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Extremal metrics on the total space of destabilising test configurations

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Abstract
We construct extremal metrics on the total space of certain destabilising test configurations for strictly semistable Kähler manifolds. This produces infinitely many new examples of manifolds admitting extremal Kähler metrics. It also shows for such metrics a new phenomenon of jumping of the complex structure along fibres.

Mathematics Subject Classification 53C55 · 32Q15 · 32Q20 · 32Q26

1 Introduction

A central theme in complex geometry is the search for canonical metrics, such as Kähler–Einstein metrics, constant scalar curvature Kähler (cscK) metrics, or, more generally, extremal Kähler metrics. Such metrics are solutions to highly complicated non-linear PDEs, and their existence question is very involved. There are examples of Kähler manifolds that admit extremal metrics in all, some, or none of its Kähler classes. The main question one would like to answer is if there is an extremal metric in a given Kähler class on a Kähler manifold.

A central conjecture in the field is the Yau–Tian–Donaldson (YTD) conjecture, which links this purely differential geometric PDE question to algebraic geometry [30, 37, 38]. More precisely, this says that the existence of a cscK metric should be equivalent to a notion of stability called K-(poly)stability. It was extended to the

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extremal setting by Székelyhidi [33], where the relevant algebraic notion is relative K-stability. Despite a huge amount of work over many years, the conjecture remains open apart from a few cases, such as the Kähler–Einstein case [14–16] and the toric case [12, 32].

Producing extremal metrics is in general very hard. Even if the YTD conjecture is proved, this will continue to be the case, as the stability criterion is also difficult to check with the present technology, at least outside the case of Fano varieties. One main avenue for actually producing extremal metrics has been to directly solve the equation, working with an ansatz that simplifies the PDE (see e.g. [1, 8, 26]). This requires one to work under certain symmetry assumptions. Another method is to construct new extremal metrics from old ones, via perturbative techniques. There have been many different such constructions, for example on blowups via gluing [2–4, 35], or on the total spaces of fibrations, working in so-called adiabatic Kähler classes, that make the base direction large compared to the fibres [7, 18, 20, 23, 25]. The current work falls into the latter category.

Previous fibration constructions have all been focused on constructing cscK or extremal metrics when the fibration has cscK fibres. The main reason for this assumption is that an asymptotic expansion of the scalar curvature with respect to metrics of the form $\omega + k\omega_B$, where ω is a relatively Kähler metric on the total space, and ω_B is pulled back from the base, yields the scalar curvature of the induced metric on the fibres as the leading order term. Thus, if working with a fixed relatively Kähler metric initially, one needs a relatively cscK metric in order to have constant scalar curvature to leading order.

In [23], it was assumed that the total space and the fibres have no automorphisms. In fact, when there is a discrepancy between the automorphism group of the fibres and the automorphisms of the total space of the fibration, the existence question becomes much more subtle. For example, on projective bundles, any hermitian metric on the bundle gives a fibrewise Fubini-Study metric on the projectivised bundle. Ideas going back to Hong in [25] say that one should use a Hermite–Einstein metric as a good choice of relatively cscK metric, in this setting. This issue was considered by many authors before Dervan and the first named author in [18] introduced an equation for the fibrewise cscK metric, called the optimal symplectic connection (OSC) equation, that picks out a canonical choice of fibrewise cscK metric, when the fibres have automorphism.

In the present work, we relax the condition that every fibre admits a cscK metric, and consider a special type of fibration where the general fibre does not admit such metrics. In fact, in our construction, all but one fibre does not admit a cscK metric. We will construct extremal metrics on the total space of this fibration, and this shows that relative stability of the fibres is not a necessary condition (although relative *semistability* is, see [19]).

Suppose that (X, L) is an *analytically K-semistable manifold*, meaning that there exists a degeneration of (X, L) to some (X_0, L_0) which admits a cscK metric. That is, we have a smooth test configuration

$$\pi : \mathcal{X} \to \mathbb{P}^1$$

and a $\mathbb{Q}$-line bundle $\mathcal{L}$ on $\mathcal{X}$ whose restriction to non-zero fibres equals L and whose central fibre is (X_0, L_0), where X_0 admits a cscK metric in $c_1(L_0)$. Note that analytic K-semistability implies K-semistability [31].

