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Foliated Hopf hypersurfaces in complex hyperbolic quadrics

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Abstract
This paper deals with a limiting case motivated by contact geometry. The limiting case of a tensorial characterization of contact hypersurfaces in Kähler manifolds leads to Hopf hypersurfaces whose maximal complex subbundle of the tangent bundle is integrable. It is known that in non-flat complex space forms and in complex quadrics such real hypersurfaces do not exist, but the existence problem in other irreducible Kähler manifolds is open. In this paper we construct explicitly a one-parameter family of homogeneous Hopf hypersurfaces, whose maximal complex subbundle of the tangent bundle is integrable, in a Hermitian symmetric space of non-compact type and rank two. These are the first known examples of such real hypersurfaces in irreducible Kähler manifolds.

Keywords Kähler manifold · Hermitian symmetric space · Complex hyperbolic quadric · Real hypersurface · Hopf hypersurface · Homogeneous real hypersurface · Contact hypersurface · Maximal complex subbundle · Riemannian foliation

Mathematics Subject Classification Primary 53C15 · 53C35 · 53C40 · 53C55 · Secondary 53C12 · 53D10

1 Introduction

We start with the motivation for this paper. A contact manifold is a smooth odd-dimensional manifold M together with a 1-form η on M satisfying $\eta \wedge (d\eta)^{n-1} \neq 0$, where $\dim_{\mathbb{R}}(M) = 2n - 1$. Such a 1-form η is called a contact form. The kernel of η defines a hyperplane distribution $\mathcal{C}$ on M, the so-called contact distribution. The contact condition $\eta \wedge (d\eta)^{n-1} \neq 0$ means that the maximal possible dimension of a submanifold of M all of whose tangent spaces are contained in $\mathcal{C}$ is equal to $n-1$. The contact condition therefore is a measure for maximal non-integrability of $\mathcal{C}$.

Let $\bar{M}$ be a Kähler manifold with Kähler structure J, Kähler metric g and $n = \dim_{\mathbb{C}}(\bar{M}) \geq 2$. Let M be a real hypersurface in $\bar{M}$ and (ϕ, ξ, η, g) be the induced almost contact metric structure on M (see Sect. 2). The subbundle $\mathcal{C} = \ker(\eta) = TM \cap J(TM)$ of TM is the maximal complex subbundle of the tangent bundle TM. The real hypersurface

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M is said to be a contact hypersurface if there exists an everywhere non-zero smooth function $f : M \to \mathbb{R}$ so that $d\eta = 2f\omega$, where ω is the fundamental 2-form on M defined by $\omega(X, Y) = g(\phi X, Y)$ for all $X, Y \in \mathfrak{X}(M)$. The fundamental 2-form ω is always closed, which implies $\eta \wedge d\eta^{n-1} = (2f)^{n-1}(\eta \wedge \omega^{n-1}) \neq 0$ if M is a contact hypersurface. Thus every contact hypersurface in a Kähler manifold is a contact manifold. In this situation the maximal complex subbundle $\mathcal{C}$ of the tangent bundle of the contact hypersurface coincides with the contact distribution. A natural problem is to determine the contact hypersurfaces in Kähler manifolds.

The first systematic study of contact hypersurfaces in Kähler manifolds was carried out by Okumura [14]. Okumura proved the following very useful characterization of contact hypersurfaces in Kähler manifolds: A real hypersurface M in a Kähler manifold $\bar{M}$ is a contact hypersurface if and only if there exists an every non-zero smooth function $f : M \to \mathbb{R}$ so that that the shape operator A of M and the structure tensor field ϕ satisfy $A\phi + \phi A = 2f\phi$. It is not difficult to prove that the function f is constant when $n > 2$ (see [3], Proposition 3.5.4). Starting from Okumura's work, contact hypersurfaces were classified in various Hermitian symmetric spaces (see [3] for an overview). The motivation for this paper is to understand the limiting case $f = 0$. We will show (see Proposition 2.2) that the limiting case $f = 0$ characterizes Hopf hypersurfaces in Kähler manifolds for which the maximal complex subbundle $\mathcal{C}$ is integrable. For the concept of Hopf hypersurfaces see Sect. 2.

The totally geodesic real hypersurface $\mathbb{R}^{2n-1}$ in the complex Euclidean space $\mathbb{C}^n$ is an elementary example of a Hopf hypersurface whose maximal complex subbundle $\mathcal{C}$ is integrable. In contrast, it is quite remarkable and not obvious that in non-flat complex space forms there are no Hopf hypersurfaces whose maximal complex subbundle $\mathcal{C}$ is integrable. This is not difficult to prove for the complex projective space $\mathbb{C}P^n(c)$ with the Fubini-Study metric of constant holomorphic sectional curvature $c > 0$, but the proof is quite involved for the complex hyperbolic space $\mathbb{C}H^n(c)$ with the Bergman metric of constant holomorphic sectional curvature $c < 0$. A detailed discussion of these two cases can be found in Section 2 of [13]. These non-existence results raise the existence question for other irreducible Kähler manifolds. In [3], the geometry of real hypersurfaces in some irreducible Hermitian symmetric spaces of rank 2 was investigated. One of these Hermitian symmetric spaces is the Grassmann manifold $SO_{2+n}/(SO_2 \times SO_n)$ of oriented 2-planes in $\mathbb{R}^{2+n}$, which is isometric to the complex quadric Q^n in $\mathbb{C}P^{n+1}(c)$ (with a suitable normalization of the metric). From the investigations in [3], Section 6.4, we can conclude that there are no Hopf hypersurfaces in this Hermitian symmetric space for which $\mathcal{C}$ is integrable.

In this paper we investigate the existence question in the dual Hermitian symmetric space of non-compact type, the complex hyperbolic quadric $Q^{n*} = SO^o_{2,n}/(SO_2 \times SO_n)$. Surprisingly, we can construct a one-parameter family of pairwise non-congruent homogeneous Hopf hypersurfaces in Q^{n*} whose maximal complex subbundle $\mathcal{C}$ is integrable.

Theorem 1.1 *There exists a one-parameter family M^{2n-1}_α, $0 \leq \alpha < \infty$, of (pairwise non-congruent) homogeneous Hopf hypersurfaces, whose maximal complex subbundle of the tangent bundle is integrable, in the Hermitian symmetric space $Q^{n*} = SO^o_{2,n}/(SO_2 \times SO_n)$, $n \geq 3$.*

We give a brief geometric description of these real hypersurfaces. We normalize the Riemannian metric on Q^{n*} so that the minimum of the sectional curvature is equal to -4. The complex hyperbolic quadric Q^{n*} is equipped with a circle bundle $\mathfrak{A}_0$ of real structures (see Sect. 3). This circle bundle determines a maximal $\mathfrak{A}_0$-invariant subbundle $\mathcal{Q}$ of the tangent bundle TM of M. The maximal Satake compactification of $Q^{n*} = SO^o_{2,n}/(SO_2 \times SO_n)$ has two boundary components of rank 1, namely a complex hyperbolic line $B_1 \cong \mathbb{C}H^1(-4)$ of constant (holomorphic) sectional curvature -4 and a real hyperbolic space $B_2 \cong \mathbb{R}H^{n-2}(-2)$ of constant sectional curvature -2. It is an interesting fact that all nonzero tangent vectors of B_1 are singular tangent vectors of Q^{n*} of a particular type (that is, tangent vectors that are contained in more than one maximal flat of Q^{n*}). In [4], we developed a technique, the so-called canonical extension method, for extending isometric actions on boundary components of irreducible Riemannian symmetric spaces of noncompact type to isometric actions on the entire symmetric space. This method can be used to extend submanifolds in boundary components. By extending a point in the boundary component $B_1 \cong \mathbb{C}H^1(-4)$ we obtain an isometric embedding P^{n-1} of the complex hyperbolic space $\mathbb{C}H^{n-1}(-4)$ into Q^{n*} as a homogeneous complex hypersurface. This construction will be explained in detail in Sect. 4, where we will also investigate the geometry of this homogeneous complex hypersurface.

This homogeneous complex hypersurface P^{n-1} will appear as the integral manifolds of the integrable distribution $\mathcal{C}$ in our examples. The Langlands decomposition of the parabolic subgroup of $SO^o_{2,n}$ with boundary component $B_1 \cong \mathbb{C}H^1(-4)$ induces a horospherical decomposition $B_1 \times \mathbb{R} \times H^{2n-3}$ of Q^{n*}, where H^{2n-3} is the $(2n-3)$-dimensional Heisenberg group with 1-dimensional center. The product $\mathbb{R} \times H^{2n-3}$ corresponds to the homogeneous complex hypersurface $P^{n-1} \cong \mathbb{C}H^{n-1}(-4)$. Now take any complete curve γ in the boundary component $B_1 \cong \mathbb{C}H^1(-4)$ with constant geodesic curvature $\alpha \geq 0$. The curve γ is a geodesic in $\mathbb{C}H^1(-4)$ if $\alpha = 0$, an equidistant curve to a geodesic in $\mathbb{C}H^1(-4)$ if $0 < \alpha < 2$, a horocycle if $\alpha = 2$, or a closed circle in $\mathbb{C}H^1(-4)$ if $2 < \alpha < \infty$. Sliding the homogeneous complex hypersurface $P^{n-1} \cong \mathbb{C}H^{n-1}(-4)$ along the curve γ in a suitable way, we obtain a homogeneous Hopf hypersurface M^{2n-1}_α in Q^{n*} whose maximal complex subbundle $\mathcal{C}$ is integrable. We will see that the homogeneous real hypersurface M^{2n-1}_α has constant principal curvatures $\alpha, 0, +1, -1$ with multiplicities $1, 2, n-1, n-1$, respectively. The principal curvature space T_α is equal to the orthogonal complement $\mathcal{C}^\perp$ of $\mathcal{C}$ in TM. The principal curvature space T_0 is equal to the orthogonal complement $\mathcal{C} \ominus \mathcal{Q}$ of $\mathcal{Q}$ in $\mathcal{C}$. The principal curvature spaces T_1 and T_{-1} span $\mathcal{Q}$ are mapped into each other by the structure tensor field ϕ and are equal to the ±1-eigenspaces of the restriction to $\mathcal{Q}$ of a suitable real structure in $\mathfrak{A}_0$. The hypersurfaces M^{2n-1}_α will in fact be constructed through an algebraic process, and the "sliding" description is a geometric interpretation of this algebraic construction, which will be explained thoroughly during the construction process. The homogeneous real hypersurface M^{2n-1}_α is diffeomorphic to $\mathbb{R}^{2n-1}$ for $0 \leq \alpha \leq 2$ and diffeomorphic to $S^1 \times \mathbb{R}^{2n-2}$ for $2 < \alpha < \infty$.

We point out that none of the homogeneous real hypersurfaces M^{2n-1}_α in Theorem 1.1 arises as a limit of contact hypersurfaces in Q^{n*}. The classification of contact hypersurfaces in Q^{n*} can be found in Section 7.8 of [3]. For every real number $f > 0$ there exists, up to isometric congruence, a unique connected complete contact hypersurface $\tilde{M}^{2n-1}_f$ in Q^{n*} satisfying $A\phi + \phi A = 2f\phi$. This family $\tilde{M}^{2n-1}_f$ of contact hypersurfaces collapses to a totally geodesic complex embedding of the complex hyperbolic quadric Q^{n-1^*} into Q^{n*} when taking the limit $f \to 0$, and so $\lim_{f\to 0} \tilde{M}^{2n-1}_f = Q^{n-1^*}$ is not a real hypersurface.

The paper is organized as follows. In Sect. 2 we introduce basic concepts from almost contact metric geometry in Kähler manifolds and provide characterizations of Hopf hypersurfaces and of real hypersurfaces satisfying $A\phi + \phi A = 0$. In Sect. 3 we present two models for the complex hyperbolic quadric $Q^{n*} = SO^o_{2,n}/(SO_2 \times SO_n)$. The first one is the standard symmetric space model, and the second one is the solvable Lie group model originating from an Iwasawa decomposition of $SO^o_{2,n}$. The interplay between both models allows us to switch between geometric and algebraic interpretations of relevant concepts. In Sect. 4 we construct the isometric embedding of the complex hyperbolic space $\mathbb{C}H^{n-1}(-4)$ as a homogeneous complex hypersurface P^{n-1} in Q^{n*} and discuss aspects of the geometry of this embedding. The homogeneous real hypersurfaces M^{2n-1}_α $(2 < \alpha)$ will be constructed in Sect. 5 as the tubes around the homogeneous complex hypersurface P^{n-1} in Q^{n*}. In Sect. 6 we use the theory of parabolic subalgebras of real semisimple Lie algebras for the construction of the minimal homogeneous real hypersurface M^{2n-1}_0. In Sect. 7 we construct the homogeneous real hypersurfaces M^{2n-1}_α $(0 < \alpha < 2)$ as the equidistant hypersurfaces to M^{2n-1}_0. The homogeneous real hypersurface M^{2n-1}_2 will be constructed in Sect. 8 as the canonical extension of a horocycle in the boundary component $B_1 \cong \mathbb{C}H^1(-4)$. We will also investigate the geometry of the homogeneous real hypersurfaces M^{2n-1}_α in the corresponding sections. In Sect. 9 we investigate the curvature of the homogeneous real hypersurfaces M^{2n-1}_α.

2 The maximal complex subbundle of the tangent bundle

Let $\bar{M}$ be a Kähler manifold with Kähler structure J and Kähler metric g. We always assume $n = \dim_{\mathbb{C}}(\bar{M}) \geq 2$. Let M be a real hypersurface in $\bar{M}$. We will denote the induced Riemannian metric on M also by g. The Levi Civita covariant derivative of $\bar{M}$ and M is denoted by $\bar{\nabla}$ and ∇, respectively. The Lie algebra of smooth vector fields on M is denoted by $\mathfrak{X}(M)$.

Let ζ be a (local) unit normal vector field on M. We denote by $A = A_\zeta$ the shape operator of M with respect to ζ. The unit vector field

$$\xi = -J\zeta$$

is the Reeb vector field on M. The flow of the Reeb vector field ξ is the Reeb flow on M. We define a 1-form η on M by

$$\eta(X) = g(X, \xi)$$

for all $X \in \mathfrak{X}(M)$ and a skew-symmetric tensor field ϕ on M by decomposing JX into its tangential component ϕX and its normal component $g(JX, \zeta)\zeta$, that is,

$$JX = \phi X + g(JX, \zeta)\zeta = \phi X + \eta(X)\zeta$$

for all $X \in \mathfrak{X}(M)$. The 1-form η is the almost contact form on M, and the skew-symmetric tensor field ϕ is the structure tensor field on M. The quadruple (ϕ, ξ, η, g) is the induced almost contact metric structure on M. Note that

$$\eta(\xi) = 1,\ \phi\xi = 0 \text{ and } \phi^2 X = -X + \eta(X)\xi$$

for all $X \in \mathfrak{X}(M)$. Using the Kähler property $\bar{\nabla}J = 0$ and the Weingarten formula we obtain

$$0 = (\bar{\nabla}_X J)\zeta = \bar{\nabla}_X J\zeta - J\bar{\nabla}_X \zeta = -\bar{\nabla}_X \xi + JAX$$

for all $X \in \mathfrak{X}(M)$. The tangential component of this equation induces the useful equation

$$\nabla_X \xi = \phi AX$$

for all $X \in \mathfrak{X}(M)$.

The subbundle

$$\mathcal{C} = \ker(\eta) = TM \cap J(TM)$$

of the tangent bundle TM of M is the maximal complex subbundle of TM. We denote by $\Gamma(\mathcal{C})$ the set of all vector fields X on M with values in $\mathcal{C}$, that is,

$$\begin{aligned}\Gamma(\mathcal{C}) &= \{X \in \mathfrak{X}(M) \,:\, X_p \in \mathcal{C}_p \text{ for all } p \in M\}\\ &= \{X \in \mathfrak{X}(M) \,:\, \eta(X) = 0\}.\end{aligned}$$

The real hypersurface M is called a Hopf hypersurface if the Reeb flow on M is a geodesic flow, that is, if the integral curves of the Reeb vector field ξ are geodesics in M. We have the following characterization of Hopf hypersurfaces.

Proposition 2.1 *Let M be a real hypersurface in a Kähler manifold $\bar{M}$ with induced almost contact metric structure (ϕ, ξ, η, g). The following statements are equivalent*:

(i) *M is a Hopf hypersurface in $\bar{M}$*;
(ii) $\nabla_\xi \xi = 0$;
(iii) *The Reeb vector field ξ is a principal curvature vector of M at every point*;
(iv) *The maximal complex subbundle $\mathcal{C}$ of TM is invariant under the shape operator A* of M, *that is*, $A\mathcal{C} \subseteq \mathcal{C}$.

Proof Let $p \in M$ and $c : I \to M$ be an integral curve of the Reeb vector field ξ with $0 \in I$ and $c(0) = p$. Then we have $\nabla_{\xi_p} \xi = \nabla_{\dot{c}(0)} \xi = (\xi \circ c)'(0) = \dot{c}'(0)$. If M is a Hopf hypersurface, then we have $\dot{c}'(0) = 0$ by definition and therefore $\nabla_{\xi_p} \xi = 0$. Since this holds at any point $p \in M$, we obtain $\nabla_\xi \xi = 0$. Conversely, if $\nabla_\xi \xi = 0$, then $\dot{c}' = \nabla_{\dot{c}} \xi = \nabla_{\xi \circ c} \xi = 0$ for any integral curve c of ξ. Thus any integral curve of ξ is a geodesic in M and hence M is a Hopf hypersurface. This establishes the equivalence of (i) and (ii)

The kernel $\ker(\phi)$ of the structure tensor field ϕ is spanned by the Reeb vector field, that is, $\ker(\phi) = \mathbb{R}\xi$. Since $\nabla_\xi \xi = \phi A \xi$, we therefore see that $\nabla_\xi \xi = 0$ if and only if $A\xi \in \mathbb{R}\xi$, which shows that (ii) and (iii) are equivalent.

