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Multiplication operators in BV spaces

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Abstract
The aim of this paper is to provide necessary and sufficient conditions on the generator of a multiplication operator acting in the spaces of functions of bounded Young and Riesz variation so that it is, among other things, invertible, continuous, finite rank, compact, Fredholm or has closed range. Furthermore, we characterize various spectra of such operators and give some estimates on their measure of non-compactness.

Keywords Bijective operators · Closed range operators · Continuous operators · Compact operators · Finite-rank operators · Fredholm operators · Injective operators · Jordan variation · Measure of non-compactness of an operator · Multiplication operators · Riesz variation · Spaces of functions of bounded variation · Spectrum of an operator · Surjective operators · Wiener variation · Young variation

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1 Introduction

Multiplication operators are among the most basic linear operators acting in function spaces, or, in a more general setting, in (normed) algebras. Because of that and because of the fact that they are also building blocks for other important linear and nonlinear operators (such as, for example, weighted composition and superposition operators), multiplication operators have attracted great interest of analysts.

It seems impossible to summarize all the recent research concerning multiplication operators and related topics. Therefore, let us only draw the readers' attention to a few articles. Multiplication operators between two classical spaces of Lebesgue integrable functions were studied by, for example, Takagi et al.in [37]. Those results were extended to

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Orlicz spaces by Chawziuk et al. (see [17–19] and the references therein). On the other hand, properties of multiplication operators in Köthe spaces were investigated, for example, by Drewnowski et al. in [22] as well as by Castillo et al. in [15] (see also [34]). Let us also mention the papers by Bonet et al. [11] and de Jager et al. [25], where such operators were studied on, respectively, weighted Banach spaces of analytic functions and non-commutative spaces.

It may, therefore, come as a surprise that there are only two papers dealing with both the function-theoretic and topological properties of multiplication operators in the spaces of functions of bounded variation. Multiplication operators in the space BV of functions of bounded Jordan variation were exhaustively studied in [8], while those acting in the space WBV_p of functions of bounded Wiener variation were investigated in [10] (for the definitions of the Jordan, Wiener and other variations see Section 2).

One of the reasons for this situation may be the fact that the multiplier classes Y/X, consisting of those elements g for which the multiplication $x \mapsto gx$ is a well-defined operator from X to Y (for a formal definition of Y/X see Section 3), have been fully characterized for various BV-type spaces only very recently (see [12, 13]; cf. also [16]). This said, let us make a small digression. If X is a linear space of real-valued functions defined on the interval [0, 1] which is closed under multiplication and contains constant functions (for example, if X coincides with C, that is, the space of continuous functions on [0, 1], or with B, that is, the space of bounded functions on the same interval), then it is easy to check that $X/X = X$. However, in other classes of functions which are not necessarily closed under multiplication, a characterization of X/X can be much harder. For instance, the proof that D/D consists only of constant functions requires a lot more work; here D stands for the class of Darboux functions[1]. (An elementary proof of this fact, together with some brief historical background concerning this result, can be found in [12]). Adding another space Y to the mix complicates things even more. In some cases, a full determination of Y/X is extremely difficult. For instance, the class D/C satisfies the chain of inclusions $C \subsetneq D \cap \mathcal{B}_1 \subsetneq D/C \subsetneq D$, where $\mathcal{B}_1$ denotes the class of Baire-1 functions, but its exact characterization is—at least to our knowledge—unknown. A detailed discussion of these and many more multiplier classes can be found in [12, 13].

The aim of this paper is twofold. First is to extend the results of [8, 10] to multiplication operators acting between not necessarily equal spaces YBV_φ of functions of bounded Young variation and to provide necessary and sufficient conditions guaranteeing that such operators are, among other things, bijective, continuous, finite rank, compact, Fredholm or have closed range. Second is to check whether multiplication operators in different BV spaces also enjoy similar properties. Therefore, beside the spaces YBV_φ we chose to investigate also the spaces RBV_p of functions of bounded Riesz variation, which, roughly speaking, are situated on the other end of the "spectrum" of BV-type spaces (RBV_p spaces are contained in BV, while YBV_φ spaces contain BV; all functions in RBV_p are continuous, when $1 < p < +\infty$, while YBV_φ contain also some functions which are discontinuous; finally, RBV_p spaces are decreasing with respect to p, while WBV_p spaces, which are a special case of YBV_φ for $\varphi(u) = u^p$, increase with respect to the parameter). Finally, whenever it was possible we decided to prove abstract results concerning multiplication operators acting in general linear/normed spaces of real-valued functions defined on the interval [0, 1].

[1] Let us recall that a function $x : [0, 1] \to \mathbb{R}$ is called a *Darboux function* if it has the intermediate value property, that is, x attains any real number between $x(a)$ and $x(b)$ for any choice of $a, b \in [0, 1]$.

As it would only obscure the paper and make the readers experience the *déjà vu* phenomenon, we decided not to include the information concerning the relation between our results and the results of [8, 10] after each theorem. However, following the saying "give credit where credit is due," we would like to underline once again that invertibility, continuity, compactness, Fredholmness and several other properties of multiplication operators $M_g : \mathrm{BV} \to \mathrm{BV}$ and $M_g : \mathrm{WBV}_p \to \mathrm{WBV}_p$ were first characterized in, respectively, [8] and [10]. Furthermore, let us add that although we clearly used and adapted some ideas and methods introduced in [8, 10], we also employed several new techniques (especially when multiplication operators in the spaces RBV_p were considered).

The paper is organized as follows. In Sect. 2, we gather basic definitions and facts concerning functions of bounded Young and Riesz variation which will be needed throughout the article. In Sect. 3, we briefly discuss multiplier classes of some BV-type spaces. Sect. 4 is devoted to studying function-theoretic properties (such as injectivity, surjectivity and bijectivity) of multiplication operators in spaces of functions of bounded Young and Riesz variation as well as in some abstract function spaces. Finally, in Sect. 5 we discuss in detail topological properties of such operators. We begin with continuity and compactness; in particular, we provide some estimates on the measure of non-compactness of multiplication operators and explain how similar estimates for the essential norm can be obtained. Furthermore, we describe multiplication operators in the spaces of bounded Young and Riesz variation which have closed range and are Fredholm operators. Finally, we provide also a full characterization of various spectra of those operators.

1.1 Added in the proof

After we had submitted our paper to the journal, we found that Astudillo-Villalba et al. had just published a paper concerning multiplication operators between different spaces of functions of Wiener bounded variation (see [9]). It is worth noting that the results established in [9] are contained in our results on multiplication operators acting in the spaces of functions of Young bounded variation.

2 Preliminaries

The aim of this section is to introduce the notation used in the paper and recall some basic definitions and facts concerning functions of bounded variation. Since we will try to be as brief as possible, we refer readers who are not familiar with various generalizations of the classical Jordan variation to a very nice monograph on that subject [6].

2.1 Notation

Let us begin with some notation and conventions. If $A \subseteq \mathbb{R}$, then by χ_A we will denote the characteristic function of the set A, that is, $\chi_A(t) = 0$ for $t \in \mathbb{R} \setminus A$ and $\chi_A(t) = 1$ for $t \in A$. Moreover, if A is finite, then by $\#A$ we will denote the number of elements of the set A; clearly, $\#\emptyset = 0$. We will also set $\#A = +\infty$ if A is infinite. Let us also recall that a set is called countable if it is equinumerous with some subset of positive integers $\mathbb{N}$. In particular, the empty set is both countable and finite. We will call a closed interval $I \subseteq \mathbb{R}$ degenerate if $I = [a, a] = \{a\}$ for some $a \in \mathbb{R}$. Often in the paper, we will write that X is a linear space of real-valued functions defined on the interval [0, 1]. We will always assume that the linear structure of X is inherited from the field of the real numbers $\mathbb{R}$, that is, for $x, y \in X$ and $\alpha \in \mathbb{R}$ the functions $x + y$ and αx will be given by $t \mapsto x(t) + y(t)$ and $t \mapsto \alpha x(t)$, respectively. In particular, the zero function, i.e., the constant function taking the value 0, will be always the additive identity. Similarly, $(xy)(t) = x(t)y(t)$, although the product xy may not always be an element of X even though x, y will be. As regards division, by x/y we will mean the function $t \mapsto x(t)/y(t)$, provided that it is well-defined. We will also understand that two functions in X are equal if they attain the same values at each point in the interval [0, 1]. As usual by C and B, we will denote the Banach spaces of all, respectively, continuous and bounded real-valued functions defined on the interval [0, 1], endowed with the supremum norm $\|x\|_\infty := \sup_{t\in[0,1]} |x(t)|$. Furthermore, the symbol L_p will stand for the Banach space of all (equivalence classes of) functions which are Lebesgue integrable with p-th power on the interval [0, 1], where $1 \leq p < +\infty$, endowed with the norm

$$\|x\|_{L_p} := \left(\int_0^1 |x(t)|^p \mathrm{d}t \right)^{1/p}.$$

If $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are two normed spaces, we will say that X is *embedded* into Y (and write $X \hookrightarrow Y$) when $X \subseteq Y$ (as sets) and the identity mapping $\mathrm{I} : X \to Y$ is continuous, that is, when there is a constant $c > 0$ such that $\|x\|_Y \leq c\|x\|_X$ for all $x \in X$. We will call the constant c an *embedding constant*. Throughout the paper by $\operatorname{Ker} L$ and $\operatorname{Im} L$ we will denote, respectively, the kernel and range of the linear operator L.

2.2 Support of a function

We follow the notation introduced in [8] and for a function $x : [0, 1] \to \mathbb{R}$ we write $\operatorname{supp}(x) := \{t \in [0, 1] | x(t) \neq 0\}$ for its *support*. Note that in contrast to the standard definition of the support of a function, we do not take the closure here. For $\delta \geq 0$ we also write $\operatorname{supp}_\delta(x) := \{t \in [0, 1] | |x(t)| > \delta\}$. Observe that $\operatorname{supp}_\delta(x)$ decreases with respect to δ for any fixed function $x : [0, 1] \to \mathbb{R}$, that is, $\operatorname{supp}_{\delta_2}(x) \subseteq \operatorname{supp}_{\delta_1}(x)$ if $\delta_2 \geq \delta_1$. Moreover, we clearly have $\operatorname{supp}_0(x) = \operatorname{supp}(x)$ and $\operatorname{supp}(x) = \bigcup_{\delta>0} \operatorname{supp}_\delta(x) = \bigcup_{n\in\mathbb{N}} \operatorname{supp}_{1/n}(x)$. In particular, this implies that if $\operatorname{supp}_\delta(x)$ is countable for each $\delta > 0$, then so is $\operatorname{supp}(x)$ (as a countable union of countable sets). In other words, if $\operatorname{supp}(x)$ is uncountable, then $\operatorname{supp}_\delta(x)$ is uncountable (and hence infinite) for some $\delta > 0$.

In the sequel, we will also need a notion complementary to the concept of the support of a function. For a given function $x : [0, 1] \to \mathbb{R}$ by Z_x, we will denote the set of zeros of x, that is, $Z_x := [0, 1] \setminus \operatorname{supp}(x)$.

2.3 Functions of bounded Young variation

Before we are able to define a function of bounded variation in the sense of Young, we need to recall the definition of a Young function.

Definition 1 A function $\varphi : [0, +\infty) \to [0, +\infty)$ is said to be a *Young function* (or *φ-function*) if it is convex and such that $\varphi(t) = 0$ if and only if $t = 0$.

Remark 2 Note that according to this definition every Young function is continuous and strictly increasing so that $\varphi(t) \to +\infty$ as $t \to +\infty$. Moreover, due to convexity, $\varphi(st) \leq s\varphi(t)$ for all $s \in [0,1], t \in [0,+\infty)$, as well as $\varphi(st) \geq s\varphi(t)$ for all $s \in [1,+\infty), t \in [0,+\infty)$.

With this definition at hand, one can define the variation in the sense of Young, which, as the name suggests, was introduced by Laurence Chisholm Young in 1937 (see [39]).

Definition 3 Let x be a real-valued function defined on $[a, b]$ with $a < b$ and let φ be a given Young function. The (possibly infinite) number

$$\operatorname{var}_{\varphi}(x;[a,b]) = \sup \sum_{i=1}^{n} \varphi\big(|x(t_i) - x(t_{i-1})|\big),$$

where the supremum is taken over all finite partitions $a = t_0 < t_1 < \ldots < t_n = b$ of $[a, b]$, is called the *Young variation* (or *φ-variation*) of the function x over $[a, b]$. (For $[a,b] = [0,1]$ we just write $\operatorname{var}_{\varphi}(x) := \operatorname{var}_{\varphi}(x;[0,1])$.)

By YBV_{φ}, we will denote the class of all functions $x : [0,1] \to \mathbb{R}$ such that λx is of bounded Young variation for some $\lambda > 0$, that is,

$$\mathrm{YBV}_{\varphi} := \{x : [0,1] \to \mathbb{R} \,\big|\, \operatorname{var}_{\varphi}(\lambda x) < +\infty \text{ for some } \lambda > 0\}.$$

It is well-known that YBV_{φ} is a linear space and becomes a Banach space when endowed with the norm $\|x\|_{\mathrm{YBV}_{\varphi}} := \|x\|_{\infty} + |x|_{\mathrm{YBV}_{\varphi}}$, where

$$|x|_{\mathrm{YBV}_{\varphi}} := \inf\{\lambda > 0 \,\big|\, \operatorname{var}_{\varphi}(x/\lambda) \leq 1\}$$

(cf. [32]). Note that the definition of the norm $\|\cdot\|_{\mathrm{YBV}_{\varphi}}$ is meaningful as each function of bounded Young variation is bounded.

Remark 4 The special case when $\varphi_p(t) = t^p$ for $p \geq 1$ will be of particular interest for us. If $p = 1$, the Young variation coincides with the classic variation, which goes back to Camille Jordan (see [26, 27]). In this case, we will simply write BV instead of YBV_{φ_1}, $\operatorname{var}(x)$ instead of $\operatorname{var}_{\varphi_1}(x)$ and $\|\cdot\|_{BV}$ instead of $\|\cdot\|_{\mathrm{YBV}_{\varphi_1}}$. For $p \geq 1$, the Young variation becomes the Wiener variation, which was introduced by Wiener in [38]; we will write WBV_p instead of YBV_{φ_p}, $\operatorname{var}_p(x)$ instead of $\operatorname{var}_{\varphi_p}(x)$ and $\|\cdot\|_{\mathrm{WBV}_p}$ instead of $\|\cdot\|_{\mathrm{YBV}_{\varphi_p}}$. It is easy to check that in those special cases, $|x|_{\mathrm{YBV}_{\varphi_p}} = (\operatorname{var}_p(x))^{1/p}$ for $p \geq 1$; in particular, $|x|_{\mathrm{YBV}_{\varphi_1}} = \operatorname{var}(x)$.

To study multiplication operators acting between different spaces of functions of bounded Young variation, we need to introduce a relation between Young functions. For two Young functions φ and ψ, we will write $\varphi \prec \psi$ if and only if

$$\limsup_{t \to 0^+} \frac{\varphi(\lambda t)}{\psi(t)} < +\infty \quad \text{for some } \lambda > 0$$

(see [13, Section 6]). Equivalently, $\varphi \prec \psi$ if and only if

$$\text{there exist positive constants } \mu, d, T \text{ such that } \varphi(t) \leq d\psi(\mu t) \text{ for all } t \in [0,T] \tag{1}$$

(see [21, Condition 2.2.2 (*)]).

Although the following embedding result is known in the literature (see, for example, [21, Theorem 4.1.1]), the estimates on the embedding constant are often omitted. Therefore, for readers' convenience, we will provide its short proof.

Proposition 5 *Let* $\varphi, \psi : [0, +\infty) \to [0, +\infty)$ *be two Young functions such that* $\varphi \prec \psi$. *Then,* $\mathrm{YBV}_\psi \hookrightarrow \mathrm{YBV}_\varphi$ *and* $\|x\|_{\mathrm{YBV}_\varphi} \leq c\|x\|_{\mathrm{YBV}_\psi}$ *for every* $x \in \mathrm{YBV}_\psi$, *where* $1 \leq c \leq \max\{1 + 2T^{-1}, \mu, d\mu\}$ *and the constants* μ, d, T *appear in* (1).

Proof Fix $x \in \mathrm{YBV}_\psi$. Notice that we may assume that $|x|_{\mathrm{YBV}_\psi} > 0$, since otherwise x is a constant function and $x \in \mathrm{YBV}_\varphi$ with $\|x\|_{\mathrm{YBV}_\varphi} = \|x\|_{\mathrm{YBV}_\psi}$. Set $\lambda := \max\{2T^{-1}\|x\|_\infty, \varepsilon\mu|x|_{\mathrm{YBV}_\psi}, \varepsilon\mu d|x|_{\mathrm{YBV}_\psi}\}$, where ε is an arbitrary number greater than 1. If $0 = t_0 < \ldots < t_n = 1$ is a finite partition of the interval [0, 1], then

$$\begin{aligned}\sum_{i=1}^{n} \varphi\left(\frac{|x(t_i) - x(t_{i-1})|}{\lambda}\right) &\leq d\sum_{i=1}^{n} \psi\left(\frac{\mu|x(t_i) - x(t_{i-1})|}{\lambda}\right)\\ &\leq \sum_{i=1}^{n} \psi\left(\frac{|x(t_i) - x(t_{i-1})|}{\varepsilon|x|_{\mathrm{YBV}_\psi}}\right) \leq \operatorname{var}_\psi\left(\frac{x}{\varepsilon|x|_{\mathrm{YBV}_\psi}}\right) \leq 1;\end{aligned}$$

to obtain the second inequality one has to consider two cases: $d \in (0, 1]$ and $d > 1$. Therefore, $\operatorname{var}_\varphi(x/\lambda) \leq 1$, and so $x \in \mathrm{YBV}_\varphi$. Moreover, in view of the arbitrariness of ε and the continuity of the max function, this also implies that $|x|_{\mathrm{YBV}_\varphi} \leq \max\{2T^{-1}\|x\|_\infty, \mu|x|_{\mathrm{YBV}_\psi}, \mu d|x|_{\mathrm{YBV}_\psi}\}$. Hence, $\|x\|_{\mathrm{YBV}_\varphi} = \|x\|_\infty + |x|_{\mathrm{YBV}_\varphi} \leq \max\{1 + 2T^{-1}, \mu, d\mu\}\|x\|_{\mathrm{YBV}_\psi}$. The proof is complete. □

Remark 6 The estimate for the embedding constant appearing in Proposition 5, in general, may be not optimal. For example, it is easy to check that the condition (1) is satisfied with $\mu = d = T = 1$ for $\varphi(t) = t^2$ and $\psi(t) = t$, and so we get the estimate $1 \leq c \leq 3$. However, it can be proved that $(\operatorname{var}_q(x))^{1/q} \leq (\operatorname{var}_p(x))^{1/p}$ for $x \in \mathrm{WBV}_p$, where $1 \leq p \leq q$ (see, for example, [6, Proposition 1.38], [20, Remark 2.5] or [29, p. 55]), which means that $\mathrm{WBV}_p \hookrightarrow \mathrm{WBV}_q$ with the embedding constant $c = 1$.