We will let $\mathcal{L}_k = \mathcal{L} + \pi^*(\mathcal{O}(k))$. Under certain conditions, we produce an extremal metric on $\mathcal{X}$ in $c_1(\mathcal{L}_k)$ for $k \gg 0$.

Theorem 1 *Suppose (X, L) is a smooth analytically strictly K-semistable manifold such that the automorphism group of (X, L) is discrete and the cscK central fibre $(\mathcal{X}_0, \mathcal{L}_0)$ has $\mathbb{C}^*$ as its maximal torus. Then there exists an extremal metric on $\mathcal{X}$ in $c_1(\mathcal{L} + \pi^*(\mathcal{O}(k)))$ for all $k \gg 0$.*

The key issue to overcome in this situation is that working with a fixed relatively Kähler metric on the total space will never yield a good approximate solution to the extremal equation, as the leading order term in the expansion of the scalar curvature will not be a holomorphy potential. Thus one has to work with a *sequence* of relatively Kähler metrics, and constructing a good such sequence is a key step in the argument.

The fact that one should be able to choose such a sequence for a general K-semistable manifold follows from a conjecture of Donaldson [31], which states that for a polarised manifold (Y, H),

$$\inf_{\omega \in c_1(H)} \|S(\omega) - \hat{S}\| = \sup_{\mathcal{Y} \text{ t.c.}} \frac{-DF(\mathcal{Y})}{\|\mathcal{Y}\|},$$

where the supremum on the right hand side goes over all test configurations for (Y, H) and $\|\cdot\|$ is a norm on test configurations. In particular, if (Y, H) is strictly semistable, this supremum should be zero, and so we should be able to find metrics that are *arbitrarily close to being cscK*, even if no actual cscK metric exists. Under our assumption that (X, L) is *analytically* K-semistable, it is automatic that we can produce such a sequence near the central fibre. Thus the main step for us in this part of the argument is to extend this relatively Kähler metric defined near the central fibre, to the full test configuration.

Our assumption on the automorphism group can be thought of as analogous to the assumptions of Fine mentioned above. While the total space of the test configuration always has to have non-trivial automorphism group, our assumption says that there is no discrepancy between the automorphism group of the central fibre and that of the total space of the test configuration. In particular, we see no issue related to the *choice* of cscK metric on the central fibre, and so see nothing like the OSC equation coming in. It should be possible to relax these conditions, introducing an OSC like equation in the semistable setting, and, indeed, such a theory is currently under development by Annamaria Ortu [29]. See Remark 4 for further details.

The result also has some consequences and applications. A natural question to ask is about what happens from a metric viewpoint to the extremal metrics when $k \to \infty$. Our construction shows that a phenomenon of *jumping of the complex structures* within the fibres is occurring. More precisely:

Proposition 2 *For* any *$p \in \mathcal{X}$ the pointed Gromov–Hausdorff limits based at p is isometric to $X_0 \times \mathbb{R}^2$, where X_0 is the cscK central fibre.*

Importantly, we can use our result to actually produce many new manifolds admitting extremal metrics. Often in the construction of extremal metrics on fibrations, it is difficult to actually generate examples. This is because the constructions requires one to solve a complicated PDE on the base of the fibration, at least when the fibres are not all isomorphic. In our case, the fibres are not all isomorphic, as there is a jump of biholomorphism class between the central fibre and the general fibre. However, the base equation just reduces to the equation for the Fubini-Study metric on $\mathbb{P}^1$, and so we see no issues coming from the base.

To obtain examples, we therefore have to find explicit families of K-semistable manifolds degenerating to a cscK manifold. In general, this is a hard question, but a plethora of examples are known to exist, due to massive recent progress on the study of K-stability for Fano manifolds, and in particular for threefolds (see [5] and the references therein). This allows us to produce many new manifolds admitting extremal metrics, see Theorem 25.