We have the orthogonal decomposition $TM = \mathcal{C} \oplus \mathbb{R}\xi$. Since the shape operator A is self-adjoint, the equivalence of (iii) and (iv) is obvious. □

The next result provides a characterization of real hypersurfaces when taking the limit $f \to 0$ in Okumura's characterization $A\phi + \phi A = 2f\phi$ of contact hypersurfaces in Kähler manifolds.

Proposition 2.2 *Let M be a real hypersurface in a Kähler manifold $\bar{M}$ with induced almost contact metric structure (ϕ, ξ, η, g). The following statements are equivalent*:

(i) *The almost contact form η is closed, that is*, $d\eta = 0$.
(ii) *The shape operator A* of *M and the structure tensor field ϕ satisfy*

$$A\phi + \phi A = 0.$$

(iii) *The real hypersurface M is a Hopf hypersurface, and the maximal complex subbundle $\mathcal{C}$* of *TM is integrable.*

Proof Using the equation $\nabla_X \xi = \phi AX$, the exterior derivative $d\eta$ of η is

$$\begin{aligned} d\eta(X,Y) &= d(\eta(Y))(X) - d(\eta(X))(Y) - \eta([X,Y]) \\ &= Xg(Y,\xi) - Yg(X,\xi) - g([X,Y],\xi) \\ &= g(\nabla_X Y,\xi) + g(Y,\nabla_X \xi) - g(\nabla_Y X,\xi) - g(X,\nabla_Y \xi) - g([X,Y],\xi) \\ &= g(Y,\phi AX) - g(X,\phi AY) \\ &= g((A\phi + \phi A)X,Y) \end{aligned}$$

for all $X, Y \in \mathfrak{X}(M)$. It follows that η is closed if and only if $A\phi + \phi A = 0$, which shows that (i) and (ii) are equivalent.

The above calculations imply that for $X, Y \in \Gamma(\mathcal{C})$ we have

$$\eta([X,Y]) = -d\eta(X,Y) = -g((A\phi + \phi A)X,Y).$$

It follows that the distribution $\mathcal{C}$ is involutive if and only if $g((A\phi + \phi A)X,Y) = 0$ holds for all $X, Y \in \Gamma(\mathcal{C})$. We have $g((A\phi + \phi A)\xi,Y) = g(\phi A\xi,Y) = 0$ for all $Y \in \Gamma(\mathcal{C})$ if and only if $A\xi \in \mathbb{R}\xi$, that is, if and only if M is a Hopf hypersurface. We always have $g((A\phi + \phi A)\xi,\xi) = 0$. Using Frobenius Theorem we can now conclude the equivalence of (ii) and (iii). □

3 The complex hyperbolic quadric

The complex hyperbolic quadric is the Riemannian symmetric space

$$Q^{n*} = SO^o_{2,n}/(SO_2 \times SO_n),\ n \geq 1,$$

where $SO^o_{2,n}$ denotes the identity component of the indefinite special orthogonal group $SO_{2,n}$ and $SO_2 \times SO_n$ is embedded canonically into $SO^o_{2,n}$. The complex hyperbolic quadric $SO^o_{2,n}/(SO_2 \times SO_n)$ is the non-compact dual symmetric space of the complex quadric $SO_{2+n}/(SO_2 \times SO_n)$. We put $G = SO^o_{2,n}$, $K = SO_2 \times SO_n$ and denote by $o \in Q^{n*}$ the "base point" $I_{2+n}K$ of the homogeneous space G/K, where $I_{2+n} \in G$ is the identity $((2+n) \times (2+n))$-matrix. Then K is the isotropy group of G at o. We now describe the construction of the complex hyperbolic quadric as a Riemannian symmetric space in some more detail.

We denote by $M_{2,n}(\mathbb{R})$ the real vector space of $(2 \times n)$-matrices with real coefficients. Let

$$\mathfrak{g} = \mathfrak{so}_{2,n} = \left\{ \begin{pmatrix} A_1 & B \\ B^\top & A_2 \end{pmatrix} : A_1 \in \mathfrak{so}_2,\ A_2 \in \mathfrak{so}_n,\ B \in M_{2,n}(\mathbb{R}) \right\}$$

be the Lie algebra of $G = SO^o_{2,n}$ and

$$\mathfrak{k} = \mathfrak{so}_2 \oplus \mathfrak{so}_n = \left\{ \begin{pmatrix} A_1 & 0_{2,n} \\ 0_{n,2} & A_2 \end{pmatrix} : A_1 \in \mathfrak{so}_2,\ A_2 \in \mathfrak{so}_n \right\}$$

be the Lie algebra of $K = SO_2 \times SO_n$. Let

$$B : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R},\ (X,Y) \mapsto \mathrm{tr}(\mathrm{ad}(X)\mathrm{ad}(Y)) = n\,\mathrm{tr}(XY)$$

be the Killing form of $\mathfrak{g}$ and

$$\mathfrak{p} = \left\{ \begin{pmatrix} 0_{2,2} & B \\ B^\top & 0_{n,n} \end{pmatrix} : B \in M_{2,n}(\mathbb{R}) \right\}$$

be the orthogonal complement of $\mathfrak{k}$ in $\mathfrak{g}$ with respect to B. The resulting decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is a Cartan decomposition of $\mathfrak{g}$. We identify the tangent space $T_o Q^{n*}$ of Q^{n*} at o with $\mathfrak{p}$ in the usual way.

The Cartan involution $\theta \in \mathrm{Aut}(\mathfrak{g})$ on $\mathfrak{g}$ is given by

$$\theta(X) = I_{2,n} X I_{2,n} \text{ with } I_{2,n} = \begin{pmatrix} -I_2 & 0_{2,n} \\ 0_{n,2} & I_n \end{pmatrix},$$

where I_2 and I_n are the identity (2×2)-matrix and $(n \times n)$-matrix, respectively. Then

$$B_\theta : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}, \ (X, Y) = -B(X, \theta(Y))$$

is a positive definite $\mathrm{Ad}(K)$-invariant inner product on $\mathfrak{g}$. The Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ is orthogonal with respect to B_θ. The restriction of B_θ to $\mathfrak{p} \times \mathfrak{p}$ induces a G-invariant Riemannian metric g_{B_θ} on Q^{n*}, which is often referred to as the standard homogeneous metric on Q^{n*}. The complex hyperbolic quadric (Q^{n*}, g_{B_θ}) is an Einstein manifold with Einstein constant $-\frac{1}{2}$ (see [18] and use duality between Riemannian symmetric spaces of compact type and of non-compact type). We renormalize the standard homogeneous metric g_{B_θ} so that the Einstein constant of the renormalized Riemannian metric g is equal to $-2n$, that is,

$$g_{B_\theta} = 4ng.$$

This renormalization implies that the minimum of the sectional curvature of (Q^{n*}, g) is equal to -4. Note that (Q^{1^*}, g) is isometric to the complex hyperbolic line $\mathbb{C}H^1(-4)$ and (Q^{2^*}, g) is isometric to the Riemannian product $\mathbb{C}H^1(-4) \times \mathbb{C}H^1(-4)$ of two complex hyperbolic lines. For $n \geq 3$, (Q^{n*}, g) is an irreducible Riemannian symmetric space of non-compact type and rank 2. We assume $n \geq 3$ in the following.

The Lie algebra $\mathfrak{k}$ decomposes orthogonally into $\mathfrak{k} = \mathfrak{so}_2 \oplus \mathfrak{so}_n$. The first factor $\mathfrak{so}_2$ is the 1-dimensional center of $\mathfrak{k}$. The adjoint action of

$$Z = \begin{pmatrix} 0 & -1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in SO_2 \subset SO_2 \times SO_n = K$$

on $\mathfrak{p}$ induces a Kähler structure J on Q^{n*}. In this way (Q^{n*}, g, J) becomes a Hermitian symmetric space.

We define

$$c_0 = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix} \in O_2 \times SO_n.$$

Note that $c_0 \notin K$, but c_0 is in the isotropy group at o of the full isometry group of (Q^{n*}, g). The adjoint transformation $\mathrm{Ad}(c_0)$ leaves $\mathfrak{p}$ invariant and $C_0 = \mathrm{Ad}(c_0)|_{\mathfrak{p}}$ is an anti-linear involution on $\mathfrak{p} \cong T_oQ^{n*}$ satisfying $C_0J + JC_0 = 0$. In other words, C_0 is a real structure on T_oQ^{n*}. The involution C_0 commutes with $\mathrm{Ad}(g)$ for all $g \in SO_n \subset K$ but not for all $g \in K$. More precisely, for $g = (g_1, g_2) \in K$ with $g_1 \in SO_2$ and $g_2 \in SO_n$, say $g_1 = \begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}$ with $\varphi \in \mathbb{R}$, so that $\mathrm{Ad}(g_1)$ corresponds to multiplication with the complex number $\mu = e^{i\varphi}$, we have

$$C_0 \circ \mathrm{Ad}(g) = \mu^{-2}\mathrm{Ad}(g) \circ C_0.$$

It follows that we have a circle of real structures

$$\{\cos(\varphi)C_0 + \sin(\varphi)JC_0 \,:\, \varphi \in \mathbb{R}\}.$$

This set is $\mathrm{Ad}(K)$-invariant and therefore generates an $\mathrm{Ad}(G)$-invariant S^1-subbundle $\mathfrak{A}_0$ of the endomorphism bundle $\mathrm{End}(TQ^{n*})$, consisting of real structures (or conjugations) on the tangent spaces of Q^{n*}. This S^1-bundle naturally extends to an $\mathrm{Ad}(G)$-invariant vector subbundle $\mathfrak{A}$ of $\mathrm{End}(TQ^{n*})$ with $\mathrm{rk}(\mathfrak{A}) = 2$, which is parallel with respect to the induced connection on $\mathrm{End}(TQ^{n*})$. For any real structure $C \in \mathfrak{A}_0$ the tangent line to the fiber of $\mathfrak{A}$ through C is spanned by JC. For every $p \in Q^{n*}$ and real structure $C \in \mathfrak{A}_p$ we have an orthogonal decomposition

$$T_pQ^{n*} = V(C) \oplus JV(C)$$

into two totally real subspaces of T_pQ^{n*}. Here $V(C)$ and $JV(C)$ are the $(+1)$- and (-1) -eigenspaces of C, respectively. By construction, we have

$$V(C_0) = \left\{ \begin{pmatrix} 0 & 0 & u_1 & \cdots & u_n \\ 0 & 0 & 0 & \cdots & 0 \\ u_1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ u_n & 0 & 0 & \cdots & 0 \end{pmatrix} : u \in \mathbb{R}^n \right\}$$

and

$$JV(C_0) = \left\{ \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & v_1 & \cdots & v_n \\ 0 & v_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & v_n & 0 & \cdots & 0 \end{pmatrix} : v \in \mathbb{R}^n \right\}.$$

For

$$C = \cos(\varphi)C_0 + \sin(\varphi)JC_0$$

and $u \in V(C_0)$ we have

$$\begin{aligned}
&C(\cos(\varphi/2)u + \sin(\varphi/2)Ju)\\
&= \cos(\varphi/2)Cu + \sin(\varphi/2)CJu\\
&= \cos(\varphi/2)Cu - \sin(\varphi/2)JCu\\
&= \cos(\varphi/2)(\cos(\varphi)C_0 + \sin(\varphi)JC_0)u - \sin(\varphi/2)J(\cos(\varphi)C_0 + \sin(\varphi)JC_0)u\\
&= (\cos(\varphi/2)\cos(\varphi) + \sin(\varphi/2)\sin(\varphi))u + (\cos(\varphi/2)\sin(\varphi) - \sin(\varphi/2)\cos(\varphi))Ju\\
&= \cos(\varphi/2)u + \sin(\varphi/2)Ju.
\end{aligned}$$

It follows that

$$V(C) = \{\cos(\varphi/2)u + \sin(\varphi/2)Ju \,:\, u \in V(C_0)\}.$$

Geometrically this tells us that, if we rotate a real structure by angle φ, then the ± 1-eigenspaces rotate by angle $\varphi/2$.

The Riemannian metric g, the Kähler structure J and a real structure C on Q^{n*} can be used to give an explicit expression of the Riemannian curvature tensor $\bar{R}$ of (Q^{n*}, g) (see [15] and use duality). More precisely, we have

$$\begin{aligned}
\bar{R}(X,Y)Z = {} & g(X,Z)Y - g(Y,Z)X + g(JX,Z)JY - g(JY,Z)JX + 2g(JX,Y)JZ\\
& + g(CX,Z)CY - g(CY,Z)CX + g(JCX,Z)JCY - g(JCY,Z)JCX
\end{aligned}$$

for all $X, Y, Z \in \mathfrak{X}(Q^{n*})$, where C is an arbitrary real structure in $\mathfrak{A}_0$.

For every non-zero tangent vector $v \in \mathfrak{p} \cong T_oQ^{n*}$ there exists a maximal abelian subspace $\mathfrak{a} \subset \mathfrak{p}$ with $v \in \mathfrak{a}$. If $\mathfrak{a}$ is unique, then v is said to be a regular tangent vector, otherwise v is said to be a singular tangent vector. From the explicit expression of the Riemannian curvature tensor it is straightforward to find the singular tangent vectors of Q^{n*}. There are exactly two types of singular tangent vectors $v \in T_oQ^{n*}$, which can be characterized as follows:

(i) If there exists a real structure $C \in \mathfrak{A}_0$ such that $v \in V(C)$, then v is singular. Such a singular tangent vector is called $\mathfrak{A}$-principal.
(ii) If there exist a real structure $C \in \mathfrak{A}_0$ and orthonormal vectors $u, w \in V(C)$ such that $\frac{v}{||v||} = \frac{1}{\sqrt{2}}(u + Jw)$, then v is singular. Such a singular tangent vector is called $\mathfrak{A}$ -isotropic.

For every unit tangent vector $v \in T_oQ^{n*}$ there exist a real structure $C \in \mathfrak{A}_0$ and orthonormal vectors $u, w \in V(C)$ such that

$$v = \cos(t)u + \sin(t)Jw$$

for some $t \in [0, \frac{\pi}{4}]$. The singular tangent vectors correspond to the boundary values $t = 0$ and $t = \frac{\pi}{4}$.

Let v be a unit tangent vector of Q^{n*} and consider the Jacobi operator $\bar{R}_v$ defined by

$$\bar{R}_vX = \bar{R}(X, v)v.$$

We have

$$\bar{R}_vX = -X + g(X,v)v - 3g(X,Jv)Jv + g(X,Cv)Cv - g(Cv,v)CX + g(X,JCv)JCv.$$

By a straightforward computation we obtain the eigenvalues and eigenspaces of $\bar{R}_v$ (see also [15]). The eigenvalues are

$$0, -1+\cos(2t), -1-\cos(2t), -2+2\sin(2t), -2-2\sin(2t)$$

with corresponding eigenspaces

$$\begin{aligned} E_0 &= \mathbb{R}u \oplus \mathbb{R}w \cong \mathbb{R}^2, \\ E_{-1+\cos(2t)} &= V(C) \ominus (\mathbb{R}u \oplus \mathbb{R}w) \cong \mathbb{R}^{n-2}, \\ E_{-1-\cos(2t)} &= JV(C) \ominus J(\mathbb{R}u \oplus \mathbb{R}w) \cong \mathbb{R}^{n-2}, \\ E_{-2+2\sin(2t)} &= \mathbb{R}(Ju+w) \cong \mathbb{R}, \\ E_{-2-2\sin(2t)} &= \mathbb{R}(Ju-w) \cong \mathbb{R}, \end{aligned}$$

where C is a suitable real structure, and $u, w \in V(C)$ are orthonormal vectors such that

$$v = \cos(t)u + \sin(t)Jw$$

for some $t \in [0, \frac{\pi}{4}]$. The five eigenvalues are distinct unless $t \in \{0, \tan^{-1}(\frac{1}{2}), \frac{\pi}{4}\}$.

