[\omega_1\left(f,\frac{1}{n^\beta}\right)+\frac{2\left\|\|f\|_\gamma\right\|_\infty}{e^{\mu\left(n^{1-\beta}-2\right)}}\right] =: \lambda_1(n), \tag{38}$$

and

(2)

$$\left\|\|L_n(f)-f\|_\gamma\right\|_\infty \leq \lambda_1(n). \tag{39}$$

We notice that $\lim_{n\to\infty} L_n(f) \overset{\|\cdot\|_\gamma}{=} f$, *pointwise and uniformly.*

Above ω_1 *is with respect to* $p=\infty$ *and the speed of convergence is* $\max\left(\frac{1}{n^\beta}, \frac{2}{e^{\mu\left(n^{1-\beta}-2\right)}}\right)=\frac{1}{n^\beta}$.

Proof We observe that

$$\begin{aligned}\Delta(x) &:= L_n^*(f,x)-f(x)\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} Z(nx-k) \\ &= \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f\left(\frac{k}{n}\right)Z(nx-k)-\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor} f(x)Z(nx-k) \\ &= \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left(f\left(\frac{k}{n}\right)-f(x)\right)Z(nx-k).\end{aligned} \tag{40}$$

Thus

$$\begin{aligned}\|\Delta(x)\|_\gamma &\leq \sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|_\gamma Z(nx-k) \\ &= \sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty \leq \frac{1}{n^\beta}}}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|_\gamma Z(nx-k) \\ &+ \sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb\rfloor}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|_\gamma Z(nx-k) \\ &\overset{(13)}{\leq} \omega_1\left(f,\frac{1}{n^\beta}\right)+2\left\|\|f\|_\gamma\right\|_\infty \sum_{\substack{k=\lceil na\rceil \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}}}^{\lfloor nb\rfloor} Z(nx-k) \\ &\overset{(18)}{\leq} \omega_1\left(f,\frac{1}{n^\beta}\right)+\frac{2\left\|\|f\|_\gamma\right\|_\infty}{e^{\mu\left(n^{1-\beta}-2\right)}}, \quad 0<\beta<1, \mu>0.\end{aligned} \tag{41}$$

So that

$$\|\Delta(x)\|_{\gamma} \leq \omega_1\left(f, \frac{1}{n^{\beta}}\right) + \frac{2\left\|\|f\|_{\gamma}\right\|_{\infty}}{e^{\mu\left(n^{1-\beta}-2\right)}}. \tag{42}$$

Now, using (31), we finish the proof. □

We make

Remark 3.2 [15, pp. 263–266] Let $\left(\mathbb{R}^N, \|\cdot\|_p\right)$, $N \in \mathbb{N}$; where $\|\cdot\|_p$ is the L_p-norm, $1 \leq p \leq \infty$. $\mathbb{R}^N$ is a Banach space, and $\left(\mathbb{R}^N\right)^j$ denotes the j-fold product space $\mathbb{R}^N \times \cdots \times \mathbb{R}^N$ endowed with the max-norm $\|x\|_{\left(\mathbb{R}^N\right)^j} := \max_{1\leq\lambda\leq j} \|x_\lambda\|_p$, where $x := \left(x_1, \ldots, x_j\right) \in \left(\mathbb{R}^N\right)^j$.

Let $\left(X, \|\cdot\|_\gamma\right)$ be a general Banach space. Then, the space $V_j := V_j\left(\left(\mathbb{R}^N\right)^j; X\right)$ of all j-multilinear continuous maps $g : \left(\mathbb{R}^N\right)^j \to X$, $j = 1, \ldots, m$, is a Banach space with norm

$$\|g\| := \|g\|_{V_j} := \sup_{\left(\|x\|_{\left(\mathbb{R}^N\right)^j}=1\right)} \|g(x)\|_\gamma = \sup \frac{\|g(x)\|_\gamma}{\|x_1\|_p \cdots \left\|x_j\right\|_p}. \tag{43}$$

Let M be a non-empty convex and compact subset of $\mathbb{R}^N$ and $x_0 \in M$ is fixed.

Let O be an open subset of $\mathbb{R}^N$: $M \subset O$. Let $f : O \to X$ be a continuous function, whose Fréchet derivatives (see [29]) $f^{(j)} : O \to V_j = V_j\left(\left(\mathbb{R}^N\right)^j; X\right)$ exist and are continuous for $1 \leq j \leq m$, $m \in \mathbb{N}$.

Call $(x - x_0)^j := (x - x_0, \ldots, x - x_0) \in \left(\mathbb{R}^N\right)^j$, $x \in M$.

We will work with $f|_M$.

Then, by Taylor's formula [21], [29, p. 124], we get

$$f(x) = \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!} + R_m(x, x_0), \quad \text{all } x \in M, \tag{44}$$

where the remainder is the Riemann integral

$$R_m(x, x_0) := \int_0^1 \frac{(1-u)^{m-1}}{(m-1)!}\left(f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0)\right)(x - x_0)^m \mathrm{d}u; \tag{45}$$

here, we set $f^{(0)}(x_0)(x - x_0)^0 = f(x_0)$.

We consider

$$w := \omega_1\left(f^{(m)}, h\right) := \sup_{\substack{x,y\in M:\\ \|x-y\|_p\leq h}} \left\|f^{(m)}(x) - f^{(m)}(y)\right\|, \tag{46}$$

$h > 0$.

We obtain

$$\begin{aligned} &\left\|\left(f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0)\right)(x - x_0)^m\right\|_\gamma \\ &\leq \left\|f^{(m)}(x_0 + u(x - x_0)) - f^{(m)}(x_0)\right\| \cdot \|x - x_0\|_p^m \\ &\leq w\|x - x_0\|_p^m \left\lceil \frac{u\|x - x_0\|_p}{h} \right\rceil, \end{aligned} \tag{47}$$

by Lemma 7.1.1, [1, p. 208], where $\lceil\cdot\rceil$ is the ceiling.

Therefore, for all $x \in M$ (see [1, pp. 121–122])

$$\begin{aligned} \|R_m(x, x_0)\|_\gamma &\leq w\|x - x_0\|_p^m \int_0^1 \left\lceil \frac{u\|x - x_0\|_p}{h} \right\rceil \frac{(1-u)^{m-1}}{(m-1)!}\mathrm{d}u \\ &= w\Phi_m\left(\|x - x_0\|_p\right) \end{aligned} \tag{48}$$

by a change of variable, where

$$\Phi_m(t) := \int_0^{|t|} \left\lceil \frac{s}{h} \right\rceil \frac{(|t|-s)^{m-1}}{(m-1)!} ds = \frac{1}{m!} \left(\sum_{j=0}^{\infty} (|t| - jh)_+^m \right), \quad \forall\, t \in \mathbb{R} \tag{49}$$

is a (polynomial) spline function; see [1, p. 210–211].