Outline

In Sect. 2 we recall some basic theory on cscK metrics relevant to our problem. In Sect. 3, we provide a starting point for the construction, by defining a good relatively Kähler metric on the test configuration. Our main result, Theorem 1, is proved in Sect. 4. In Sect. 5 we provide applications to Gromov–Hausdorff limits of K-semistable manifolds, and finally in Sect. 6, we provide multiple examples of manifolds to which our construction applies.

2 Background on cscK metrics

In this section we recall some general theory regarding the cscK/extremal equation, its linearisation, and the deformation theory for cscK metrics. A good reference for most of the material in this section is the book of Székelyhidi [36].

2.1 Constant scalar curvature and extremal metrics

Let X be a compact Kähler manifold of dimension n, with ω the Kähler form of a Kähler metric on X. Any other Kähler form on X in the same class as ω can be described via a function, called a Kähler potential. More precisely, if we let ω_ϕ for a real-valued function ϕ denote

$$\omega_\phi = \omega + i\partial\bar{\partial}\phi,$$

then the space of Kähler potentials $\mathcal{K}$ parametrising the Kähler forms in the class $[\omega]$ is given by

$$\mathcal{K} = \{\phi : \omega_\phi > 0\}.$$

Associated to ω_ϕ is the Ricci form

$$\rho_\phi = -i\partial\bar{\partial}\log(\omega_\phi^n).$$

The corresponding bilinear form is the Ricci curvature of the metric associated to ω_ϕ. The scalar curvature of this metric is given by contracting with respect to the metric,

$$S(\omega_\phi) = \Lambda_{\omega_\phi}(\rho_\phi).$$

We say ω_ϕ has *constant scalar curvature* if there exists a constant c such that

$$S(\omega_\phi) = c.$$

The constant c is predetermined by the topology of X and the class $[\omega]$.

A generalisation of constant scalar curvature metrics is *extremal* Kähler metrics, introduced by Calabi ([8]). These are defined as the critical points of the functional

$$\phi \mapsto \int_X S(\omega_\phi)\omega_\phi^n.$$

Since X is compact, this is equivalent to

$$\mathcal{D}(S(\omega_\phi)) = 0,$$

where $\mathcal{D}(f) = \bar{\partial}(\nabla^{1,0}_{\omega_\phi}(f))$. This says that the gradient of the scalar curvature is a holomorphic vector field. Such functions are referred to as *holomorphy potentials*.

The holomorphic vector fields that admit a *complex-valued* holomorphy potential are precisely given by the holomorphic vector fields with a zero somewhere [28, Theorem 1]. On the other hand, the space of holomorphic vector fields with a real holomorphy potential may depend on the metric. However, if the metric is invariant under the action of a fixed maximal torus, then the space of real holomorphic vector field does not depend on the metric, and the holomorphy potentials change in a predictable way. If we let $\overline{\mathfrak{h}}$ denote the space of real holomorphy potentials with respect to ω, then a holomorphy potential for $\nabla^{1,0}h$ with respect to ω_ϕ is given by

$$h + \frac{1}{2}\langle\nabla h, \nabla\phi\rangle.$$

Thus, if our background metric is torus invariant, this means that to solve the extremal equation we seek a torus invariant function ϕ and a holomorphy potential h with respect to ω such that

$$S(\omega_\phi) = h + \frac{1}{2}\langle\nabla h, \nabla\phi\rangle.$$

2.2 The linearisation of the scalar curvature operator

The linearisation of the scalar curvature operator

$$\phi \mapsto S(\omega_\phi)$$

will be key in our construction. This operator is closely linked to the *Lichnerowicz operator*, defined as follows.