If $t = 0$, then $Cv = v$ and hence v is $\mathfrak{A}$-principal. In this case $\bar{R}_v$ has two eigenvalues $0, -2$ with corresponding eigenspaces

$$\begin{aligned} E_0 &= \mathbb{R}v \oplus J(V(C) \ominus \mathbb{R}v) \cong \mathbb{R}^n, \\ E_{-2} &= \mathbb{R}Jv \oplus (V(C) \ominus \mathbb{R}v) \cong \mathbb{R}^n. \end{aligned}$$

If $t = \frac{\pi}{4}$, then $v = \frac{1}{\sqrt{2}}(u + Jw)$ and hence v is $\mathfrak{A}$-isotropic. In this case $\bar{R}_v$ has three eigenvalues $0, -1, -4$ with corresponding eigenspaces

$$\begin{aligned} E_0 &= \mathbb{R}v \oplus \mathbb{R}Cv \oplus \mathbb{R}JCv = \mathbb{R}v \oplus \mathbb{C}Cv \cong \mathbb{R} \oplus \mathbb{C}, \\ E_{-1} &= \mathfrak{p} \ominus (\mathbb{C}v \oplus \mathbb{C}Cv) \cong \mathbb{C}^{n-2}, \\ E_{-4} &= \mathbb{R}Jv \cong \mathbb{R}. \end{aligned}$$

If $t = \tan^{-1}(\frac{1}{2})$, then $\cos(t) = \frac{2}{\sqrt{5}}$, $\sin(t) = \frac{1}{\sqrt{5}}$, and hence $\cos(2t) = \frac{3}{5}$ and $\sin(2t) = \frac{4}{5}$. In this case $\bar{R}_v$ has four eigenvalues $0, -\frac{2}{5}, -\frac{8}{5}, -\frac{18}{5}$.

Let $\mathfrak{a}$ be a maximal abelian subspace of $\mathfrak{p}$ and $\mathfrak{a}^*$ be the dual vector space of $\mathfrak{a}$. For each $\alpha \in \mathfrak{a}^*$ we define

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g} : \operatorname{ad}(H)X = \alpha(H)X \text{ for all } H \in \mathfrak{a}\}.$$

If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$, then α is a restricted root and $\mathfrak{g}_\alpha$ is a restricted root space. Let $\Sigma \subset \mathfrak{a}^*$ be the set of restricted roots. The restricted root spaces provide a restricted root space decomposition

$$\mathfrak{g} = \mathfrak{g}_0 \oplus \left(\bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \right)$$

of $\mathfrak{g}$, where $\mathfrak{g}_0 = \mathfrak{k}_0 \oplus \mathfrak{a}$ and $\mathfrak{k}_0 \cong \mathfrak{so}_{n-2}$ is the centralizer of $\mathfrak{a}$ in $\mathfrak{k}$. The restricted root spaces $\mathfrak{g}_\alpha$ and $\mathfrak{g}_0$ are pairwise orthogonal with respect to B_θ. The corresponding restricted root system is of type B_2. We choose a set $\Lambda = \{\alpha_1, \alpha_2\}$ of simple roots of Σ such that α_1 is the longer root of the two simple roots and denote by Σ^+ the resulting set of positive restricted roots. If we write, as usual, $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = \epsilon_2$, the positive restricted roots are

$$\alpha_1 = \epsilon_1 - \epsilon_2,\ \alpha_2 = \epsilon_2,\ \alpha_1 + \alpha_2 = \epsilon_1,\ \alpha_1 + 2\alpha_2 = \epsilon_1 + \epsilon_2.$$

The multiplicities of the two long roots α_1 and $\alpha_1 + 2\alpha_2$ are equal to 1, and the multiplicities of the two short roots α_2 and $\alpha_1 + \alpha_2$ are equal to $n-2$, respectively. Explicitly, the positive restricted root spaces and $\mathfrak{g}_0$ are:

$$\mathfrak{g}_0 = \left\{ \begin{pmatrix} 0 & 0 & a_1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & a_2 & 0 & \cdots & 0 \\ a_1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & a_2 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & & & \\ \vdots & \vdots & \vdots & \vdots & & B & \\ 0 & 0 & 0 & 0 & & & \end{pmatrix} : a_1, a_2 \in \mathbb{R},\ B \in \mathfrak{so}_{n-2} \right\} \cong \mathbb{R}^2 \oplus \mathfrak{so}_{n-2},$$

$$\mathfrak{g}_{\alpha_1+\alpha_2} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & v_1 & \cdots & v_{n-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & v_1 & \cdots & v_{n-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ v_1 & 0 & -v_1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & 0 & -v_{n-2} & 0 & 0 & \cdots & 0 \end{pmatrix} : v \in \mathbb{R}^{n-2} \right\} \cong \mathbb{R}^{n-2},$$

$$\mathfrak{g}_{\alpha_2} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & w_1 & \cdots & w_{n-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & w_1 & \cdots & w_{n-2} \\ 0 & w_1 & 0 & -w_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & w_{n-2} & 0 & -w_{n-2} & 0 & \cdots & 0 \end{pmatrix} : w \in \mathbb{R}^{n-2} \right\} \cong \mathbb{R}^{n-2},$$

$$\mathfrak{g}_{\alpha_1} = \left\{ \begin{pmatrix} 0 & x & 0 & x & 0 & \cdots & 0 \\ -x & 0 & x & 0 & 0 & \cdots & 0 \\ 0 & x & 0 & x & 0 & \cdots & 0 \\ x & 0 & -x & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} : x \in \mathbb{R} \right\} \cong \mathbb{R},$$

$$\mathfrak{g}_{\alpha_1+2\alpha_2} = \left\{ \begin{pmatrix} 0 & y & 0 & -y & 0 & \cdots & 0 \\ -y & 0 & y & 0 & 0 & \cdots & 0 \\ 0 & y & 0 & -y & 0 & \cdots & 0 \\ -y & 0 & y & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} : y \in \mathbb{R} \right\} \cong \mathbb{R}.$$

The negative restricted root spaces can be computed easily from the positive restricted root spaces using the fact that $\mathfrak{g}_{-\alpha} = \theta(\mathfrak{g}_\alpha)$.

For each $\alpha \in \Sigma$ we define

$$\mathfrak{k}_\alpha = \mathfrak{k} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}),\ \mathfrak{p}_\alpha = \mathfrak{p} \cap (\mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}).$$

Then we have $\mathfrak{p}_\alpha = \mathfrak{p}_{-\alpha}$, $\mathfrak{k}_\alpha = \mathfrak{k}_{-\alpha}$ and $\mathfrak{p}_\alpha \oplus \mathfrak{k}_\alpha = \mathfrak{g}_\alpha \oplus \mathfrak{g}_{-\alpha}$ for all $\alpha \in \Sigma$.

We define a nilpotent subalgebra $\mathfrak{n}$ of $\mathfrak{g}$ by

$$\begin{aligned}
\mathfrak{n} &= \mathfrak{g}_{\alpha_1} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2} \\
&= \left\{ \begin{pmatrix} 0 & x+y & 0 & x-y & v_1 & \cdots & v_{n-2} \\ -x-y & 0 & x+y & 0 & w_1 & \cdots & w_{n-2} \\ 0 & x+y & 0 & x-y & v_1 & \cdots & v_{n-2} \\ x-y & 0 & -x+y & 0 & w_1 & \cdots & w_{n-2} \\ v_1 & w_1 & -v_1 & -w_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & w_{n-2} & -v_{n-2} & -w_{n-2} & 0 & \cdots & 0 \end{pmatrix} : \begin{array}{l} x, y \in \mathbb{R}, \\ v, w \in \mathbb{R}^{n-2} \end{array} \right\}.
\end{aligned}$$

Then $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{a} \oplus \mathfrak{n}$ is an Iwasawa decomposition of $\mathfrak{g}$, which induces a corresponding Iwasawa decomposition $G = KAN$ of G. Here, A and N are the connected closed subgroups of G with Lie algebras $\mathfrak{a}$ and $\mathfrak{n}$, respectively.

The subalgebra

$$\mathfrak{a} \oplus \mathfrak{n} = \left\{ \begin{pmatrix} 0 & x+y & a_1 & x-y & v_1 & \cdots & v_{n-2} \\ -x-y & 0 & x+y & a_2 & w_1 & \cdots & w_{n-2} \\ a_1 & x+y & 0 & x-y & v_1 & \cdots & v_{n-2} \\ x-y & a_2 & -x+y & 0 & w_1 & \cdots & w_{n-2} \\ v_1 & w_1 & -v_1 & -w_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & w_{n-2} & -v_{n-2} & -w_{n-2} & 0 & \cdots & 0 \end{pmatrix} : \begin{array}{l} a_1, a_2, x, y \in \mathbb{R}, \\ v, w \in \mathbb{R}^{n-2} \end{array} \right\}$$

of $\mathfrak{g}$ is solvable and the corresponding connected closed subgroup AN of G with Lie algebra $\mathfrak{a} \oplus \mathfrak{n}$ is solvable, simply connected and acts simply transitively on Q^{n*}. Then (Q^{n*}, g) is isometric to the solvable Lie group AN equipped with the left-invariant Riemannian metric $\langle \cdot, \cdot \rangle$ defined by

$$\begin{aligned}
\langle H_1 + \hat{X}_1, H_2 + \hat{X}_2 \rangle &= -\frac{1}{4n} B(H_1, \theta(H_2)) - \frac{1}{8n} B(\hat{X}_1, \theta(\hat{X}_2)) \\
&= -\frac{1}{4} \operatorname{tr}(H_1 \theta(H_2)) - \frac{1}{8} \operatorname{tr}(\hat{X}_1 \theta(\hat{X}_2)) \\
&= \frac{1}{4} \operatorname{tr}(H_1 H_2) - \frac{1}{8} \operatorname{tr}(\hat{X}_1 \theta(\hat{X}_2))
\end{aligned}$$

with $H_1, H_2 \in \mathfrak{a}$ and $\hat{X}_1, \hat{X}_2 \in \mathfrak{n}$. For each $\hat{X} \in \mathfrak{n}$, the orthogonal projection X onto $\mathfrak{p}$ with respect to B_θ is

$$X = \frac{1}{2}(\hat{X} - \theta(\hat{X})) \in \mathfrak{p}.$$

By construction, we have $\langle \hat{X}, \hat{X} \rangle = g(X, X)$ and

$$\langle H_1 + \hat{X}_1, H_2 + \hat{X}_2 \rangle = g(H_1 + X_1, H_2 + X_2).$$

Let $H^1, H^2 \in \mathfrak{a}$ be the dual basis of $\alpha_1, \alpha_2 \in \mathfrak{a}^*$ defined by $\alpha_\nu(H^\mu) = \delta_{\nu\mu}$. Since $\alpha_1 = \epsilon_1 - \epsilon_2$ and $\alpha_2 = \epsilon_2$, we have

$$H^1 = \begin{pmatrix} 0&0&1&0&0&\cdots&0\\ 0&0&0&0&0&\cdots&0\\ 1&0&0&0&0&\cdots&0\\ 0&0&0&0&0&\cdots&0\\ 0&0&0&0&0&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&0&0&\cdots&0 \end{pmatrix},\ H^2 = \begin{pmatrix} 0&0&1&0&0&\cdots&0\\ 0&0&0&1&0&\cdots&0\\ 1&0&0&0&0&\cdots&0\\ 0&1&0&0&0&\cdots&0\\ 0&0&0&0&0&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&0&0&\cdots&0 \end{pmatrix}.$$

Note that

$$\langle H^1, H^1\rangle = \frac{1}{4}\mathrm{tr}(H^1H^1) = \frac{1}{2},\ \langle H^2, H^2\rangle = \frac{1}{4}\mathrm{tr}(H^2H^2) = 1.$$

For each α in Σ we define the root vector $H_\alpha \in \mathfrak{a}$ of α by $\langle H_\alpha, H\rangle = \alpha(H)$ for all $H \in \mathfrak{a}$. Note that

$$[H, X_\alpha] = \mathrm{ad}(H)X_\alpha = \alpha(H)X_\alpha = \langle H_\alpha, H\rangle X_\alpha$$

for all $H \in \mathfrak{a}$ and $X_\alpha \in \mathfrak{g}_\alpha$. If we put

$$H_\alpha = \begin{pmatrix} 0&0&x_1&0&0&\cdots&0\\ 0&0&0&x_2&0&\cdots&0\\ x_1&0&0&0&0&\cdots&0\\ 0&x_2&0&0&0&\cdots&0\\ 0&0&0&0&0&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&0&0&\cdots&0 \end{pmatrix},\ H = \begin{pmatrix} 0&0&a_1&0&0&\cdots&0\\ 0&0&0&a_2&0&\cdots&0\\ a_1&0&0&0&0&\cdots&0\\ 0&a_2&0&0&0&\cdots&0\\ 0&0&0&0&0&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&0&0&\cdots&0 \end{pmatrix},$$

then

$$\langle H_\alpha, H\rangle = \frac{1}{4}\mathrm{tr}(H_\alpha H) = \frac{1}{2}(x_1a_1 + x_2a_2).$$

It follows that

$$H_{\alpha_1} = \begin{pmatrix} 0&0&2&0&0&\cdots&0\\ 0&0&0&-2&0&\cdots&0\\ 2&0&0&0&0&\cdots&0\\ 0&-2&0&0&0&\cdots&0\\ 0&0&0&0&0&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&0&0&\cdots&0 \end{pmatrix},\ H_{\alpha_1+2\alpha_2} = \begin{pmatrix} 0&0&2&0&0&\cdots&0\\ 0&0&0&2&0&\cdots&0\\ 2&0&0&0&0&\cdots&0\\ 0&2&0&0&0&\cdots&0\\ 0&0&0&0&0&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&0&0&\cdots&0 \end{pmatrix},$$

$$H_{\alpha_2} = \begin{pmatrix} 0&0&0&0&0&\cdots&0\\ 0&0&0&2&0&\cdots&0\\ 0&0&0&0&0&\cdots&0\\ 0&2&0&0&0&\cdots&0\\ 0&0&0&0&0&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&0&0&\cdots&0 \end{pmatrix},\ H_{\alpha_1+\alpha_2} = \begin{pmatrix} 0&0&2&0&0&\cdots&0\\ 0&0&0&0&0&\cdots&0\\ 2&0&0&0&0&\cdots&0\\ 0&0&0&0&0&\cdots&0\\ 0&0&0&0&0&\cdots&0\\ \vdots&\vdots&\vdots&\vdots&\vdots&\ddots&\vdots\\ 0&0&0&0&0&\cdots&0 \end{pmatrix}.$$

We have

$$\begin{aligned}\langle H_{\alpha_1}, H_{\alpha_1}\rangle &= \frac{1}{4}\mathrm{tr}(H_{\alpha_1}H_{\alpha_1}) = 4,\\ \langle H_{\alpha_1+2\alpha_2}, H_{\alpha_1+2\alpha_2}\rangle &= \frac{1}{4}\mathrm{tr}(H_{\alpha_1+2\alpha_2}H_{\alpha_1+2\alpha_2}) = 4,\\ \langle H_{\alpha_2}, H_{\alpha_2}\rangle &= \frac{1}{4}\mathrm{tr}(H_{\alpha_2}H_{\alpha_2}) = 2,\\ \langle H_{\alpha_1+\alpha_2}, H_{\alpha_1+\alpha_2}\rangle &= \frac{1}{4}\mathrm{tr}(H_{\alpha_1+\alpha_2}H_{\alpha_1+\alpha_2}) = 2,\end{aligned}$$

and

$$2H^1 = H_{\alpha_1+\alpha_2} \qquad \text{and} \qquad 2H^2 = H_{\alpha_1+2\alpha_2}.$$

4 The homogeneous complex hypersurface

In this section we construct a homogeneous complex hypersurface $P^{n-1} \cong \mathbb{C}H^{n-1}(-4)$ in (Q^{n*}, g) and compute its shape operator. We define

$$\mathfrak{h}^{2n-3} = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2}.$$

It is easy to verify that $\mathfrak{h}^{2n-3}$ is a nilpotent subalgebra of $\mathfrak{n}$ and isomorphic to the $(2n-3)$-dimensional Heisenberg algebra with 1-dimensional center.

We have

$$[H^2, \hat{X}] = \begin{cases} \hat{X} & , \text{ if } \hat{X} \in \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2},\\ 2\hat{X} & , \text{ if } \hat{X} \in \mathfrak{g}_{\alpha_1+2\alpha_2}.\end{cases}$$

It follows that

$$\mathfrak{d} = \mathbb{R}H^2 \oplus \mathfrak{h}^{2n-3} = \mathbb{R}H_{\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2}$$

is a solvable subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$. (Note that $\mathbb{R}H^2$ denotes here the real span of H^2 and not the real hyperbolic plane!) In fact, this subalgebra is the standard solvable extension of the Heisenberg algebra $\mathfrak{h}^{2n-3}$ and isomorphic to the solvable Lie algebra of the solvable part of the Iwasawa decomposition of the isometry group of the complex hyperbolic space $\mathbb{C}H^{n-1}(-4)$ (see, e.g., [5] or [17]).

This construction leads to an isometric embedding $\hat{P}^{n-1}$ of the $(n-1)$-dimensional complex hyperbolic space $\mathbb{C}H^{n-1}(-4)$ with constant holomorphic sectional curvature -4 into $(AN, \langle\cdot,\cdot\rangle)$. By construction, $\hat{P}^{n-1}$ is a homogeneous submanifold of $(AN, \langle\cdot,\cdot\rangle)$. Let $\hat{J}$ be the complex structure on $(AN, \langle\cdot,\cdot\rangle)$ corresponding to the complex structure J on (Q^{n*}, g). We have $\hat{J}\mathfrak{g}_{\alpha_2} = \mathfrak{g}_{\alpha_1+\alpha_2}$ and $\hat{J}H^2 \in \mathfrak{g}_{\alpha_1+2\alpha_2}$, which shows that the tangent space

$$T_o\hat{P}^{n-1} = \mathbb{R}H_{\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2}$$

is a complex subspace of T_oAN. Since AN is contained in the identity component $SO^o_{2,n}$ of the full isometry group of Q^{n*}, it consists of holomorphic isometries, which implies that $\hat{P}^{n-1}$ is a complex submanifold of $(AN, \langle\cdot,\cdot\rangle)$.