Also, from there, we get

$$\Phi_m(t) \leq \left(\frac{|t|^{m+1}}{(m+1)!h} + \frac{|t|^m}{2m!} + \frac{h\,|t|^{m-1}}{8(m-1)!} \right), \quad \forall\, t \in \mathbb{R}, \tag{50}$$

with equality true only at $t = 0$.

Therefore, it holds

$$\|R_m(x, x_0)\|_\gamma \leq w \left(\frac{\|x - x_0\|_p^{m+1}}{(m+1)!h} + \frac{\|x - x_0\|_p^m}{2m!} + \frac{h\,\|x - x_0\|_p^{m-1}}{8(m-1)!} \right), \quad \forall\, x \in M. \tag{51}$$

We have found that

$$\left\| f(x) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(x - x_0)^j}{j!} \right\|_\gamma \leq \omega_1\left(f^{(m)}, h\right) \left(\frac{\|x - x_0\|_p^{m+1}}{(m+1)!h} + \frac{\|x - x_0\|_p^m}{2m!} + \frac{h\,\|x - x_0\|_p^{m-1}}{8(m-1)!} \right) < \infty, \tag{52}$$

$\forall\, x, x_0 \in M$.

Here, $0 < \omega_1\left(f^{(m)}, h\right) < \infty$, by M being compact and $f^{(m)}$ being continuous on M.

One can rewrite (52) as follows:

$$\left\| f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \right\|_\gamma \leq \omega_1\left(f^{(m)}, h\right) \left(\frac{\|\cdot - x_0\|_p^{m+1}}{(m+1)!h} + \frac{\|\cdot - x_0\|_p^m}{2m!} + \frac{h\,\|\cdot - x_0\|_p^{m-1}}{8(m-1)!} \right), \quad \forall\, x_0 \in M, \tag{53}$$

a pointwise functional inequality on M.

Here, $(\cdot - x_0)^j$ maps M into $\left(\mathbb{R}^N\right)^j$ and it is continuous, and also, $f^{(j)}(x_0)$ maps $\left(\mathbb{R}^N\right)^j$ into X and it is continuous. Hence, their composition $f^{(j)}(x_0)(\cdot - x_0)^j$ is continuous from M into X.

Clearly, $f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \in C(M, X)$, and hence, $\left\| f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \right\|_\gamma \in C(M)$.

Let $\left\{\widetilde{S}_N\right\}_{N \in \mathbb{N}}$ be a sequence of positive linear operators' mapping $C(M)$ into $C(M)$.

Therefore, we obtain

$$\left(\widetilde{S}_N \left(\left\| f(\cdot) - \sum_{j=0}^{m} \frac{f^{(j)}(x_0)(\cdot - x_0)^j}{j!} \right\|_\gamma \right) \right)(x_0) \leq \omega_1\left(f^{(m)}, h\right) \left[\frac{\left(\widetilde{S}_N\left(\|\cdot - x_0\|_p^{m+1}\right)\right)(x_0)}{(m+1)!h} + \frac{\left(\widetilde{S}_N\left(\|\cdot - x_0\|_p^{m}\right)\right)(x_0)}{2m!} + \frac{h\left(\widetilde{S}_N\left(\|\cdot - x_0\|_p^{m-1}\right)\right)(x_0)}{8(m-1)!} \right] \tag{54}$$

$\forall\, N \in \mathbb{N}$, $\forall\, x_0 \in M$.

Clearly, (54) is valid when $M=\prod_{i=1}^{N}[a_i,b_i]$ and $\widetilde{S}_n=\widetilde{L}_n$, see (23).

All the above is preparation for the following theorem, where we assume Fr échet differentiability of functions.

This will be a direct application of Theorem 10.2, [15, pp. 268–270]. The operators L_n, $\widetilde{L}_n$ fulfill its assumptions; see (22), (23), (25), (26), and (27).

We present the following high-order approximation results.

Theorem 3.3 *Let O open subset of $\left(\mathbb{R}^N,\|\cdot\|_p\right)$, $p\in[1,\infty]$, such that $\prod_{i=1}^{N}[a_i,b_i]\subset O\subseteq\mathbb{R}^N$, and let $\left(X,\|\cdot\|_\gamma\right)$ be a general Banach space. Let $m\in\mathbb{N}$ and $f\in C^m(O,X)$, the space of m-times continuously Fréchet differentiable functions from O into X. We study the approximation of $f|_{\prod_{i=1}^{N}[a_i,b_i]}$. Let $x_0\in\left(\prod_{i=1}^{N}[a_i,b_i]\right)$ and $r>0$. Then*

(1)

$$\left\|(L_n(f))(x_0)-\sum_{j=0}^{m}\frac{1}{j!}\left(L_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\gamma \le\frac{\omega_1\left(f^{(m)},r\left(\left(\widetilde{L}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{L}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}\times\left[\frac{1}{(m+1)}+\frac{r}{2}+\frac{mr^2}{8}\right]; \tag{55}$$

(2) *in addition, if $f^{(j)}(x_0)=0$, $j=1,\ldots,m$, we have*

$$\|(L_n(f))(x_0)-f(x_0)\|_\gamma \le\frac{\omega_1\left(f^{(m)},r\left(\left(\widetilde{L}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{L}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}\times\left[\frac{1}{(m+1)}+\frac{r}{2}+\frac{mr^2}{8}\right]; \tag{56}$$

(3)

$$\|(L_n(f))(x_0)-f(x_0)\|_\gamma\le\sum_{j=1}^{m}\frac{1}{j!}\left\|\left(L_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\gamma +\frac{\omega_1\left(f^{(m)},r\left(\left(\widetilde{L}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\frac{1}{m+1}}\right)}{rm!}\left(\left(\widetilde{L}_n\left(\|\cdot-x_0\|_p^{m+1}\right)\right)(x_0)\right)^{\left(\frac{m}{m+1}\right)}\times\left[\frac{1}{(m+1)}+\frac{r}{2}+\frac{mr^2}{8}\right]; \tag{57}$$

and

(4)

$$\left\| \left\| L_n(f) - f \right\|_\gamma \right\|_{\infty, \prod_{i=1}^N [a_i, b_i]}$$
$$\leq \sum_{j=1}^m \frac{1}{j!} \left\| \left\| \left(L_n \left(f^{(j)}(x_0) (\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}$$
$$+ \frac{\omega_1 \left(f^{(m)}, r \left\| \left(\widetilde{L}_n \left(\|\cdot - x_0\|_p^{m+1} \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}^{\frac{1}{m+1}} \right)}{rm!}$$
$$\times \left\| \left(\widetilde{L}_n \left(\|\cdot - x_0\|_p^{m+1} \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}^{\left(\frac{m}{m+1}\right)}$$
$$\times \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right]. \tag{58}$$

We need the following.

Lemma 3.4 *The function* $\left(\widetilde{L}_n \left(\|\cdot - x_0\|_p^m \right) \right)(x_0)$ *is continuous in* $x_0 \in \left(\prod_{i=1}^N [a_i, b_i] \right)$, $m \in \mathbb{N}$.