Let $\mathcal{D} = \mathcal{D}_\omega : C^\infty(X) \to \Gamma(T^{1,0}X \otimes \Lambda^{0,1}X)$ be the operator

$$\mathcal{D}(f) = \bar{\partial}(\nabla^{1,0}_\omega(f)),$$

as above. Let $\mathcal{D}^*_\omega : \Gamma(T^{1,0}X \otimes \Lambda^{0,1}X) \to C^\infty(X)$ be its formal adjoint, with respect to ω. The Lichnerowicz operator is then the fourth order self-adjoint operator

$$L_\omega : C^\infty(X) \to C^\infty(X)$$

defined as

$$L_\omega = \mathcal{D}^*_\omega \mathcal{D}_\omega.$$

This operator admits an expansion

$$L_\omega(f) = \Delta^2(f) + \langle \mathrm{Ric}(\omega), i\partial\bar{\partial}(f)\rangle_\omega + \frac{1}{2}\langle \nabla S(\omega), \nabla f\rangle_\omega,$$

which in particular shows that it is elliptic. Note also that the kernel, and hence the cokernel, of L_ω consists precisely of the space $\overline{\mathfrak{h}}$ of real holomorphy potentials with respect to ω.

The linearisation of the scalar curvature map at 0 is then given by

$$dS_0(f) = -L_\omega(f) + \frac{1}{2}\langle \nabla_\omega S(\omega), \nabla_\omega f\rangle_\omega.$$

At a non-zero value of ϕ, the linearisation is given by the same formula, but with respect to ω_ϕ instead of ω.

2.3 Kuranishi theory

The construction relies on Kuranishi theory [27]. We follow the exposition of Székelyhidi [34]. Let (M, ω) be a symplectic manifold. We will denote by

$$\mathcal{J}(M, \omega) = \{J \in \mathrm{End}(TM) : J^2 = -\mathrm{Id},\, J^*\omega = \omega\}$$

the space of ω-compatible almost complex structures on M. Here $J^*\omega(v, w) = \omega(Jv, Jw)$. The tangent space to $\mathcal{J}(M, \omega)$ at J is given by

$$T_J\mathcal{J}(M, \omega) = \{A \in \mathrm{End}(TM) : A \circ J = -J \circ A, \omega(A(\cdot), \cdot) = -\omega(\cdot, A(\cdot))\}.$$

The space $\mathcal{J}(M, \omega)$ is in itself a complex manifold, with complex structure at J given by $A \mapsto J \circ A$.

Kuranishi used the above framework to construct a *versal deformation space* of a cscK $(\mathcal{X}_0, \mathcal{L}_0)$. This is a complex space V together with a holomorphic map

$$f : V \to \mathcal{J}(M, \omega)$$

coming from a universal family

$$\mathcal{Y} \to V.$$

If T_0 is a maximal compact subgroup T_0 of $\mathrm{Aut}_0(\mathcal{X}_0, \mathcal{L}_0)$, then this can be taken to be equivariant under the T_0 action. We assume that $(\mathcal{X}_0, \mathcal{L}_0)$ is the central fibre of a test configuration of some semistable (X, L) and we can then also assume that T_0 contains a maximal torus of the reduced automorphism group of (X, L).

As a smooth manifold, $\mathcal{Y}$ is simply $V \times M$, and the action is the product action. If we have some initial deformation of (X, L) to $(\mathcal{X}_0, \mathcal{L}_0)$, this then arises, near the central fibre, from a map to V. That is, there exists a map

$$\Phi : \Delta \to V$$

such that, if we let J_0 be the complex structure of the central fibre, then

$$\begin{aligned} f \circ \Phi(0) &= J_0 \\ f \circ \Phi(t) &\cong J \text{ if } t \neq 0. \end{aligned}$$

In fact, one obtains a family

$$(\mathcal{X}, \mathcal{L}) \to \Delta$$

over a disk $\Delta \subset \mathbb{C}$, containing 0, such that the central fibre is $(\mathcal{X}_0, \mathcal{L}_0)$, and the complex structure of the non-zero fibres $\mathcal{X}_t$ are $J_t = f \circ \Phi(t)$. Without loss of generality, we may assume the disk Δ has size 2. The fact that such a family exists follows by [34] and the fact that the central fibre remains $(\mathcal{X}_0, \mathcal{L}_0)$ follows by uniqueness of cscK degenerations [13]. Note that this is a smooth family, so $J_t = f \circ \Phi(t) \to J_0$ as $t \to 0$.