Altogether we conclude that the solvable subalgebra

$$\mathfrak{d} = \mathbb{R}H_{\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2}$$

of $\mathfrak{a} \oplus \mathfrak{n}$ induces an isometric embedding $\hat{P}^{n-1}$ of the complex hyperbolic space $\mathbb{C}H^{n-1}(-4)$ with constant holomorphic sectional curvature -4 into $(AN, \langle\cdot,\cdot\rangle)$ as a homogeneous complex hypersurface. This induces an isometric embedding P^{n-1} of the complex hyperbolic space $\mathbb{C}H^{n-1}(-4)$ with constant holomorphic sectional curvature -4 into (Q^{n*}, g) as a homogeneous complex hypersurface.

Remark 4.1 Smyth [16] proved that every homogeneous complex hypersurface in the complex hyperbolic space $\mathbb{C}H^n$ is a complex hyperbolic hyperplane $\mathbb{C}H^{n-1}$ embedded in $\mathbb{C}H^n$ as a totally geodesic submanifold. As we have just seen, up to congruency, there are at least two homogeneous complex hypersurfaces in the complex hyperbolic quadric Q^{n*}, namely the complex hyperbolic quadric Q^{n-1^*} and the complex hyperbolic space $P^{n-1} \cong \mathbb{C}H^{n-1}(-4)$. The first one is totally geodesic (see [8, 11] and use duality between Riemannian symmetric spaces of compact type and of non-compact type), the second one is not. The classification of the homogeneous complex hypersurfaces in the complex hyperbolic quadric Q^{n*} remains an open problem.

We now compute the shape operator $\hat{A}$ of $\hat{P}^{n-1} \cong \mathbb{C}H^{n-1}(-4)$ in $(AN, \langle\cdot,\cdot\rangle)$. Let

$$\hat{\zeta} \in (\mathfrak{a} \ominus \mathbb{R}H^2) \oplus \mathfrak{g}_{\alpha_1} = \mathbb{R}H_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1}$$

be a unit normal vector of $\hat{P}^{n-1}$ at o. The Weingarten equation tells us that

$$\langle \hat{A}_{\hat{\zeta}}\hat{X}, \hat{Y}\rangle = -\langle \hat{\nabla}_{\hat{X}}\hat{\zeta}, \hat{Y}\rangle,$$

where $\hat{\nabla}$ is the Levi Civita covariant derivative of $(AN, \langle\cdot,\cdot\rangle)$ and $\hat{X}, \hat{Y} \in \mathfrak{d}$. We consider $\hat{\zeta}, \hat{X}, \hat{Y}$ as left-invariant vector fields. Since $\langle\cdot,\cdot\rangle$ is a left-invariant Riemannian metric, the Koszul formula for $\hat{\nabla}$ implies

$$2\langle \hat{A}_{\hat{\zeta}}\hat{X}, \hat{Y}\rangle = 2\langle \hat{\nabla}_{\hat{X}}\hat{Y}, \hat{\zeta}\rangle = \langle [\hat{X}, \hat{Y}], \hat{\zeta}\rangle + \langle [\hat{\zeta}, \hat{X}], \hat{Y}\rangle + \langle [\hat{\zeta}, \hat{Y}], \hat{X}\rangle.$$

Since $\mathfrak{d}$ is a subalgebra of $\mathfrak{a} \oplus \mathfrak{n}$, we have $[\hat{X}, \hat{Y}] \in \mathfrak{d}$ and hence $\langle [\hat{X}, \hat{Y}], \hat{\zeta}\rangle = 0$. Moreover, since $\mathrm{ad}(\hat{\zeta})^* = -\mathrm{ad}(\theta(\hat{\zeta}))$, we have

$$\langle [\hat{\zeta}, \hat{Y}], \hat{X}\rangle = -\langle [\theta(\hat{\zeta}), \hat{X}], \hat{Y}\rangle.$$

Altogether this implies

$$2\langle \hat{A}_{\hat{\zeta}}\hat{X}, \hat{Y}\rangle = \langle [\hat{\zeta} - \theta(\hat{\zeta}), \hat{X}], \hat{Y}\rangle.$$

Thus, the shape operator $\hat{A}_{\hat{\zeta}}$ of $\hat{P}^{n-1}$ is given by

$$\hat{A}_{\hat{\zeta}}\hat{X} = [\zeta, \hat{X}]_{\mathfrak{d}},$$

where

$$\zeta = \frac{1}{2}(\hat{\zeta} - \theta(\hat{\zeta})) \in \mathfrak{p}$$

is the orthogonal projection of $\hat{\zeta}$ onto $\mathfrak{p}$ and $[\,\cdot\,]_{\mathfrak{d}}$ is the orthogonal projection onto $\mathfrak{d}$.

The normal space $\nu_o\hat{P}^{n-1}$ of $\hat{P}^{n-1}$ at the point o is given by

$$v_o\hat{P}^{n-1} = \mathbb{R}H_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1} = \left\{ \begin{pmatrix} 0 & x & a & x & 0 & \cdots & 0 \\ -x & 0 & x & -a & 0 & \cdots & 0 \\ a & x & 0 & x & 0 & \cdots & 0 \\ x & -a & -x & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} : a, x \in \mathbb{R} \right\},$$

and the tangent space $T_o\hat{P}^{n-1}$ of $\hat{P}^{n-1}$ at the point o is given by

$$T_o\hat{P}^{n-1} = \mathfrak{d} = \left\{ \begin{pmatrix} 0 & y & b & -y & v_1 & \cdots & v_{n-2} \\ -y & 0 & y & b & w_1 & \cdots & w_{n-2} \\ b & y & 0 & -y & v_1 & \cdots & v_{n-2} \\ -y & b & y & 0 & w_1 & \cdots & w_{n-2} \\ v_1 & w_1 & -v_1 & -w_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & w_{n-2} & -v_{n-2} & -w_{n-2} & 0 & \cdots & 0 \end{pmatrix} : \begin{array}{l} b, y \in \mathbb{R}, \\ v, w \in \mathbb{R}^{n-2} \end{array} \right\}.$$

The vector $\hat{\zeta} = \frac{1}{2}H_{\alpha_1} \in \mathfrak{a}$ is a unit normal vector of $\hat{P}^{n-1}$ at o. We have

$$\theta(\hat{\zeta}) = \frac{1}{2}\theta(H_{\alpha_1}) = -\frac{1}{2}H_{\alpha_1} = -\hat{\zeta}$$

and thus

$$\zeta = \frac{1}{2}(\hat{\zeta} - \theta(\hat{\zeta})) = \hat{\zeta}.$$

A straightforward matrix computation gives

$$\left[\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & -1 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}, \begin{pmatrix} 0 & y & b & -y & v_1 & \cdots & v_{n-2} \\ -y & 0 & y & b & w_1 & \cdots & w_{n-2} \\ b & y & 0 & -y & v_1 & \cdots & v_{n-2} \\ -y & b & y & 0 & w_1 & \cdots & w_{n-2} \\ v_1 & w_1 & -v_1 & -w_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & w_{n-2} & -v_{n-2} & -w_{n-2} & 0 & \cdots & 0 \end{pmatrix} \right]$$
$$= \begin{pmatrix} 0 & 0 & 0 & 0 & v_1 & \cdots & v_{n-2} \\ 0 & 0 & 0 & 0 & -w_1 & \cdots & -w_{n-2} \\ 0 & 0 & 0 & 0 & v_1 & \cdots & v_{n-2} \\ 0 & 0 & 0 & 0 & -w_1 & \cdots & -w_{n-2} \\ v_1 & -w_1 & -v_1 & w_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & -w_{n-2} & -v_{n-2} & w_{n-2} & 0 & \cdots & 0 \end{pmatrix} \in \mathfrak{d}.$$

Since the latter matrix is in $\mathfrak{d}$, we conclude that

$$\hat{A}_{\hat{\zeta}}\hat{X} = \begin{pmatrix} 0 & 0 & 0 & 0 & v_1 & \cdots & v_{n-2} \\ 0 & 0 & 0 & 0 & -w_1 & \cdots & -w_{n-2} \\ 0 & 0 & 0 & 0 & v_1 & \cdots & v_{n-2} \\ 0 & 0 & 0 & 0 & -w_1 & \cdots & -w_{n-2} \\ v_1 & -w_1 & -v_1 & w_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & -w_{n-2} & -v_{n-2} & w_{n-2} & 0 & \cdots & 0 \end{pmatrix}$$

with

$$\hat{X} = \begin{pmatrix} 0 & y & b & -y & v_1 & \cdots & v_{n-2} \\ -y & 0 & y & b & w_1 & \cdots & w_{n-2} \\ b & y & 0 & -y & v_1 & \cdots & v_{n-2} \\ -y & b & y & 0 & w_1 & \cdots & w_{n-2} \\ v_1 & w_1 & -v_1 & -w_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & w_{n-2} & -v_{n-2} & -w_{n-2} & 0 & \cdots & 0 \end{pmatrix} \in T_o\hat{P}^{n-1}.$$

It follows that the principal curvatures of $\hat{P}^{n-1}$ with respect to the unit normal vector $\hat{\zeta}$ are 0, 1 and -1, with corresponding principal curvature spaces

$$\hat{T}_0^{\hat{\zeta}} = \mathbb{R}H_{\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2},\ \hat{T}_1^{\hat{\zeta}} = \mathfrak{g}_{\alpha_1+\alpha_2},\ \hat{T}_{-1}^{\hat{\zeta}} = \mathfrak{g}_{\alpha_2}.$$

We now compute the shape operator of $\hat{P}^{n-1}$ at o for other unit normal vectors. Since $\nu_o\hat{P}^{n-1}$ is $\hat{J}$-invariant, the vector $\hat{J}\hat{\zeta} \in \mathfrak{g}_{\alpha_1}$ is a unit normal vector of $\hat{P}^{n-1}$ at o. Moreover, $\hat{\zeta}, \hat{J}\hat{\zeta}$ is an orthonormal basis of the normal space $\nu_o\hat{P}^{n-1}$. Using a well-known formula for the shape operator of a complex submanifold of a Kähler manifold (see, e.g., [7], Lemma 7.4), we have

$$\hat{A}_{\hat{J}\hat{\zeta}} = \hat{J}\hat{A}_{\hat{\zeta}}.$$

Since every unit normal vector of $\hat{P}^{n-1}$ at o is of the form

$$\cos(\varphi)\hat{\zeta} + \sin(\varphi)\hat{J}\hat{\zeta},$$

the shape operator $\hat{A}_{\hat{\zeta}}$ therefore completely determines the shape operator for every other unit normal vector of $\hat{P}^{n-1}$ at o. More precisely, we have

$$\hat{A}_{\cos(\varphi)\hat{\zeta}+\sin(\varphi)\hat{J}\hat{\zeta}} = \cos(\varphi)\hat{A}_{\hat{\zeta}} + \sin(\varphi)J\hat{A}_{\hat{\zeta}}.$$

This readily implies that the principal curvatures of $\hat{P}^{n-1}$ with respect to the unit normal vector $\cos(\varphi)\hat{\zeta} + \sin(\varphi)\hat{J}\hat{\zeta}$ are 0, 1 and -1, with corresponding principal curvature spaces

$$\begin{aligned} \hat{T}_0^{\cos(\varphi)\hat{\zeta}+\sin(\varphi)\hat{J}\hat{\zeta}} &= \mathbb{R}H_{\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2}, \\ \hat{T}_1^{\cos(\varphi)\hat{\zeta}+\sin(\varphi)\hat{J}\hat{\zeta}} &= \{\cos\left(\tfrac{\varphi}{2}\right)\hat{X} + \sin\left(\tfrac{\varphi}{2}\right)J\hat{X} : \hat{X} \in \mathfrak{g}_{\alpha_1+\alpha_2}\}, \\ \hat{T}_{-1}^{\cos(\varphi)\hat{\zeta}+\sin(\varphi)\hat{J}\hat{\zeta}} &= \{\sin\left(\tfrac{\varphi}{2}\right)\hat{X} - \cos\left(\tfrac{\varphi}{2}\right)J\hat{X} : \hat{X} \in \mathfrak{g}_{\alpha_1+\alpha_2}\}. \end{aligned}$$

Using orthogonal projections onto $\mathfrak{p}$ we obtain the corresponding description of the shape operator A of P^{n-1} at o.

Recall that

$$\nu_o\hat{P}^{n-1} = \mathbb{R}H_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1}.$$

The orthogonal projection of $\nu_o\hat{P}^{n-1}$ onto $\mathfrak{p}$ is

$$\nu_oP^{n-1} = \mathbb{C}H_{\alpha_1} = \mathbb{R}H_{\alpha_1} \oplus \mathfrak{p}_{\alpha_1} = \left\{ \begin{pmatrix} 0 & 0 & a & x & 0 & \cdots & 0 \\ 0 & 0 & x & -a & 0 & \cdots & 0 \\ a & x & 0 & 0 & 0 & \cdots & 0 \\ x & -a & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} : a, x \in \mathbb{R} \right\}.$$

The complex line $\mathbb{C}H_{\alpha_1}$ is a Lie triple system in $\mathfrak{p}$ and therefore determines a totally geodesic complex submanifold B_1 of Q^{n*}. The (non-zero) tangent vectors of B_1 are $\mathfrak{A}$-isotropic, which implies that the sectional curvature of B_1 is equal to -4. Thus B_1 is isometric to the complex hyperbolic line $\mathbb{C}H^1(-4)$ of constant (holomorphic) sectional curvature -4. We will encounter B_1 again later, where it appears in a horospherical decomposition of the complex hyperbolic quadric.

We now apply the standard real structure C_0 to the normal space $\nu_oP^{n-1} = T_oB_1$,

$$C_0(T_oB_1) = \left\{ \begin{pmatrix} 0 & 0 & a & x & 0 & \cdots & 0 \\ 0 & 0 & -x & a & 0 & \cdots & 0 \\ a & -x & 0 & 0 & 0 & \cdots & 0 \\ x & a & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} : a, x \in \mathbb{R} \right\}$$
$$= \mathbb{R}H_{\alpha_1+2\alpha_2} \oplus \mathfrak{p}_{\alpha_1+2\alpha_2} = \mathbb{C}H_{\alpha_1+2\alpha_2}.$$

Note that $C_0(T_oB_1) = C(T_oB_1)$ for any real structure C at o and therefore the construction is independent of the choice of real structure. The complex line $\mathbb{C}H_{\alpha_1+2\alpha_2}$ is also a Lie triple system in $\mathfrak{p}$ and determines a totally geodesic complex submanifold Σ_1 of Q^{n*}. The (non-zero) tangent vectors of Σ_1 are also $\mathfrak{A}$-isotropic, which implies that the sectional curvature of Σ_1 is equal to -4. Thus Σ_1 is isometric to the complex hyperbolic line $\mathbb{C}H^1(-4)$ of constant (holomorphic) sectional curvature -4. The tangent space $T_o\Sigma_1$ is the kernel of the shape operator of the homogeneous complex hypersurface $P^{n-1} \cong \mathbb{C}H^{n-1}(-4)$. Since $\mathbb{R}H_{\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2}$ is a subalgebra of $\mathfrak{d}$, this implies geometrically that $\hat{P}^{n-1}$, and hence also P^{n-1}, is foliated by totally geodesic complex hyperbolic lines $\mathbb{C}H^1(-4)$ whose tangent spaces are obtained by rotating the normal spaces of $\hat{P}^{n-1}$ (resp. P^{n-1}) via a real structure $\hat{C}$ (resp. C).

The Riemannian product $B_1 \times \Sigma_1 \cong \mathbb{C}H^1(-4) \times \mathbb{C}H^1(-4)$ is isometric to the complex hyperbolic quadric Q^{2*} and describes the standard isometric embedding of Q^{2*} into Q^{n*}.

We have

$$[\mathfrak{p}_{\alpha_1+\alpha_2} \oplus \mathfrak{p}_{\alpha_2}, \mathfrak{p}_{\alpha_1+\alpha_2} \oplus \mathfrak{p}_{\alpha_2}] \subset \mathfrak{k}_0 \oplus \mathfrak{k}_{\alpha_1} \oplus \mathfrak{k}_{\alpha_1+2\alpha_2}$$

and

$$
\begin{aligned}
[\mathfrak{k}_0, \mathfrak{p}_{\alpha_1+\alpha_2}] &\subset \mathfrak{p}_{\alpha_1+\alpha_2},\\
[\mathfrak{k}_{\alpha_1}, \mathfrak{p}_{\alpha_1+\alpha_2}] &\subset \mathfrak{p}_{\alpha_2},\\
[\mathfrak{k}_{\alpha_1+2\alpha_2}, \mathfrak{p}_{\alpha_1+\alpha_2}] &\subset \mathfrak{p}_{\alpha_2},\\
[\mathfrak{k}_0, \mathfrak{p}_{\alpha_2}] &\subset \mathfrak{p}_{\alpha_2},\\
[\mathfrak{k}_{\alpha_1}, \mathfrak{p}_{\alpha_2}] &\subset \mathfrak{p}_{\alpha_1+\alpha_2},\\
[\mathfrak{k}_{\alpha_1+2\alpha_2}, \mathfrak{p}_{\alpha_2}] &\subset \mathfrak{p}_{\alpha_1+\alpha_2}.
\end{aligned}
$$

Altogether we conclude that $\mathfrak{p}_{\alpha_1+\alpha_2} \oplus \mathfrak{p}_{\alpha_2}$ is a Lie triple system. It is easy to see that this Lie triple system is J-invariant. The complex totally geodesic submanifold of Q^{n*} generated by this Lie triple system is isometric to Q^{n-2*}. However, the only complex totally geodesic submanifolds of a complex hyperbolic space are again complex hyperbolic spaces (see [19] and use duality). It follows that there exists a totally geodesic submanifold $\Sigma^{n-2} \cong \mathbb{C}H^{n-2}(-4)$ of $P \cong \mathbb{C}H^{n-1}(-4)$ with $T_o\Sigma^{n-2} = \mathfrak{p}_{\alpha_1+\alpha_2} \oplus \mathfrak{p}_{\alpha_2}$. We have

$$T_oP^{n-1} = T_o\Sigma_1 \oplus T_o\Sigma^{n-2},\ \nu_oP^{n-1} = T_oB_1.$$

The tangent space $T_o\Sigma^{n-2} = \mathfrak{p}_{\alpha_1+\alpha_2} \oplus \mathfrak{p}_{\alpha_2}$ and the normal space $\nu_o\Sigma^{n-2} = \mathfrak{a} \oplus \mathfrak{p}_{\alpha_1+2\alpha_2} \oplus \mathfrak{p}_{\alpha_1}$ are Lie triple systems in $\mathfrak{p}$.