Proof By Lemma 10.3, [15, p. 272]. □

Remark 3.5 By Remark 10.4 [15, p. 273], we get that

$$\left\| \left(\widetilde{L}_n \left(\|\cdot - x_0\|_p^k \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]} \leq \left\| \left(\widetilde{L}_n \left(\|\cdot - x_0\|_p^{m+1} \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}^{\left(\frac{k}{m+1}\right)} \tag{59}$$

for all $k = 1, \ldots, m$. □

We give the following.

Corollary 3.6 (To Theorem 3.3, case of $m = 1$) *Then*

(1)

$$\|(L_n(f))(x_0) - f(x_0)\|_\gamma \leq \left\| \left(L_n \left(f^{(1)}(x_0)(\cdot - x_0) \right) \right)(x_0) \right\|_\gamma$$
$$+ \frac{1}{2r} \omega_1 \left(f^{(1)}, r \left(\left(\widetilde{L}_n \left(\|\cdot - x_0\|_p^2 \right) \right)(x_0) \right)^{\frac{1}{2}} \right) \left(\left(\widetilde{L}_n \left(\|\cdot - x_0\|_p^2 \right) \right)(x_0) \right)^{\frac{1}{2}}$$
$$\times \left[1 + r + \frac{r^2}{4} \right], \tag{60}$$

and

(2)

$$\left\| \left\| (L_n(f)) - f \right\|_\gamma \right\|_{\infty, \prod_{i=1}^N [a_i, b_i]}$$
$$\leq \left\| \left\| \left(L_n \left(f^{(1)}(x_0)(\cdot - x_0) \right) \right)(x_0) \right\|_\gamma \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}$$
$$+ \frac{1}{2r} \omega_1 \left(f^{(1)}, r \left\| \left(\widetilde{L}_n \left(\|\cdot - x_0\|_p^2 \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}^{\frac{1}{2}} \right)$$
$$\times \left\| \left(\widetilde{L}_n \left(\|\cdot - x_0\|_p^2 \right) \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}^{\frac{1}{2}} \left[1 + r + \frac{r^2}{4} \right], \tag{61}$$

$r > 0$.

We make the following.

Remark 3.7 We estimate $0 < \alpha < 1$, $\mu > 0$, $m, n \in \mathbb{N} : n^{1-\alpha} > 2$,

$$\widetilde{L}_n \left(\|\cdot - x_0\|_\infty^{m+1} \right)(x_0) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx_0 - k)}$$

$$\overset{(19)}{<} \left(\frac{4\left(1 + e^{-2\mu}\right)}{1 - e^{-2\mu}} \right)^N \sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k) \tag{62}$$

$$= \left(\frac{4\left(1 + e^{-2\mu}\right)}{1 - e^{-2\mu}} \right)^N \left\{ \sum_{\substack{k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_\infty \le \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k) \right.$$

$$\left. + \sum_{\substack{k = \lceil na \rceil \\ : \left\| \frac{k}{n} - x_0 \right\|_\infty > \frac{1}{n^\alpha}}}^{\lfloor nb \rfloor} \left\| \frac{k}{n} - x_0 \right\|_\infty^{m+1} Z(nx_0 - k) \right\}$$

$$\overset{(20)}{\le} \left(\frac{4\left(1 + e^{-2\mu}\right)}{1 - e^{-2\mu}} \right)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{\|b - a\|_\infty^{m+1}}{e^{\mu\left(n^{1-\beta} - 2\right)}} \right\} \tag{63}$$

(where $b - a = (b_1 - a_1, \ldots, b_N - a_N)$).

We have proved that ($\forall\, x_0 \in \prod_{i=1}^N [a_i, b_i]$)

$$\widetilde{L}_n \left(\|\cdot - x_0\|_\infty^{m+1} \right)(x_0) < \left(\frac{4\left(1 + e^{-2\mu}\right)}{1 - e^{-2\mu}} \right)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{\|b - a\|_\infty^{m+1}}{e^{\mu\left(n^{1-\beta} - 2\right)}} \right\} =: \Lambda_1(n) \tag{64}$$

($0 < \alpha < 1$, $m, n \in \mathbb{N} : n^{1-\alpha} > 2$, $\mu > 0$).

And, consequently, it holds

$$\left\| \widetilde{L}_n \left(\|\cdot - x_0\|_\infty^{m+1} \right)(x_0) \right\|_{\infty, x_0 \in \prod_{i=1}^N [a_i, b_i]}$$
$$< \left(\frac{4\left(1 + e^{-2\mu}\right)}{1 - e^{-2\mu}} \right)^N \left\{ \frac{1}{n^{\alpha(m+1)}} + \frac{\|b - a\|_\infty^{m+1}}{e^{\mu\left(n^{1-\beta} - 2\right)}} \right\} = \Lambda_1(n) \to 0, \quad \text{as } n \to +\infty. \tag{65}$$

Therefore, we have that $\Lambda_1(n) \to 0$, as $n \to +\infty$. Thus, when $p \in [1, \infty]$, from Theorem 3.3, we have the convergence to zero in the right-hand sides of parts (1), (2).

Next, we estimate $\left\| \left(\widetilde{L}_n \left(f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma$.

We have that

$$\left(\widetilde{L}_n \left(f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) = \frac{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j Z(nx_0 - k)}{\sum_{k=\lceil na \rceil}^{\lfloor nb \rfloor} Z(nx_0 - k)}. \tag{66}$$

When $p = \infty$, $j = 1, \ldots, m$, we obtain

$$\left\| f^{(j)}(x_0) \left(\frac{k}{n} - x_0 \right)^j \right\|_\gamma \le \left\| f^{(j)}(x_0) \right\| \left\| \frac{k}{n} - x_0 \right\|_\infty^j. \tag{67}$$

We further have that

$$\left\|\left(\widetilde{L}_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\gamma$$
$$\overset{(19)}{<}\left(\frac{4\left(1+e^{-2\mu}\right)}{1-e^{-2\mu}}\right)^N\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f^{(j)}(x_0)\left(\frac{k}{n}-x_0\right)^j\right\|_\gamma Z(nx_0-k)\right)$$
$$\leq\left(\frac{4\left(1+e^{-2\mu}\right)}{1-e^{-2\mu}}\right)^N\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|f^{(j)}(x_0)\right\|\left\|\frac{k}{n}-x_0\right\|_\infty^j Z(nx_0-k)\right) \tag{68}$$
$$=\left(\frac{4\left(1+e^{-2\mu}\right)}{1-e^{-2\mu}}\right)^N\left\|f^{(j)}(x_0)\right\|\left(\sum_{k=\lceil na\rceil}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^j Z(nx_0-k)\right)$$
$$=\left(\frac{4\left(1+e^{-2\mu}\right)}{1-e^{-2\mu}}\right)^N\left\|f^{(j)}(x_0)\right\|\left\{\sum_{\substack{k=\lceil na\rceil\\ \left\{:\left\|\frac{k}{n}-x_0\right\|_\infty\leq\frac{1}{n^\alpha}\right.}}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^j Z(nx_0-k)\right.$$
$$\left.+\sum_{\substack{k=\lceil na\rceil\\ \left\{:\left\|\frac{k}{n}-x_0\right\|_\infty>\frac{1}{n^\alpha}\right.}}^{\lfloor nb\rfloor}\left\|\frac{k}{n}-x_0\right\|_\infty^j Z(nx_0-k)\right\}$$
$$\overset{(20)}{\leq}\left(\frac{4\left(1+e^{-2\mu}\right)}{1-e^{-2\mu}}\right)^N\left\|f^{(j)}(x_0)\right\|\left\{\frac{1}{n^{\alpha j}}+\frac{\|b-a\|_\infty^j}{e^{\mu\left(n^{1-\beta}-2\right)}}\right\}\to 0,\quad \text{as } n\to\infty. \tag{69}$$