Among all Young functions especially important are those which satisfy a certain growth condition; namely, the so-called δ_2-condition. We say that the Young function $\varphi : [0, +\infty) \to [0, +\infty)$ satisfies the δ_2-*condition* and write $\varphi \in \delta_2$ if

$$\limsup_{t\to 0+} \frac{\varphi(2t)}{\varphi(t)} < +\infty.$$

It can be shown that to every Young function φ satisfying the δ_2-condition one can assign the non-decreasing function $\Lambda : (0, +\infty) \to [1, +\infty)$ given by

$$\Lambda(T) := \sup_{0<t\leq T} \frac{\varphi(2t)}{\varphi(t)} \tag{2}$$

(see [6, p. 115]); for simplicity, we also extend the function Λ over the whole nonnegative half-axis putting $\Lambda(0) := 0$. The significance of Young functions satisfying the δ_2

-condition comes from the following result, which was proved by Musielak and Orlicz (cf. [32, Theorems 1.01 and 3.11]).

Proposition 7 *Let* $\varphi : [0, +\infty) \to [0, +\infty)$ *be a Young function. If* $\varphi \in \delta_2$, *then*

$$\mathrm{YBV}_\varphi = \{x : [0,1] \to \mathbb{R} \mid \operatorname{var}_\varphi(x) < +\infty\}.$$

Moreover, for any sequence $(x_n)_{n\in\mathbb{N}}$ *in* YBV_φ *which is bounded in B we have* $\lim_{n\to\infty} \operatorname{var}_\varphi(x_n) = 0$ *if and only if* $\lim_{n\to\infty} |x_n|_{\mathrm{YBV}_\varphi} = 0$.

2.4 Functions of bounded Riesz variation

The second type of variation we are going to deal with in this paper is the Riesz variation. It was introduced in 1910 by Frigyes Riesz (see [35]), and its definition reads as follows. (This time we will deal with functions defined on the interval [0, 1] only.)

Definition 8 Let $1 \le p < +\infty$ and let x be a real-valued function defined on [0, 1]. The (possibly infinite) number

$$\operatorname{var}_p^R(x) = \sup \sum_{i=1}^{n} \frac{|x(t_i) - x(t_{i-1})|^p}{(t_i - t_{i-1})^{p-1}},$$

where the supremum is taken over all finite partitions $0 = t_0 < \ldots < t_n = 1$ of [0, 1], is called the *Riesz variation* of the function x over [0, 1].

By RBV_p, we will denote the space of all functions $x : [0,1] \to \mathbb{R}$ with bounded Riesz variation, that is,

$$\mathrm{RBV}_p := \{x : [0,1] \to \mathbb{R} \mid \operatorname{var}_p^R(x) < +\infty\}.$$

It can be proved that RBV_p is a Banach space when endowed with the norm

$$\|x\|_{\mathrm{RBV}_p} := \|x\|_\infty + (\operatorname{var}_p^R(x))^{1/p}$$

(cf. [6, Proposition 2.51]). As in the case of the Young variation, the definition of the norm $\|\cdot\|_{\mathrm{RBV}_p}$ is meaningful, since it is easy to show that each function $x \in \mathrm{RBV}_p$ is bounded.

In the sequel, we will frequently use the following result characterizing functions of bounded Riesz variation which was first proved in [35] (see also [6, Theorem 3.34]).

Theorem 9 (Riesz) *Let* $1 < p < +\infty$. *Then, a function* $x : [0,1] \to \mathbb{R}$ *belongs to* RBV_p *if and only if it is absolutely continuous and its derivative* x' (*which then exists almost everywhere on* [0, 1]) *is in* L_p. *Moreover, in this case the following equality holds*

$$\operatorname{var}_p^R(x) = \int_0^1 |x'(t)|^p \,\mathrm{d}t. \tag{3}$$

Remark 10 Although for $p = 1$, the Riesz variation reduces to the Jordan variation, the formula (3) is then no longer true, because there are functions of bounded Jordan variation which are not continuous, let alone absolutely continuous (any characteristic function of a proper subset of [0, 1] is such an example). However, it still holds for the class of all absolutely continuous functions which is a subclass of BV; of course, then $\mathrm{var}\,_1^R(x)$ reduces to $\mathrm{var}\,(x)$—cf. Remark 4.

Using the Riesz theorem we can also prove some embedding results for spaces of functions of bounded Riesz variation.

Proposition 11 *If* $1 \le q \le p < +\infty$, *then* $\mathrm{RBV}_p \hookrightarrow \mathrm{RBV}_q$ *with the embedding constant* $c = 1$.

Proof If $p = q$, then the claim is trivial. So, we may assume that $1 \le q < p$. The proof follows from Theorem 9, Remark 10 and the well-known estimates between L_p-norms

$$\left(\int_0^1 |y(t)|^q \,\mathrm{d}t\right)^{1/q} \le \left(\int_0^1 |y(t)|^p \,\mathrm{d}t\right)^{1/p}$$

holding for any $y \in L_p$ (see [24, Theorem 13.17]). □

2.5 Banach algebras

A natural habitat for multiplication operators is algebras. Let us recall that an algebra X is called a *normed algebra* if it is a normed space with a norm $\|\cdot\|$ satisfying the estimate of the form $\|xy\| \le \|x\|\|y\|$ for all $x, y \in X$. If, additionally, the norm $\|\cdot\|$ is complete, then X is called a *Banach algebra*. Some authors assume also that a Banach algebra X must contain a unit e, that is, an element of norm 1 such that $xe = ex = x$ for all $x \in X$ (cf., for example, [36, Part III]). It turns out that the spaces[2] YBV_φ and RBV_p are Banach algebras when endowed with the norms $\|\cdot\|_{\mathrm{YBV}_\varphi}$ and $\|\cdot\|_{\mathrm{RBV}_p}$, respectively (as a unit we take the constant function $e : [0, 1] \to \mathbb{R}$ given by $e(t) = 1$ for $t \in [0, 1]$). This result was first proved by Maligranda and Orlicz (see [29, Theorems 2 and 3]). The main ingredient in the proof are the estimates of the form

$$|xy|_{\mathrm{YBV}_\varphi} \le \|x\|_\infty |y|_{\mathrm{YBV}_\varphi} + \|y\|_\infty |x|_{\mathrm{YBV}_\varphi}$$

and

$$\left(\mathrm{var}\,_p^R(xy)\right)^{1/p} \le \|x\|_\infty \left(\mathrm{var}\,_p^R(y)\right)^{1/p} + \|y\|_\infty \left(\mathrm{var}\,_p^R(x)\right)^{1/p},$$

which hold for any functions x, y belonging to an appropriate space. It is worth noting here that the formula (2.91) in [6], which reads as follows

$$\mathrm{var}\,_p^R(xy) \le \|x\|_\infty \,\mathrm{var}\,_p^R(y) + \|y\|_\infty \,\mathrm{var}\,_p^R(x), \tag{4}$$

is in general incorrect. To see this let us take a look at the following example.

[2] For completeness, let us add that if we do not specify explicitly any constraints on the parameter p or the Young function φ, it means that the statement is true for any such parameter or φ-function.

Example 12 Let $p \geq \frac{3}{2}$ and consider the functions $x, y : [0,1] \to \mathbb{R}$ given by the formula $x(t) = y(t) = e^t$ for $t \in [0,1]$. Then, using (3), it is easy to see that $\operatorname{var}_p^R(x) = \operatorname{var}_p^R(y) = p^{-1}(e^p - 1)$ and $\operatorname{var}_p^R(xy) = 2^{p-1}p^{-1}(e^p + 1)(e^p - 1)$. And so

$$\begin{aligned}\operatorname{var}_p^R(xy) &= 2^{p-1}p^{-1}(e^p+1)(e^p-1) \geq 2^{1/2}p^{-1}(e^{3/2}+1)(e^p-1)\\ &> 2p^{-1}e(e^p-1) = ep^{-1}(e^p-1) + ep^{-1}(e^p-1) = \|x\|_\infty \operatorname{var}_p^R(y) + \|y\|_\infty \operatorname{var}_p^R(x).\end{aligned}$$

This shows that the formula (4) does not hold for all $x, y \in \mathrm{RBV}_p$.

3 Multiplier spaces

The main objects of our study are multiplication operators and their properties. If X and Y are linear spaces of real-valued functions defined on the interval [0, 1], and $g : [0,1] \to \mathbb{R}$ is a given function, then the *multiplication operator* $M_g : X \to Y$, generated by the function g, is given by $M_g(x)(t) := g(t)x(t)$ for $t \in [0,1]$. In order to guarantee that M_g is well-defined, we have to make sure (by imposing appropriate conditions on g) that $M_g(X) \subseteq Y$, that is, the product gx must belong to Y, whenever x belongs to X. If we write

$$Y/X := \{g : [0,1] \to \mathbb{R} \,\big|\, gx \in Y \text{ for all } x \in X\},$$

then $M_g(X) \subseteq Y$ if and only if $g \in Y/X$. The set *Y*/*X* is often called the *multiplier class of Y over X* (or simply, the *multiplier class*, when the starting and target spaces are known). Notice that *Y*/*X* is a non-empty set, as it always contains the zero function.

In the recent paper [13] multiplier classes of various BV-type spaces were characterized. Let us briefly recall a few of those results:

$$\mathrm{YBV}_\varphi/\mathrm{YBV}_\psi = \begin{cases} \mathrm{YBV}_\varphi & \text{for } \varphi \prec \psi,\\ \mathrm{YBV}_\varphi \cap S_c & \text{for } \varphi \not\prec \psi,\end{cases}$$

$$\mathrm{WBV}_q/\mathrm{WBV}_p = \begin{cases} \mathrm{WBV}_q & \text{for } 1 \leq p \leq q < +\infty,\\ \mathrm{WBV}_q \cap S_c & \text{for } 1 \leq q < p < +\infty,\end{cases}$$

$$\mathrm{RBV}_q/\mathrm{RBV}_p = \begin{cases} \mathrm{RBV}_q & \text{for } 1 \leq q \leq p < +\infty,\\ \{0\} & \text{for } 1 \leq p < q < +\infty,\end{cases}$$

where S_c denotes the set of real-valued functions which are zero everywhere on [0, 1] except at countably many points. For completeness, let us also add that the proofs of the above-mentioned characterizations of multiplier classes presented in [13] contain some minor flaws, which can be fixed with almost no effort.

4 Function-theoretic properties

We are going to start our investigations by giving general criteria for injectivity, surjectivity and bijectivity of the multiplication operator $M_g : X \to Y$.

Since M_g is a linear operator, we immediately obtain a criterion for injectivity.

Proposition 13 *Let X, Y be two linear spaces of real-valued functions defined on the interval* $[0, 1]$ *and let* $M_g : X \to Y$ *be the multiplication operator generated by* $g \in Y/X$. *The operator* M_g *is injective if and only if for each* $x \in X \setminus \{0\}$ *there is some* $t \in \operatorname{supp}(g)$ *such that* $x(t) \neq 0$. *In particular, if* $\operatorname{supp}(g) = [0, 1]$, *then* M_g *is injective.*

The above criterion is too broad to be useful, but it shows that the injectivity of M_g in general does not only depend on g but also on X. In some cases, namely if the space X is sufficiently "large," it turns out that the dependency on X is redundant. To make our considerations as general as possible, let us state the following somewhat technical definition.

Definition 14 We say that a linear space X of real-valued functions defined on $[0, 1]$

- *separates points* if for each $t \in [0, 1]$ there is some $x \in X$ such that $x(t) \neq 0$,
- *strongly separates points* if X contains all characteristic functions of singletons,
- *uniformly separates points* if $X \subseteq C$ and if for each $t \in [0, 1]$ and each $\delta > 0$ there is some $x \in X$ such that $t \in \operatorname{supp}(x) \subseteq [t-\delta, t+\delta]$.

Remark 15 Note that each space which separates points uniformly or strongly also separates points. Other relations, however, do not hold. For instance, the spaces C, B, BV, WBV_p, YBV_φ and RBV_p separate points. On the other hand, the spaces B, BV, WBV_p and YBV_φ separate points strongly, but not uniformly, whereas the spaces C and RBV_p (with $1 < p < +\infty$) separate points uniformly, but not strongly. Finally, the space of constant functions defined on the interval $[0, 1]$ only separates points, but neither strongly nor uniformly. One of the reasons that this space cannot separate points either strongly or uniformly is that it is one-dimensional, and spaces which strongly/uniformly separate points are necessarily infinite-dimensional.

Now, we are ready to prove the injectivity criterion for multiplication operators, which associates the injectivity of a given operator M_g with the number of zeros of its generator g.

Theorem 16 *Let X, Y be two linear spaces of real-valued functions defined on the interval* $[0, 1]$ *and let* $M_g : X \to Y$ *be the multiplication operator generated by* $g \in Y/X$.

(a) *If X strongly separates points (especially, if X is one of the spaces BV,* WBV_p *or* YBV_φ*), then* $\dim \operatorname{Ker} M_g = \#Z_g$. *In particular,* M_g *is injective if and only if* $\operatorname{supp}(g) = [0, 1]$.
(b) *If X uniformly separates points (especially, if X is the space* RBV_p *for some* $1 < p < +\infty$*), then* $\dim \operatorname{Ker} M_g = \#\Big([0, 1] \setminus \overline{\operatorname{supp}(g)}\Big)$. *In particular,* M_g *is injective if and only if* $\overline{\operatorname{supp}(g)} = [0, 1]$.

Remark 17 Before we proceed to the proof of Theorem 16 let us notice that, since $\overline{\operatorname{supp}(g)}$ is a closed subset of $[0, 1]$, there are in fact only two possibilities in (b): either $\dim \operatorname{Ker} M_g = 0$, or $\dim \operatorname{Ker} M_g = +\infty$, depending on whether $\overline{\operatorname{supp}(g)} = [0, 1]$ or not.

Proof of Theorem 16 We begin with the proof of (a). First, we show that $\dim \operatorname{Ker} M_g \geq \#Z_g$. Clearly, we may assume that $Z_g \neq \emptyset$. Fix any $n \in \mathbb{N}$ such that $n \leq \#Z_g$. Then, there exist n (distinct) points $t_1, \ldots, t_n \in [0, 1]$ such that $g(t_i) = 0$ for $i = 1, \ldots, n$. So,

$x_i := \chi_{\{t_i\}} \in \operatorname{Ker} M_g$. As the set $\{x_1, \ldots, x_n\}$ is linearly independent, this means that $\dim \operatorname{Ker} M_g \geq n$. Since this is true for any number $n \leq \# Z_g$, we get $\dim \operatorname{Ker} M_g \geq \# Z_g$.

Now, we will prove the opposite inequality. This time we may assume that $\# Z_g < +\infty$. If $\# Z_g = 0$, that is $\operatorname{supp}(g) = [0,1]$, then clearly $\dim \operatorname{Ker} M_g = 0$. Suppose now that $\# Z_g = n$ for some $n \in \mathbb{N}$. Then, we can write $Z_g = \{t_1, \ldots, t_n\}$ for some distinct points $t_1, \ldots, t_n \in [0,1]$. If $x \in \operatorname{Ker} M_g$, then for $t \notin \{t_1, \ldots, t_n\}$ we have

$$x(t) = \frac{1}{g(t)} \cdot g(t)x(t) = \frac{1}{g(t)} \cdot 0 = 0.$$

This implies that $x = \sum_{i=1}^{n} x(t_i)\chi_{\{t_i\}}$, that is, $\operatorname{Ker} M_g \subseteq \operatorname{lin}\{\chi_{\{t_1\}}, \ldots, \chi_{\{t_n\}}\} \subseteq X$; here and throughout the paper by $\operatorname{lin} A$ we denote the *linear span* (or *hull*) of the set A. So, $\dim \operatorname{Ker} M_g \leq n = \# Z_g$. This ends the first part of the proof.

The proof of (b) is slightly different from the above one. Suppose that $\#[0,1] \setminus \overline{\operatorname{supp}(g)} = 0$, that is, $\operatorname{supp}(g)$ is dense in $[0,1]$. Let $x \in \operatorname{Ker} M_g$ and fix $t \in [0,1]$. If $g(t) \neq 0$, then $x(t) = 0$. On the other hand, if $g(t) = 0$, then since $\operatorname{supp}(g)$ is dense in $[0,1]$, there is a sequence $(t_n)_{n \in \mathbb{N}}$ of elements of the support of g which converges to t. Then, clearly, $x(t_n) = 0$ for each $n \in \mathbb{N}$. But the function x is continuous by definition, and so $x(t) = 0$. Consequently, $x = 0$, which means that $\dim \operatorname{Ker} M_g = 0$.

Now, let us assume that $\overline{\operatorname{supp}(g)}$ is a proper subset of $[0,1]$, and let us fix $n \in \mathbb{N}$. Since $\overline{\operatorname{supp}(g)}$ is closed in $[0,1]$, we can find n points $t_1, \ldots, t_n$ together with open and pairwise disjoint subsets $U_1, \ldots, U_n$ of $[0,1]$ such that $t_i \in U_i$ and $U_i \cap \operatorname{supp}(g) = \emptyset$ for $i = 1, \ldots, n$. As X uniformly separates points, this means that there exist n continuous functions $x_1, \ldots, x_n$ with the property that $x_i(t_i) = 1$ and $\operatorname{supp}(x_i) \subseteq U_i$ for $i = 1, \ldots, n$. It is clear that $\{x_1, \ldots, x_n\}$ is a linearly independent subset of $\operatorname{Ker} M_g$. Therefore, $\dim \operatorname{Ker} M_g \geq n$. As this is true for any positive integer n, we get $\dim \operatorname{Ker} M_g = +\infty$. In view of Remark 17, this ends the proof. □

Note that there is no simple analogue of the above result in the case when X is assumed to separate points, but neither strongly nor uniformly.

Example 18 To see this it suffices to take X to be the space of all constant real-valued functions, $Y = C$ and define the function $g : [0,1] \to \mathbb{R}$ by the formula $g(t) = \max\left\{t - \frac{1}{2}, 0\right\}$ for $t \in [0,1]$, as then $\dim \operatorname{Ker} M_g = 0$, but $\# Z_g = \#\Big([0,1] \setminus \overline{\operatorname{supp}(g)}\Big) = +\infty$.