We summarize the previous discussion in the following theorem.

Theorem 4.2 *There exists a homogeneous complex hypersurface P^{n-1} in (Q^{n*}, g) which is isometric to the complex hyperbolic space $\mathbb{C}H^{n-1}(-4)$ of constant holomorphic sectional curvature -4. In terms of root spaces and root vectors, the tangent space and normal space of P^{n-1}* at o is

$$T_oP^{n-1} = \mathbb{R}H_{\alpha_1+2\alpha_2} \oplus \mathfrak{p}_{\alpha_1+2\alpha_2} \oplus \mathfrak{p}_{\alpha_1+\alpha_2} \oplus \mathfrak{p}_{\alpha_2},\ \nu_oP^{n-1} = \mathbb{R}H_{\alpha_1} \oplus \mathfrak{p}_{\alpha_1}.$$

The normal space ν_oP^{n-1} is a Lie triple system, and the totally geodesic submanifold B_1 of *Q^{n*} generated by this Lie triple system is isometric to a complex hyperbolic line $\mathbb{C}H^1(-4)$ of constant (holomorphic) sectional curvature -4. The (non-zero) tangent vectors of B_1* are *$\mathfrak{A}$-isotropic. In particular, the (non-zero) normal vectors of P^{n-1} are $\mathfrak{A}$-isotropic singular tangent vectors of Q^{n*}.*

The tangent space T_oP^{n-1} decomposes orthogonally into

$$T_oP^{n-1} = C(\nu_oP^{n-1}) \oplus (\mathfrak{p}_{\alpha_1+\alpha_2} \oplus \mathfrak{p}_{\alpha_2}),$$

where C is any real structure in $\mathfrak{A}_0$ at o. *The subspace $C(\nu_oP^{n-1})$ is a Lie triple system, and the totally geodesic submanifold Σ_1* of *Q^{n*} generated by this Lie triple system is isometric to a complex hyperbolic line $\mathbb{C}H^1(-4)$ of constant (holomorphic) sectional curvature -4. The (non-zero) tangent vectors of Σ_1* are *$\mathfrak{A}$-isotropic. The subspace $\mathfrak{p}_{\alpha_1+\alpha_2} \oplus \mathfrak{p}_{\alpha_2}$ is a Lie triple system in $\mathfrak{p}$ and a complex subspace of T_oP^{n-1}. The totally geodesic submanifold of Q^{n*} generated by this Lie triple system is isometric to the complex hyperbolic quadric Q^{n-2*}, and the totally geodesic submanifold of $P^{n-1} \cong \mathbb{C}H^{n-1}(-4)$ generated by this complex subspace is isometric to the complex hyperbolic space $\mathbb{C}H^{n-2}(-4)$.*

Let $\zeta \in \nu_oP^{n-1}$ be a unit normal vector of P^{n-1}. Then, ζ is of the form

$$\zeta = \frac{1}{2}\cos(\varphi)H_{\alpha_1} + \frac{1}{2}\sin(\varphi)JH_{\alpha_1}$$

and the principal curvatures of P^{n-1} *with respect to* ζ *are* $0, 1, -1$ *with corresponding principal curvature spaces*

$$T_0^\zeta = C(\nu_o P^{n-1}) = T_o\Sigma_1,$$
$$T_1^\zeta = \{\cos\left(\tfrac{\varphi}{2}\right)X + \sin\left(\tfrac{\varphi}{2}\right)JX \,:\, X \in \mathfrak{p}_{\alpha_1+\alpha_2}\} \subset V(C),$$
$$T_{-1}^\zeta = \{\sin\left(\tfrac{\varphi}{2}\right)X - \cos\left(\tfrac{\varphi}{2}\right)JX \,:\, X \in \mathfrak{p}_{\alpha_1+\alpha_2}\} \subset JV(C),$$

where $C = \cos(\varphi)C_0 + \sin(\varphi)JC_0$. *The* 0*-eigenspace is independent of the choice of unit normal vector* ζ *and coincides with the kernel* T_0 *of the shape operator of* P^{n-1}.

Let M be a submanifold of a Riemannian manifold $\bar{M}$ and $\zeta \in \nu_p M$ be a normal vector of M. Consider the Jacobi operator $\bar{R}_\zeta = \bar{R}(\cdot, \zeta)\zeta \,:\, T_p\bar{M} \to T_p\bar{M}$. If $\bar{R}_\zeta(T_pM) \subseteq T_pM$, then the restriction $\mathcal{K}_\zeta$ of $\bar{R}_\zeta$ to T_pM is a self-adjoint endomorphism of T_pM, the so-called normal Jacobi operator of M with respect to ζ. The family $\mathcal{K} = (\mathcal{K}_\zeta)_{\zeta\in\nu M}$ is called the normal Jacobi operator of M.

A submanifold M of a Riemannian manifold $\bar{M}$ is curvature-adapted if for every normal vector $\zeta \in \nu_pM$, $p \in M$, the following two conditions are satisfied:

(i) $\bar{R}_\zeta(T_pM) \subseteq T_pM$;
(ii) the normal Jacobi operator $\mathcal{K}_\zeta$ and the shape operator A_ζ of M are simultaneously diagonalizable, that is,

$$\mathcal{K}_\zeta A_\zeta = A_\zeta \mathcal{K}_\zeta.$$

Since $\bar{R}_{\lambda\zeta} = \lambda^2\bar{R}_\zeta$ for all $\lambda > 0$, it suffices to check conditions (i) and (ii) only for unit normal vectors. Curvature-adapted submanifolds were introduced in [6]. They were also studied by Gray in [10] using the notion of compatible submanifolds. Curvature-adapted submanifolds form a very useful class of submanifolds in the context of focal sets and tubes.

Corollary 4.3 *The homogeneous complex hypersurface* $P^{n-1} \cong \mathbb{C}H^{n-1}(-4)$ in (Q^{n*}, g) *is curvature-adapted.*

Proof Let ζ be a unit normal vector of P^{n-1} at o. Then ζ is an $\mathfrak{A}$-isotropic singular tangent vector of Q^{n*}. We already computed the eigenvalues and eigenspaces of the Jacobi operator $\bar{R}_\zeta$ in Sect. 3. It follows from this that $\bar{R}_\zeta$ has three eigenvalues $0, -1, -4$ with corresponding eigenspaces

$$E_0^\zeta = \mathbb{R}\zeta \oplus \mathbb{C}C_0\zeta,\ \ E_{-1}^\zeta = \mathfrak{p} \ominus (\mathbb{C}\zeta \oplus \mathbb{C}C_0\zeta),\ \ E_{-4}^\zeta = \mathbb{R}J\zeta.$$

Note that E_{-1}^ζ is independent of the choice of the unit normal vector ζ and hence we can denote this space by E_{-1}. The tangent space T_oP^{n-1} is given by

$$T_oP^{n-1} = C_0(\nu_oP^{n-1}) \oplus E_{-1}.$$

From Theorem 4.2 we see that $T_0^\zeta = T_0 = C_0(\nu_oP^{n-1}) = \mathbb{C}C_0\zeta \subset E_0^\zeta$ and $E_{-1} = T_1^\zeta \oplus T_{-1}^\zeta$, which implies that $\mathcal{K}_\zeta$ and A_ζ commute. Since this holds for all unit normal vectors ζ, it follows that P^{n-1} is curvature-adapted. □

5 Tubes around the homogeneous complex hypersurface

In this section we discuss the geometry of the tubes around the homogeneous complex hypersurface $P^{n-1} \cong \mathbb{C}H^{n-1}(-4)$ in (Q^{n*}, g). We first observe that the tubes around P^{n-1} are homogeneous real hypersurfaces in Q^{n*}. In fact, the connected closed subgroup H of $G = SO^0_{2,n}$ with Lie algebra

$$\mathfrak{h} = \mathfrak{k}_{\alpha_1} \oplus \mathfrak{d} = \mathfrak{k}_{\alpha_1} \oplus \mathbb{R}H_{\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2}$$

acts on Q^{n*} with cohomogeneity one (see [2], Theorem 8). By construction, the orbit of H containing o is the homogeneous complex hypersurface P^{n-1} and the principal orbits are the tubes around P^{n-1}. We denote by P^{2n-1}_r the tube with radius $r \in \mathbb{R}_+$ around P^{n-1} in Q^{n*}. Note that P^{2n-1}_r is a homogeneous real hypersurface in Q^{n*} and hence $\dim_{\mathbb{R}}(P^{2n-1}_r) = 2n-1$.

By Corollary 4.3, the homogeneous complex hypersurface P^{n-1} is curvature-adapted. Since tubes around curvature-adapted submanifolds in Riemannian symmetric spaces are again curvature-adapted (see [10], Theorem 6.14, or [6], Theorem 6), Corollary 4.3 implies:

Proposition 5.1 *The tube P^{2n-1}_r with radius $r \in \mathbb{R}_+$ around the homogeneous complex hypersurface $P^{n-1} \cong \mathbb{C}H^{n-1}(-4)$ in (Q^{n*}, g) is curvature-adapted.*

We can therefore use Jacobi field theory to compute the principal curvatures and principal curvature spaces of P^{2n-1}_r (see, e.g., [1], Section 10.2.3, for a detailed description of the methodology). Since P^{2n-1}_r is a homogeneous real hypersurface in Q^{n*}, it suffices to compute the principal curvatures and principal curvature spaces at one point. Let $\zeta \in \nu_o P^{n-1}$ be a unit normal vector and $\gamma : \mathbb{R} \to Q^{n*}$ the geodesic in Q^{n*} with $\gamma(0) = o$ and $\dot\gamma(0) = \zeta$. Then $p = \gamma(r) \in P^{2n-1}_r$ and $\zeta_r = \dot\gamma(r)$ is a unit normal vector of P^{2n-1}_r at o. Since ζ is $\mathfrak{A}$-isotropic, also ζ_r is $\mathfrak{A}$-isotropic. Thus the normal bundle of P^{2n-1}_r consists of $\mathfrak{A}$-isotropic singular tangent vectors of Q^{n*}.

We denote by $\gamma^\perp$ the parallel subbundle of the tangent bundle of Q^{n*} along γ that is defined by the orthogonal complements of $\mathbb{R}\dot\gamma(t)$ in $T_{\gamma(t)}Q^{n*}$, $t \in \mathbb{R}$, and put

$$\bar R^\perp_\gamma = \bar R_\gamma|_{\gamma^\perp} = \bar R(\cdot, \dot\gamma)\dot\gamma|_{\gamma^\perp}.$$

Let D be the $\mathrm{End}(\gamma^\perp)$-valued tensor field along γ solving the Jacobi equation

$$D'' + \bar R^\perp_\gamma \circ D = 0,\ D(0) = \begin{pmatrix} \mathrm{id}_{T_oP^{n-1}} & 0 \\ 0 & 0 \end{pmatrix},\ D'(0) = \begin{pmatrix} -A_\zeta & 0 \\ 0 & \mathrm{id}_{\mathbb{R}J\zeta} \end{pmatrix},$$

where the decomposition of the matrices is with respect to the decomposition $\gamma^\perp(0) = T_oP^{n-1} \oplus \mathbb{R}J\zeta$ and A_ζ is the shape operator of P^{n-1} with respect to ζ. If $v \in T_oP^{n-1}$ and B_v is the parallel vector field along γ with $B_v(0) = v$, then $Z_v = DB_v$ is the Jacobi field along γ with initial values $Z_v(0) = v$ and $Z'_v(0) = -A_\zeta v$. If $v \in \mathbb{R}J\zeta$ and B_v is the parallel vector field along γ with $B_v(0) = v$, then $Z_v = DB_v$ is the Jacobi field along γ with initial values $Z_v(0) = 0$ and $Z'_v(0) = v$. We decompose T_oP^{n-1} orthogonally into $T_oP^{n-1} = T^\zeta_0 \oplus T^\zeta_1 \oplus T^\zeta_{-1}$ (see Theorem 4.2).

Since ζ is $\mathfrak{A}$-isotropic, the Jacobi operator $\bar R^\perp_\gamma$ at o is of matrix form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

with respect to the decomposition $T_0^\zeta \oplus T_1^\zeta \oplus T_{-1}^\zeta \oplus \mathbb{R}J\zeta$. Since (Q^{n*}, g) is a Riemannian symmetric space, the Jacobi operator $\bar{R}_\gamma^\perp$ is parallel along γ. By solving the above second-order initial value problem explicitly we obtain

$$D(r) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-r} & 0 & 0 \\ 0 & 0 & e^r & 0 \\ 0 & 0 & 0 & \frac{1}{2}\sinh(2r) \end{pmatrix}$$

with respect to the parallel translate of the decomposition $T_0^\zeta \oplus T_1^\zeta \oplus T_{-1}^\zeta \oplus \mathbb{R}J\zeta$ along γ from o to $\gamma(r)$. The shape operator $A_{\zeta_r}^r$ of P_r^{2n-1} with respect to the unit normal vector $\zeta_r = \dot{\gamma}(r)$ satisfies the equation

$$A_{\zeta_r}^r = -D'(r) \circ D^{-1}(r).$$

The matrix representation of $A_{\zeta_r}^r$ with respect to the parallel translate of the decomposition $T_0^\zeta \oplus T_1^\zeta \oplus T_{-1}^\zeta \oplus \mathbb{R}J\zeta$ along γ from o to $\gamma(r)$ therefore is

$$A_{\zeta_r}^r = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2\coth(2r) \end{pmatrix}.$$

It is remarkable that the principal curvatures of the tubes P_r^{2n-1}, corresponding to the maximal complex subbundle $\mathcal{C}$, are the same as those for the focal set P^{n-1}. The only additional principal curvature comes from the circles in P_r^{2n-1} generated by the unit normal bundle of P^{n-1}, which in fact is the Hopf principal curvature function α. We change the orientation of the unit normal vector field of P_r^{2n-1} so that α becomes positive, that is, $\alpha = 2\coth(2r)$.

Since the Kähler structure J is parallel along γ, the condition $JT_1^\zeta = T_{-1}^\zeta$ is preserved by parallel translation along γ. From this we easily see that the shape operator $A_{\zeta_r}^r$ of P_r^{2n-1} satisfies $A_{\zeta_r}^r \phi + \phi A_{\zeta_r}^r = 0$. We summarize the previous discussion in the following result.

Theorem 5.2 *Let P_r^{2n-1} be the tube with radius $r \in \mathbb{R}_+$ around the homogeneous complex hypersurface $P^{n-1} \cong \mathbb{C}H^{n-1}(-4)$ in (Q^{n*}, g). The normal bundle of P_r^{2n-1} consists of $\mathfrak{A}$-isotropic singular tangent vectors of (Q^{n*}, g). The homogeneous real hypersurface P_r^{2n-1} has four distinct constant principal curvatures*

$$0, 1, -1, 2\coth(2r)$$

with multiplicities $2, n-2, n-2, 1$, respectively, with respect to a suitable orientation of the unit normal vector field ζ_r of P_r^{2n-1}. In particular, the mean curvature of P_r^{2n-1} is *equal to* $2\coth(2r)$. *The corresponding principal curvature spaces are*

$$T_0^{\zeta_r} = \mathbb{C}C\zeta_r = \mathcal{C} \ominus \mathcal{Q}, \; T_{2\coth(2r)}^{\zeta_r} = \mathbb{R}J\zeta_r,$$

where C is an arbitrary real structure on Q^{n*}. The principal curvature spaces $T_1^{\zeta_r}$ and $T_{-1}^{\zeta_r}$ are mapped into each other by the complex structure J (or equivalently, by the structure tensor field ϕ) and are contained in the ± 1-eigenspaces of a suitable real structure C. Moreover, the shape operator A^r and the structure tensor field ϕ of P_r^{2n-1} *satisfy*

$$A^r\phi + \phi A^r = 0.$$

To put this into the context of Theorem 1.1, we define $M_\alpha^{2n-1} = P_r^{2n-1}$ with $\alpha = 2\coth(2r)$. Recall that the normal space $\nu_o P^{n-1}$ is a Lie triple system and the totally geodesic submanifold B_1 of Q^{n*} generated by this Lie triple system is isometric to a complex hyperbolic line $\mathbb{C}H^1(-4)$ of constant holomorphic sectional curvature -4. The same is true for all the other normal spaces of P^{n-1}. It follows that, by construction, the integral curves of the Reeb vector field $\xi = -J\zeta$ are circles of radius r in a complex hyperbolic line of constant sectional curvature -4. Such a circle has constant geodesic curvature $\alpha = 2\coth(2r)$. We thus see that the integral curves of the Reeb vector field are circles with radius r in a complex hyperbolic line $\mathbb{C}H^1(-4)$. This clarifies the geometric construction discussed in the introduction.