Since the J_t are all isomorphic to J and we can trivialise the test configuration over Δ^*, we can pull back the J_t with $t \neq 0$ to the fixed complex structure J, using diffeomorphisms of the underlying smooth manifold. Pulling back ω via these maps, we then obtain symplectic forms α_t on M such that the Kähler structures (M, J_t, ω) and (M, J, α_t) are isomorphic. To pass between these two points of view, we use

Moser's trick. We have that α_t and ω are in the same cohomology class, which in particular means we can write

$$\alpha_t = \omega + i\partial\bar{\partial}(\phi_t)$$

for some function ϕ_t. We can realise the diffeomorphism $f_t : M \to M$ such that $J_t = f_t^* J$ as the time 1 flow of the vector field ν_t satisfying $J d\phi_t + \iota_{\nu_t}\alpha_t = 0$. The equation comes from the fact that ω and α_t are related by $\alpha_t = \omega + d(J d\phi_t)$, see e.g. [17, Sect. 7.2].

The upshot is that we have two points of view of the family over Δ^*. The first is simply as the restriction of $(\mathcal{X}, \mathcal{L}) \to \Delta$, equipped with the relatively Kähler form ω, to Δ^*. The other is as $X \times \Delta^*$ with a relatively Kähler form α such that the restriction α_t to $X \times \{t\}$ is $\omega + i\partial\bar{\partial}(\phi_t)$. The Kähler structures on the fibres over Δ^* in the two points of view are therefore:

- (M, J_t, ω), viewed as the fibre over t of $(\mathcal{X}, \mathcal{L}) \to \Delta$;
- (M, J, α_t).

The two are related by

$$(M, J_t, \omega) = (M, f_t^* J, f_t^* \alpha_t),$$

where f_t is the time 1 flow of the vector field ν_t satisfying

$$J d\phi_t + \iota_{\nu_t}\alpha_t = 0.$$

Ultimately, we are interested in constructing extremal metrics on the total space of the test configuration over $\mathbb{P}^1$ we obtain from the above. The family $(\mathcal{X}, \mathcal{L})$ has a local $\mathbb{C}^*$ action, and we can equivariantly glue the restriction of $\mathcal{X}$ to $\Delta \setminus \{0\}$ to $X \times \mathbb{C}$, with the trivial action on the first component. We thus obtain a test configuration

$$\pi : (\mathcal{X}, \mathcal{L}) \to \mathbb{P}^1$$

over $\mathbb{P}^1$. Over the disk Δ, we have a relatively Kähler metric on $\mathcal{X}$, given simply by the symplectic form ω, which is cscK on the central fibre. In the next section, we will modify this metric so that it extends to the whole total space of this test configuration, not just the preimage of the disk Δ. Note that the stabiliser of the T_0 action on V at the point corresponding to (X, L) is a torus T inside the automorphism group of (X, L), and if we assume $T_0/T \cong \mathbb{C}^*$, which is the $\mathbb{C}^*$ action on the total space of the test configuration, then we have a T_0 action on the total space $\mathcal{X}$ of the test configuration as well.

3 Extending metrics

Here we will explain how to obtain a family of global S^1-invariant metrics on $\mathcal{X}$ from a local deformation. We will also study their fibrewise properties, and in particular the goal is to show that we can do the extension so that the fibrewise average scalar curvature goes to 0 in the family.

The point is to take the family we have over the disk Δ and modify it so that it is independent of t outside a smaller disk. We can then trivially extend it to all of $\mathbb{P}^1$. The initial Kähler metrics we have over $t \in \Delta$ are $O(|t|)$ away from a cscK metric. We now want to preserve this globally. That is, for every $\varepsilon > 0$, we want to construct a fibrewise Kähler metric ω_ε on the family over $\mathbb{P}^1$ such that fibrewise scalar curvature of ω_ε is $O(\varepsilon)$ is $O(\varepsilon)$ on *every* fibre. The key to this extension procedure will be to go between the two points of view on the family given in Sect. 2.3.