That is

$$\left\|\left(\widetilde{L}_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\gamma\to 0,\quad \text{as } n\to\infty.$$

Therefore, when $p=\infty$, for $j=1,\ldots,m$, we have proved

$$\left\|\left(\widetilde{L}_n\left(f^{(j)}(x_0)(\cdot-x_0)^j\right)\right)(x_0)\right\|_\gamma$$
$$<\left(\frac{4\left(1+e^{-2\mu}\right)}{1-e^{-2\mu}}\right)^N\left\|f^{(j)}(x_0)\right\|\left\{\frac{1}{n^{\alpha j}}+\frac{\|b-a\|_\infty^j}{e^{\mu\left(n^{1-\beta}-2\right)}}\right\}$$
$$\leq\left(\frac{4\left(1+e^{-2\mu}\right)}{1-e^{-2\mu}}\right)^N\left\|f^{(j)}(x_0)\right\|_\infty\left\{\frac{1}{n^{\alpha j}}+\frac{\|b-a\|_\infty^j}{e^{\mu\left(n^{1-\beta}-2\right)}}\right\}=:\Lambda_{2j}(n)<\infty, \tag{70}$$

and converges to zero, as $n\to\infty$.

We conclude the following:
In Theorem 3.3, the right-hand sides of (57) and (58) converge to zero as $n\to\infty$, for any $p\in[1,\infty]$.
Also in Corollary 3.6, the right-hand sides of (60) and (61) converge to zero as $n\to\infty$, for any $p\in[1,\infty]$.

Conclusion 3.8 *We have proved that the left-hand sides of* (55), (56), (57), (58), *and* (60), (61) *converge to zero as* $n\to\infty$, *for* $p\in[1,\infty]$. *Consequently,* $L_n\to I$ *(unit operator) pointwise and uniformly, as* $n\to\infty$, *where* $p\in[1,\infty]$. *In the presence of initial conditions, we achieve a higher speed of convergence; see* (56). *Higher speed of convergence happens also to the left-hand side of* (55).

We further give the following:

Corollary 3.9 (To Theorem 3.3) *Let O open subset of $\left(\mathbb{R}^N, \|\cdot\|_\infty\right)$, such that $\prod_{i=1}^N [a_i, b_i] \subset O \subseteq \mathbb{R}^N$, and let $\left(X, \|\cdot\|_\gamma\right)$ be a general Banach space. Let $m \in \mathbb{N}$ and $f \in C^m(O, X)$, the space of m-times continuously Fréchet differentiable functions from O into X. We study the approximation of $f|_{\prod_{i=1}^N [a_i,b_i]}$. Let $x_0 \in \left(\prod_{i=1}^N [a_i, b_i]\right)$ and $r > 0$. Here, $\Lambda_1(n)$ as in (65) and $\Lambda_{2j}(n)$ as in (70), where $n \in \mathbb{N} : n^{1-\alpha} > 2$, $0 < \alpha < 1$, $\mu > 0$, $j = 1, \ldots, m$. Then*

(1)

$$\left\| (L_n(f))(x_0) - \sum_{j=0}^{m} \frac{1}{j!} \left(L_n \left(f^{(j)}(x_0)(\cdot - x_0)^j \right) \right)(x_0) \right\|_\gamma \leq \frac{\omega_1\left(f^{(m)}, r(\Lambda_1(n))^{\frac{1}{m+1}}\right)}{rm!} (\Lambda_1(n))^{\left(\frac{m}{m+1}\right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right]; \tag{71}$$

(2) *in addition, if $f^{(j)}(x_0) = 0$, $j = 1, \ldots, m$, we have*

$$\|(L_n(f))(x_0) - f(x_0)\|_\gamma \leq \frac{\omega_1\left(f^{(m)}, r(\Lambda_1(n))^{\frac{1}{m+1}}\right)}{rm!} (\Lambda_1(n))^{\left(\frac{m}{m+1}\right)} \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right]; \tag{72}$$

(3)

$$\left\| \|L_n(f) - f\|_\gamma \right\|_{\infty, \prod_{i=1}^N [a_i,b_i]} \leq \sum_{j=1}^{m} \frac{\Lambda_{2j}(n)}{j!} + \frac{\omega_1\left(f^{(m)}, r(\Lambda_1(n))^{\frac{1}{m+1}}\right)}{rm!} (\Lambda_1(n))^{\left(\frac{m}{m+1}\right)} \times \left[\frac{1}{(m+1)} + \frac{r}{2} + \frac{mr^2}{8} \right] =: \Lambda_3(n) \to 0, \quad \text{as } n \to \infty. \tag{73}$$

We continue with the following.

Theorem 3.10 *Let $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1$, $\mu > 0$, $x \in \mathbb{R}^N$, $N, n \in \mathbb{N}$ with $n^{1-\beta} > 2$, ω_1 is for $p = \infty$. Then*

(1)

$$\|B_n(f, x) - f(x)\|_\gamma \leq \omega_1\left(f, \frac{1}{n^\beta}\right) + \frac{2\left\| \|f\|_\gamma \right\|_\infty}{e^{\mu\left(n^{1-\beta}-2\right)}} =: \lambda_2(n); \tag{74}$$

(2)

$$\left\| \|B_n(f) - f\|_\gamma \right\|_\infty \leq \lambda_2(n). \tag{75}$$

Given that $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$, we obtain $\lim_{n\to\infty} B_n(f) = f$, uniformly. The speed of convergence above is $\max\left(\frac{1}{n^\beta}, \frac{2}{e^{\left(n^{1-\beta}-2\right)}}\right) = \frac{1}{n^\beta}$.