6 The minimal homogeneous Hopf hypersurface

In this section we construct the minimal homogeneous real hypersurface M_0^{2n-1} in (Q^{n*}, g). The construction is a special case of the canonical extension technique developed by the author and Tamaru in [4].

We start by defining the reductive subalgebra

$$\mathfrak{l}_1 = \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_{\alpha_1} \cong \mathfrak{su}_{1,1} \oplus \mathbb{R} \oplus \mathfrak{so}_{n-2}$$

and the nilpotent subalgebra

$$\mathfrak{n}_1 = \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2} \cong \mathfrak{h}^{2n-3}$$

of $\mathfrak{g} = \mathfrak{so}_{2,n}$. Here, $\mathfrak{h}^{2n-3}$ is the $(2n-3)$-dimensional Heisenberg algebra with 1-dimensional center. Note that $\mathfrak{n}_1$ already appeared in the construction of the homogeneous complex hypersurface P^{n-1} in Sect. 4 as part of the subalgebra $\mathfrak{d} = \mathbb{R}H_{\alpha_1+2\alpha_2} \oplus \mathfrak{n}_1$. We define

$$\mathfrak{a}_1 = \ker(\alpha_1) = \mathbb{R}H_{\alpha_1+2\alpha_2},\ \mathfrak{a}^1 = \mathbb{R}H_{\alpha_1},$$

which gives an orthogonal decomposition of $\mathfrak{a}$ into $\mathfrak{a} = \mathfrak{a}_1 \oplus \mathfrak{a}^1$. The reductive subalgebra $\mathfrak{l}_1$ is the centralizer and the normalizer of $\mathfrak{a}_1$ in $\mathfrak{g}$. Since $[\mathfrak{l}_1, \mathfrak{n}_1] \subseteq \mathfrak{n}_1$,

$$\mathfrak{q}_1 = \mathfrak{l}_1 \oplus \mathfrak{n}_1$$

is a subalgebra of $\mathfrak{g}$, the so-called parabolic subalgebra of $\mathfrak{g}$ associated with the simple root α_1. The subalgebra $\mathfrak{l}_1 = \mathfrak{q}_1 \cap \theta(\mathfrak{q}_1)$ is a reductive Levi subalgebra of $\mathfrak{q}_1$ and $\mathfrak{n}_1$ is the unipotent radical of $\mathfrak{q}_1$. Therefore the decomposition $\mathfrak{q}_1 = \mathfrak{l}_1 \oplus \mathfrak{n}_1$ is a semidirect sum of the Lie algebras $\mathfrak{l}_1$ and $\mathfrak{n}_1$. The decomposition $\mathfrak{q}_1 = \mathfrak{l}_1 \oplus \mathfrak{n}_1$ is the Chevalley decomposition of the parabolic subalgebra $\mathfrak{q}_1$.

Next, we define a reductive subalgebra $\mathfrak{m}_1$ of $\mathfrak{g}$ by

$$\mathfrak{m}_1 = \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{a}^1 \oplus \mathfrak{g}_{\alpha_1} \oplus \mathfrak{k}_0 \cong \mathfrak{su}_{1,1} \oplus \mathfrak{so}_{n-2}.$$

The subalgebra $\mathfrak{m}_1$ normalizes $\mathfrak{a}_1 \oplus \mathfrak{n}_1$. The decomposition

$$\mathfrak{q}_1 = \mathfrak{m}_1 \oplus \mathfrak{a}_1 \oplus \mathfrak{n}_1$$

is the Langlands decomposition of the parabolic subalgebra $\mathfrak{q}_1$. We define a subalgebra $\mathfrak{k}_1$ of $\mathfrak{k}$ by

$$\mathfrak{k}_1 = \mathfrak{q}_1 \cap \mathfrak{k} = \mathfrak{l}_1 \cap \mathfrak{k} = \mathfrak{m}_1 \cap \mathfrak{k} = \mathfrak{k}_{\alpha_1} \oplus \mathfrak{k}_0 \cong \mathfrak{so}_2 \oplus \mathfrak{so}_{n-2}.$$

Next, we define the semisimple subalgebra

$$\mathfrak{g}_1 = \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{a}^1 \oplus \mathfrak{g}_{\alpha_1} \cong \mathfrak{su}_{1,1}.$$

It is easy to see that the subspaces

$$\mathfrak{a} \oplus \mathfrak{p}_{\alpha_1} = \mathfrak{l}_1 \cap \mathfrak{p}, \; \mathfrak{a}^1 \oplus \mathfrak{p}_{\alpha_1} = \mathfrak{m}_1 \cap \mathfrak{p} = \mathfrak{g}_1 \cap \mathfrak{p}$$

are Lie triple systems in $\mathfrak{p}$. Then $\mathfrak{g}_1 = \mathfrak{k}_{\alpha_1} \oplus (\mathfrak{a}^1 \oplus \mathfrak{p}_{\alpha_1})$ is a Cartan decomposition of the semisimple subalgebra $\mathfrak{g}_1$ of $\mathfrak{g}$ and $\mathfrak{a}^1$ is a maximal abelian subspace of $\mathfrak{a}^1 \oplus \mathfrak{p}_{\alpha_1}$. Moreover, $\mathfrak{g}_1 = (\mathfrak{k}_{\alpha_1} \oplus \mathfrak{a}^1) \oplus \mathfrak{g}_{-\alpha_1} \oplus \mathfrak{g}_{\alpha_1}$ is the restricted root space decomposition of $\mathfrak{g}_1$ with respect to $\mathfrak{a}^1$ and $\{\pm\alpha_1\}$ is the corresponding set of restricted roots.

We now relate these algebraic constructions to the geometry of the complex hyperbolic quadric Q^{n*}. We denote by $A_1 \cong \mathbb{R}$ the connected abelian subgroup of G with Lie algebra $\mathfrak{a}_1$ and by $N_1 \cong H^{2n-3}$ the connected nilpotent subgroup of G with Lie algebra $\mathfrak{n}_1 \cong \mathfrak{h}^{2n-3}$. Here, H^{2n-3} is the $(2n-3)$-dimensional Heisenberg group with 1-dimensional center. The centralizer $L_1 = Z_G(\mathfrak{a}_1) \cong SU_{1,1} \times \mathbb{R} \times SO_{n-2}$ of $\mathfrak{a}_1$ in G is a reductive subgroup of G with Lie algebra $\mathfrak{l}_1$. The subgroup A_1 is contained in the center of L_1. The subgroup L_1 normalizes N_1 and $Q_1 = L_1N_1$ is a subgroup of G with Lie algebra $\mathfrak{q}_1$. The subgroup Q_1 coincides with the normalizer $N_G(\mathfrak{l}_1 \oplus \mathfrak{n}_1)$ of $\mathfrak{l}_1 \oplus \mathfrak{n}_1$ in G and hence Q_1 is a closed subgroup of G. The subgroup Q_1 is the parabolic subgroup of G associated with the simple root α_1.

Let $G_1 \cong SU_{1,1}$ be the connected subgroup of G with Lie algebra $\mathfrak{g}_1 \cong \mathfrak{su}_{1,1}$. The intersection K_1 of L_1 and K, i.e., $K_1 = L_1 \cap K \cong SO_2 \times SO_{n-2}$, is a maximal compact subgroup of L_1 and $\mathfrak{k}_1$ is the Lie algebra of K_1. The adjoint group $\mathrm{Ad}(L_1)$ normalizes $\mathfrak{g}_1$, and consequently $M_1 = K_1G_1 \cong SU_{1,1} \times SO_{n-2}$ is a subgroup of L_1. The Lie algebra of M_1 is $\mathfrak{m}_1$ and L_1 is isomorphic to the Lie group direct product $M_1 \times A_1$, i.e., $L_1 = M_1 \times A_1 \cong (SU_{1,1} \times SO_{n-2}) \times \mathbb{R}$. The parabolic subgroup Q_1 acts transitively on Q^{n*} and the isotropy subgroup at o is K_1, that is, $Q^{n*} \cong Q_1/K_1$.

Since $\mathfrak{g}_1 = \mathfrak{k}_{\alpha_1} \oplus (\mathfrak{a}^1 \oplus \mathfrak{p}_{\alpha_1})$ is a Cartan decomposition of the semisimple subalgebra $\mathfrak{g}_1$, we have $[\mathfrak{a}^1 \oplus \mathfrak{p}_{\alpha_1}, \mathfrak{a}^1 \oplus \mathfrak{p}_{\alpha_1}] = \mathfrak{k}_{\alpha_1}$. Thus $G_1 \cong SU_{1,1}$ is the connected closed subgroup of G with Lie algebra $[\mathfrak{a}^1 \oplus \mathfrak{p}_{\alpha_1}, \mathfrak{a}^1 \oplus \mathfrak{p}_{\alpha_1}] \oplus (\mathfrak{a}^1 \oplus \mathfrak{p}_{\alpha_1})$. Since $\mathfrak{a}^1 \oplus \mathfrak{p}_{\alpha_1}$ is a Lie triple system in $\mathfrak{p}$, the orbit $B_1 = G_1 \cdot o$ of the G_1-action on Q^{n*} containing o is a connected totally geodesic submanifold of Q^{n*} with $T_oB_1 = \mathfrak{a}^1 \oplus \mathfrak{p}_{\alpha_1}$. Moreover, B_1 is a Riemannian symmetric space of non-compact type and rank 1, and

$$B_1 = G_1 \cdot o = G_1/(G_1 \cap K_1) \cong SU_{1,1}/SO_2 \cong \mathbb{C}H^1(-4),$$

where $\mathbb{C}H^1(-4)$ is a complex hyperbolic line of constant (holomorphic) sectional curvature -4. The submanifold B_1 is a boundary component of Q^{n*} in the context of the maximal Satake compactification of Q^{n*}. This boundary component coincides with the totally geodesic submanifold B_1 that we constructed in Sect. 4.

Clearly, $\mathfrak{a}_1$ is a Lie triple system and the corresponding totally geodesic submanifold is a Euclidean line $\mathbb{R} = A_1 \cdot o$. Since the action of A_1 on M is free and A_1 is simply connected, we can identify $\mathbb{R}$ and A_1 canonically.

Finally, $\mathfrak{f}_1 = \mathfrak{a} \oplus \mathfrak{p}_{\alpha_1}$ is a Lie triple system and the corresponding totally geodesic submanifold F_1 is the symmetric space

$$F_1 = L_1 \cdot o = L_1/K_1 = (M_1 \times A_1)/K_1 = B_1 \times \mathbb{R} \cong \mathbb{C}H^1(-4) \times \mathbb{R}.$$

The submanifolds F_1 and B_1 have a natural geometric interpretation. Denote by $\bar{C}^+(\Lambda) \subset \mathfrak{a}$ the closed positive Weyl chamber that is determined by the two simple roots α_1 and α_2. Let Z be non-zero vector in $\bar{C}^+(\Lambda)$ such that $\alpha_1(Z) = 0$ and $\alpha_2(Z) > 0$, and consider the geodesic $\gamma_Z(t) = \mathrm{Exp}(tZ) \cdot o$ in Q^{n*} with $\gamma_Z(0) = o$ and $\dot{\gamma}_Z(0) = Z$. The totally geodesic submanifold F_1 is the union of all geodesics in Q^{n*} parallel to γ_Z, and B_1 is the semisimple part of F_1 in the de Rham decomposition of F_1 (see, e.g., [9], Proposition 2.11.4 and Proposition 2.20.10).

The parabolic group Q_1 is diffeomorphic to the product $M_1 \times A_1 \times N_1$. This analytic diffeomorphism induces an analytic diffeomorphism between

$$B_1 \times \mathbb{R} \times N_1 \cong \mathbb{C}H^1(-4) \times \mathbb{R} \times H^{2n-3}$$

and Q^{n*}, giving a horospherical decomposition of the complex hyperbolic quadric Q^{n*},

$$\mathbb{C}H^1(-4) \times \mathbb{R} \times H^{2n-3} \cong Q^{n*}.$$

The factor $\mathbb{R} \times H^{2n-3}$ corresponds to the homogeneous complex hypersurface $P^{n-1} \cong \mathbb{C}H^{n-1}(-4)$ that we discussed in Sect. 4.

We have $\mathbb{R}H_{\alpha_1} = \mathfrak{a}^1 \subset \mathfrak{g}_1$ and $G_1 \cdot o = B_1$. It follows from Theorem 4.2 that $\mathfrak{a}^1$ consists of $\mathfrak{A}$-isotropic tangent vectors of Q^{n*}. Let $A^1 \cong \mathbb{R}$ be the abelian subalgebra of $\mathfrak{a}$ with Lie algebra $\mathfrak{a}^1$. Then the orbit $A^1 \cdot o$ is the path of an $\mathfrak{A}$-isotropic geodesic γ (determined by the root vector H_{α_1}) in the complex hyperbolic quadric Q^{n*}. Moreover, by construction, this geodesic is contained in the boundary component $B_1 \cong \mathbb{C}H^1(-4)$. The action of A^1 on $\mathbb{C}H^1(-4)$ is of cohomogeneity one. The orbit containing o is the geodesic γ, and the other orbits are the equidistant curves to γ.

The canonical extension of the cohomogeneity one action of A^1 on the boundary component $B_1 \cong \mathbb{C}H^1(-4)$ is defined as follows. We first define the solvable subalgebra

$$\begin{aligned}\mathfrak{s}_1 &= \mathfrak{a}^1 \oplus \mathfrak{a}_1 \oplus \mathfrak{n}_1 = \mathfrak{a} \oplus \mathfrak{n}_1 \\ &= \left\{ \begin{pmatrix} 0 & y & a_1 & -y & v_1 & \cdots & v_{n-2} \\ -y & 0 & y & a_2 & w_1 & \cdots & w_{n-2} \\ a_1 & y & 0 & -y & v_1 & \cdots & v_{n-2} \\ -y & a_2 & y & 0 & w_1 & \cdots & w_{n-2} \\ v_1 & w_1 & -v_1 & -w_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & w_{n-2} & -v_{n-2} & -w_{n-2} & 0 & \cdots & 0 \end{pmatrix} : \begin{array}{l} a_1, a_2, y \in \mathbb{R}, \\ v, w \in \mathbb{R}^{n-2} \end{array} \right\}\end{aligned}$$

of $\mathfrak{a} \oplus \mathfrak{n}$. Let S_1 be the connected solvable subgroup of AN with Lie algebra $\mathfrak{s}_1$. Then the action of S_1 on AN (resp. Q^{n*}) is of cohomogeneity one (see [4]). By construction, all orbits of the S_1-action on AN (resp. Q^{n*}) are homogeneous real hypersurfaces in $(AN, \langle\cdot,\cdot\rangle)$ (resp. (Q^{n*}, g)). Let $\hat{M}_0^{2n-1}$ (resp. M_0^{2n-1}) be the orbit containing the point o. Geometrically, we can describe this orbit as the canonical extension of an $\mathfrak{A}$-isotropic geodesic in the boundary component $B_1 \cong \mathbb{C}H^1(-4)$.

We will now compute the shape operator of the homogeneous real hypersurface $\hat{M}_0^{2n-1}$ in $(AN, \langle\cdot,\cdot\rangle)$. Since $\hat{M}_0^{2n-1}$ is homogeneous, it suffices to make the computations at the point o. We define

$$\hat{\zeta} = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathfrak{g}_{\alpha_1} \subset \mathfrak{n}.$$

Then

$$\theta(\hat{\zeta}) = \begin{pmatrix} 0 & 1 & 0 & -1 & 0 & \cdots & 0 \\ -1 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & -1 & 0 & 1 & 0 & \cdots & 0 \\ -1 & 0 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathfrak{g}_{-\alpha_1}$$

and

$$\langle \hat{\zeta}, \hat{\zeta} \rangle = -\frac{1}{8}\operatorname{tr}(\hat{\zeta}\theta(\hat{\zeta})) = 1.$$

Thus $\hat{\zeta}$ is a unit normal vector of $\hat{M}_0^{2n-1}$ at o. Let $\hat{A}$ be the shape operator of $\hat{M}_0^{2n-1}$ in $(AN, \langle\cdot,\cdot\rangle)$ with respect to $\hat{\zeta}$. As in Sect. 4, using arguments involving the Weingarten and Koszul formulas, we can show that

$$\hat{A}\hat{X} = [\zeta, \hat{X}]_{\mathfrak{s}_1}$$

for all $\hat{X} \in \mathfrak{s}_1$, where $\zeta = \frac{1}{2}(\hat{\zeta} - \theta(\hat{\zeta}))$ is the orthogonal projection of $\hat{\zeta}$ onto $\mathfrak{p}$ and $[\,\cdot\,]_{\mathfrak{s}_1}$ is the orthogonal projection onto $\mathfrak{s}_1$.