The following corollary for the spaces of our interest follows immediately from Theorem 16.

Corollary 19

(a) *Let X be any of the spaces BV,* WBV_p *or* YBV_φ *and let $g \in X$. Then, $M_g : X \to X$ is injective if and only if* $\operatorname{supp}(g) = [0,1]$.

(b) *Let X be the space* RBV_p *for* $1 < p < +\infty$ *and let $g \in X$. Then, $M_g : X \to X$ is injective if and only if* $\overline{\operatorname{supp}(g)} = [0,1]$.

Our next task is to characterize surjectivity of multiplication operators. We begin with some abstract results. It should not come as a surprise that this time we will need to assume some additional conditions on the target space.

Theorem 20 *Let X, Y be two linear spaces of real-valued functions defined on the interval* $[0, 1]$ *with Y separating points. Moreover, let $M_g : X \to Y$ be the multiplication operator generated by $g \in Y/X$. Then, M_g is surjective if and only if* $\operatorname{supp}(g) = [0,1]$ *and* $1/g \in X/Y$. *In particular, if M_g is surjective, then it is also injective.*

Proof Assume first that M_g is a surjective operator. Clearly, $\operatorname{supp}(g) \subseteq [0,1]$. Now, fix $t \in [0,1]$. Since Y separates points, there is some $y \in Y$ such that $y(t) \neq 0$. As M_g is surjective, we can find some $x \in X$ such that $M_g(x) = y$. In particular, $g(t) \neq 0$, and consequently $g(t) \neq 0$ for all $t \in [0,1]$ as t was arbitrary. This shows that $\operatorname{supp}(g) = [0,1]$. In particular, the function $1/g$ is well-defined. To prove the second condition, note that for any $y \in Y$ we can again find $x \in X$ such that $gx = y$. Hence, $y/g = x \in X$. This proves that $1/g \in X/Y$.

For the converse assume that $\operatorname{supp}(g) = [0,1]$ and $1/g \in X/Y$. For $y \in Y$ the function $x := y/g$ belongs to X and satisfies $gx = y$, i.e., M_g is a surjection.

The fact that surjectivity of M_g implies its injectivity is a consequence of the first part of the proof and Proposition 13. □

It turns out that if $Y = X$, then surjectivity of the multiplication operator always implies its injectivity; in other words, we do not need the additional assumption that Y (which in this case coincides with X) separates points. Indeed, let X be any space of real-valued functions defined on [0, 1] and let $M_g : X \to X$ be a surjective operator generated by a function $g \in X/X$. Further, suppose on the contrary that there is some $x \in X$ with $M_g(x) = 0$, but $x \not\equiv 0$. That means there is some $t \in [0,1]$ such that $x(t) \neq 0$. Due to the fact that $gx \equiv 0$, we get $g(t) = 0$. But since M_g is assumed to be surjective, we must find some $z \in X$ such that $M_g(z) = x$; in particular, $0 = g(t)z(t) = x(t) \neq 0$, a contradiction.

In the general case, however, that is, when X and Y do not necessarily coincide, if we do not assume that Y separates points, it is easy to give an example of a multiplication operator $M_g : X \to Y$ which is surjective but not injective.

Example 21 For instance, take $X = C$, $Y = \{c\chi_{\{0\}} : [0,1] \to \mathbb{R} | c \in \mathbb{R}\}$ and $g : [0,1] \to \mathbb{R}$ given by $g = \chi_{\{0\}}$. Then, Y cannot separate points as for any $t \in (0,1]$ and any $y \in Y$ we have $y(t) = 0$. Moreover, M_g cannot be injective, because for the two constant functions $x_1 \equiv a$ and $x_2 \equiv b$ with $a \neq b$ we have $x_1, x_2 \in C$, but $M_g(x_1) = a\chi_{\{0\}} \neq b\chi_{\{0\}} = M_g(x_2)$. Finally, if $y \in Y$ is given, then $y = y(0)\chi_{\{0\}}$. The constant function $x \equiv y(0)$ belongs to C and satisfies $M_g(x) = x(0)\chi_{\{0\}} = y(0)\chi_{\{0\}} = y$ showing that M_g is indeed surjective.

From Theorem 20 we obtain two corollaries for the BV-type spaces we are interested in.

Corollary 22 *Let X be a linear space of real-valued functions defined on the interval* [0, 1] *such that $Y \subseteq X \subseteq B$, where Y is one of the spaces BV,* WBV_p, YBV_φ *or* RBV_p. *Moreover, let $M_g : X \to Y$ be the multiplication operator generated by $g \in Y/X$. Then, M_g is surjective if and only if* $\inf_{t\in[0,1]} |g(t)| > 0$.

Proof Let us assume that M_g is surjective. Then, from Theorem 20 we obtain that $\operatorname{supp}(g) = [0,1]$ and $1/g \in X/Y$. Note that $y \equiv 1$ belongs to all of the considered spaces BV, WBV_p, YBV_φ, RBV_p, and so we obtain $X/Y \subseteq X \subseteq B$. Thus, $1/g$ is bounded, and this is possible only if $\inf_{t\in[0,1]} |g(t)| > 0$.

Conversely, assume that $\inf_{t\in[0,1]} |g(t)| > 0$. Then, $\operatorname{supp}(g) = [0,1]$. It is also easy to see that $1/g \in Y$. Since Y is closed under multiplication, $Y \subseteq Y/Y \subseteq X/Y$, and hence $1/g \in X/Y$. Again from Theorem 20, we obtain that M_g is surjective. □

Corollary 23 *Let X be one of the spaces BV, WBV_p, YBV_φ or RBV_p and let $M_g : X \to X$ be the multiplication operator generated by $g \in X$. Then, the following conditions are equivalent:*

(a) $\inf_{t\in[0,1]} |g(t)| > 0$,
(b) *M_g is bijective with $M_g^{-1} = M_{1/g}$,*
(c) *M_g is surjective,*
(d) *$\operatorname{Im} M_g$ is dense in X.*

Proof Only the implication (d) ⇒ (a) requires a proof. Suppose that the range of M_g is dense in X but $\inf_{t\in[0,1]} |g(t)| = 0$. Then, it is possible to find a sequence $(t_n)_{n\in\mathbb{N}}$ in $[0, 1]$ such that $|g(t_n)| \to 0$ as $n \to +\infty$. Since $\operatorname{Im} M_g$ is dense in X, for $e \equiv 1$ there exists a function $x \in X$ such that $\frac{1}{2} \geq \|M_g(x) - e\|_X \geq \|M_g(x) - e\|_\infty \geq |g(t_n)x(t_n) - 1|$ for each $n \in \mathbb{N}$. As functions in X are bounded, passing with $n \to +\infty$ in the above inequality yields $\frac{1}{2} \geq 1$, which clearly is impossible. Hence, $\inf_{t\in[0,1]} |g(t)| > 0$. □

Let us also take a look at the following qualitative version of Theorem 20.

Theorem 24 *Let X, Y be two linear spaces of real-valued functions defined on the interval $[0, 1]$. Assume that X separates points strongly or uniformly (in particular, X is one of the spaces BV, WBV_p, YBV_φ and RBV_p). Moreover, let $M_g : X \to Y$ be the multiplication operator generated by $g \in Y/X$. Then, $\dim \operatorname{Im} M_g = \#\operatorname{supp}(g)$.*

Proof First, we show the inequality $\dim \operatorname{Im} M_g \geq \#\operatorname{supp}(g)$ which is obviously true for $g \equiv 0$. Thus, we assume that $g \not\equiv 0$, which implies that $\#\operatorname{supp}(g) \geq 1$. Fix $n \in \mathbb{N}$ with $\#\operatorname{supp}(g) \geq n$. Then, we can find n distinct numbers $t_1, \ldots, t_n \in [0,1]$ such that $g(t_j) \neq 0$ for $1 \leq j \leq n$. If X separates points strongly, let $y_j := g\chi_{\{t_j\}}$. On the other hand, if X separates points uniformly, then we can find n continuous functions $x_1, \ldots, x_n \in X$ such that $x_j(t_j) = 1$ for $j \in \{1, \ldots, n\}$ and $\operatorname{supp}(x_i) \cap \operatorname{supp}(x_j) = \emptyset$ for any distinct indices $i, j \in \{1, \ldots, n\}$; and we set $y_j = gx_j$. Now, for $j = 1, \ldots, n$, let $\lambda_j \in \mathbb{R}$ be so that $\sum_{j=1}^n \lambda_j y_j \equiv 0$. By evaluating this equation at each $t = t_k$, we get that $0 = \sum_{j=1}^n \lambda_j y_j(t_k) = \lambda_k y_k(t_k) = \lambda_k g(t_k)$, where $1 \leq k \leq n$. This implies that $\lambda_k = 0$ for $1 \leq k \leq n$. Thus, $\{y_1, \ldots, y_n\}$ is a linearly independent subset of $\operatorname{Im} M_g$; in particular, $\dim \operatorname{Im} M_g \geq n$. Since this is true for each n such that $\#\operatorname{supp}(g) \geq n$, we obtain $\dim \operatorname{Im} M_g \geq \#\operatorname{supp}(g)$.

In order to show the remaining inequality $\dim \operatorname{Im} M_g \leq \#\operatorname{supp}(g)$, we may assume that $\#\operatorname{supp}(g) < +\infty$, because otherwise this inequality is clearly true. Moreover, if

$\mathrm{supp}\,(g) = \emptyset$, then $\dim\,\mathrm{Im}\,M_g = 0$. Hence, we may assume that $n = \#\,\mathrm{supp}\,(g)$ for some positive integer n and write $\mathrm{supp}\,(g) = \{t_1, \dots, t_n\}$. Since X separates points strongly/uniformly, we can find n functions $x_1, \dots, x_n$ in X such that $x_j(t_j) = 1$ for $j = 1, \dots, n$ and $\mathrm{supp}\,(x_i) \cap \mathrm{supp}\,(x_j) = \emptyset$ for $i \neq j$. Define $y_j := gx_j$. Let $y \in \mathrm{Im}\,M_g$. Then, there is some $x \in X$ such that $y = gx$. Moreover, $y = gx = \sum_{j=1}^{n} x(t_j)y_j$, which shows that the linear span of $\{y_1, \dots, y_n\}$ contains $\mathrm{Im}\,M_g$; in particular, $\dim\,\mathrm{Im}\,M_g \leq n = \#\,\mathrm{supp}\,(g)$. This completes the proof. □

Remark 25 Note that we cannot drop the phrase "strongly or uniformly" in Theorem 24. For instance, let X be the space of constant functions and let $Y = C$. Consider $M_g : X \to Y$, generated by $g(t) = t$. Then, $\mathrm{supp}\,(g) = (0, 1]$ is even uncountable, but $\mathrm{Im}\,M_g = \{y : [0, 1] \to \mathbb{R} \,|\, y(t) = at \text{ for some } a \in \mathbb{R}\}$ is a one-dimensional subspace of C.

5 Topological properties

We now turn to analytic properties of the multiplication operators $M_g : X \to Y$. Here, we are particularly interested in continuity and compactness for X and Y being one of the spaces BV, WBV_p, YBV_φ or RBV_p.

5.1 Continuity

Recall that for a linear operator $L : X \to Y$ between two normed spaces $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ the operator norm is defined by $\|L\|_{X\to Y} = \sup_{\|x\|_X \leq 1} \|L(x)\|_Y$, and L is *bounded* (*continuous*) if and only if $\|L\|_{X\to Y}$ is finite.

Although multiplication operators are one of the simplest operators one can imagine, they are not always bounded. In particular, in [8] the authors remarked (see page 106) that multiplication operators in Köthe spaces are (well-defined) and continuous if and only if they are generated by an essentially bounded function (we refer to the paper [22] for more information on the boundedness of multiplication operators in such spaces; see also [15]). For readers' convenience we provide yet another example of a discontinuous multiplication operator acting from a linear subspace of C^1 (which is not a Köthe space); here by C^1 we denote the space of all real-valued continuously differentiable functions defined on the interval [0, 1].

Example 26 Consider the space $C_0^1 := \{x \in C^1 \,|\, x(0) = 0\}$ equipped with the norm $\|\cdot\|_\infty$, and the function $g : [0, 1] \to \mathbb{R}$ defined by

$$g(t) = \begin{cases} t^{-1} & \text{for } 0 < t \leq 1, \\ 0 & \text{for } t = 0. \end{cases}$$

Then, $g \in L_1/C_0^1$, because for $x \in C_0^1$ the function gx is almost everywhere equal to the continuous (and thus Lebesgue integrable) function $y : [0, 1] \to \mathbb{R}$ given by

$$y(t) = \begin{cases} t^{-1}x(t) & \text{for } 0 < t \leq 1, \\ x'(0) & \text{for } t = 0. \end{cases}$$

Consequently, the operator $M_g : C_0^1 \to L_1$ is well-defined. However, M_g is not bounded. To see this let us consider the functions $x_n : [0, 1] \to \mathbb{R}$, where $n \in \mathbb{N}$, defined by

$$x_n(t) = \begin{cases} 2nt - n^2t^2 & \text{for } 0 \le t \le 1/n, \\ 1 & \text{for } 1/n < t \le 1. \end{cases}$$

It is easy to check that $x_n \in C_0^1$ and $\|x_n\|_\infty = 1$ for every $n \in \mathbb{N}$. But

$$\|M_g(x_n)\|_{L_1} = \int_0^1 |g(t)x_n(t)|\,\mathrm{d}t \ge \int_{1/n}^1 t^{-1}\,\mathrm{d}t = \ln n,$$

which means that the sequence $\big(M_g(x_n)\big)_{n\in\mathbb{N}}$ is unbounded in L_1. Thus, M_g cannot be continuous.

On a more positive note, we have the following simple, but quite general, result, which we will use to prove some norm estimates for the multiplication operator acting in the BV spaces.

Proposition 27 *Let $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ be two normed spaces of real-valued functions defined on $[0, 1]$ and assume that the constant function $e \equiv 1$ is contained in X with $\|e\|_X = 1$. Moreover, let $M_g : X \to Y$ be the multiplication operator generated by a function $g \in Y/X$. Then, the following statements hold.*

(a) *The operator norm of M_g is bounded from below by $\|g\|_Y$, that is, $\|g\|_Y \le \|M_g\|_{X\to Y}$.*
(b) *If $X \hookrightarrow Y$ with the embedding constant $c > 0$ and if Y is a normed algebra in the norm $\|\cdot\|_Y$, then the operator norm of M_g is bounded from above by $c\|g\|_Y$, that is, $\|M_g\|_{X\to Y} \le c\|g\|_Y$.*

Proof Note that since $e \in X$, we have $g = M_g(e) \in Y$, which ensures that the quantity $\|g\|_Y$ appearing in (a) and (b) makes sense. The proof of (a) is obvious, because $\|g\|_Y = \|M_g(e)\|_Y = \|M_g(e)\|_Y/\|e\|_X \le \|M_g\|_{X\to Y}$.

Now, let us move to (b). If $X \hookrightarrow Y$ with the embedding constant $c > 0$ (i.e., $\|x\|_Y \le c\|x\|_X$ for all $x \in X$) and if Y is a normed algebra, then $\|M_g(x)\|_Y = \|gx\|_Y \le \|g\|_Y\|x\|_Y \le c\|g\|_Y\|x\|_X$ for $x \in X$. And so $\|M_g\|_{X\to Y} \le c\|g\|_Y$. This shows (b) and completes the proof.

Remark 28 Observe that Example 26 does not contradict Proposition 27, because $e \equiv 1$ is not contained in C_0^1.

All the BV spaces considered in this paper are Banach algebras and contain the constant function $e \equiv 1$, which has norm 1. Therefore, in the special case when X and Y coincide and are one of our BV spaces, we have the following corollary.

Corollary 29 *Let X be one of the spaces BV, WBV_p, YBV_φ or RBV_p and let $M_g : X \to X$ be the multiplication operator generated by a function $g \in X$. Then, the operator M_g is bounded and*

$$\|M_g\|_{X\to X} = \|g\|_X.$$

Now, we would like to study the continuity of the multiplication operator acting between different spaces of functions of bounded variation. We start with the Young variation and a technical lemma.

Lemma 30 *Assume that* $\varphi : [0,+\infty) \to [0,+\infty)$ *is a given Young function. Moreover, let* $x : [0,1] \to \mathbb{R}$ *be a real-valued function with countable support and let* $\lambda > 0$. *Then,*

$$\sum_{t\in \operatorname{supp}(x)} \varphi(\lambda|x(t)|) \le \operatorname{var}_{\varphi}(\lambda x) \le \sum_{t\in \operatorname{supp}(x)} \varphi(2\lambda|x(t)|).$$

Before we provide a proof of Lemma 30 one remark is in order: throughout the paper, we adopt a useful convention that summing over an empty set always gives zero.

Proof of Lemma 30 Of course, we may assume that $\operatorname{supp}(x) \neq \emptyset$, because otherwise there is nothing to prove. First, we will show the left inequality. Let $0 \le t_1 < \ldots < t_n \le 1$ be arbitrary n points in the support of x. Moreover, if $\#\operatorname{supp}(g) = 1$, take any $s_1 \notin \operatorname{supp}(g)$. Similarly, if $\#\operatorname{supp}(g) \ge 2$, take any n points $s_1, \ldots, s_n$ not belonging to $\operatorname{supp}(x)$ such that $s_i \in (t_i, t_{i+1})$ for $i = 1, \ldots, n-1$ and $s_n \in (s_{n-1}, t_n)$; this is clearly possible as the support of x is countable. Then, $\sum_{i=1}^{n} \varphi\big(\lambda|x(t_i)|\big) = \sum_{i=1}^{n} \varphi\big(\lambda|x(t_i) - x(s_i)|\big) \le \operatorname{var}_{\varphi}(\lambda x)$. Taking the supremum over all possible finite subsets $\{t_1, \ldots, t_n\}$ of $\operatorname{supp}(x)$, we get $\sum_{t\in \operatorname{supp}(x)} \varphi(\lambda|x(t)|) \le \operatorname{var}_{\varphi}(\lambda x)$.