Proof We have that

$$B_n(f, x) - f(x) \overset{(13)}{=} \sum_{k=-\infty}^{\infty} f\left(\frac{k}{n}\right) Z(nx - k) - f(x) \sum_{k=-\infty}^{\infty} Z(nx - k) = \sum_{k=-\infty}^{\infty} \left(f\left(\frac{k}{n}\right) - f(x) \right) Z(nx - k). \tag{76}$$

Hence

$$\|B_n(f,x)-f(x)\|_\gamma \le \sum_{k=-\infty}^{\infty}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|_\gamma Z(nx-k)$$
$$= \sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty \le \frac{1}{n^\beta}\end{cases}}^{\infty}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|_\gamma Z(nx-k)$$
$$+ \sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}\end{cases}}^{\infty}\left\|f\left(\frac{k}{n}\right)-f(x)\right\|_\gamma Z(nx-k)$$
$$\overset{(13)}{\le} \omega_1\left(f,\frac{1}{n^\beta}\right)+2\left\|\|f\|_\gamma\right\|_\infty \sum_{\begin{cases} k=-\infty \\ \left\|\frac{k}{n}-x\right\|_\infty > \frac{1}{n^\beta}\end{cases}}^{\infty} Z(nx-k)$$
$$\overset{(20)}{\le} \omega_1\left(f,\frac{1}{n^\beta}\right)+\frac{2\left\|\|f\|_\gamma\right\|_\infty}{e^{\mu\left(n^{1-\beta}-2\right)}}, \tag{77}$$

proving the claim. □

We give the following.

Theorem 3.11 *Let* $f\in C_B\left(\mathbb{R}^N,X\right)$, $0<\beta<1$, $x\in\mathbb{R}^N$, $\mu>0$, $N,n\in\mathbb{N}$ *with* $n^{1-\beta}>2$, ω_1 *is for* $p=\infty$. *Then*

(1)

$$\|C_n(f,x)-f(x)\|_\gamma \le \omega_1\left(f,\frac{1}{n}+\frac{1}{n^\beta}\right)+\frac{2\left\|\|f\|_\gamma\right\|_\infty}{e^{\mu\left(n^{1-\beta}-2\right)}} =: \lambda_3(n); \tag{78}$$

(2)

$$\left\|\|C_n(f)-f\|_\gamma\right\|_\infty \le \lambda_3(n). \tag{79}$$

Given that $f\in\left(C_U\left(\mathbb{R}^N,X\right)\cap C_B\left(\mathbb{R}^N,X\right)\right)$, *we obtain* $\lim_{n\to\infty}C_n(f)=f$, *uniformly.*

Proof We notice that

$$\int_{\frac{k}{n}}^{\frac{k+1}{n}} f(t)\,dt = \int_{\frac{k_1}{n}}^{\frac{k_1+1}{n}}\int_{\frac{k_2}{n}}^{\frac{k_2+1}{n}}\cdots\int_{\frac{k_N}{n}}^{\frac{k_N+1}{n}} f(t_1,t_2,\ldots,t_N)\,dt_1dt_2\ldots dt_N$$
$$=\int_0^{\frac{1}{n}}\int_0^{\frac{1}{n}}\cdots\int_0^{\frac{1}{n}} f\left(t_1+\frac{k_1}{n},t_2+\frac{k_2}{n},\ldots,t_N+\frac{k_N}{n}\right)dt_1\cdots dt_N=\int_0^{\frac{1}{n}} f\left(t+\frac{k}{n}\right)dt. \tag{80}$$

Thus, it holds (by (35))

$$C_n(f,x)=\sum_{k=-\infty}^{\infty}\left(n^N\int_0^{\frac{1}{n}} f\left(t+\frac{k}{n}\right)dt\right)Z(nx-k). \tag{81}$$

We observe that

$$
\begin{aligned}
&\|C_n(f,x)-f(x)\|_\gamma\\
&=\left\|\sum_{k=-\infty}^{\infty}\left(n^N\int_0^{\frac{1}{n}} f\left(t+\frac{k}{n}\right)\mathrm{d}t\right)Z(nx-k)-\sum_{k=-\infty}^{\infty} f(x)\,Z(nx-k)\right\|_\gamma\\
&=\left\|\sum_{k=-\infty}^{\infty}\left(\left(n^N\int_0^{\frac{1}{n}} f\left(t+\frac{k}{n}\right)\mathrm{d}t\right)-f(x)\right)Z(nx-k)\right\|_\gamma\\
&=\left\|\sum_{k=-\infty}^{\infty}\left(n^N\int_0^{\frac{1}{n}}\left(f\left(t+\frac{k}{n}\right)-f(x)\right)\mathrm{d}t\right)Z(nx-k)\right\|_\gamma \qquad (82)\\
&\le\sum_{k=-\infty}^{\infty}\left(n^N\int_0^{\frac{1}{n}}\left\|f\left(t+\frac{k}{n}\right)-f(x)\right\|_\gamma\mathrm{d}t\right)Z(nx-k)\\
&=\sum_{\begin{cases}k=-\infty\\ \left\|\frac{k}{n}-x\right\|_\infty\le\frac{1}{n^\beta}\end{cases}}^{\infty}\left(n^N\int_0^{\frac{1}{n}}\left\|f\left(t+\frac{k}{n}\right)-f(x)\right\|_\gamma\mathrm{d}t\right)Z(nx-k)\\
&\quad+\sum_{\begin{cases}k=-\infty\\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}\end{cases}}^{\infty}\left(n^N\int_0^{\frac{1}{n}}\left\|f\left(t+\frac{k}{n}\right)-f(x)\right\|_\gamma\mathrm{d}t\right)Z(nx-k)\\
&\le\sum_{\begin{cases}k=-\infty\\ \left\|\frac{k}{n}-x\right\|_\infty\le\frac{1}{n^\beta}\end{cases}}^{\infty}\left(n^N\int_0^{\frac{1}{n}}\omega_1\left(f,\|t\|_\infty+\left\|\frac{k}{n}-x\right\|_\infty\right)\mathrm{d}t\right)Z(nx-k)\\
&\quad+2\left\|\|f\|_\gamma\right\|_\infty\left(\sum_{\begin{cases}k=-\infty\\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}\end{cases}}^{\infty}Z(|nx-k|)\right)\\
&\le\omega_1\left(f,\frac{1}{n}+\frac{1}{n^\beta}\right)+\frac{2\left\|\|f\|_\gamma\right\|_\infty}{e^{\mu\left(n^{1-\beta}-2\right)}}, \qquad (83)
\end{aligned}
$$

proving the claim. □

We also present the following.

Theorem 3.12 *Let $f\in C_B\left(\mathbb{R}^N,X\right)$, $0<\beta<1$, $x\in\mathbb{R}^N$, $\mu>0$, $N,n\in\mathbb{N}$ with $n^{1-\beta}>2$, ω_1 is for $p=\infty$. Then*

(1)

$$\|D_n(f,x)-f(x)\|_\gamma\le\omega_1\left(f,\frac{1}{n}+\frac{1}{n^\beta}\right)+\frac{2\left\|\|f\|_\gamma\right\|_\infty}{e^{\mu\left(n^{1-\beta}-2\right)}}=:\lambda_4(n); \qquad (84)$$

(2)

$$\left\|\|D_n(f)-f\|_\gamma\right\|_\infty\le\lambda_4(n). \qquad (85)$$

Given that $f\in\left(C_U\left(\mathbb{R}^N,X\right)\cap C_B\left(\mathbb{R}^N,X\right)\right)$, we obtain $\lim_{n\to\infty}D_n(f)=f$, uniformly.