We have

$$\zeta = \frac{1}{2}(\hat{\zeta} - \theta(\hat{\zeta})) = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in \mathfrak{p}_{\alpha_1}.$$

For

$$\hat{X} = \begin{pmatrix} 0 & y & a_1 & -y & v_1 & \cdots & v_{n-2} \\ -y & 0 & y & a_2 & w_1 & \cdots & w_{n-2} \\ a_1 & y & 0 & -y & v_1 & \cdots & v_{n-2} \\ -y & a_2 & y & 0 & w_1 & \cdots & w_{n-2} \\ v_1 & w_1 & -v_1 & -w_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & w_{n-2} & -v_{n-2} & -w_{n-2} & 0 & \cdots & 0 \end{pmatrix} \in \mathfrak{s}_1$$

we then compute

$$[\zeta, \hat{X}] = \begin{pmatrix} 0 & a_2 - a_1 & 0 & 0 & w_1 & \cdots & w_{n-2} \\ a_1 - a_2 & 0 & 0 & 0 & v_1 & \cdots & v_{n-2} \\ 0 & 0 & 0 & a_2 - a_1 & w_1 & \cdots & w_{n-2} \\ 0 & 0 & a_1 - a_2 & 0 & v_1 & \cdots & v_{n-2} \\ w_1 & v_1 & -w_1 & -v_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{n-2} & v_{n-2} & -w_{n-2} & -v_{n-2} & 0 & \cdots & 0 \end{pmatrix}.$$

The orthogonal projection of $[\zeta, \hat{X}]$ onto $\mathfrak{s}_1$ is

$$[\zeta, \hat{X}]_{\mathfrak{s}_1} = \begin{pmatrix} 0 & 0 & 0 & 0 & w_1 & \cdots & w_{n-2} \\ 0 & 0 & 0 & 0 & v_1 & \cdots & v_{n-2} \\ 0 & 0 & 0 & 0 & w_1 & \cdots & w_{n-2} \\ 0 & 0 & 0 & 0 & v_1 & \cdots & v_{n-2} \\ w_1 & v_1 & -w_1 & -v_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{n-2} & v_{n-2} & -w_{n-2} & -v_{n-2} & 0 & \cdots & 0 \end{pmatrix}.$$

We conclude that the shape operator $\hat{A}$ of $\hat{M}_0^{2n-1}$ in $(AN, \langle \cdot, \cdot \rangle)$ with respect to $\hat{\zeta}$ is given by

$$\hat{A}\hat{X} = \begin{pmatrix} 0 & 0 & 0 & 0 & w_1 & \cdots & w_{n-2} \\ 0 & 0 & 0 & 0 & v_1 & \cdots & v_{n-2} \\ 0 & 0 & 0 & 0 & w_1 & \cdots & w_{n-2} \\ 0 & 0 & 0 & 0 & v_1 & \cdots & v_{n-2} \\ w_1 & v_1 & -w_1 & -v_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{n-2} & v_{n-2} & -w_{n-2} & -v_{n-2} & 0 & \cdots & 0 \end{pmatrix}$$

with

$$\hat{X} = \begin{pmatrix} 0 & y & a_1 & -y & v_1 & \cdots & v_{n-2} \\ -y & 0 & y & a_2 & w_1 & \cdots & w_{n-2} \\ a_1 & y & 0 & -y & v_1 & \cdots & v_{n-2} \\ -y & a_2 & y & 0 & w_1 & \cdots & w_{n-2} \\ v_1 & w_1 & -v_1 & -w_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & w_{n-2} & -v_{n-2} & -w_{n-2} & 0 & \cdots & 0 \end{pmatrix} \in \mathfrak{s}_1.$$

From this we deduce that 0 is a principal curvature of $\hat{M}_0^{2n-1}$ with multiplicity 3 and corresponding principal curvature space

$$\hat{T}_0 = \mathfrak{a} \oplus \mathfrak{g}_{\alpha_1 \oplus 2\alpha_2}.$$

On the orthogonal complement $\mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_2}$ the shape operator is of the form

$$\hat{A} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

with respect to the orthogonal decomposition $\mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_2}$. The characteristic polynomial of this matrix is $x^2 - 1$, and hence the eigenvalues of $\hat{A}$ restricted to $\mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_2}$ are 1 and -1. The corresponding eigenspaces are

$$\hat{T}_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & u_1 & \cdots & u_{n-2} \\ 0 & 0 & 0 & 0 & u_1 & \cdots & u_{n-2} \\ 0 & 0 & 0 & 0 & u_1 & \cdots & u_{n-2} \\ 0 & 0 & 0 & 0 & u_1 & \cdots & u_{n-2} \\ u_1 & u_1 & -u_1 & -u_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{n-2} & u_{n-2} & -u_{n-2} & -u_{n-2} & 0 & \cdots & 0 \end{pmatrix} \right\} \cong \mathbb{R}^{n-2}$$

and

$$\hat{T}_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 & u_1 & \cdots & u_{n-2} \\ 0 & 0 & 0 & 0 & -u_1 & \cdots & -u_{n-2} \\ 0 & 0 & 0 & 0 & u_1 & \cdots & u_{n-2} \\ 0 & 0 & 0 & 0 & -u_1 & \cdots & -u_{n-2} \\ u_1 & -u_1 & -u_1 & u_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ u_{n-2} & -u_{n-2} & -u_{n-2} & u_{n-2} & 0 & \cdots & 0 \end{pmatrix} \right\} \cong \mathbb{R}^{n-2}.$$

All of the above calculations are with respect to the metric $\langle \cdot, \cdot \rangle$ on AN. We now switch to the Riemannian metric g on Q^{n*} and the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Recall that, by construction, $(AN, \langle \cdot, \cdot \rangle)$ and (Q^{n*}, g) are isometric and the metrics are related by

$$\langle H_1 + \hat{X}_1, H_2 + \hat{X}_2 \rangle = g(H_1, H_2) + g(X_1, X_2)$$

with $H_1, H_2 \in \mathfrak{a}$ and $\hat{X}_1, \hat{X}_2 \in \mathfrak{n}$.

Since $\hat{\zeta}$ is a unit vector in $\mathfrak{g}_{\alpha_1}$, the vector $\zeta = \frac{1}{2}(\hat{\zeta} - \theta(\hat{\zeta}))$ is a unit vector in $\mathfrak{p}_{\alpha_1}$. Since $\mathfrak{p}_{\alpha_1} \subset T_o B_1$ and all (non-zero) tangent vectors of the boundary component B_1 are $\mathfrak{A}$-isotropic (see Theorem 4.2), we conclude that the normal bundle of M_0^{2n-1} consists of $\mathfrak{A}$-isotropic singular tangent vectors of (Q^{n*}, g).

Let A be the shape operator of M_0^{2n-1} in (Q^{n*}, g) with respect to ζ. The above calculations imply that

$$AX = \begin{pmatrix} 0 & 0 & 0 & 0 & w_1 & \cdots & w_{n-2} \\ 0 & 0 & 0 & 0 & v_1 & \cdots & v_{n-2} \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ w_1 & v_1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ w_{n-2} & v_{n-2} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with

$$X = \begin{pmatrix} 0 & 0 & a_1 & -y & v_1 & \cdots & v_{n-2} \\ 0 & 0 & y & a_2 & w_1 & \cdots & w_{n-2} \\ a_1 & y & 0 & 0 & 0 & \cdots & 0 \\ -y & a_2 & 0 & 0 & 0 & \cdots & 0 \\ v_1 & w_1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & w_{n-2} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in T_o M_0^{2n-1} \subset \mathfrak{p}.$$

From this, we easily deduce the following result.

Theorem 6.1 *Let M_0^{2n-1} be the homogeneous real hypersurface in (Q^{n*}, g) obtained by canonical extension of the geodesic that is tangent to the root vector H_{α_1} in the boundary component $B_1 \cong \mathbb{C}H^1(-4)$ of (Q^{n*}, g). The normal bundle of M_0^{2n-1} consists of $\mathfrak{A}$-isotropic singular tangent vectors of (Q^{n*}, g) and M_0^{2n-1} has three distinct constant principal curvatures 0, 1, -1 with multiplicities 3, $n-2$, $n-2$, respectively. The principal curvature spaces T_0, T_1 and T_{-1} are*

$$\begin{aligned} T_0 &= \mathfrak{a} \oplus \mathfrak{p}_{\alpha_1+2\alpha_2} = \mathbb{R}J\zeta \oplus (\mathcal{C} \ominus \mathcal{Q}), \\ T_1 &= \{X - JX : X \in \mathfrak{p}_{\alpha_2}\} = \{X + JX : X \in \mathfrak{p}_{\alpha_1+\alpha_2}\}, \\ T_{-1} &= \{X + JX : X \in \mathfrak{p}_{\alpha_2}\} = \{X - JX : X \in \mathfrak{p}_{\alpha_1+\alpha_2}\}. \end{aligned}$$

We have $T_1 \oplus T_{-1} = \mathcal{Q}$ and $JT_1 = T_{-1}$. The shape operator A of M_0^{2n-1} satisfies

$$A\phi + \phi A = 0.$$

Note that

$$T_1 \subset V\left(\frac{1}{\sqrt{2}}(C_0 + JC_0)\right),\ T_{-1} \subset JV\left(\frac{1}{\sqrt{2}}(C_0 + JC_0)\right).$$

We immediately see from Theorem 6.1 that $\mathrm{tr}(A) = 0$.

Corollary 6.2 *The homogeneous Hopf hypersurface M_0^{2n-1} in (Q^{n*}, g) is minimal.*

The eigenspaces T_0, T_1 and T_{-1} of the shape operator A and the eigenspaces E_0, E_{-1} and E_{-4} of the normal Jacobi operator $\mathcal{K}_\zeta$ satisfy

$$T_0 = E_0 \oplus E_{-4},\ \ T_{-1} \oplus T_1 = E_{-1}.$$

It follows that A and $\mathcal{K} = \mathcal{K}_\zeta$ are simultaneously diagonalizable and hence $A\mathcal{K} = \mathcal{K}A$. This implies that M_0^{2n-1} is curvature-adapted.

Corollary 6.3 *The homogeneous Hopf hypersurface M_0^{2n-1} in (Q^{n*}, g) is curvature-adapted.*

We finally relate this construction to the discussion in the introduction. The subalgebra

$$\mathfrak{s}_1 = \mathfrak{a} \oplus \mathfrak{n}_1 = \mathfrak{a} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2}$$

of $\mathfrak{a} \oplus \mathfrak{n}$ contains the subalgebra

$$\mathfrak{d} = \mathbb{R}H_{\alpha_1+2\alpha_2} \oplus \mathfrak{g}_{\alpha_2} \oplus \mathfrak{g}_{\alpha_1+\alpha_2} \oplus \mathfrak{g}_{\alpha_1+2\alpha_2}.$$

The subalgebra $\mathfrak{d}$ induces the homogeneous complex hypersurface $\hat{P}^{n-1} \cong \mathbb{C}H^{n-1}(-4)$, as discussed in Sect. 4. Since the construction is left-invariant, it follows that the homogeneous real hypersurface $\hat{M}_0^{2n-1}$ in $(AN, \langle\cdot,\cdot\rangle)$ is foliated by isometric copies of the homogeneous complex hypersurface $\hat{P}^{n-1} \cong \mathbb{C}H^{n-1}(-4)$. This implies that the homogeneous complex hypersurface M_0^{2n-1} in (Q^{n*}, g) is foliated by isometric copies of the homogeneous complex hypersurface $P^{n-1} \cong \mathbb{C}H^{n-1}(-4)$. The normal space $\nu_o P^{n-1}$ is a Lie triple system, and the totally geodesic submanifold B_1 of Q^{n*} generated by this Lie triple system is a complex hyperbolic line $\mathbb{C}H^1(-4)$ of constant holomorphic sectional curvature -4. The same is true for all the normal spaces of P^{n-1} at other points. It follows that, by construction, the integral curves of the Reeb vector field $\xi = -J\zeta$ are geodesics in a complex hyperbolic line of constant (holomorphic) sectional curvature -4. Such a geodesic has constant geodesic curvature 0. This clarifies the geometric construction explained in the introduction.

7 Equidistant real hypersurfaces

In this section we compute the shape operator of the other orbits of the cohomogeneity one action on (Q^{n*}, g) that we discussed in Sect. 6. Recall that M_0^{2n-1} is the orbit of this action containing o. Since the action is isometric, the other orbits are the equidistant real hypersurfaces to M_0^{2n-1}. For $r \in \mathbb{R}_+$ we denote by M_α^{2n-1} the equidistant real hypersurface to M_0^{2n-1} at oriented distance $r \in \mathbb{R}_+$, where we put $\alpha = 2\tanh(2r)$.

From Corollary 6.3 we know that M_0^{2n-1} is a curvature-adapted real hypersurface in Q^{n*}. We can therefore use Jacobi field theory to compute the principal curvatures and principal curvature spaces of M_α^{2n-1} (see, e.g., [1], Section 10.2.2). Since M_α^{2n-1} is a homogeneous real hypersurface in Q^{n*}, it suffices to compute the principal curvatures and principal curvature spaces at one point. Let $\zeta \in \nu_o M_0^{2n-1}$ be the unit normal vector of M_0^{2n-1} as defined in Sect. 6 and A_ζ be the shape operator of M_0^{2n-1} at o with respect to ζ. We denote by T_0, T_1 and T_{-1} the principal curvature spaces as in Theorem 6.1. Let $\gamma : \mathbb{R} \to Q^{n*}$ be the geodesic in Q^{n*} with $\gamma(0) = o$ and $\dot{\gamma}(0) = \zeta$. Then $p = \gamma(r) \in M_\alpha^{2n-1}$ and $\zeta_r = \dot{\gamma}(r)$ is a unit normal vector of M_α^{2n-1} at p. We denote by $\gamma^\perp$ the parallel subbundle of the tangent bundle of Q^{n*} along γ that is defined by the orthogonal complements of $\mathbb{R}\dot{\gamma}(t)$ in $T_{\gamma(t)}Q^{n*}$, and put

$$\bar{R}_\gamma^\perp = \bar{R}_\gamma|_{\gamma^\perp} = \bar{R}(\cdot, \dot{\gamma})\dot{\gamma}|_{\gamma^\perp}.$$

Let D be the $\mathrm{End}(\gamma^\perp)$-valued tensor field along γ solving the Jacobi equation

$$D'' + \bar{R}_\gamma^\perp \circ D = 0,\ D(0) = \mathrm{id}_{T_o M_0^{2n-1}},\ D'(0) = -A_\zeta.$$

If $v \in T_o M_0^{2n-1}$ and B_v is the parallel vector field along γ with $B_v(0) = v$, then $Y = DB_v$ is the Jacobi field along γ with initial values $Y(0) = v$ and $Y'(0) = -A_\zeta v$.

Since ζ is $\mathfrak{A}$-isotropic, the Jacobi operator $\bar{R}_\gamma^\perp$ at o is of matrix form

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{pmatrix}$$

with respect to the decomposition $\mathbb{C}C\zeta \oplus T_1 \oplus T_{-1} \oplus \mathbb{R}J\zeta$. Note that $T_0 = \mathbb{C}C\zeta \oplus \mathbb{R}J\zeta$. Since (Q^{n*}, g) is a Riemannian symmetric space, the Jacobi operator $\bar{R}^{\perp}_{\gamma}$ is parallel along γ. By solving the above second-order initial value problem explicitly we obtain

$$D(r) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & e^{-r} & 0 & 0 \\ 0 & 0 & e^{r} & 0 \\ 0 & 0 & 0 & \cosh(2r) \end{pmatrix}$$

with respect to the parallel translate of the decomposition $T_0 \oplus T_1 \oplus T_{-1} \oplus \mathbb{R}J\zeta$ along γ from o to $\gamma(r)$. The shape operator $A^{\alpha}_{\zeta_r}$ of M^{2n-1}_{α} with respect to $\zeta_r = \dot{\gamma}(r)$ satisfies the equation

$$A^{\alpha}_{\zeta_r} = -D'(r) \circ D^{-1}(r).$$

The matrix representation of $A^{\alpha}_{\zeta_r}$ with respect to the parallel translate of the decomposition $T_0 \oplus T_1 \oplus T_{-1} \oplus \mathbb{R}J\zeta$ along γ from o to $\gamma(r)$ therefore is

$$A^{\alpha}_{\zeta_r} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2\tanh(2r) \end{pmatrix}.$$

It is remarkable that the principal curvatures of the equidistant real hypersurfaces to M^{2n-1}_0 are preserved along the parallel translate of the maximal complex subspace $\mathcal{C}_o \subset T_o M^{2n-1}_0$. The only additional principal curvature arises in direction of the Reeb vector field, which is the Hopf principal curvature. We change the orientation of the unit normal vector field ζ_r so that the Hopf principal curvature is positive, that is, is equal to α. Thus we have proved:

Theorem 7.1 *Let M^{2n-1}_0 be the minimal homogeneous Hopf hypersurface in (Q^{n*}, g) as in Sect. 6 and M^{2n-1}_{α} be the equidistant real hypersurface at oriented distance $r \in \mathbb{R}_+$ from M^{2n-1}_0, where $\alpha = 2\tanh(2r)$. Then M^{2n-1}_{α} is a homogeneous Hopf hypersurface with four distinct constant principal curvatures* $0, 1, -1, 2\tanh(2r)$ *with multiplicities* $2, n-2, n-2, 1$, *respectively. The principal curvature spaces $T_0, T_1, T_{-1}, T_{2\tanh(2r)}$ satisfy*

$$T_0 = \mathcal{C} \ominus \mathcal{Q},$$
$$T_{2\tanh(2r)} = \mathbb{R}J\zeta_r = \mathcal{C}^{\perp},$$
$$T_1 \oplus T_{-1} = \mathcal{Q} \text{ and } JT_1 = T_{-1}.$$

Moreover, the shape operator A^{α} and the structure tensor field ϕ of M^{2n-1}_{α} *satisfy*

$$A^{\alpha}\phi + \phi A^{\alpha} = 0.$$

The principal curvature spaces $T_{2\tanh(2r)}$, T_0, T_1 and T_{-1} of the shape operator A^{α} and the eigenspaces E_0, E_{-1} and E_{-4} of the normal Jacobi operator $\mathcal{K}^{\alpha} = \mathcal{K}_{\zeta_r}$ satisfy

$$T_{2\tanh(2r)} = E_{-4},\ T_0 = E_0,\ T_{-1} \oplus T_1 = E_{-1}.$$

It follows that A^α and $\mathcal{K}^\alpha$ are simultaneously diagonalizable and hence $A^\alpha \mathcal{K}^\alpha = \mathcal{K}^\alpha A^\alpha$. This implies:

Corollary 7.2 *The equidistant real hypersurface* M^{2n-1}_α, $0 < \alpha < 2$, *to the minimal homogeneous Hopf hypersurface* M^{2n-1}_0 *in* (Q^{n*}, g) *is curvature-adapted.*

By construction, the integral curves of the Reeb vector field on M^{2n-1}_α are congruent to an equidistant curve at distance $r = \frac{1}{2}\tanh^{-1}(\frac{\alpha}{2})$ to a geodesic in a complex hyperbolic line $\mathbb{C}H^1(-4)$. Such an equidistant curve has constant geodesic curvature $2\tanh(2r)$. As in previous cases, this leads to the geometric interpretation of M^{2n-1}_α given by attaching copies of the homogeneous complex hypersurface $P^{n-1} \cong \mathbb{C}H^{n-1}(-4)$ to such an equidistant curve to a geodesic in the boundary component $B_1 \cong \mathbb{C}H^1(-4)$. Equivalently, M^{2n-1}_α is the canonical extension of an equidistant curve at distance $r = \frac{1}{2}\tanh^{-1}(\frac{\alpha}{2})$ to a geodesic in the boundary component $B_1 \cong \mathbb{C}H^1(-4)$.