Now, let us pass to the proof of the second inequality. If $0 = \tau_0 < \ldots < \tau_n = 1$ is an arbitrary finite partition of the interval [0, 1], then, by the convexity and monotonicity of φ, we have

$$\begin{aligned}\sum_{i=1}^{n} \varphi\big(\lambda|x(\tau_i) - x(\tau_{i-1})|\big) &\le \sum_{i=1}^{n} \varphi\Big(\tfrac{1}{2}\cdot 2\lambda|x(\tau_i)| + \tfrac{1}{2}\cdot 2\lambda|x(\tau_{i-1})|\Big)\\ &\le \sum_{i=0}^{n} \varphi\big(2\lambda|x(\tau_i)|\big) \le \sum_{t\in \operatorname{supp}(x)} \varphi(2\lambda|x(t)|).\end{aligned}$$

Hence, $\operatorname{var}_{\varphi}(\lambda x) \le \sum_{t\in \operatorname{supp}(x)} \varphi(2\lambda|x(t)|)$. The proof is complete. □

With the above lemma at hand, we are ready to prove two results on the continuity of multiplication operators in YBV_{φ} spaces.

Theorem 31 *Let* $\varphi, \psi : [0,+\infty) \to [0,+\infty)$ *be two Young functions. The multiplication operator* $M_g : \mathrm{YBV}_{\psi} \to \mathrm{YBV}_{\varphi}$ *generated by a function* $g \in \mathrm{YBV}_{\varphi}/\mathrm{YBV}_{\psi}$ *is bounded. Moreover, there exists a constant* $c \ge 1$ *(depending only on the functions* φ, ψ*) such that*

$$\|g\|_{\mathrm{YBV}_{\varphi}} \le \|M_g\|_{\mathrm{YBV}_{\psi}\to\mathrm{YBV}_{\varphi}} \le c\|g\|_{\mathrm{YBV}_{\varphi}}.$$

Proof The proof in the case $\varphi \prec \psi$ is a direct consequence of Propositions 5 and 27 and the fact that the spaces of functions of bounded Young variation are Banach algebras (in their respective norms) and contain the constant function $e \equiv 1$.

So now, let $\varphi \not\prec \psi$. Then, $\mathrm{YBV}_{\varphi}/\mathrm{YBV}_{\psi} = \mathrm{YBV}_{\varphi} \cap S_c$. Clearly, we may assume that $\operatorname{supp}(g) \neq \emptyset$; then, in particular, $|g|_{\mathrm{YBV}_{\varphi}} > 0$. Fix a nonzero function x in YBV_{ψ} and let $\lambda := 2\varepsilon\|x\|_{\infty}|g|_{\mathrm{YBV}_{\varphi}}$, where $\varepsilon > 1$. Applying Lemma 30, we get

$$\operatorname{var}_{\varphi}\Big(\frac{gx}{\lambda}\Big) \le \sum_{t\in \operatorname{supp}(g)} \varphi\Big(\frac{2|g(t)||x(t)|}{\lambda}\Big) = \sum_{t\in \operatorname{supp}(g)} \varphi\bigg(\frac{|g(t)||x(t)|}{\varepsilon\|x\|_\infty |g|_{\mathrm{YBV}_\varphi}}\bigg)$$
$$\le \sum_{t\in \operatorname{supp}(g)} \varphi\bigg(\frac{|g(t)|}{\varepsilon |g|_{\mathrm{YBV}_\varphi}}\bigg) \le \operatorname{var}_{\varphi}\bigg(\frac{g}{\varepsilon |g|_{\mathrm{YBV}_\varphi}}\bigg) \le 1.$$

Thus, $|M_g(x)|_{\mathrm{YBV}_\varphi} \le \lambda$. In view of the arbitrariness of ε, we obtain $\|M_g(x)\|_{\mathrm{YBV}_\varphi} \le \|x\|_\infty \|g\|_\infty + 2\|x\|_\infty |g|_{\mathrm{YBV}_\varphi} \le 2\|x\|_{\mathrm{YBV}_\psi} \|g\|_{\mathrm{YBV}_\varphi}$, and so $\|M_g\|_{\mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi} \le 2\|g\|_{\mathrm{YBV}_\varphi}$. The other inequality follows from Proposition 27 (a). □

Observe that from the proof it follows that for the constant c in Theorem 31 we have: $c \le 2$ if $\varphi \not\prec \psi$ and $c \le \max\{1 + 2T^{-1}, \mu, d\mu\}$ when $\varphi \prec \psi$, where the constants μ, d, T appear in (1). In some cases, however, it is possible to give optimal estimates on the constant c. Let us look at one such situation.

Proposition 32 *Let* $1 \le p \le q < +\infty$. *Then, the multiplication operator* $M_g : \mathrm{WBV}_p \to \mathrm{WBV}_q$ *generated by a function* $g \in \mathrm{WBV}_q$ *is continuous and* $\|M_g\|_{\mathrm{WBV}_p \to \mathrm{WBV}_q} = \|g\|_{\mathrm{WBV}_q}$.

Proof We use Proposition 27, but this time instead of the general embedding result, that is Proposition 5, we use the fact that $\mathrm{WBV}_p \hookrightarrow \mathrm{WBV}_q$ with the embedding constant $c = 1$ (cf. Remark 6). □

Finally, we will deal with the continuity of multiplication operators acting in the spaces of functions of bounded Riesz variation. This time, however, the situation (and the proofs) will be much simpler.

Theorem 33 *Let* $1 \le p, q < +\infty$ *and let* $M_g : \mathrm{RBV}_p \to \mathrm{RBV}_q$ *be the multiplication operator generated by a function* $g \in \mathrm{RBV}_q/\mathrm{RBV}_p$. *Then,* M_g *is continuous and* $\|M_g\|_{\mathrm{RBV}_p \to \mathrm{RBV}_q} = \|g\|_{\mathrm{RBV}_q}$.

Proof First, let us assume that $1 \le p < q$. Then, $\mathrm{RBV}_q/\mathrm{RBV}_p = \{0\}$. Consequently, M_g is the zero operator. In particular, it is continuous and the formula for its norm holds. If, on the other hand, $1 \le q \le p$, then the proof follows from Propositions 11 and 27 and the fact that *{\text{RBV}}* spaces are Banach algebras. □

Remark 34 It is worth noting that other cases that are not covered by Proposition 27 are sometimes also known. For instance, one can show with the help of Hölder's inequality that for $g \in L_{pq/(p-q)}$, where $1 \le q < p < +\infty$, the multiplication operator $M_g : L_p \to L_q$ is well-defined and continuous with $\|M_g\|_{L_p \to L_q} = \|g\|_{L_{pq/(p-q)}}$.

5.2 Spectra

In this short section, we will show how to apply the results established in the previous parts of the paper to characterize various spectra of multiplication operators acting in BV-type spaces.

Let $L : X \to X$ be a continuous linear operator acting in a real Banach space X. Set

$$\sigma(L) := \{\lambda \in \mathbb{R} | \lambda \mathrm{I} - L \text{ is not bijective}\},$$
$$\sigma_p(L) := \{\lambda \in \mathbb{R} | \lambda \mathrm{I} - L \text{ is not injective}\},$$
$$\sigma_r(L) := \{\lambda \in \mathbb{R} | \lambda \mathrm{I} - L \text{ is injective but } \operatorname{Im}(\lambda \mathrm{I} - L) \text{ is not dense in } X\},$$
$$\sigma_c(L) := \left\{ \lambda \in \mathbb{R} \,\middle|\, \begin{array}{l} \lambda \mathrm{I} - L \text{ is injective and } \operatorname{Im}(\lambda \mathrm{I} - L) \text{ is dense in } X, \\ \text{but } (\lambda \mathrm{I} - L)^{-1} \text{ is not bounded on } \operatorname{Im}(\lambda \mathrm{I} - L) \end{array} \right\}.$$

The above sets are, respectively, called the *spectrum*, *point spectrum*, *residual spectrum* and *continuous spectrum* of the operator L. It is well-known that $\sigma(L)$ is a disjoint union of $\sigma_p(L)$, $\sigma_r(L)$ and $\sigma_c(L)$. (For more information on various spectra of linear operators we refer the reader to, for example, [7, Chapter 1] or [23, Chapter VI].) In the case when X is one of our BV spaces and L is a multiplication operator, then L is continuous by Corollary 29, and we get the following two results.

Theorem 35 *Let X be one of the spaces BV, WBV_p or YBV_φ, and let $M_g : X \to X$ be the multiplication operator generated by a function $g \in X$. Then,*

(a) $\sigma(M_g) = \overline{g([0,1])}$; *in particular*, $\rho(M_g) := \sup\{|\lambda| \,|\, \lambda \in \sigma(M_g)\} = \|g\|_\infty$,
(b) $\sigma_p(M_g) = g([0,1])$,
(c) $\sigma_r(M_g) = \overline{g([0,1])} \setminus g([0,1])$,
(d) $\sigma_c(M_g) = \emptyset$.

Proof Notice that $\lambda \mathrm{I} - M_g = M_{\lambda - g}$ for any $\lambda \in \mathbb{R}$; by a slight abuse of notation, we identify the constant function with its value. To prove the above equalities, we will use Corollaries 19, 23 and 29. Note that $\lambda \in \sigma(M_g)$ if and only if $\inf_{t\in[0,1]} |\lambda - g(t)| = 0$, which, in turn, is equivalent to $\lambda \in \overline{g([0,1])}$. Hence, $\sigma(M_g) = \overline{g([0,1])}$. The equality $\rho(M_g) = \|g\|_\infty$ is now obvious.

Further, $\lambda \in \sigma_p(M_g)$ if and only if the operator $M_{\lambda-g}$ is not injective. This happens exactly when $\lambda = g(t)$ for some $t \in [0,1]$, which implies that $\sigma_p(M_g) = g([0,1])$.

To show that $\sigma_c(M_g)$ is empty, it suffices to observe that if the range of $\lambda \mathrm{I} - M_g$ is dense in X, then this operator must be bijective by Corollary 23. This, in turn, together with the inverse mapping theorem and Corollary 29, implies that its inverse $(\lambda \mathrm{I} - M_g)^{-1}$ must be continuous on X. And so the conditions "$\operatorname{Im}(\lambda \mathrm{I} - M_g)$ is dense in X" and "$(\lambda \mathrm{I} - M_g)^{-1}$ is not bounded" cannot be satisfied simultaneously.

Finally, the equality for $\sigma_r(M_g)$ follows from the fact that $\sigma(M_g)$ is a disjoint union of $\sigma_p(M_g)$, $\sigma_r(M_g)$ and $\sigma_c(M_g)$. □

Theorem 36 *Let $1 < p < +\infty$ and let $M_g : \mathrm{RBV}_p \to \mathrm{RBV}_p$ be the multiplication operator generated by a function $g \in \mathrm{RBV}_p$. Then,*

(a) $\sigma(M_g) = g([0,1])$; *in particular*, $\rho(M_g) := \sup\{|\lambda| \,|\, \lambda \in \sigma(M_g)\} = \|g\|_\infty$,
(b) $\sigma_p(M_g)$ *consists of all the numbers* $\lambda \in \mathbb{R}$ *such that the set* $\{t \in [0,1] | g(t) = \lambda\}$ *contains a non-empty open interval*,
(c) $\sigma_r(M_g) = g([0,1]) \setminus \sigma_p(M_g)$,
(d) $\sigma_c(M_g) = \emptyset$.

Proof Only (b) requires a proof. Notice also that we may write $g([0,1])$ instead of $\overline{g([0,1])}$ as the function g is continuous. In view of Corollary 19, $\lambda \in \sigma_p(M_g)$ if and only if $[0,1] \setminus \overline{\operatorname{supp}(\lambda - g)} \neq \emptyset$. As $\overline{\operatorname{supp}(\lambda - g)}$ is a closed subset of $[0,1]$, this can happen exactly when there is a non-empty and open interval U such that $U \subseteq [0,1] \setminus \overline{\operatorname{supp}(\lambda - g)}$. In other words, $\lambda \in \sigma_p(M_g)$ if and only if the set $\{t \in [0,1] | g(t) = \lambda\}$ contains a non-empty open interval. The proof is complete. □

Remark 37 It is worth noting here that the spectral behavior of the multiplication operator acting in RBV_p spaces for $1 < p < +\infty$ is identical to the behavior of the multiplication operator acting in the space of continuous functions C (for more details see [7, Example 1.6]).

5.3 Compactness

Now, we turn our attention to studying compactness of multiplication operators. Let us recall that a linear operator $L : X \to Y$ between Banach spaces is *compact* if the image $L(\overline{B}_X(0,1))$ of the closed unit ball (or, in fact, any bounded set) in X is a relatively compact subset of Y. Clearly, not every multiplication operator is compact. The simplest example is probably the identity operator on an infinite-dimensional normed space of real-valued functions defined on $[0,1]$ (cf. also Example 26). One important family of compact operators is the class of operators of finite rank. A continuous operator $L : X \to Y$ between Banach spaces is said to be of *finite rank* if the range $\operatorname{Im} L$ is a finite-dimensional subspace of Y. As the properties of compact and finite-rank operators are classical and well-known, we will not dwell on this issue any longer. We refer the reader to, for example, the monograph [30] for more information.

Without further ado, let us move to the main topic of this section. A simple rewording of Theorem 24, together with the continuity results from Section 5.1, leads to the following characterization of finite-rank operators for functions of bounded Young variation.

Theorem 38 *Let $\varphi, \psi : [0,+\infty) \to [0,+\infty)$ be two Young functions and let $M_g : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$ be the multiplication operator generated by a function $g \in \mathrm{YBV}_\varphi / \mathrm{YBV}_\psi$. Then, M_g has finite rank if and only if $\# \operatorname{supp}(g) < +\infty$.*

So far, the properties of multiplication operators acting in YBV_φ and RBV_p spaces have been similar. This is the first time when the two theories depart slightly from one another.

Theorem 39 *Let $1 < p, q < +\infty$ and let $M_g : \mathrm{RBV}_p \to \mathrm{RBV}_q$ be the multiplication operator generated by a function $g \in \mathrm{RBV}_q / \mathrm{RBV}_p$.*

(a) *If $q > p$, then M_g has always finite rank (as the zero operator).*
(b) *If $q \leq p$, then M_g is of finite rank if and only if $g \equiv 0$.*

Proof In view of the fact that $\mathrm{RBV}_q / \mathrm{RBV}_p = \{0\}$ if $q > p$, we need to address the case $q \leq p$ only. According to Theorem 24 the operator M_g has finite-dimensional range if and only if $\# \operatorname{supp}(g) < +\infty$. But since $\mathrm{RBV}_q / \mathrm{RBV}_p = \mathrm{RBV}_q \subseteq C$ for $1 < q \leq p < +\infty$, the support of g consists of at most finitely many elements if and only if it is empty. Thus, if

$M_g : \mathrm{RBV}_p \to \mathrm{RBV}_q$ for $1 < q \le p < +\infty$ is of finite rank, then $g \equiv 0$. The other implication is obvious. □

Note that in the preceding theorem we excluded the situation when either the starting or the target space coincides with RBV_1. We did this because the nature of the spaces RBV_p is different for $p > 1$ and $p = 1$ (cf. Remark 10), and it turns out that those two cases need to be treated separately.

Theorem 40 *Let* $1 \le p < +\infty$. *The multiplication operator* $M_g : \mathrm{RBV}_p \to \mathrm{BV}$ *generated by a function* $g \in \mathrm{BV}$ *has finite rank if and only if* $\#\,\mathrm{supp}\,(g) < +\infty$.

Proof The proof follows from Theorem 24 and the fact that the multiplication operator $M_g : \mathrm{RBV}_p \to \mathrm{BV}$, where $1 \le p < +\infty$, is continuous (see Theorem 33). □

Remark 41 Let us explain why we did not study the multiplication operator $M_g : BV \to \mathrm{RBV}_p$ for $1 < p < +\infty$ in the above theorem. The reason is simple. Such an operator must be generated by the zero function (cf. Section 3), so it has trivially finite rank.

Now, let us move to the study of compactness. We begin with abstract results providing a necessary condition for a multiplication operator to be compact.

Proposition 42 *Let X be a normed space of real-valued functions defined on* $[0, 1]$ *which strongly separates points and in which the set of all characteristic functions of singletons is bounded. Moreover, let Y be another normed space of real-valued functions defined on* $[0, 1]$ *such that* $Y \hookrightarrow B$. *If the multiplication operator* $M_g : X \to Y$, *generated by a function* $g \in Y/X$, *is compact, then* $\mathrm{supp}\,(g)$ *is countable.*

Proof Suppose on the contrary that $M_g : X \to Y$ is compact but $\mathrm{supp}\,(g)$ is not countable. This implies that for some $\delta > 0$ there is a sequence $(t_n)_{n\in\mathbb{N}}$ of distinct points in the interval $[0, 1]$ such that $|g(t_n)| \ge \delta$ for all $n \in \mathbb{N}$ (cf. Section 2.2). The functions $x_n := \chi_{\{t_n\}}$ form a bounded sequence in X, but for $m, n \in \mathbb{N}$ with $m \ne n$ we have

$$c\|M_g(x_m) - M_g(x_n)\|_Y \ge \|g \cdot (x_m - x_n)\|_\infty \ge \Big|g(t_n)\big(x_m(t_n) - x_n(t_n)\big)\Big| = |g(t_n)| \ge \delta > 0,$$

where the positive constant c is such that $\|y\|_\infty \le c\|y\|_Y$ for all $y \in Y$. Hence, $(M_g(x_n))_{n\in\mathbb{N}}$ cannot have a subsequence converging in Y, and thus M_g cannot be compact—contradiction. □

It turns out that the necessary condition described in Proposition 42 is also a sufficient one in many situations. However, before we will be able to prove this in the case of the spaces of functions of bounded Young variation we need the following technical lemma.

Lemma 43 *Let* $\varphi : [0, +\infty) \to [0, +\infty)$ *be a Young function satisfying the* δ_2*-condition and let* $x \in \mathrm{YBV}_\varphi$ *with* $\mathrm{supp}\,(x) \subseteq \{t_1, t_2, \dots\} \subseteq [0, 1]$. *Then*, $\mathrm{var}_{\varphi}(x) \le \Lambda(\|x\|_\infty) \sum_{j=1}^{\infty} \varphi\big(|x(t_j)|\big)$, *where* Λ *is defined by* (2).

Proof We may clearly assume that $\operatorname{supp}(x) \neq \emptyset$ as otherwise the claim is trivial. (Let us recall that in this case the quantity $\Lambda(\|x\|_\infty)$ is also meaningful since we put $\Lambda(0) := 0$.) If $0 = \tau_0 < \ldots < \tau_n = 1$ is an arbitrary finite partition of the interval $[0, 1]$, then reasoning as in the proof of Lemma 30 we can show that $\sum_{j=1}^{n} \varphi\big(|x(\tau_j) - x(\tau_{j-1})|\big) \leq \sum_{j=1}^{\infty} \varphi\big(2|x(t_j)|\big)$. Now, it suffices to observe that $\varphi\big(2|x(t_j)|\big) \leq \Lambda(\|x\|_\infty)\varphi\big(|x(t_j)|\big)$ for any $j \in \mathbb{N}$ and use the fact that Λ is non-decreasing. □

Now, we are in position to prove a characterization of those multiplication operators acting in the spaces of functions of bounded Young variation which are compact. Note that we will require the φ-function of the target space to satisfy the δ_2-condition.