Proof We have that [by (37)]

$$\|D_n(f,x)-f(x)\|_\gamma = \left\|\sum_{k=-\infty}^{\infty}\delta_{nk}(f)\,Z(nx-k)-\sum_{k=-\infty}^{\infty}f(x)\,Z(nx-k)\right\|_\gamma$$

$$=\left\|\sum_{k=-\infty}^{\infty}(\delta_{nk}(f)-f(x))\,Z(nx-k)\right\|_\gamma = \left\|\sum_{k=-\infty}^{\infty}w_r\left(f\left(\frac{k}{n}+\frac{r}{n\theta}\right)-f(x)\right)Z(nx-k)\right\|_\gamma$$

$$\leq \sum_{k=-\infty}^{\infty}\left(\sum_{r=0}^{\theta}w_r\left\|f\left(\frac{k}{n}+\frac{r}{n\theta}\right)-f(x)\right\|_\gamma\right)Z(nx-k)$$

$$=\sum_{\substack{k=-\infty\\ \left\|\frac{k}{n}-x\right\|_\infty\leq\frac{1}{n^\beta}}}^{\infty}\left(\sum_{r=0}^{\theta}w_r\left\|f\left(\frac{k}{n}+\frac{r}{n\theta}\right)-f(x)\right\|_\gamma\right)Z(nx-k)$$

$$+\sum_{\substack{k=-\infty\\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}}}^{\infty}\left(\sum_{r=0}^{\theta}w_r\left\|f\left(\frac{k}{n}+\frac{r}{n\theta}\right)-f(x)\right\|_\gamma\right)Z(nx-k)$$

$$\leq\sum_{\substack{k=-\infty\\ \left\|\frac{k}{n}-x\right\|_\infty\leq\frac{1}{n^\beta}}}^{\infty}\left(\sum_{r=0}^{\theta}w_r\left\|f\left(\frac{k}{n}+\frac{r}{n\theta}\right)-f(x)\right\|_\gamma\right)Z(nx-k)$$

$$+2\left\|\|f\|_\gamma\right\|_\infty\left(\sum_{\substack{k=-\infty\\ \left\|\frac{k}{n}-x\right\|_\infty>\frac{1}{n^\beta}}}^{\infty}(Z(nx-k))\right)$$

$$\leq\omega_1\left(f,\frac{1}{n}+\frac{1}{n^\beta}\right)+\frac{2\left\|\|f\|_\gamma\right\|_\infty}{e^{\mu(n^{1-\beta}-2)}}=\lambda_4(n),$$

proving the claim. □

We make the following.

Definition 3.13 Let $f\in C_B(\mathbb{R}^N,X)$, $N\in\mathbb{N}$, where $(X,\|\cdot\|_\gamma)$ is a Banach space. We define the general neural network operator

$$F_n(f,x):=\sum_{k=-\infty}^{\infty}l_{nk}(f)\,Z(nx-k)$$
$$=\begin{cases}B_n(f,x), & \text{if } l_{nk}(f)=f\left(\frac{k}{n}\right),\\ C_n(f,x), & \text{if } l_{nk}(f)=n^N\int_{\frac{k}{n}}^{\frac{k+1}{n}}f(t)\,dt,\\ D_n(f,x), & \text{if } l_{nk}(f)=\delta_{nk}(f).\end{cases} \quad (86)$$

Clearly, $l_{nk}(f)$ is an X-valued bounded linear functional, such that $\|l_{nk}(f)\|_\gamma\leq\left\|\|f\|_\gamma\right\|_\infty$.
Hence, $F_n(f)$ is a bounded linear operator with $\left\|\|F_n(f)\|_\gamma\right\|_\infty\leq\left\|\|f\|_\gamma\right\|_\infty$.
We need the following.

Theorem 3.14 *Let $f\in C_B(\mathbb{R}^N,X)$, $N\geq 1$. Then, $F_n(f)\in C_B(\mathbb{R}^N,X)$.*

Proof Lengthy and similar to the proof of Theorem 10 of [18], as such is omitted. □

Remark 3.15 By (22), it is obvious that $\left\| \| L_n(f) \|_\gamma \right\|_\infty \leq \left\| \| f \|_\gamma \right\|_\infty < \infty$, and $L_n(f) \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$, given that $f \in C\left(\prod_{i=1}^N [a_i, b_i], X\right)$.

Call K_n any of the operators L_n, B_n, C_n, D_n.

Clearly, then

$$\left\| \| K_n^2(f) \|_\gamma \right\|_\infty = \left\| \| K_n(K_n(f)) \|_\gamma \right\|_\infty \leq \left\| \| K_n(f) \|_\gamma \right\|_\infty \leq \left\| \| f \|_\gamma \right\|_\infty, \tag{87}$$

etc.

Therefore, we get

$$\left\| \left\| K_n^k(f) \right\|_\gamma \right\|_\infty \leq \left\| \| f \|_\gamma \right\|_\infty, \quad \forall\, k \in \mathbb{N}, \tag{88}$$

the contraction property.

Also, we see that

$$\left\| \left\| K_n^k(f) \right\|_\gamma \right\|_\infty \leq \left\| \left\| K_n^{k-1}(f) \right\|_\gamma \right\|_\infty \leq \cdots \leq \left\| \| K_n(f) \|_\gamma \right\|_\infty \leq \left\| \| f \|_\gamma \right\|_\infty. \tag{89}$$

Here, K_n^k are bounded linear operators.

Notation 3.16 *Here, $N \in \mathbb{N}$, $0 < \beta < 1$. Denote by*

$$c_N := \begin{cases} \left(\frac{4\left(1+e^{-2\mu}\right)}{1-e^{-2\mu}} \right)^N, & \text{if } K_n = L_n, \\ 1, & \text{if } K_n = B_n, C_n, D_n, \end{cases} \tag{90}$$

$$\Lambda(n) := \begin{cases} \frac{1}{n^\beta}, & \text{if } K_n = L_n, B_n, \\ \frac{1}{n} + \frac{1}{n^\beta}, & \text{if } K_n = C_n, D_n, \end{cases} \tag{91}$$

$$\Omega := \begin{cases} C\left(\prod_{i=1}^N [a_i, b_i], X\right), & \text{if } K_n = L_n, \\ C_B\left(\mathbb{R}^N, X\right), & \text{if } K_n = B_n, C_n, D_n, \end{cases} \tag{92}$$

and

$$Y := \begin{cases} \prod_{i=1}^N [a_i, b_i], & \text{if } K_n = L_n, \\ \mathbb{R}^N, & \text{if } K_n = B_n, C_n, D_n. \end{cases} \tag{93}$$

We give the condensed.