8 The homogeneous Hopf hypersurface of horocyclic type

In this section we discuss the canonical extension of a horocycle in the boundary component $B_1 \cong \mathbb{C}H^1(-4)$, which leads to the homogeneous real hypersurface M^{2n-1}_2 in Theorem 1.1. We first define the solvable subalgebra

$$\mathfrak{h}_1 = \mathfrak{a}_1 \oplus \mathfrak{n} = \left\{ \begin{pmatrix} 0 & x+y & a & x-y & v_1 & \cdots & v_{n-2} \\ -x-y & 0 & x+y & a & w_1 & \cdots & w_{n-2} \\ a & x+y & 0 & x-y & v_1 & \cdots & v_{n-2} \\ x-y & a & -x+y & 0 & w_1 & \cdots & w_{n-2} \\ v_1 & w_1 & -v_1 & -w_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & w_{n-2} & -v_{n-2} & -w_{n-2} & 0 & \cdots & 0 \end{pmatrix} : \begin{array}{l} a, x, y \in \mathbb{R}, \\ v, w \in \mathbb{R}^{n-2} \end{array} \right\}$$

of $\mathfrak{a} \oplus \mathfrak{n}$. Recall that $\mathfrak{a}^1 \oplus \mathfrak{g}_{\alpha_1} = \mathbb{R}H_{\alpha_1} \oplus \mathfrak{g}_{\alpha_1}$ generates the boundary component $B_1 \cong \mathbb{C}H^1(-4)$. The orbit containing o of the 1-dimensional Lie group generated by $\mathfrak{g}_{\alpha_1}$ is a horocycle in the boundary component B_1. Since the tangent vectors of B_1 are $\mathfrak{A}$-isotropic, the horocycle is $\mathfrak{A}$-isotropic. The canonical extension of this cohomogeneity one action on B_1 is the cohomogeneity one action on Q^{n*} by the subgroup H_1 of AN with Lie algebra $\mathfrak{h}_1$. Let $\hat{M}^{2n-1}_2 = H_1 \cdot o \cong H_1$ be the orbit of the H_1-action on $(AN, \langle\cdot,\cdot\rangle)$ containing o and $M^{2n-1}_2 = H_1 \cdot o$ be the orbit of the H_1-action on (Q^{n*}, g) containing o.

The normal space $\nu_o\hat{M}^{2n-1}_2$ of $\hat{M}^{2n-1}_2$ at o is

$$\nu_o\hat{M}^{2n-1}_2 = \mathfrak{a}^1 = \mathbb{R}H_{\alpha_1}.$$

Since $\langle H_{\alpha_1}, H_{\alpha_1}\rangle = 4$, the vector $\hat{\zeta} = \frac{1}{2}H_{\alpha_1} \in \mathfrak{a}$ is a unit normal vector of $\hat{M}^{2n-1}_2$ at o. Let $\hat{A}$ be the shape operator of $\hat{M}^{2n-1}_2$ in $(AN, \langle\cdot,\cdot\rangle)$ with respect to $\hat{\zeta}$. As in previous sections, we can show that the shape operator $\hat{A}$ of $\hat{M}^{2n-1}_2$ is given by

$$\hat{A}\hat{X} = [\zeta, \hat{X}]_{\mathfrak{h}_1}$$

for all $\hat{X} \in \mathfrak{h}_1$, where

$$\zeta = \frac{1}{2}(\hat{\xi} - \theta(\hat{\xi})) = \hat{\xi} = \frac{1}{2} H_{\alpha_1}$$

and $[\,\cdot\,]_{\mathfrak{h}_1}$ is the orthogonal projection onto $\mathfrak{h}_1$.

For

$$\hat{X} = \begin{pmatrix} 0 & x+y & a & x-y & v_1 & \cdots & v_{n-2} \\ -x-y & 0 & x+y & a & w_1 & \cdots & w_{n-2} \\ a & x+y & 0 & x-y & v_1 & \cdots & v_{n-2} \\ x-y & a & -x+y & 0 & w_1 & \cdots & w_{n-2} \\ v_1 & w_1 & -v_1 & -w_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & w_{n-2} & -v_{n-2} & -w_{n-2} & 0 & \cdots & 0 \end{pmatrix} \in \mathfrak{h}_1$$

we then compute

$$[\zeta, \hat{X}] = \begin{pmatrix} 0 & 2x & 0 & 2x & v_1 & \cdots & v_{n-2} \\ -2x & 0 & 2x & 0 & -w_1 & \cdots & -w_{n-2} \\ 0 & 2x & 0 & 2x & v_1 & \cdots & v_{n-2} \\ 2x & 0 & -2x & 0 & -w_1 & \cdots & -w_{n-2} \\ v_1 & -w_1 & -v_1 & w_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & -w_{n-2} & -v_{n-2} & w_{n-2} & 0 & \cdots & 0 \end{pmatrix}.$$

Since the last matrix is in $\mathfrak{h}_1$, the orthogonal projection of $[\zeta, \hat{X}]$ onto $\mathfrak{h}_1$ is $[\zeta, \hat{X}]$. We conclude that the shape operator $\hat{A}$ of $\hat{M}_2^{2n-1}$ in $(AN, \langle \cdot, \cdot \rangle)$ is given by

$$\hat{A}\hat{X} = \begin{pmatrix} 0 & 2x & 0 & 2x & v_1 & \cdots & v_{n-2} \\ -2x & 0 & 2x & 0 & -w_1 & \cdots & -w_{n-2} \\ 0 & 2x & 0 & 2x & v_1 & \cdots & v_{n-2} \\ 2x & 0 & -2x & 0 & -w_1 & \cdots & -w_{n-2} \\ v_1 & -w_1 & -v_1 & w_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & -w_{n-2} & -v_{n-2} & w_{n-2} & 0 & \cdots & 0 \end{pmatrix}$$

with

$$\hat{X} = \begin{pmatrix} 0 & x+y & a & x-y & v_1 & \cdots & v_{n-2} \\ -x-y & 0 & x+y & a & w_1 & \cdots & w_{n-2} \\ a & x+y & 0 & x-y & v_1 & \cdots & v_{n-2} \\ x-y & a & -x+y & 0 & w_1 & \cdots & w_{n-2} \\ v_1 & w_1 & -v_1 & -w_1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & w_{n-2} & -v_{n-2} & -w_{n-2} & 0 & \cdots & 0 \end{pmatrix} \in \mathfrak{h}_1.$$

From this we deduce that the principal curvatures of $\hat{M}_2^{2n-1}$ are $2, 0, 1, -1$ with corresponding multiplicities $1, 2, n-2, n-2$, respectively. The corresponding principal curvature spaces are

$$\hat{T}_2 = \mathfrak{g}_{\alpha_1}\ \hat{T}_0 = \mathfrak{a}_1 \oplus \mathfrak{g}_{\alpha_1 \oplus 2\alpha_2}\ \hat{T}_1 = \mathfrak{g}_{\alpha_1+\alpha_2}\ \hat{T}_{-1} = \mathfrak{g}_{\alpha_2}.$$

All of the above calculations are with respect to the metric $\langle \cdot, \cdot \rangle$ on AN. We now switch to the Riemannian metric g on Q^{n*} and the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Recall that, by construction, $(AN, \langle \cdot, \cdot \rangle)$ and (Q^{n*}, g) are isometric and the metrics are related by

$$\langle H_1 + \hat{X}_1, H_2 + \hat{X}_2 \rangle = g(H_1, H_2) + g(X_1, X_2)$$

with $H_1, H_2 \in \mathfrak{a}$ and $\hat{X}_1, \hat{X}_2 \in \mathfrak{n}$.

Let A be the shape operator of M_2^{2n-1} in (Q^{n*}, g) with respect to ζ. The above calculations then imply

$$AX = \begin{pmatrix} 0 & 0 & 0 & 2x & v_1 & \cdots & v_{n-2} \\ 0 & 0 & 2x & 0 & -w_1 & \cdots & -w_{n-2} \\ 0 & 2x & 0 & 0 & 0 & \cdots & 0 \\ 2x & 0 & 0 & 0 & 0 & \cdots & 0 \\ v_1 & -w_1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & -w_{n-2} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

with

$$X = \begin{pmatrix} 0 & 0 & a & x-y & v_1 & \cdots & v_{n-2} \\ 0 & 0 & x+y & a & w_1 & \cdots & w_{n-2} \\ a & x+y & 0 & 0 & 0 & \cdots & 0 \\ x-y & a & 0 & 0 & 0 & \cdots & 0 \\ v_1 & w_1 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ v_{n-2} & w_{n-2} & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} \in T_o M_2^{2n-1} \subset \mathfrak{p}.$$

From this we deduce the following result.

Theorem 8.1 *The homogeneous real hypersurface M_2^{2n-1} in (Q^{n*}, g) has four distinct constant principal curvatures* $2, 0, 1, -1$ *with multiplicities* $1, 2, n-2, n-2$, *respectively. The principal curvature spaces* T_2, T_0, T_1 and T_{-1} *are*

$$T_2 = \mathfrak{p}_{\alpha_1} = \mathbb{R}J\zeta,\ T_0 = \mathbb{R}H_{\alpha+2\alpha_2} \oplus \mathfrak{p}_{\alpha_1 \oplus 2\alpha_2} = \mathcal{C} \ominus \mathcal{Q},\ T_1 = \mathfrak{p}_{\alpha_1+\alpha_2},\ T_{-1} = \mathfrak{p}_{\alpha_2}.$$

In particular, T_1 and T_{-1} *are mapped into each other by the structure tensor field ϕ. Moreover, the shape operator A and the structure tensor field ϕ* of M_2^{2n-1} *satisfy*

$$A\phi + \phi A = 0.$$

Note that $T_1 \subset V(C_0)$ and $T_{-1} \subset JV(C_0)$.

The eigenspaces T_2, T_0, T_1 and T_{-1} of the shape operator A and the eigenspaces E_0, E_{-1} and E_{-4} of the normal Jacobi operator $\mathcal{K} = \mathcal{K}_\zeta$ satisfy

$$T_2 = E_{-4},\ T_0 = E_0,\ T_{-1} \oplus T_1 = E_{-1}.$$

It follows that A and $\mathcal{K}$ are simultaneously diagonalizable and hence $A\mathcal{K} = \mathcal{K}A$. Thus we have proved the following.

Corollary 8.2 *The homogeneous Hopf hypersurface M_2^{2n-1} in (Q^{n*}, g) is curvature-adapted.*

By construction, the integral curves of the Reeb vector field ξ are congruent to a horocycle in a complex hyperbolic line $\mathbb{C}H^1(-4)$. Such a horocycle has constant geodesic curvature 2. As in previous cases, this leads to the geometric interpretation of M_2^{2n-1} being obtained by attaching isometric copies of the homogeneous complex hypersurface $P^{n-1} \cong \mathbb{C}H^{n-1}(-4)$ to the horocycle in a suitable way. Equivalently, M_2^{2n-1} is the canonical extension of a horocycle in the boundary component $B_1 \cong \mathbb{C}H^1(-4)$.

9 Curvature

In this section we compute the Ricci tensor Ric_α and the scalar curvature s_α of the homogeneous Hopf hypersurface M_α^{2n-1} in (Q^{n*}, g). Let R_α, Ric_α, s_α be the Riemannian curvature tensor, Ricci tensor, scalar curvature of M_α^{2n-1}, respectively. Let A_α and $\mathcal{K}_\alpha$ be the shape operator and normal Jacobi operator of M_α^{2n-1} with respect to the unit normal vector ζ_α, respectively. The Gauss equation tells us that

$$g(\bar{R}(X, Y)Z, W) = g(R_\alpha(X, Y)Z, W) - g(A_\alpha Y, Z)g(A_\alpha X, W) + g(A_\alpha X, Z)g(A_\alpha Y, W)$$

for all $X, Y, Z, W \in \mathfrak{X}(M_\alpha^{2n-1})$. Contracting the Gauss equation gives, after some straightforward computations, the expression

$$\mathrm{Ric}_\alpha X = -2nX - \mathcal{K}_\alpha X + \alpha A_\alpha X - A_\alpha^2 X,$$

where we used the fact that the Ricci tensor of (Q^{n*}, g) is equal to $-2ng$ and $\mathrm{tr}(A_\alpha) = \alpha$ by Theorem 1.1. Since the unit normal vector ζ_α of M_α^{2n-1} is $\mathfrak{A}$-isotropic, the normal Jacobi operator $\mathcal{K}_\alpha$ of M_α^{2n-1} satisfies

$$\mathcal{K}_\alpha X = \begin{cases} 0 & , \text{ if } X \in \mathcal{C} \ominus \mathcal{Q} = T_0, \\ -X & , \text{ if } X \in \mathcal{Q} = T_{-1} \oplus T_1, \\ -4X & , \text{ if } X \in \mathcal{C}^\perp = \mathbb{R}\xi = T_\alpha \end{cases}$$

by Theorem 1.1 and the description of the Jacobi operator in Sect. 3. It follows that

$$\mathrm{Ric}_\alpha X = \begin{cases} -2nX & , \text{ if } X \in \mathcal{C} \ominus \mathcal{Q} = T_0, \\ (-2n - \alpha)X & , \text{ if } X \in T_{-1}, \\ (-2n + \alpha)X & , \text{ if } X \in T_1, \\ (-2n + 4)X & , \text{ if } X \in \mathcal{C}^\perp = \mathbb{R}\xi = T_\alpha. \end{cases}$$

It follows that the Ricci tensor of M_α^{2n-1} has two (if $\alpha = 0$), three (if $\alpha = 4$) or four (if $\alpha \notin \{0, 4\}$) constant eigenvalues. More specifically, for $\alpha = 0$ we obtain

$$\mathrm{Ric}_0 X = -2nX + 4\eta(X)\xi,$$

which means that M_0^{2n-1} is pseudo-Einstein (see [12]).

Proposition 9.1 *The minimal homogeneous real hypersurface M_0^{2n-1} is a pseudo-Einstein Hopf hypersurface in (Q^{n*}, g). In particular, the Ricci tensor* Ric_0 of M_0^{2n-1} *is ϕ-invariant, that is*, $\mathrm{Ric}_0 \circ \phi = \phi \circ \mathrm{Ric}_0$.

We also see that

$$\mathrm{Ric}_\alpha \circ \phi + \phi \circ \mathrm{Ric}_\alpha = -4n\phi.$$

This equation is motivated by Ricci solitons (see [3], Lemma 3.3.11). However, none of the homogeneous Hopf hypersurfaces M_α^{2n-1} is a Ricci soliton.

By contracting the Ricci tensor we see that the scalar curvature of M_α^{2n-1} is independent of α.

Proposition 9.2 *The scalar curvature s_α of the homogeneous Hopf hypersurface* M_α^{2n-1} in (Q^{n*}, g) *does not depend on α and satisfies*

$$s_\alpha = 4 - 2n(2n-1).$$

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