Theorem 44 *Let $\varphi, \psi : [0, +\infty) \to [0, +\infty)$ be two Young functions with $\varphi \in \delta_2$ and let $M_g : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$ be the multiplication operator generated by a function $g \in \mathrm{YBV}_\varphi / \mathrm{YBV}_\psi$.*

(a) *If $\varphi \nprec \psi$, then M_g is always compact.*
(b) *If $\varphi \prec \psi$, then M_g is compact if and only if* $\operatorname{supp}(g)$ *is countable.*

Proof Note that $\mathrm{YBV}_\varphi / \mathrm{YBV}_\psi = \mathrm{YBV}_\varphi \cap S_c$ if $\varphi \nprec \psi$. This, together with Proposition 42, implies that we need to show only that the countability of $\operatorname{supp}(g)$ guarantees the compactness of M_g. We will prove both cases $\varphi \prec \psi$ and $\varphi \nprec \psi$ simultaneously. If $\operatorname{supp}(g)$ is finite, then M_g has finite rank by Theorem 38, and hence is compact. On the other hand, if $\operatorname{supp}(g)$ is infinite, we can write $E := \{t_1, t_2, t_3, \ldots\} = \operatorname{supp}(g) \subseteq [0, 1]$. Setting $E_n := \{t_1, t_2, \ldots, t_n\}$, we see that the functions $g_n := \chi_{E_n} g$ have finite support and thus belong to $\mathrm{YBV}_\varphi \cap S_c$. By Theorem 38 the operators $M_{g_n} : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$ have finite rank, and hence are compact. To end the proof it suffices now to show that $M_{g_n} \to M_g$ with $n \to +\infty$ in the operator norm. But, in view of Theorem 31, there exists a constant $c \geq 1$ such that $\|g_n - g\|_{\mathrm{YBV}_\varphi} \leq \|M_{g_n} - M_g\|_{\mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi} \leq c\|g_n - g\|_{\mathrm{YBV}_\varphi}$ for all $n \in \mathbb{N}$. So, equivalently, we need to show that $\|g_n - g\|_{\mathrm{YBV}_\varphi} \to 0$ as $n \to +\infty$.

Observe that

$$\varphi\big(\|g_n - g\|_\infty\big) = \varphi\left(\sup_{j>n} |g(t_j)|\right) = \sup_{j>n} \varphi\big(|g(t_j)|\big) \leq \sum_{j=n+1}^{\infty} \varphi\big(|g(t_j)|\big) \leq \operatorname{var}_\varphi(g_n - g);$$

in the last inequality we used Lemma 30. Consequently, $\|g_n - g\|_\infty \leq \varphi^{-1}\big(\operatorname{var}_\varphi(g_n - g)\big)$. Furthermore, as the function Λ given by (2) is non-decreasing and $\|g_n - g\|_\infty \leq 2\|g\|_\infty$, by Lemma 43, we have

$$\operatorname{var}_\varphi(g_n - g) \leq \Lambda(2\|g\|_\infty) \sum_{j=1}^{\infty} \varphi\big(|g_n(t_j) - g(t_j)|\big) = \Lambda(2\|g\|_\infty) \sum_{j=n+1}^{\infty} \varphi\big(|g(t_j)|\big).$$

Notice, however, that the series $\sum_{j=1}^{\infty} \varphi\big(|g(t_j)|\big)$ is (absolutely) convergent, because $\sum_{j=1}^{\infty} \varphi\big(|g(t_j)|\big) \leq \operatorname{var}_\varphi(g) < +\infty$ (see Lemma 30 and Proposition 7). In particular, $\sum_{j=n+1}^{\infty} \varphi\big(|g(t_j)|\big) \to 0$ as $n \to +\infty$. Therefore, applying Proposition 7 once again and using the fact that φ^{-1} is continuous, we see that $\|g_n - g\|_{\mathrm{YBV}_\varphi} \to 0$ as $n \to +\infty$.

This shows that the operator M_g is compact and ends the proof. □

Remark 45 It is worth noting here that from the proof of Theorem 44 it follows that each compact multiplication operator between YBV_ψ and YBV_φ with $\varphi \in \delta_2$ is the limit of a sequence of finite-rank multiplication operators.

The following example shows that in general the requirement $\varphi \in \delta_2$ cannot be dropped. The idea to use the Young function φ given by (5) comes from Example 2.3 in [6]. There, Appell et al. used the same function φ to illustrate that without the δ_2-condition the set $\{x : [0,1] \to \mathbb{R} \mid \operatorname{var}_\varphi(x) < +\infty\}$ is not linear. Also, the function g below and f in [6] are similar. However, for readers' convenience we decided to provide all the details, not only those connected with the lack of compactness of the multiplication operator generated by g.

Example 46 Let us consider the Young function $\varphi : [0,+\infty) \to [0,+\infty)$ given by

$$\varphi(t) = \begin{cases} 0, & \text{if } t = 0, \\ e^{-1/t}, & \text{if } 0 < t < \frac{1}{4}, \\ (16t-3)e^{-4}, & \text{if } t \geq \frac{1}{4}. \end{cases} \tag{5}$$

It can be easily checked that φ does not satisfy the δ_2-condition. Furthermore, let $g : [0,1] \to \mathbb{R}$ be defined by

$$g(t) = \begin{cases} \frac{1}{\ln n}, & \text{if } t = \frac{1}{n} \text{ for some } n \in \mathbb{N} \text{ with } n \geq e^4, \\ 0, & \text{otherwise.} \end{cases}$$

Using Lemma 30 we obtain

$$\operatorname{var}_\varphi\Big(\tfrac{1}{4}g\Big) \leq \sum_{n \geq e^4} \varphi\Big(\tfrac{1}{2}g\Big(\tfrac{1}{n}\Big)\Big) = \sum_{n \geq e^4} \varphi\Big(\tfrac{1}{2\ln n}\Big) = \sum_{n \geq e^4} \frac{1}{n^2} \leq \frac{\pi^2}{6}.$$

Hence, $g \in \mathrm{YBV}_\varphi \cap S_c$. In particular, the multiplication operator $M_g : \mathrm{YBV}_\varphi \to \mathrm{YBV}_\varphi$ is well-defined and continuous. However, as we are going to show, it is not compact. Consider the sequence $(x_n)_{n\in\mathbb{N}}$, where $x_n := \frac{1}{4}\chi_{(0,\frac{1}{n})}$ for $n \in \mathbb{N}$. It is not hard to check that $x_n \in \mathrm{YBV}_\varphi$ and $|x_n|_{\mathrm{YBV}_\varphi} \leq \frac{1}{2}$ (and thus $\|x_n\|_{\mathrm{YBV}_\varphi} \leq \frac{3}{4}$).

Now, let us suppose that $(M_g(x_n))_{n\in\mathbb{N}}$ has a subsequence $(M_g(x_{n_k}))_{k\in\mathbb{N}}$ convergent to some $y \in \mathrm{YBV}_\varphi$. Since the norm convergence in YBV_φ is stronger than the uniform convergence and the sequence $(x_n)_{n\in\mathbb{N}}$ converges pointwise to zero on the interval [0, 1], the sole candidate for y is the zero function. Let $K \in \mathbb{N}$ be such that $\|M_g(x_{n_k})\|_{\mathrm{YBV}_\varphi} < \frac{1}{4}$ for all $k \geq K$. Then, it is not difficult to check that $\operatorname{var}_\varphi(4gx_{n_k}) \leq 1$ for all $k \geq K$. As $\operatorname{var}_\varphi(4gx_{n_k};[0,1]) \geq \operatorname{var}_\varphi(4gx_{n_k};[0,1/n_k])$, this means that $\operatorname{var}_\varphi(4gx_{n_k};[0,1/n_k]) \leq 1$ for all $k \geq K$; here by $\operatorname{var}_\varphi(f;[a,b])$, where $[a,b] \subseteq [0,1]$, we mean the φ-variation of the function f over the interval $[a,b]$—cf. Definition 3. But for each $k \geq e^4$ we have

$$\operatorname{var}_\varphi\big(4gx_{n_k};[0,1/n_k]\big) \geq \sup_{m \geq n_k+1} \sum_{i=n_k+1}^{m} \varphi\Big(4\big|(gx_{n_k})\Big(\tfrac{1}{i}\Big) - (gx_{n_k})(s_i^m)\big|\Big),$$

where $s_i^m := \frac{1}{2}(\frac{1}{i} + \frac{1}{i-1})$. Thus,

$$\operatorname{var}_{\varphi}\big(4gx_{n_k};[0,1/n_k]\big) \geq \sum_{i=n_k+1}^{\infty} \varphi\Big(g\Big(\tfrac{1}{i}\Big)\Big) = \sum_{i=n_k+1}^{\infty} \frac{1}{i} = +\infty \quad \text{for } k \geq e^4.$$

The obtained contradiction shows that the sequence $(M_g(x_n))_{n\in\mathbb{N}}$ does not contain a convergent subsequence. In other words, the multiplication operator $M_g : \mathrm{YBV}_\varphi \to \mathrm{YBV}_\varphi$, generated by g, is not compact.

We will end this part with a result providing a sufficient condition for a multiplication operator between general spaces of functions of bounded Young variation to be compact. Naturally, this time, we will not require the φ-function φ of the target space to satisfy the δ_2-condition. Instead, we will require the generator g of the multiplication operator not to oscillate too much; namely, we will assume that $\operatorname{var}_\varphi(\lambda g) < +\infty$ for each $\lambda > 0$. Note that there are plenty of such functions; for example, each nonzero function x in BV satisfies this condition, as

$$\sum_{i=1}^{n} \varphi\big(\lambda|x(t_i) - x(t_{i-1})|\big) \leq \frac{\varphi(2\lambda\|x\|_\infty)}{2\|x\|_\infty} \cdot \sum_{i=1}^{n} |x(t_i) - x(t_{i-1})| \leq \frac{\varphi(2\lambda\|x\|_\infty)}{2\|x\|_\infty} \cdot \operatorname{var}(x)$$

for any finite partition $0 = t_0 < \ldots < t_n = 1$ of the interval [0, 1]. Moreover, the condition in question is also satisfied by any $g \in \mathrm{YBV}_\varphi$ if $\varphi \in \delta_2$ (cf. Proposition 7). Unfortunately, we do not know whether the assumption "$\operatorname{var}_\varphi(\lambda g) < +\infty$ for each $\lambda > 0$," besides being sufficient, is also necessary in the general setting.

Theorem 47 *Let $\varphi, \psi : [0,+\infty) \to [0,+\infty)$ be two Young functions and let $M_g : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$ be the multiplication operator generated by a function $g \in \mathrm{YBV}_\varphi \cap S_c$. If $\operatorname{var}_\varphi(\lambda g) < +\infty$ for each $\lambda > 0$, then M_g is compact.*

Proof Note that the assumption $g \in \mathrm{YBV}_\varphi \cap S_c$ guarantees that the multiplication operator $M_g : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$ is well-defined regardless of whether $\varphi \prec \psi$ or not. Of course, we may additionally assume that g is nonzero, since otherwise there is nothing to prove.

Let $(x_n)_{n\in\mathbb{N}}$ be an arbitrary sequence in YBV_ψ with elements in the closed unit ball. By Helly's selection theorem (cf. [32, Theorem 1.3]) the sequence $(x_n)_{n\in\mathbb{N}}$ has a subsequence $(x_{n_k})_{k\in\mathbb{N}}$ pointwise convergent on [0, 1] to a function $x \in \mathrm{YBV}_\psi$. We are going to show that $\|M_g(x_{n_k}) - M_g(x)\|_{\mathrm{YBV}_\varphi} \to 0$ as $k \to +\infty$, which would clearly mean that the operator M_g is compact. Fix any $\lambda > 0$ and $\varepsilon > 0$. By Lemma 30, we have $\sum_{t\in \operatorname{supp}(g)} \varphi(4\lambda|g(t)|) \leq \operatorname{var}_\varphi(4\lambda g)$, which in view of the assumption implies that the series $\sum_{t\in \operatorname{supp}(g)} \varphi(4\lambda|g(t)|)$ is (absolutely) convergent. In particular, there is a finite set $T := \{t_1, \ldots, t_m\} \subseteq \operatorname{supp}(g)$ of distinct points such that $\sum_{t\in \operatorname{supp}(g)\setminus T} \varphi(4\lambda|g(t)|) \leq \frac{1}{2}\varepsilon$. Let $N \in \mathbb{N}$ be such that

$$\left|x_{n_k}(t_i) - x(t_i)\right| \leq \frac{1}{2\lambda\|g\|_\infty} \cdot \varphi^{-1}\Big(\frac{\varepsilon}{2m}\Big)$$

for all $k \geq N$ and $i = 1, \ldots, m$. Note also that $\operatorname{supp}(gx_{n_k} - gx) \subseteq \operatorname{supp}(g)$ for all $k \in \mathbb{N}$. Thus, by Lemma 30, for $k \geq N$ we have

$$\begin{aligned}\operatorname{var}_{\varphi}\left[\lambda(M_g(x_{n_k}) - M_g(x))\right] &\le \sum_{t\in \operatorname{supp}(g)} \varphi\big(2\lambda|g(t)x_{n_k}(t) - g(t)x(t)|\big)\\ &\le \sum_{t\in \operatorname{supp}(g)\setminus T} \varphi(4\lambda|g(t)|) + \sum_{t\in T} \varphi\big(2\lambda\|g\|_\infty |x_{n_k}(t) - x(t)|\big)\\ &\le \frac{1}{2}\varepsilon + \frac{1}{2}\varepsilon = \varepsilon\end{aligned}$$

which shows that $\operatorname{var}_{\varphi}\left[\lambda(M_g(x_{n_k}) - M_g(x))\right] \to 0$ as $k \to +\infty$ for each $\lambda > 0$. This, in turn, implies that $|M_g(x_{n_k}) - M_g(x)|_{\mathrm{YBV}_\varphi} \to 0$ as $k \to +\infty$ (cf. [31, Theorem 1.6]).

To end the proof it suffices now to show that $\|M_g(x_{n_k}) - M_g(x)\|_\infty \to 0$. Take any $t \in \operatorname{supp}(g)$. As $\operatorname{supp}(g)$ is countable, and in particular $Z_g \neq \emptyset$, we get

$$\varphi\big(|M_g(x_{n_k})(t) - M_g(x)(t)|\big) \le \operatorname{var}_{\varphi}(M_g(x_{n_k}) - M_g(x)).$$

If $t \notin \operatorname{supp}(g)$, the above inequality is trivially satisfied. Thus,

$$\|M_g(x_{n_k}) - M_g(x)\|_\infty \le \varphi^{-1}\big(\operatorname{var}_{\varphi}(M_g(x_{n_k}) - M_g(x))\big)$$

which, in view of the first part of the proof, shows that $\|M_g(x_{n_k}) - M_g(x)\|_\infty \to 0$ as $k \to +\infty$. This completes the proof. □

For the Riesz spaces RBV_p we have a result of a similar (yet distinct) flavor. Since each function in RBV_p for $1 < p < +\infty$ is continuous, compactness of M_g leads to a stronger degeneracy. Notice the resemblance of Theorem 39 and the following theorem. However, now the proof will require a bit more work and a compactness result proved recently by Bugajewski and Gulgowski in [14], which says that if a non-empty subset A of BV is relatively compact, then it is *equivariated*, meaning that for each $\varepsilon > 0$ there is a finite partition $0 = t_0^\varepsilon < t_1^\varepsilon < \ldots < t_m^\varepsilon = 1$ of the interval $[0, 1]$ such that $\operatorname{var}(x) \le \varepsilon + \sum_{i=1}^m \left|x(t_i^\varepsilon) - x(t_{i-1}^\varepsilon)\right|$ for every $x \in A$.

Theorem 48 *Let $1 < p, q < +\infty$ and let $M_g : \mathrm{RBV}_p \to \mathrm{RBV}_q$ be the multiplication operator generated by a function $g \in \mathrm{RBV}_q/\mathrm{RBV}_p$.*

(a) *If $q > p$, then M_g is always compact (as the zero operator).*
(b) *If $q \le p$, then M_g is compact if and only if $g \equiv 0$.*

Proof Since $\mathrm{RBV}_q/\mathrm{RBV}_p = \{0\}$ for $1 < p < q$, we need only to prove the necessity part of (b). So, let $M_g : \mathrm{RBV}_p \to \mathrm{RBV}_q$ be a compact operator and let us assume that $1 < q \le p$. Then, in particular, $\mathrm{RBV}_q/\mathrm{RBV}_p = \mathrm{RBV}_q$. Suppose now that g is not identically zero. Since it is continuous, there must exist some interval $[a, b] \subseteq [0, 1]$ of positive length and a positive constant δ such that $|g(t)| > \delta$ for all $t \in [a, b]$. For any fixed $n \in \mathbb{N}$ let $s_k := a + \frac{k}{2n} \cdot (b - a)$ for $k = 0, 1, \ldots, 2n$ and define $x_n : [0, 1] \to \mathbb{R}$ to be the real-valued function whose graph is a simple polygonal line with nodes at the points (0, 0), (1, 0), $(s_k, 0)$ for $k \in \{0, \ldots, 2n\}$ even, and $(s_k, \frac{1}{2n})$ for $k \in \{0, \ldots, 2n\}$ odd. It can be easily checked that $x_n \in \mathrm{RBV}_p$ with $\|x_n\|_{\mathrm{RBV}_p} = \frac{1}{2n} + (b - a)^{1/p-1}$ for $n \in \mathbb{N}$, meaning that the set $A := \{x_n | n \in \mathbb{N}\} \subseteq \mathrm{RBV}_p$ is bounded. In view of our assumption, this implies that $M_g(A)$

is a relatively compact subset of RBV_q. Since RBV_q is continuously embedded into $\mathrm{RBV}_1 = BV$ (see Proposition 11), $M_g(A)$ is a relatively compact subset of BV. Therefore, $M_g(A)$ is equivariated, that is, for each $\varepsilon > 0$ there exists a finite partition $0 = t_0^\varepsilon < t_1^\varepsilon < \ldots < t_m^\varepsilon = 1$ of the interval $[0, 1]$ such that $\operatorname{var}(gx_n) \leq \varepsilon + \sum_{i=1}^m \left|(gx_n)(t_i^\varepsilon) - (gx_n)(t_{i-1}^\varepsilon)\right|$ for every $n \in \mathbb{N}$. In particular, for $\varepsilon = \frac{1}{8}\delta$, we can find a finite partition $0 = t_0^\delta < \ldots < t_l^\delta = 1$ such that for every positive integer $n \geq 8l\delta^{-1}\|g\|_\infty$ we have

$$\operatorname{var}(gx_n) \leq \frac{1}{8}\delta + \sum_{i=1}^{l} \left|(gx_n)(t_i^\delta) - (gx_n)(t_{i-1}^\delta)\right| \leq \frac{1}{8}\delta + \frac{l}{n}\|g\|_\infty \leq \frac{1}{4}\delta.$$

But, for any $n \in \mathbb{N}$, we have

$$\operatorname{var}(gx_n) \geq \sum_{k=0}^{n-1} \left|(gx_n)(s_{2k+1}) - (gx_n)(s_{2k})\right| = \frac{1}{2n}\sum_{k=0}^{n-1} \left|g(s_{2k+1})\right| \geq \frac{1}{2}\delta,$$

which leads to a contradiction. Thus, g is identically equal to zero. □

In the special cases when $p = q$, it is also possible to give another proof of Theorem 48, which does not require any knowledge about (relatively) compact subsets of BV; for more details consult [33].