The test configurations we consider arise in the manner described in Sect. 2.3. We have a trivialisation of $\mathcal{X}$ over $\mathbb{C}^*$ and diffeomorphisms $f_t : M \to M$ such that $f_t^* J_t = J$ and Kähler metrics

$$\alpha_t = f_t^* \omega = \omega_0 + i\partial\bar{\partial}\phi_t,$$

for some background Kähler form ω_0 on X, which as in Sect. 2.3 we may assume is ω. Let $\chi(s) : [0, \infty) \to [0, 1]$ be a cut-off function which is 1 when $s \leq \frac{1}{2}$ and 0 when $s \geq \frac{3}{4}$. Define a new 2-form ω_ε by

$$\omega_\varepsilon = \omega + i\partial\bar{\partial}\left(\chi\left(\frac{|t|}{\varepsilon}\right)\phi_t + (1-\chi\left(\frac{|t|}{\varepsilon}\right))\phi_\varepsilon\right).$$

The form ω_ε is a smooth relative Kähler form over a disk around the origin. The Kähler condition on the fibres $\mathcal{X}_t$ of the test configuration is immediate: on a given fibre, we are taking a convex combination of Kähler forms (χ is only dependent on the base variable, and so its derivatives won't enter into showing positivity in the vertical direction).

Note that this form is equal to α_t when $|t| \leq \frac{\varepsilon}{2}$ and equal to α_ε when $|t| \geq \frac{3\varepsilon}{4}$. With respect to the original structure, we then have a path I_t of complex structures such that $I_t = J_t$ when $|t| \leq \frac{\varepsilon}{2}$ and equal to J_ε when $|t| \geq \frac{3\varepsilon}{4}$. Thus we can fill in to a family over $\mathbb{P}^1$ by putting $I_t = I_\varepsilon$ when $|t| > \varepsilon$ too. Put differently, the relatively Kähler form ω_ε defined originally on the test configuration over a disk around the origin in $\mathbb{C}$, can be extended to a relatively Kähler form, still denoted ω_ε, on the total space of the compactified test configuration over $\mathbb{P}^1$.

The initial metrics determined by $(J, \alpha_t) \cong (J_t, \omega)$ on the fibres $\mathcal{X}_t$ approach a cscK metric as $t \to 0$, namely the one determined by (J_0, ω), since $J_t \to J_0$ as $t \to 0$. We want to see that this holds for ω_ε on *every* fibre as $\varepsilon \to 0$. A key estimate in order to establish this is the following comparison between the Kähler potentials ϕ_t for the $\alpha_t = \omega + i\partial\bar{\partial}(\phi_t)$.

Proposition 3 *Under a suitable normalisation for the* ϕ_t*, there exists a positive constant* $C > 0$ *such that for all* $\varepsilon > 0$ *sufficiently small,*

$$\|\phi_t - \phi_\varepsilon\|_{C^{4,\alpha}(X)} \leq C\varepsilon$$

for $t \in [\frac{\varepsilon}{2}, \varepsilon]$.

The key to establishing this result is to relate $\phi_t - \phi_\varepsilon$ to $J_t - J_\varepsilon$, where we already know such bounds.

Proof From Sect. 2.3, we have that $(J_t, \omega) = f_t^*(J, \alpha_t)$, for a diffeomorphism $f_t : M \to M$ produced from Moser's trick as the time 1 flow of the vector field ν_t satisfying

$$J d\phi_t + \iota_{\nu_t} \alpha_t = 0. \tag{1}$$

Now,

$$J_t - J_\varepsilon = (f_t^* - f_\varepsilon^*)(J),$$

so $f_t^* - f_\varepsilon^*$ is mutually bounded with $J_t - J_\varepsilon$. On the other hand, the norm of f_t^* (computed with respect to any fixed Kähler metric on X) can be mutually bounded with the norm of ν_t, since f_t is the time 1 flow of this vector field. This vector field can in turn be mutually bounded with the C^1 and C^2-semi-norms of ϕ_t, since ν_t is determined by Eq. (1). In particular, the higher semi-norms of $\phi_t - \phi_\varepsilon$ are mutually bounded the norms of $J_t - J_\varepsilon$. Since the latter goes to zero as both complex structures approach J_0 (smoothly, so we get bounds in any $C^{k,\alpha}$-norm we would like), we get the same behaviour for the $C^{k+1,\alpha}$ and $C^{k+2,\alpha}$-semi-norms of $\phi_t - \phi_\varepsilon$.