Theorem 3.17 *Let $f \in \Omega$, $0 < \beta < 1$, $x \in Y$; $n, \mu > 0$; $N \in \mathbb{N}$ with $n^{1-\beta} > 2$. Then*

(i)

$$\| K_n(f, x) - f(x) \|_\gamma \leq c_N \left[\omega_1(f, \Lambda(n)) + \frac{2 \left\| \| f \|_\gamma \right\|_\infty}{e^{\mu\left(n^{1-\beta}-2\right)}} \right] =: \tau(n), \tag{94}$$

where ω_1 is for $p = \infty$,
and

(ii)

$$\left\| \| K_n(f) - f \|_\gamma \right\|_\infty \leq \tau(n) \to 0, \quad \text{as } n \to \infty. \tag{95}$$

For f uniformly continuous and in Ω, we obtain

$$\lim_{n \to \infty} K_n(f) = f,$$

pointwise and uniformly.

Proof By Theorems 3.1, 3.10, 3.11 and 3.12. □

Next, we do iterated neural network approximation (see also [10]).
We make the following.

Remark 3.18 Let $r \in \mathbb{N}$ and K_n as above. We observe that

$$K_n^r f - f = \left(K_n^r f - K_n^{r-1} f\right) + \left(K_n^{r-1} f - K_n^{r-2} f\right) + \left(K_n^{r-2} f - K_n^{r-3} f\right) + \cdots + \left(K_n^2 f - K_n f\right) + \left(K_n f - f\right).$$

Then

$$\begin{aligned}\left\| \left\| K_n^r f - f \right\|_\gamma \right\|_\infty &\leq \left\| \left\| K_n^r f - K_n^{r-1} f \right\|_\gamma \right\|_\infty + \left\| \left\| K_n^{r-1} f - K_n^{r-2} f \right\|_\gamma \right\|_\infty \\ &+ \left\| \left\| K_n^{r-2} f - K_n^{r-3} f \right\|_\gamma \right\|_\infty + \cdots + \left\| \left\| K_n^2 f - K_n f \right\|_\gamma \right\|_\infty + \left\| \left\| K_n f - f \right\|_\gamma \right\|_\infty \\ &= \left\| \left\| K_n^{r-1} \left(K_n f - f\right) \right\|_\gamma \right\|_\infty + \left\| \left\| K_n^{r-2} \left(K_n f - f\right) \right\|_\gamma \right\|_\infty + \left\| \left\| K_n^{r-3} \left(K_n f - f\right) \right\|_\gamma \right\|_\infty \\ &+ \cdots + \left\| \left\| K_n \left(K_n f - f\right) \right\|_\gamma \right\|_\infty + \left\| \left\| K_n f - f \right\|_\gamma \right\|_\infty \leq r \left\| \left\| K_n f - f \right\|_\gamma \right\|_\infty .\end{aligned}$$

That is

$$\left\| \left\| K_n^r f - f \right\|_\gamma \right\|_\infty \leq r \left\| \left\| K_n f - f \right\|_\gamma \right\|_\infty . \tag{96}$$

We give the following.

Theorem 3.19 *All here as in Theorem 3.17 and $r \in \mathbb{N}$, $\tau(n)$ as in (94). Then*

$$\left\| \left\| K_n^r f - f \right\|_\gamma \right\|_\infty \leq r \tau(n). \tag{97}$$

So that the speed of convergence to the unit operator of K_n^r is not worse than of K_n.

Proof As similar to [18] is omitted. □

Remark 3.20 Let $m_1, m_2, \ldots, m_r \in \mathbb{N} : m_1 \leq m_2 \leq \cdots \leq m_r$, $0 < \beta < 1$, $\mu > 0$, $f \in \Omega$. Then

$$\Lambda(m_1) \geq \Lambda(m_2) \geq \cdots \geq \Lambda(m_r), \quad \Lambda \text{ as in (91)}.$$

Therefore

$$\omega_1(f, \Lambda(m_1)) \geq \omega_1(f, \Lambda(m_2)) \geq \cdots \geq \omega_1(f, \Lambda(m_r)).$$

Assıme further that $m_i^{1-\beta} > 2$, $i = 1, \ldots, r$. Then

$$\frac{1}{e^{\mu\left(m_1^{1-\beta}-2\right)}} \geq \frac{1}{e^{\mu\left(m_2^{1-\beta}-2\right)}} \geq \cdots \geq \frac{1}{e^{\mu\left(m_r^{1-\beta}-2\right)}}.$$

Let K_{m_i} as above, $i = 1, \ldots, r$, all of the same kind. We write

$$\begin{aligned}&K_{m_r}\left(K_{m_{r-1}}\left(\ldots K_{m_2}\left(K_{m_1 f}\right)\right)\right) - f \\ &= K_{m_r}\left(K_{m_{r-1}}\left(\ldots K_{m_2}\left(K_{m_1 f}\right)\right)\right) - K_{m_r}\left(K_{m_{r-1}}\left(\ldots K_{m_2} f\right)\right) \\ &+ K_{m_r}\left(K_{m_{r-1}}\left(\ldots K_{m_2} f\right)\right) - K_{m_r}\left(K_{m_{r-1}}\left(\ldots K_{m_3} f\right)\right) \\ &+ K_{m_r}\left(K_{m_{r-1}}\left(\ldots K_{m3} f\right)\right) - K_{m_r}\left(K_{m_{r-1}}\left(\ldots K_{m_4} f\right)\right) \\ &+ \cdots + K_{m_r}\left(K_{m_{r-1}} f\right) - K_{m_r} f + K_{m_r} f - f \\ &= K_{m_r}\left(K_{m_{r-1}}\left(\ldots K_{m_2}\right)\right)\left(K_{m_1} f - f\right) + K_{m_r}\left(K_{m_{r-1}}\left(\ldots K_{m_3}\right)\right)\left(K_{m_2} f - f\right) \\ &+ K_{m_r}\left(K_{m_{r-1}}\left(\ldots K_{m_4}\right)\right)\left(K_{m3} f - f\right) + \cdots + K_{m_r}\left(K_{m_{r-1}} f - f\right) + K_{m_r} f - f.\end{aligned}$$

Hence, by the triangle inequality of $\left\| \|\cdot\|_\gamma \right\|_\infty$, we get

$$\begin{aligned}
&\left\| \left\| K_{m_r}\left(K_{m_{r-1}}\left(\ldots K_{m_2}\left(K_{m_1} f\right)\right)\right) - f \right\|_\gamma \right\|_\infty \\
&\le \left\| \left\| K_{m_r} K_{m_{r-1}} \ldots K_{m_2}\left(K_{m_1} f - f\right) \right\|_\gamma \right\|_\infty \\
&\quad + \left\| \left\| K_{m_r} K_{m_{r-1}} \ldots K_{m_2}\left(K_{m_1} f - f\right) \right\|_\gamma \right\|_\infty \\
&\quad + \left\| \left\| K_{m_r}\left(K_{m_{r-1}}\left(\ldots K_{m_4}\right)\right)\left(K_{m_3} f - f\right) \right\|_\gamma \right\|_\infty \\
&\quad + \cdots + \left\| \left\| K_{m_r}\left(K_{m_{r-1}} f - f\right) \right\|_\gamma \right\|_\infty + \left\| \left\| K_{m_r} f - f \right\|_\gamma \right\|_\infty \le
\end{aligned}$$