We end this subsection with a compactness criterion for multiplication operators acting from the space RBV_p for $1 < p < +\infty$ into BV.

Theorem 49 *Let $1 < p < +\infty$ and let $M_g : \mathrm{RBV}_p \to \mathrm{BV}$ be the multiplication operator generated by a function $g \in \mathrm{BV}$. Then, M_g is compact if and only if* $\operatorname{supp}(g)$ *is countable; moreover, M_g is then the limit of finite-rank multiplication operators acting between* RBV_p *and BV.*

Proof A reasoning similar to the one used in the proof of Theorem 44 shows that if $\operatorname{supp}(g)$ is countable then either M_g has finite rank (if $\operatorname{supp}(g)$ is finite), or is the limit of finite-rank operators (if $\operatorname{supp}(g)$ is infinite). In either case, M_g is compact.

Now, let us assume that $g \in \mathrm{BV}$ and that the multiplication operator $M_g : \mathrm{RBV}_p \to \mathrm{BV}$ is compact. If $\operatorname{supp}(g)$ were uncountable, then it would contain a point of continuity of g (because the set of points of discontinuity of a function of bounded Jordan variation is countable). In particular, there would exist an interval $[a, b] \subseteq [0, 1]$ of a positive length and a constant $\delta > 0$ such that $|g(t)| \geq \delta$ for all $t \in [a, b]$. Now, it remains to repeat the reasoning presented in the first proof of Theorem 48 to obtain a contradiction. Thus, $\operatorname{supp}(g)$ must be countable. □

Remark 50 The compactness of the multiplication operator $M_g : \mathrm{BV} \to \mathrm{RBV}_p$, where $1 < p < +\infty$, is trivial as in this case $\mathrm{RBV}_p/\mathrm{BV} = \{0\}$ (cf. Remark 41).

5.4 Measure of non-compactness and essential norm

In the previous section, we studied conditions guaranteeing compactness of multiplication operators in certain BV-type spaces. Now, we would like to look at the same problem from a more qualitative point of view. To estimate how far an operator is from being compact, one can use, for example, measures of non-compactness. Let us recall that the *measure of non-compactness* (or, the *α-norm*) $[L]$ of a bounded linear operator $L : X \to Y$ between Banach spaces is given by the formula

$$[L] := \inf\{k > 0 \big| \alpha_Y(L(A)) \le k\alpha_X(A) \text{ for every bounded subset } A \text{ of } X\};$$

here α_X and α_Y denote the Kuratowski measure of non-compactness in X and Y, respectively. (In the sequel we will be omitting the subscripts X and Y and simply write α, as the spaces involved will always be clear from the context.) Although various measures of non-compactness are widely known and used throughout nonlinear analysis, for readers' convenience let us recall the definition of the measure α. If A is a bounded subset of a Banach space (or, in general, a metric space) X, then its *Kuratowski measure of non-compactness* is given by the formula

$$\alpha(A) := \inf\left\{\varepsilon > 0 \,\middle|\, \begin{array}{l}\text{there exists a finite covering of } A\\ \text{with sets of diameter less than or}\\ \text{equal to } \varepsilon\end{array}\right\}$$

(see [28]). For basic properties of the index α we refer the reader to [2].

It is well-known that $[L] \le \|L\|_{X\to Y}$ and that the operator L is compact if and only if $[L] = 0$. Furthermore, it can be shown that

$$[L] = \inf\{k > 0 \big| \alpha(L(A)) < k\alpha(A) \text{ for every bounded subset } A \text{ of } X \text{ with } \alpha(A) > 0\}.$$

Other properties of the α-norm of a bounded linear operator with some illustrative examples can be found in, for example, [5] (cf. also [7, Section 1.2]).

Now, let us state and prove the main result of this section concerning the lower bound for $[M_g]$ in the case of the YBV_φ spaces.

Theorem 51 *Let $\varphi, \psi : [0, +\infty) \to [0, +\infty)$ be two Young functions and let $M_g : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$ be the multiplication operator generated by a function $g \in \mathrm{YBV}_\varphi/\mathrm{YBV}_\psi$. Then,*

$$\frac{\psi^{-1}\left(\frac{1}{2}\right)\left(\varphi^{-1}\left(\frac{1}{4}\right)+1\right)}{\varphi^{-1}\left(\frac{1}{4}\right)\left(\psi^{-1}\left(\frac{1}{2}\right)+2\right)} \cdot \inf\{\delta > 0 | \#\operatorname{supp}_\delta(g) < +\infty\} \le [M_g]. \tag{6}$$

Proof Notice that we may assume that the set $\{\delta > 0 | \#\operatorname{supp}_\delta(g) = +\infty\}$ is non-empty, since otherwise there is nothing to prove. So, let $\delta > 0$ be an arbitrary number such that the set $\operatorname{supp}_\delta(g)$ is infinite. Then, there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $(0, 1)$ of distinct points with the property that $|g(t_n)| > \delta$ for $n \in \mathbb{N}$. Let $x_n := \chi_{\{t_n\}}$ and $A := \{x_n | n \in \mathbb{N}\}$. Clearly, $A \subseteq \mathrm{YBV}_\psi$. Moreover, by Lemma 30 applied with $\lambda := 2/\psi^{-1}(\frac{1}{2})$, for $n \ne m$ we have $\operatorname{var}_\psi(\lambda^{-1}(x_n - x_m)) \le 2\psi(2\lambda^{-1}) = 1$, which implies that $\|x_n - x_m\|_{\mathrm{YBV}_\psi} \le 1 + 2/\psi^{-1}(\frac{1}{2})$. Thus, $\alpha(A) \le \operatorname{diam} A \le 1 + 2/\psi^{-1}(\frac{1}{2})$. Let us also note that $\|x_n - x_m\|_{\mathrm{YBV}_\psi} \ge \|x_n - x_m\|_\infty = 1$ for distinct $n, m \in \mathbb{N}$. Consequently, the sequence $(x_n)_{n\in\mathbb{N}}$ does not contain a Cauchy subsequence, and so the set A cannot be relatively compact in YBV_ψ. In other words, $\alpha(A) > 0$.

Now, let us suppose that

$$\alpha(M_g(A)) < \frac{\psi^{-1}\left(\frac{1}{2}\right)\left(\varphi^{-1}\left(\frac{1}{4}\right)+1\right)}{\varphi^{-1}\left(\frac{1}{4}\right)\left(\psi^{-1}\left(\frac{1}{2}\right)+2\right)} \cdot \delta\alpha(A),$$

and for simplicity let us denote the quantity on the right-hand side of the above inequality by γ. This means that there is a finite collection of subsets $B_1, \dots, B_k$ of YBV_φ such that $M_g(A) \subseteq B_1 \cup \dots \cup B_k$ and $\operatorname{diam} B_l < \gamma$ for $l = 1, \dots, k$. Since there are only finitely many sets B_l and all the elements of $M_g(A)$ are distinct, one of those sets (say, B_1) contains infinitely many distinct elements of $M_g(A)$, and particularly two of them, say $M_g(x_i)$ and $M_g(x_j)$. If μ is a positive number such that

$$\operatorname{var}_\varphi\left(\frac{M_g(x_i) - M_g(x_j)}{\mu}\right) \le 1,$$

then we have

$$1 \ge \operatorname{var}_\varphi\left(\frac{M_g(x_i) - M_g(x_j)}{\mu}\right) \ge 2\varphi\left(\frac{|g(t_i)|}{\mu}\right) + 2\varphi\left(\frac{|g(t_j)|}{\mu}\right) \ge 4\varphi\left(\frac{\min\left\{|g(t_i)|, |g(t_j)|\right\}}{\mu}\right).$$

Thus, $\mu \ge \min\left\{|g(t_i)|, |g(t_j)|\right\}/\varphi^{-1}(\frac{1}{4})$, and

$$\begin{aligned}
\operatorname{diam} B_1 &\ge \|M_g(x_i) - M_g(x_j)\|_{\mathrm{YBV}_\varphi} \ge \left(1 + 1/\varphi^{-1}\left(\tfrac{1}{4}\right)\right) \cdot \min\left\{|g(t_i)|, |g(t_j)|\right\} \\
&> \left(1 + 1/\varphi^{-1}\left(\tfrac{1}{4}\right)\right)\delta = \frac{\psi^{-1}\left(\frac{1}{2}\right)\left(\varphi^{-1}\left(\frac{1}{4}\right)+1\right)}{\varphi^{-1}\left(\frac{1}{4}\right)\left(\psi^{-1}\left(\frac{1}{2}\right)+2\right)} \cdot \left(1 + 2/\psi^{-1}\left(\tfrac{1}{2}\right)\right)\delta \\
&\ge \frac{\psi^{-1}\left(\frac{1}{2}\right)\left(\varphi^{-1}\left(\frac{1}{4}\right)+1\right)}{\varphi^{-1}\left(\frac{1}{4}\right)\left(\psi^{-1}\left(\frac{1}{2}\right)+2\right)} \cdot \delta\alpha(A) = \gamma,
\end{aligned}$$

a contradiction, since we know that $\operatorname{diam} B_1 < \gamma$. Therefore,

$$\alpha(M_g(A)) \ge \frac{\psi^{-1}\left(\frac{1}{2}\right)\left(\varphi^{-1}\left(\frac{1}{4}\right)+1\right)}{\varphi^{-1}\left(\frac{1}{4}\right)\left(\psi^{-1}\left(\frac{1}{2}\right)+2\right)} \cdot \delta\alpha(A).$$

We have thus shown that if $\delta > 0$ does not belong to the set $\{\delta > 0 | \# \operatorname{supp}_\delta(g) < +\infty\}$, then

$$\frac{\psi^{-1}\left(\frac{1}{2}\right)\left(\varphi^{-1}\left(\frac{1}{4}\right)+1\right)}{\varphi^{-1}\left(\frac{1}{4}\right)\left(\psi^{-1}\left(\frac{1}{2}\right)+2\right)} \cdot \delta$$

does not belong to the set of all positive numbers η such that $\alpha(M_g(A)) < \eta\alpha(A)$ for every bounded subset A of YBV_ψ with $\alpha(A) > 0$. Consequently,

$$\frac{\psi^{-1}\left(\frac{1}{2}\right)\left(\varphi^{-1}\left(\frac{1}{4}\right)+1\right)}{\varphi^{-1}\left(\frac{1}{4}\right)\left(\psi^{-1}\left(\frac{1}{2}\right)+2\right)} \cdot \inf\{\delta > 0 | \# \operatorname{supp}_\delta(g) < +\infty\} \le [M_g].$$

This ends the proof. □

Remark 52 It is worth noting that in the special case of the Jordan variation, that is, when $\varphi(u) = \psi(u) = u$, the constant

$$\frac{\psi^{-1}\left(\frac{1}{2}\right)\left(\varphi^{-1}\left(\frac{1}{4}\right)+1\right)}{\varphi^{-1}\left(\frac{1}{4}\right)\left(\psi^{-1}\left(\frac{1}{2}\right)+2\right)}$$

appearing on the left-hand side of (6) equals 1.

To obtain an upper estimate for $[M_g]$ (better than $\|M\|_{\mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi}$) we will additionally assume that φ satisfies the δ_2-condition.

Proposition 53 *Let* $\varphi, \psi : [0, +\infty) \to [0, +\infty)$ *be two Young functions with* $\varphi \in \delta_2$ *and let* $M_g : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$ *be the multiplication operator generated by a function* $g \in \mathrm{YBV}_\varphi / \mathrm{YBV}_\psi$. *Then, there exists a constant* $c \ge 1$ (*depending only on the functions* φ, ψ) *such that*

$$[M_g] \le c \min\left\{ \|g_r\|_{\mathrm{YBV}_\varphi}, \|g_l\|_{\mathrm{YBV}_\varphi} \right\}, \tag{7}$$

where $g_r, g_l : [0, 1] \to \mathbb{R}$ *are the right and left regularization of* g, *respectively, and are given by*

$$g_r(t) := \begin{cases} g(t+), & \text{if } t \in [0,1), \\ g(1), & \text{if } t = 1, \end{cases} \qquad g_l(t) := \begin{cases} g(0), & \text{if } t = 0, \\ g(t-), & \text{if } t \in (0,1]. \end{cases}$$

Proof Before we proceed to the main part of the proof, let us recall that functions of bounded Young variation have one-sided limits at each point and countably many points of discontinuity (cf. [31, Theorem 10.9]). Thus, the right and left regularizations of g are well-defined. Moreover, it is fairly easy to check that $g_r, g_l \in \mathrm{YBV}_\varphi$ and $\max\left\{ \|g_r\|_{\mathrm{YBV}_\varphi}, \|g_l\|_{\mathrm{YBV}_\varphi} \right\} \le \|g\|_{\mathrm{YBV}_\varphi}$. As the functions g, g_r and g_l differ at at most countably many points, we also have $g_r, g_l \in \mathrm{YBV}_\varphi / \mathrm{YBV}_\psi$.

Now, let $h_r, h_l : [0, 1] \to \mathbb{R}$ be given by $h_r := g - g_r$ and $h_l := g - g_l$. Clearly, $h_r, h_l \in \mathrm{YBV}_\varphi \cap S_c \subseteq \mathrm{YBV}_\varphi / \mathrm{YBV}_\psi$. Therefore, by Theorem 44, the multiplication operators $M_{h_r}, M_{h_l} : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$ are compact. Using the properties of the Kuratowski measure of non-compactness, for any bounded subset A of YBV_ψ we get

$$\alpha(M_g(A)) \le \alpha(M_{g_l}(A)) + \alpha(M_{h_l}(A)) = \alpha(M_{g_l}(A)) \le \|M_{g_l}\|_{\mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi} \cdot \alpha(A),$$

and similarly $\alpha(M_g(A)) \le \|M_{g_r}\|_{\mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi} \cdot \alpha(A)$. To end the proof it suffices now to apply Theorem 31. □

Remark 54 From the proof of Proposition 53 follows an even stronger version of the estimate (7). Namely, the minimum on the right-hand side of (7) may be replaced by the infimum of all the YBV_φ-norms of functions $g_* \in \mathrm{YBV}_\varphi$ which differ from g at at most countably many points in [0, 1]. We chose to state Proposition 53 for the left and right regularization of g, because in many cases g_r and g_l make the function g nicer; for example, they smooth out isolated jumps of g. Moreover, in the special case when $\varphi \not\prec \psi$, that is, when $g \in \mathrm{YBV}_\varphi \cap S_c$, if $g(0) = g(1) = 0$, we get $g_r \equiv g_l \equiv 0$, which implies that the multiplication operator generated by g is compact (cf. Theorem 44).

Finally, let us note that similar to what we did in Section 5.1 (see Corollary 29 as well as Proposition 32 and the paragraph before it), it is possible to give some estimates on the constant c appearing in (7).

Although in some cases the upper estimate of $[M_g]$ provided in the above proposition gives satisfactory results (cf. Remark 54), in general it seems far from being optimal even when $c = 1$. It would be interesting to find the exact formula for $[M_g]$. A closer look at the results of this and the previous section may suggest that the α-norm of $M_g : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$ and the quantity $\inf\{\delta > 0 \mid \#\operatorname{supp}_\delta(g) < +\infty\}$ are equivalent, at least in the situation when $\varphi \in \delta_2$. In other words, we conjecture that in such a case there exist positive constants a, b such that

$$a \inf\{\delta > 0 \mid \#\operatorname{supp}_\delta(g) < +\infty\} \le [M_g] \le b \inf\{\delta > 0 \mid \#\operatorname{supp}_\delta(g) < +\infty\}. \tag{8}$$

It turns out that there are some clues indicating that the above estimate may be true. Let us take a look at two such instances.

Example 55 Let $\varphi : [0, +\infty) \to [0, +\infty)$ be a Young function and let $(r_n)_{n\in\mathbb{N}}$ be an arbitrary sequence of distinct numbers in (0, 1). Fix a nonnegative number λ and consider the multiplication operator $M_g : \mathrm{YBV}_\varphi \to \mathrm{YBV}_\varphi$ generated by the function $g : [0, 1] \to \mathbb{R}$ given by

$$g(t) = \begin{cases} \lambda + 1/n^2 & \text{for } t = r_n, \\ \lambda & \text{otherwise.} \end{cases}$$

Note that the operator M_g is well-defined as $g \in \mathrm{YBV}_\varphi$. Using a similar approach to the one we used in the proof of Proposition 53 it can be shown that $\alpha(M_g(A)) = \alpha(M_{g_r}(A)) = \alpha(M_{g_l}(A))$, and hence $\alpha(M_g(A)) = \alpha(\lambda A) = \lambda\alpha(A)$ for each bounded subset A of YBV_φ (the functions $g - g_r$ and $g - g_l$ belong to $BV \cap S_c$, and so we can use Theorem 47). This implies that $[M_g] = \lambda$. Furthermore, observe that $\operatorname{supp}_\delta(g) = [0, 1]$ if $\delta < \lambda$. On the other hand, if $\delta \ge \lambda$, then $\operatorname{supp}_\delta(g)$ consists of those points r_n whose indices $n \in \mathbb{N}$ satisfy the inequality $1/n^2 > \delta - \lambda$. Thus, $\#\operatorname{supp}_\delta(g) < +\infty$ if and only if $\delta > \lambda$, and so $\inf\{\delta > 0 \mid \#\operatorname{supp}_\delta(g) < +\infty\} = \lambda$. In other words, $[M_g] = \inf\{\delta > 0 \mid \#\operatorname{supp}_\delta(g) < +\infty\}$.