[repeatedly applying (87)]

$$\begin{aligned}
&\left\| \left\| K_{m_1} f - f \right\|_\gamma \right\|_\infty + \left\| \left\| K_{m_2} f - f \right\|_\gamma \right\|_\infty + \left\| \left\| K_{m_3} f - f \right\|_\gamma \right\|_\infty \\
&+ \cdots + \left\| \left\| K_{m_{r-1}} f - f \right\|_\gamma \right\|_\infty + \left\| \left\| K_{m_2} f - f \right\|_\gamma \right\|_\infty + \left\| \left\| K_{m_3} f - f \right\|_\gamma \right\|_\infty \\
&+ \cdots + \left\| \left\| K_{m_{r-1}} f - f \right\|_\gamma \right\|_\infty + \left\| \left\| K_{m_r} f - f \right\|_\gamma \right\|_\infty = \sum_{i=1}^{r} \left\| \left\| K_{m_i} f - f \right\|_\gamma \right\|_\infty .
\end{aligned}$$

That is, we proved

$$\left\| \left\| K_{m_r}\left(K_{m_{r-1}}\left(\ldots K_{m_2}\left(K_{m_1} f\right)\right)\right) - f \right\|_\gamma \right\|_\infty \le \sum_{i=1}^{r} \left\| \left\| K_{m_i} f - f \right\|_\gamma \right\|_\infty . \tag{98}$$

We also present the following.

Theorem 3.21 *Let $f \in \Omega$; $m, N, m_1, m_2, \ldots, m_r \in \mathbb{N} : m_1 \le m_2 \le \cdots \le m_r$, $0 < \beta < 1$, $\mu > 0$; $m_i^{1-\beta} > 2$, $i = 1, \ldots, r$, $x \in Y$, and let $\left(K_{m_1}, \ldots, K_{m_r}\right)$ as $\left(L_{m_1}, \ldots, L_{m_r}\right)$ or $\left(B_{m_1}, \ldots, B_{m_r}\right)$ or $\left(C_{m_1}, \ldots, C_{m_r}\right)$ or $\left(D_{m_1}, \ldots, D_{m_r}\right)$, $p = \infty$. Then*

$$\begin{aligned}
&\left\| K_{m_r}\left(K_{m_{r-1}}\left(\ldots K_{m_2}\left(K_{m_1} f\right)\right)\right)(x) - f(x) \right\|_\gamma \\
&\le \left\| \left\| K_{m_r}\left(K_{m_{r-1}}\left(\ldots K_{m_2}\left(K_{m_1} f\right)\right)\right) - f \right\|_\gamma \right\|_\infty \\
&\le \sum_{i=1}^{r} \left\| \left\| K_{m_i} f - f \right\|_\gamma \right\|_\infty \\
&\le c_N \sum_{i=1}^{r} \left[\omega_1\left(f, \Lambda\left(m_i\right)\right) + \frac{2 \left\| \|f\|_\gamma \right\|_\infty}{e^{\mu\left(m_i^{1-\beta} - 2\right)}} \right] \\
&\le r c_N \left[\omega_1\left(f, \Lambda\left(m_1\right)\right) + \frac{2 \left\| \|f\|_\gamma \right\|_\infty}{e^{\mu\left(m_1^{1-\beta} - 2\right)}} \right].
\end{aligned} \tag{99}$$

Clearly, we notice that the speed of convergence to the unit operator of the multiply iterated operator is not worse than the speed of K_{m_1}.

Proof As similar to [18] is omitted. □

We continue with the following.

Theorem 3.22 *Let all as in Corollary 3.9, and $r \in \mathbb{N}$. Here, $\Lambda_3(n)$ is as in (73). Then*

$$\left\| \left\| L_n^r f - f \right\|_\gamma \right\|_\infty \le r \left\| \|L_n f - f\|_\gamma \right\|_\infty \le r \Lambda_3(n) . \tag{100}$$

Proof As similar to [18] is omitted. □

Next, we present some L_{p_1}, $p_1 \geq 1$, approximation related results.

Theorem 3.23 *Let* $p_1 \geq 1$, $f \in C\left(\prod_{i=1}^{n}[a_i, b_i], X\right)$, $0 < \beta < 1, \mu > 0$; $N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$, *and* $\lambda_1(n)$ *as in* (38), ω_1 *is for* $p = \infty$. *Then*

$$\left\| \|L_n f - f\|_\gamma \right\|_{p_1, \prod_{i=1}^{n}[a_i, b_i]} \leq \lambda_1(n) \left(\prod_{i=1}^{n} (b_i - a_i) \right)^{\frac{1}{p_1}}. \tag{101}$$

We notice that $\lim_{n\to\infty} \left\| \|L_n f - f\|_\gamma \right\|_{p_1, \prod_{i=1}^{n}[a_i, b_i]} = 0$.

Proof Obvious, by integrating (38), etc. □

It follows:

Theorem 3.24 *Let* $p_1 \geq 1$, $f \in C_B\left(\mathbb{R}^N, X\right)$, $0 < \beta < 1, \mu > 0$; $N, n \in \mathbb{N}$ *with* $n^{1-\beta} > 2$, *and* ω_1 *is for* $p = \infty$; $\lambda_2(n)$ *as in* (74) *and* P *a compact set of* $\mathbb{R}^N$. *Then*

$$\left\| \|B_n f - f\|_\gamma \right\|_{p_1, P} \leq \lambda_2(n) |P|^{\frac{1}{p_1}}, \tag{102}$$

where $|P| < \infty$, *is the Lebesgue measure of* P. *We notice that* $\lim_{n\to\infty} \left\| \|B_n f - f\|_\gamma \right\|_{p_1, P} = 0$ *for* $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$.

Proof By integrating (74), etc. □

Next come.

Theorem 3.25 *All as in Theorem* 3.24, *but we use* $\lambda_3(n)$ *of* (78). *Then*

$$\left\| \|C_n f - f\|_\gamma \right\|_{p_1, P} \leq \lambda_3(n) |P|^{\frac{1}{p_1}}. \tag{103}$$

We have that $\lim_{n\to\infty} \left\| \|C_n f - f\|_\gamma \right\|_{p_1, P} = 0$ *for* $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$.

Proof By (78). □

Theorem 3.26 *All as in Theorem* 3.24, *but we use* $\lambda_4(n)$ *of* (84). *Then*

$$\left\| \|D_n f - f\|_\gamma \right\|_{p_1, P} \leq \lambda_4(n) |P|^{\frac{1}{p_1}}. \tag{104}$$

We have that $\lim_{n\to\infty} \left\| \|D_n f - f\|_\gamma \right\|_{p_1, P} = 0$ *for* $f \in \left(C_U\left(\mathbb{R}^N, X\right) \cap C_B\left(\mathbb{R}^N, X\right)\right)$.

Proof By (84). □

Application 3.27 *A typical application of all of our results is when* $\left(X, \|\cdot\|_\gamma\right) = (\mathbb{C}, |\cdot|)$, *where* $\mathbb{C}$ *is the set of the complex numbers.*

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