Now, let us move to the second result connected with the conjecture (8); this time, we will assume that we work with functions of bounded Jordan variation only. However, before we will be able to state it let us recall that we call $g : [0, 1] \to \mathbb{R}$ a *step function* if and only if there exist a finite collection of (not necessarily distinct) real numbers $\lambda_1, \ldots, \lambda_n, \mu_1, \ldots, \mu_m$ together with a finite collection of pairwise disjoint subintervals $I_1, \ldots, I_n$ of [0, 1] with non-empty interiors and a finite collection of distinct points $t_1, \ldots, t_m \in [0, 1]$ such that $[0, 1] \setminus \bigcup_{i=1}^n I_i = \{t_1, \ldots, t_m\}$ and $g = \sum_{i=1}^n \lambda_i \chi_{I_i} + \sum_{i=1}^m \mu_i \chi_{\{t_i\}}$.

In the proof of the following result, we will aim at expressing g in the simplest form possible, reducing the number of points t_i to the minimum zero.

Proposition 56 *If* $M_g : \mathrm{BV} \to \mathrm{BV}$ *is the multiplication operator generated by a step function* g, *then*

$$[M_g] = \inf\{\delta > 0 | \# \operatorname{supp}_\delta(g) < +\infty\}.$$

Proof The step function g is clearly of bounded Jordan variation, and so the multiplication operator $M_g : \mathrm{BV} \to \mathrm{BV}$ is well-defined.

Note that $\inf\{\delta > 0 | \# \operatorname{supp}_\delta(g) < +\infty\} = |\lambda|$, where $|\lambda| := \max_{1 \le i \le n} |\lambda_i|$. Hence, by Theorem 51 and Remark 52, we have $|\lambda| \le [M_g]$. To end the proof it suffices now to show that $\alpha(M_g(A)) \le |\lambda| \alpha(A)$ for each bounded subset A of BV. As $\alpha(M_g(A)) = \alpha(M_h(A))$, where $h : [0,1] \to \mathbb{R}$ is given by

$$h(t) = \begin{cases} g(t+), & \text{if } t \in [0,1), \\ g(1-), & \text{if } t = 1, \end{cases}$$

we may assume that g is right-continuous at each point in the interval $[0, 1)$ and left-continuous at $t = 1$ (cf. the proof of Proposition 53 and Example 55). Then, in the decomposition of g only the intervals $I_1, \dots, I_n$ will appear, that is, $g = \sum_{i=1}^n \lambda_i \chi_{I_i}$, where $I_1, \dots, I_n$ are pairwise disjoint subintervals of $[0, 1]$ with non-empty interiors such that $I_1 \cup \dots \cup I_n = [0, 1]$. If $n = 1$, then $g \equiv \lambda_1$ and $\alpha(M_g(A)) = |\lambda_1| \alpha(A) = |\lambda| \alpha(A)$ for every bounded subset A of BV, and the proof is complete. So, now let $n \ge 2$. By re-indexing the intervals $I_1, \dots, I_n$ if necessary, we may additionally assume that $\sup I_i \le \inf I_{i+1}$. Furthermore, let us denote the left and right end-point of I_i by τ_{i-1} and τ_i, respectively. In particular, $\tau_{i-1} \in I_i$ for $i = 1, \dots, n-1$ and $I_n = [\tau_{n-1}, \tau_n]$.

Now, let A be a bounded subset of BV and let us fix $\varepsilon > 0$. Then, there exist finitely many non-empty sets $A_1, \dots, A_p$ such that $A = \bigcup_{i=1}^p A_i$ and $\operatorname{diam} A_i \le \alpha(A) + \frac{\varepsilon}{2|\lambda|}$ for $i = 1, \dots, p$. (Observe that we may assume that $|\lambda| > 0$; otherwise M_g would be the zero operator and the equality $[M_g] = |\lambda|$ would follow.) Since the set A is bounded, there exists $r > 0$ such that $\|x\|_{BV} \le r$ for each $x \in A$. Let us write the interval $[-r, r]$ as a union of l (not necessarily disjoint) subintervals $J_1, \dots, J_l$ of diameters not exceeding $\frac{\varepsilon}{4(n-1)|\lambda|}$. Further, by Γ let us denote the finite set of all mappings $\gamma : \{1, \dots, n-1\} \to \{1, \dots, l\}$, and for each $\gamma \in \Gamma$ let us set

$$B_\gamma := \{x \in BV | x(\tau_i) \in J_{\gamma(i)} \text{ for each } i = 1, \dots, n-1\};$$

we borrowed the idea to consider the family Γ from Ambrosetti—cf. [4, p. 354]. Then,

$$M_g(A) \subseteq \bigcup_{k=1}^{p} \bigcup_{\gamma \in \Gamma} M_g(A_k \cap B_\gamma).$$

Moreover, if we take $x, y \in A_k \cap B_\gamma$, using the properties of the Jordan variation, we obtain

$$\begin{aligned}\operatorname{var}\left(M_g(x)-M_g(y);[0,1]\right) &= \sum_{i=1}^{n}\operatorname{var}\left(M_g(x)-M_g(y);\overline{I_i}\right)\\ &\le \sum_{i=1}^{n-1}|\lambda_i|\operatorname{var}(x-y;\overline{I_i})+\sum_{i=1}^{n-1}|\lambda_{i+1}-\lambda_i||(x-y)(\tau_i)|+|\lambda_n|\operatorname{var}(x-y;I_n)\\ &\le |\lambda|\operatorname{var}(x-y;[0,1])+\frac{1}{2}\varepsilon;\end{aligned}$$

let us recall that $\operatorname{var}(z;I)$ denotes the (Jordan) variation of the function z over the closed interval I. And so,

$$\|M_g(x)-M_g(y)\|_{BV} \le |\lambda|\,\|x-y\|_{BV}+\tfrac{1}{2}\varepsilon \le |\lambda|\alpha(A)+\varepsilon.$$

Therefore, $\alpha(M_g(A)) \le |\lambda|\alpha(A)+\varepsilon$. Since the number $\varepsilon>0$ was arbitrary, we finally get $\alpha(M_g(A)) \le |\lambda|\alpha(A)$, which shows that $[M_g] \le |\lambda|$ and completes the proof. □

Remark 57 Another measure indicating how far a bounded linear operator $L : X \to Y$ between Banach spaces is from being compact is its *essential norm* defined by the formula

$$\|L\|_e := \inf\left\{\|L-K\|_{X\to Y} \,\middle|\, K : X \to Y \text{ linear and compact}\right\}$$

(see, for example, [5, Section 3] or [7, p. 34]). Although we will spend no time investigating the estimates for the essential norm of a multiplication operator acting in BV-type space, let us mention that $[L] \le \|L\|_e \le \|L\|_{X\to Y}$. So, it is possible to obtain results similar to Theorem 51 and Proposition 53 for the essential norm of M_g.

We conclude this section with two open problems.

Openproblem 1 Provide the exact formula for the measure of non-compactness (and/or the essential norm) of the multiplication operator $M_g : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$. In particular, determine whether conjecture (8) is true.

Openproblem 2 Provide any non-trivial estimates for the measure of non-compactness (and/or the essential norm) of the multiplication operator $M_g : \mathrm{RBV}_p \to \mathrm{RBV}_q$.

5.5 Multiplication operators with closed range

In this section, we are going to discuss conditions guaranteeing that multiplication operators in BV-type spaces have closed range. We begin with a necessary condition and spaces of functions of bounded Young variation.

Proposition 58 *Let $\varphi, \psi : [0,+\infty) \to [0,+\infty)$ be two Young functions and let $M_g : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$ be the multiplication operator generated by a function $g \in \mathrm{YBV}_\varphi/\mathrm{YBV}_\psi$. If M_g has closed range, then there exists a constant $\delta>0$ such that $|g(t)| \ge \delta$ for each $t \in \operatorname{supp}(g)$.*

Note that the condition "there exists a constant $\delta > 0$ such that $|g(t)| \geq \delta$ for each $t \in \operatorname{supp}(g)$" is satisfied vacuously when $\operatorname{supp}(g) = \emptyset$. So, we do not need to exclude the situation when $g \equiv 0$ from our considerations.

Proof of Theorem 58 Clearly, we may assume that $\operatorname{supp}(g) \neq \emptyset$. Since M_g is bounded and has closed range there is a constant $c > 0$ such that for each $y \in \operatorname{Im} M_g$ there is some $\eta \in \mathrm{YBV}_\psi$ satisfying $y = M_g(\eta)$ and $\|\eta\|_{\mathrm{YBV}_\psi} \leq c\|y\|_{\mathrm{YBV}_\varphi}$ (see [1, Corollary 2.15]). Put $\delta := (c + c/\varphi^{-1}(\frac{1}{2}))^{-1}$. Let us fix $t \in \operatorname{supp}(g)$ and consider $x := \chi_{\{t\}}$. Clearly, $x \in \mathrm{YBV}_\psi$. Now, take a function $\xi \in \mathrm{YBV}_\psi$ such that $M_g(x) = M_g(\xi)$ and $\|\xi\|_{\mathrm{YBV}_\psi} \leq c\|M_g(x)\|_{\mathrm{YBV}_\varphi}$. As $t \in \operatorname{supp}(g)$, the equality $M_g(x) = M_g(\xi)$ implies that $\xi(t) = 1$, and therefore $\|\xi\|_{\mathrm{YBV}_\psi} \geq 1$. It is also not difficult to show that $|M_g(x)|_{\mathrm{YBV}_\varphi} \leq |g(t)|/\varphi^{-1}(\frac{1}{2})$. So, $1 \leq \|\xi\|_{\mathrm{YBV}_\psi} \leq c\|M_g(x)\|_{\mathrm{YBV}_\varphi} \leq |g(t)| \cdot (c + c/\varphi^{-1}(\frac{1}{2}))$, whence our claim follows. □

To obtain necessary and sufficient conditions for the operator $M_g : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$ to have closed range we need to distinguish two cases: $\varphi \nprec \psi$ and $\varphi \prec \psi$. As we will see, the former one is relatively easy to handle, whereas the latter one is more technical.

Theorem 59 *Let $\varphi, \psi : [0, +\infty) \to [0, +\infty)$ be two Young functions such that $\varphi \nprec \psi$. Moreover, let $M_g : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$ be the multiplication operator generated by a function $g \in \mathrm{YBV}_\varphi/\mathrm{YBV}_\psi$. Then, M_g has closed range if and only if $\#\operatorname{supp}(g) < +\infty$.*

Proof If $\#\operatorname{supp}(g) < +\infty$, then by Theorem 38 the range of the multiplication operator M_g is finite-dimensional, and hence closed in YBV_φ (see [3, Lemma 4.9]).

Now, let us assume that M_g has closed range. By Proposition 58 this means that there is a constant $\delta > 0$ such that $|g(t)| \geq \delta$ for all $t \in \operatorname{supp}(g)$. Note also that since $\mathrm{YBV}_\varphi/\mathrm{YBV}_\psi = \mathrm{YBV}_\varphi \cap S_c$ when $\varphi \nprec \psi$, the support of g is countable. If it were infinite, then by Lemma 30 for each $\lambda > 0$ we would have

$$\operatorname{var}_\varphi(\lambda g) \geq \sum_{t \in \operatorname{supp}(g)} \varphi(\lambda|g(t)|) \geq \sum_{t \in \operatorname{supp}(g)} \varphi(\lambda\delta) = +\infty.$$

This would, however, contradict the fact that $g \in \mathrm{YBV}_\varphi$. Hence, $\operatorname{supp}(g)$ is finite. □

If $\varphi \in \delta_2$ we can also provide an alternative proof of Theorem 59, which does not use Proposition 58.

Alternative proof of Theorem 59 when $\varphi \in \delta_2$. In view of Theorem 44 the multiplication operator M_g is compact. To end the proof it suffices now to apply Theorem 38 and a well-known result in functional analysis saying that a compact linear operator between Banach spaces has closed range if and only if it is of finite rank (see [30, Proposition 3.4.6]). □

Now, let us move to the case when $\varphi \prec \psi$.

Theorem 60 *Let $\varphi, \psi : [0, +\infty) \to [0, +\infty)$ be two Young functions such that*

$$0 < \liminf_{t \to 0^+} \frac{\varphi(\alpha t)}{\psi(t)} \leq \limsup_{t \to 0^+} \frac{\varphi(\alpha t)}{\psi(t)} < +\infty \tag{9}$$

for some $\alpha > 0$. *Moreover, let* $M_g : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$ *be the multiplication operator generated by a function* $g \in \mathrm{YBV}_\varphi/\mathrm{YBV}_\psi$. *Then,* M_g *has closed range if and only if there exists a constant* $\delta > 0$ *such that* $|g(t)| \geq \delta$ *for all* $t \in \operatorname{supp}(g)$.

Proof In view of Proposition 58, we need to prove only the sufficiency part. So let us assume that $\operatorname{supp}(g) \neq \emptyset$ (otherwise there is nothing to show) and that there exists $\delta > 0$ such that $|g(t)| \geq \delta$ for all $t \in \operatorname{supp}(g)$ and let $(y_m)_{m\in\mathbb{N}}$ be a sequence in $\operatorname{Im} M_g$ which converges in YBV_φ to a function y. Then, in particular, $y(t) = 0$ for each $t \in Z_g$.

Now, let us define $x : [0,1] \to \mathbb{R}$ by the formula

$$x(t) = \begin{cases} y(t)/g(t), & \text{if } t \in \operatorname{supp}(g), \\ 0, & \text{otherwise.} \end{cases}$$

Our aim is to show that $x \in \mathrm{YBV}_\psi$. Let $0 = t_0 < \ldots < t_n = 1$ be an arbitrary finite partition of the interval $[0,1]$ and let $\lambda := \min\left\{\frac{\delta^2\mu}{2\|g\|_\infty+1}, \frac{\delta^2\mu}{2\|y\|_\infty+1}, \delta\mu\right\}$, where the number $\mu > 0$ is such that $\operatorname{var}_\varphi(\mu g) < +\infty$ and $\operatorname{var}_\varphi(\mu y) < +\infty$. (Note that such a number μ exists since $\mathrm{YBV}_\varphi/\mathrm{YBV}_\psi = \mathrm{YBV}_\varphi$.) To estimate the sum $\sum_{k=1}^n \varphi(\lambda|x(t_k) - x(t_{k-1})|)$ let us consider four cases.

Case 1: $t_{k-1}, t_k \in Z_g$. Then, $\varphi(\lambda|x(t_k) - x(t_{k-1})|) = 0$.
Case 2: $t_{k-1} \in Z_g$ and $t_k \notin Z_g$. Then,

$$\varphi(\lambda|x(t_k) - x(t_{k-1})|) = \varphi\left(\lambda\,\frac{|y(t_k)|}{|g(t_k)|}\right) \leq \varphi\left(\frac{\lambda}{\delta}|y(t_k)|\right) \leq \varphi(\mu|y(t_k) - y(t_{k-1})|).$$

Case 3: $t_{k-1} \notin Z_g$ and $t_k \in Z_g$. Then, similarly as before,

$$\varphi(\lambda|x(t_k) - x(t_{k-1})|) \leq \varphi(\mu|y(t_k) - y(t_{k-1})|).$$

Case 4: $t_{k-1}, t_k \notin Z_g$. Then,

$$\begin{aligned}\varphi(\lambda|x(t_k) - x(t_{k-1})|) &= \varphi\left(\lambda\,\frac{|g(t_{k-1})y(t_k) - g(t_k)y(t_{k-1})|}{|g(t_{k-1})g(t_k)|}\right) \\ &\leq \varphi\left(\frac{\lambda}{\delta^2}|g(t_{k-1})||y(t_k) - y(t_{k-1})| + \frac{\lambda}{\delta^2}|y(t_{k-1})||g(t_k) - g(t_{k-1})|\right) \\ &\leq \frac{1}{2}\varphi\left(\frac{2\lambda\|g\|_\infty}{\delta^2}|y(t_k) - y(t_{k-1})|\right) + \frac{1}{2}\varphi\left(\frac{2\lambda\|y\|_\infty}{\delta^2}|g(t_k) - g(t_{k-1})|\right) \\ &\leq \frac{1}{2}\varphi\big(\mu|y(t_k) - y(t_{k-1})|\big) + \frac{1}{2}\varphi\big(\mu|g(t_k) - g(t_{k-1})|\big).\end{aligned}$$

Therefore,

$$\begin{aligned}&\sum_{k=1}^n \varphi(\lambda|x(t_k) - x(t_{k-1})|) \\ &\quad\leq \sum_{k=1}^n \varphi\big(\mu|y(t_k) - y(t_{k-1})|\big) + \sum_{k=1}^n \varphi\big(\mu|g(t_k) - g(t_{k-1})|\big) \leq \operatorname{var}_\varphi(\mu y) + \operatorname{var}_\varphi(\mu g).\end{aligned}$$

And so, $x \in \mathrm{YBV}_\varphi$. Using the condition (10) it is easy to see that $\limsup_{t\to 0^+} \psi(\alpha^{-1}t)/\varphi(t) < +\infty$, meaning that $\mathrm{YBV}_\varphi \hookrightarrow \mathrm{YBV}_\psi$ (see Proposition 5).

In particular, $x \in \mathrm{YBV}_\psi$. Since $M_g(x) = y$, this shows that $\operatorname{Im} M_g$ is closed and ends the proof. □

The situation is significantly different when there is a constant $\alpha > 0$ such that

$$\liminf_{t\to 0^+} \frac{\varphi(\alpha t)}{\psi(t)} = 0.$$

This time, however, we will need to assume the δ_2-condition.

Theorem 61 *Let $\varphi, \psi : [0, +\infty) \to [0, +\infty)$ be two Young functions such that $\varphi \in \delta_2$ and*

$$0 = \liminf_{t\to 0^+} \frac{\varphi(\alpha t)}{\psi(t)} \leq \limsup_{t\to 0^+} \frac{\varphi(\alpha t)}{\psi(t)} < +\infty \tag{10}$$

for some $\alpha > 0$. Moreover, let $M_g : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$ be the multiplication operator generated by a function $g \in \mathrm{YBV}_\varphi/\mathrm{YBV}_\psi$. Then, M_g has closed range if and only if $\#\operatorname{supp}(g) < +\infty$.

The proof of Theorem 61 is a somewhat mixture of the proofs of the previous results in this section.

Proof If $\#\operatorname{supp}(g) < +\infty$, then, by Theorem 38, M_g has finite-dimensional, and thus closed, range.

Suppose now that $M_g : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$ has closed range, but $\#\operatorname{supp}(g) = +\infty$. Then, in particular, there is a constant $c > 0$ such that for each $y \in \operatorname{Im} M_g$ there exists some $\eta \in \mathrm{YBV}_\psi$ satisfying $y = M_g(\eta)$ and $\|\eta\|_{\mathrm{YBV}_\psi} \leq c\|y\|_{\mathrm{YBV}_\varphi}$ (see [1, Corollary 2.15]). By Proposition 58 there is also a constant $\delta > 0$ such that $|g(t)| \geq \delta$ for $t \in \operatorname{supp}(g)$. Further, in view of our assumption (10), we can find a sequence $(\tau_k)_{k\in\mathbb{N}}$ in $(0, 1)$ convergent to 0 such that $\lim_{k\to\infty} \frac{\varphi(\alpha\tau_k)}{\psi(\tau_k)} = 0$. Now, fix $k \in \mathbb{N}$ and set $l := \lfloor 1 + 1/\psi(\tau_k)\rfloor$, where $\lfloor\cdot\rfloor$ denotes the floor (or entire) function. Moreover, take any $2l$ distinct points in $\operatorname{supp}(g) \cap (0, 1)$ and arrange them in an ascending order: $t_1 < t_2 < \ldots < t_{2l}$. Let us define the function $x_k : [0, 1] \to \mathbb{R}$ by the formula

$$x_k(t) = \begin{cases} \alpha\tau_k/\|g\|_\infty, & \text{if } t = t_{2i-1} \text{ for } i = 1, \ldots, l, \\ 0, & \text{otherwise.} \end{cases}$$

Clearly, $x_k \in \mathrm{YBV}_\psi$ as a step function. Let ξ_k be a function in YBV_ψ such that $M_g(x_k) = M_g(\xi_k)$ and $\|\xi_k\|_{\mathrm{YBV}_\psi} \leq c\|M_g(x_k)\|_{\mathrm{YBV}_\varphi}$. Notice that the functions x_k and ξ_k must have identical values at each point t_i for $i = 1, \ldots, 2l$ as $|g(t)| \geq \delta$ for $t \in \operatorname{supp}(g)$. In particular, $\alpha\tau_k/\|g\|_\infty = \|x_k\|_\infty \leq \|\xi_k\|_\infty$. To estimate $|\xi_k|_\psi$ from below, let us take an arbitrary positive number λ such that $\operatorname{var}_\psi(\xi_k/\lambda) \leq 1$. Then,

$$\begin{aligned} 1 \geq \operatorname{var}_\psi\left(\frac{\xi_k}{\lambda}\right) &\geq \sum_{i=1}^{l} \psi\left(\frac{|\xi_k(t_{2i}) - \xi_k(t_{2i-1})|}{\lambda}\right) \\ &= \sum_{i=1}^{l} \psi\left(\frac{\alpha\tau_k}{\lambda\|g\|_\infty}\right) = \left\lfloor 1 + \frac{1}{\psi(\tau_k)}\right\rfloor \cdot \psi\left(\frac{\alpha\tau_k}{\lambda\|g\|_\infty}\right). \end{aligned}$$

Note that $\psi^{-1}\big(\lfloor 1+1/\psi(\tau_k)\rfloor^{-1}\big) \le \psi^{-1}(\psi(\tau_k)) = \tau_k$; here, ψ^{-1} denotes the inverse function of ψ, while $\lfloor\cdot\rfloor^{-1}$ denotes the reciprocal of $\lfloor\cdot\rfloor$. Hence, $\lambda \ge \alpha/\|g\|_\infty$, and so $|\xi_k|_\psi \ge \alpha/\|g\|_\infty$.

Our next step is to show that $\|M_g(x_k)\|_{\mathrm{YBV}_\varphi} \to 0$ as $k \to +\infty$. By Lemma 30 and the δ_2-condition we have

$$\begin{aligned}\mathrm{var}_{\varphi}(M_g(x_k)) &\le \sum_{i=1}^{l} \varphi\left(\frac{2|g(t_{2i-1})|\alpha\tau_k}{\|g\|_\infty}\right) \le \lfloor 1+1/\psi(\tau_k)\rfloor \cdot \varphi(2\alpha\tau_k)\\ &\le \Lambda(\alpha)\cdot\big(\varphi(\alpha\tau_k)+\varphi(\alpha\tau_k)/\psi(\tau_k)\big);\end{aligned}$$

let us recall that the function Λ is given by the formula (2). Therefore, $\mathrm{var}_{\varphi}(M_g(x_k)) \to 0$ as $k \to +\infty$ (note that the function φ is continuous and so $\varphi(\alpha\tau_k) \to 0$). This, together with the fact that $\|M_g(x_k)\|_\infty \le \alpha\tau_k$, in view of Proposition 7, implies that $\|M_g(x_k)\|_{\mathrm{YBV}_\varphi} \to 0$ as $k \to +\infty$.

Putting all the pieces together, for any $k \in \mathbb{N}$ we finally have

$$(1+\tau_k)\alpha/\|g\|_\infty \le \|\xi_k\|_{\mathrm{YBV}_\psi} \le c\|M_g(x_k)\|_{\mathrm{YBV}_\varphi}.$$

And, passing to the limit with $k \to +\infty$, yields $\alpha/\|g\|_\infty \le 0$, which is impossible. Thus, $\mathrm{supp}\,(g)$ must be finite. The proof is complete. □

Finally, let us discuss closed range multiplication operators in spaces of functions of bounded Riesz variation.

Theorem 62 *Let* $1 < p, q < +\infty$ *and let* $M_g : \mathrm{RBV}_p \to \mathrm{RBV}_q$ *be the multiplication operator generated by a function* $g \in \mathrm{RBV}_q/\mathrm{RBV}_p$.

(a) *If* $q > p$, *then* M_g *has always closed range* (*as the zero operator*).
(b) *If* $q = p$ *and if either* $\mathrm{supp}\,(g) = \emptyset$ *or* $\mathrm{supp}\,(g) = [0,1]$, *then* M_g *has closed range*.
(c) *If* $q < p$, *then* M_g *has closed range if and only if* $\mathrm{supp}\,(g) = \emptyset$.

Proof Note that (a) is trivial as in this case $\mathrm{RBV}_q/\mathrm{RBV}_p = \{0\}$, and so $\mathrm{Im}\,M_g = \{0\}$.

Now, let $q = p$. In this case $\mathrm{RBV}_q/\mathrm{RBV}_p = \mathrm{RBV}_p$. If $g = 0$, there is nothing to prove. On the other hand, if $\mathrm{supp}\,(g) = [0,1]$, then, since g is continuous, there is a constant $\delta > 0$ such that $|g(t)| \ge \delta$ for all $t \in [0,1]$. This, by Corollary 23 means that M_g is surjective, and hence has closed range.

Finally, let us show (c). Clearly, we need only to prove the necessity part. So let us assume that M_g has closed range, which implies that there exits a constant $c > 0$ such that for each $y \in \mathrm{Im}\,M_g$ there is some $\eta \in \mathrm{RBV}_p$ satisfying $y = M_g(\eta)$ and $\|\eta\|_{\mathrm{RBV}_p} \le c\|y\|_{\mathrm{RBV}_q}$ (see [1, Corollary 2.15]). However, on the contrary, let us suppose that $\mathrm{supp}\,(g) \ne \emptyset$. Since $\mathrm{RBV}_q/\mathrm{RBV}_p = \mathrm{RBV}_q \subseteq C$ for $q < p$, there must exist a non-degenerate interval $[a,b] \subseteq [0,1]$ and a constant $\delta > 0$ such that $|g(t)| \ge \delta$ for $t \in [a,b]$. Let us fix a point $s \in (a,b)$ and for each $n \in \mathbb{N}$ let us define $x_n : [0,1] \to \mathbb{R}$ by

$$x_n(t) = \begin{cases} n^{1/p-1} - n^{1/p}|t-s|, & \text{if } t \in [s-1/n, s+1/n] \cap [0,1],\\ 0, & \text{otherwise.}\end{cases}$$

Note that for all but finitely many numbers n we have $\operatorname{supp}(x_n) \subseteq [a, b]$; let us denote the first term of the sequence $(x_n)_{n\in\mathbb{N}}$ for which this condition holds by N. Moreover,

$$\|x_n\|_{\mathrm{RBV}_p} = n^{1/p-1} + \left(\int_{s-1/n}^{s+1/n} n \,\mathrm{d}t\right)^{1/p} = n^{1/p-1} + 2^{1/p}$$

for $n \geq N$. Now, for each $n \geq N$ let $\xi_n \in \mathrm{RBV}_p$ be such that $M_g(x_n) = M_g(\xi_n)$ and $\|\xi_n\|_{\mathrm{RBV}_p} \leq c\|M_g(x_n)\|_{\mathrm{RBV}_q}$. Note that the equality $M_g(x_n) = M_g(\xi_n)$ implies that ξ_n and x_n coincide on the interval $[a, b]$. In particular, $\|x_n\|_{\mathrm{RBV}_p} \leq \|\xi_n\|_{\mathrm{RBV}_p}$.

We also need to estimate $\|M_g(x_n)\|_{\mathrm{RBV}_q}$. For any $n \geq N$ we have

$$\begin{aligned}\left(\operatorname{var}_q^R(M_g(x_n))\right)^{1/q} &= \|(gx_n)'\|_{L_q} \leq \|g'x_n\|_{L_q} + \|gx_n'\|_{L_q} \\ &\leq \|x_n\|_\infty \|g'\|_{L_q} + \|g\|_\infty \cdot \left(\int_{s-1/n}^{s+1/n} |x_n'(t)|^q \,\mathrm{d}t\right)^{1/q} \\ &= n^{1/p-1} \cdot \|g'\|_{L_q} + \|g\|_\infty \cdot 2^{1/q} n^{1/p-1/q}.\end{aligned}$$

Putting all the pieces together, for $n \geq N$, we obtain

$$n^{1/p-1} + 2^{1/p} \leq \|\xi_n\|_{\mathrm{RBV}_p} \leq c\|M_g(x_n)\|_{\mathrm{RBV}_q} \leq cn^{1/p-1} \cdot \|g'\|_{L_q} + c\|g\|_\infty \cdot 2^{1/q} n^{1/p-1/q}.$$

Since $1/p - 1 < 0$ and $1/p - 1/q < 0$, passing with $n \to +\infty$ in the above estimate, yields $2^{1/p} \leq 0$, which is an utter absurd. Thus, $\operatorname{supp}(g) = \emptyset$. □

At this point, it should not come as a surprise that we need to consider the case $\mathrm{RBV}_1 = BV$ separately.

Theorem 63 *Let $1 < p < +\infty$ and let $M_g : \mathrm{RBV}_p \to \mathrm{BV}$ be the multiplication operator generated by a function $g \in \mathrm{BV}$. Then, M_g has closed range if and only if $\#\operatorname{supp}(g) < +\infty$.*

Proof If $\#\operatorname{supp}(g) < +\infty$, then the claim follows from Theorem 40—cf. the proof of Theorem 59.

We can thus assume that M_g has closed range, that is, there is a constant $c > 0$ such that for each $y \in \operatorname{Im} M_g$ there exists some $\eta \in \mathrm{RBV}_p$ satisfying $y = M_g(\eta)$ and $\|\eta\|_{\mathrm{RBV}_p} \leq c\|y\|_{BV}$ (see [1, Corollary 2.15]). Suppose, however, that $\operatorname{supp}(g)$ is infinite. If $\operatorname{supp}(g)$ were uncountable, in view of the fact that functions of bounded Jordan variation have countably many points of discontinuity, there would exist an interval $[a, b] \subseteq [0, 1]$ and a constant $\delta > 0$ such that $|g(t)| \geq \delta$ for all $t \in [a, b]$ (cf. the proof of Theorem 49). Using exactly the same approach as in the proof of Theorem 62 (c) with exactly the same sequence $(x_n)_{n\in\mathbb{N}}$ and some appropriate sequence $(\xi_n)_{n\in\mathbb{N}}$, it could be then shown that

$$n^{1/p-1} + 2^{1/p} \leq \|\xi_n\|_{\mathrm{RBV}_p} \leq c\|M_g(x_n)\|_{BV} \leq c\|x_n\|_{BV}\|g\|_{BV} \leq 3n^{1/p-1}c\|g\|_{BV}$$

for all n sufficiently large. Letting $n \to +\infty$, this would lead to a contradiction. Thus, $\operatorname{supp}(g)$ must be countable. But then, M_g is compact by Theorem 49. Since it has closed range, we deduce that $\#\operatorname{supp}(g) < +\infty$ (cf. the alternative proof of Theorem 59). This completes the proof. □

Remark 64 The case $M_g : \mathrm{BV} \to \mathrm{RBV}_p$ for $1 < p < +\infty$ is trivial since then $\mathrm{RBV}_p/BV = \{0\}$, and so M_g, as the zero operator, has closed range.

5.6 Fredholm multiplication operators

We devote this last section to studying Fredholm multiplication operators in certain BV spaces. Let us recall that a bounded linear operator $L : X \to Y$ acting between Banach spaces is called a *Fredholm operator* if $\dim \operatorname{Ker} L < +\infty$, $\operatorname{Im} L$ is closed in Y and $\dim(Y/\operatorname{Im} L) < +\infty$. The *index* of a Fredholm operator is defined by $\operatorname{ind} L := \dim \operatorname{Ker} L - \dim(Y/\operatorname{Im} L)$. (For more information on Fredholm operators see [7] or [3, Section 11.6].)

Remark 65 As it is not possible that both the sets Z_g and $\operatorname{supp}(g)$ are finite, in view of Theorems 16, 59 and 61, there are no Fredholm multiplication operators $M_g : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$, when either $\varphi \not\prec \psi$, or $\varphi \in \delta_2$ and

$$0 = \liminf_{t\to 0^+} \frac{\varphi(\alpha t)}{\psi(t)} \le \limsup_{t\to 0^+} \frac{\varphi(\alpha t)}{\psi(t)} < +\infty$$

for some $\alpha > 0$.

On a more positive note, we have the following result.

Theorem 66 *Let $\varphi, \psi : [0, +\infty) \to [0, +\infty)$ be two Young functions such that*

$$0 < \liminf_{t\to 0^+} \frac{\varphi(\alpha t)}{\psi(t)} \le \limsup_{t\to 0^+} \frac{\varphi(\alpha t)}{\psi(t)} < +\infty$$

for some $\alpha > 0$. Moreover, let $M_g : \mathrm{YBV}_\psi \to \mathrm{YBV}_\varphi$ be the multiplication operator generated by a function $g \in \mathrm{YBV}_\varphi/\mathrm{YBV}_\psi$. Then, M_g is a Fredholm operator if and only if $\#Z_g < +\infty$ and there exists a constant $\delta > 0$ such that $|g(t)| \ge \delta$ for all $t \in \operatorname{supp}(g)$; in addition, in such a case, $\operatorname{ind} M_g = 0$.

Proof Note that we need to only prove the sufficiency, because the necessity part follows from Theorems 16 and 60.

So, let us assume that $\#Z_g = n$ for some $n \in \mathbb{N} \cup \{0\}$ and that there exists a constant $\delta > 0$ such that $|g(t)| \ge \delta$ for all $t \in \operatorname{supp}(g)$. If $n = 0$, then $\operatorname{supp}(g) = [0, 1]$, and so the operator M_g is an isomorphism of YBV_ψ and YBV_φ (cf. the proof of Theorem 60), so it is clearly a Fredholm operator. Now, let us assume that $n \ge 1$, and let $Z_g = \{t_1, \dots, t_n\}$. We may assume that those points are arranged in an ascending order, that is, $0 \le t_1 < t_2 < \dots < t_n \le 1$. From Theorems 16 and 60 we know that $\dim \operatorname{Ker} M_g = n < +\infty$ and that $\operatorname{Im} M_g$ is closed. It remains to show that $\dim(\mathrm{YBV}_\varphi/\operatorname{Im} M_g) < +\infty$. We will show that $\dim(\mathrm{YBV}_\varphi/\operatorname{Im} M_g) = n$. To this end we will prove that $\operatorname{Im} M_g$ coincides with the set $\{y \in \mathrm{YBV}_\varphi | y(t_i) = 0 \text{ for } i = 1, \dots, n\}$. As $g(t_i) = 0$ for $i = 1, \dots, n$, it is clear that $\operatorname{Im} M_g \subseteq \{y \in \mathrm{YBV}_\varphi | y(t_i) = 0 \text{ for } i = 1, \dots, n\}$.

Now, let $y \in \mathrm{YBV}_\varphi$ be an arbitrary function which vanishes at the points t_i, $i = 1, \dots, n$. Define $x : [0, 1] \to \mathbb{R}$ by

$$x(t) = \begin{cases} 0, & \text{if } t = t_i \text{ for some } i = 1, \dots, n, \\ y(t)/g(t), & \text{if } t \neq t_i \text{ for every } i = 1, \dots, n. \end{cases}$$

Using the same approach as in the proof of Theorem 60 it can be shown that $x \in \mathrm{YBV}_\psi$. Since $y = M_g(x)$, we see that $y \in \operatorname{Im} M_g$. This, in turn, implies that $\{y \in \mathrm{YBV}_\varphi | y(t_i) = 0 \text{ for } i = 1, \dots, n\} \subseteq \operatorname{Im} M_g$.

To end the proof it suffices to note that $\mathrm{YBV}_\varphi / \operatorname{Im} M_g$ is linearly isomorphic with $\mathbb{R}^n$ by the map $y + \operatorname{Im} M_g \mapsto (y(t_1), \dots, y(t_n))$. □

We end this section and the whole paper with a remark concerning Fredholm multiplication operators in RBV_p spaces.

Remark 67 Let $1 < p, q < +\infty$. Note that due to Theorem 62 and the characterization of the multiplier class $\mathrm{RBV}_q/\mathrm{RBV}_p$, there are no Fredholm multiplication operators $M_g : \mathrm{RBV}_p \to \mathrm{RBV}_q$ when $q \neq p$.

Similarly, there are no Fredholm multiplication operators acting to BV of from BV. The latter case follows from the fact that any multiplication operator $M_g : \mathrm{BV} \to \mathrm{RBV}_p$, where $1 < p < +\infty$, is the zero operator. To prove the former case, let us notice that if $M_g : \mathrm{RBV}_p \to \mathrm{BV}$, where $1 < p < +\infty$, is a Fredholm operator generated by $g \in \mathrm{BV}$, then it has closed range, and so, by Theorem 63, $\# \operatorname{supp}(g) < +\infty$. But this, in view of Theorem 16, implies that $\dim \operatorname{Ker} M_g = +\infty$ and leads to a contradiction (recall that a Fredholm operator has a finite-dimensional kernel).

The problem whether there are Fredholm multiplication operators $M_g : \mathrm{RBV}_p \to \mathrm{RBV}_p$, where $1 < p < +\infty$, other than automorphisms is still open.

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