The last remaining point is to make sure we can bound the full norms, not just the semi-norms that do not involve the C^0-norm of $\phi_t - \phi_\varepsilon$. Here we use that we have not normalised the ϕ_t's yet, i.e. we are allowed to replace ϕ_t by $\phi_t + c_t$ for a constant c_t depending on t. A-posteriori we see that we should normalise the ϕ_t so that the ϕ_t are of average 0 with respect to ω, as this allows us to bound the C^0-norm uniformly by the C^1 and C^2-semi-norms. Thus we can bound the full $C^{k+2,\alpha}$-norm of $\phi_t - \phi_\varepsilon$ in terms of the $C^{k,\alpha}$-norm of $J_t - J_\varepsilon$. Picking $k = 2$, we get the required statement. □

The next goal will be to understand the expansion of the scalar curvature on $\mathcal{X}$, when choosing a certain relationship between the rate of the parameter ε of the approximate solution and the parameter k for the polarisation. To understand the dependence on the rate ε, we use techniques reminiscent of the blowup situation, since our method of gluing metrics over an annular region is the construction being utilised there. Specifically, we are following the line of proof given in [35, Proposition 20, Lemma 21, Lemma 24].

In the vertical directions, we have the following estimate for the initial metric ω_ε.

Lemma 4 [27, 34] *Let $\hat{S}$ be the average scalar curvature of the fibres of $\mathcal{X} \to \mathbb{P}^1$. Then*

$$S(\alpha_\varepsilon) - \hat{S} = O(\varepsilon).$$

In fact, from [34, Proof of Proposition 8], this can be improved so that $S(\alpha_\varepsilon) - \hat{S} = O(\varepsilon^2)$, and, moreover, the $O(\varepsilon^2)$-term is the restriction to the fibre of a holomorphy potential of the S^1-action on the test configuration $\mathcal{X} \to \mathbb{P}^1$. However, we will not need this for our construction.

We next want to show that the perturbed Kähler forms ω_ε satisfy a similar bound. From the mean value theorem, the scalar curvature of ω_ε can be controlled using the linearisation. More precisely, we have that

$$\omega_\varepsilon|_{\mathcal{X}_t} = \alpha_\varepsilon + i\partial\bar{\partial}\left(\chi\left(\frac{|t|}{\varepsilon}\right)(\phi_t - \phi_\varepsilon)\right).$$

Therefore, by the mean value theorem,

$$S(\omega_\varepsilon) = S(\alpha_\varepsilon) + L_r\left(\chi\left(\frac{|t|}{\varepsilon}\right)(\phi_t - \phi_\varepsilon)\right),$$

where L_r is the linearisation of the scalar curvature operator at the metric

$$\alpha_\varepsilon + r \cdot i\partial\bar{\partial}\left(\chi\left(\frac{|t|}{\varepsilon}\right)(\phi_t - \phi_\varepsilon)\right),$$

for some $r \in [0, 1]$.

Our goal is to show that $S(\omega_\varepsilon)$ is sufficiently close to $S(\alpha_\varepsilon)$. The above shows that it will be key to understand the linearised operator of a perturbation of ω_ε, which we now do. We emphasise that the below is a *fibrewise* statement, as the α_ε are not even Kähler, only relatively Kähler.

Proposition 5 *There exists $c, C > 0$ such that if $\|\phi\|_{C^{4,\alpha}} \le c$, then*

$$\|L_{\alpha_\varepsilon + i\partial\bar{\partial}\phi}(f) - L_{\alpha_\varepsilon}(f)\|_{C^{0,\alpha}} \le C\|\phi\|_{C^{4,\alpha}}\|f\|_{C^{4,\alpha}}$$

on each fibre of $\mathcal{X} \to \mathbb{P}^1$.

Proof This is a direct consequence of the fact that, with respect to some/any background metric ω_0, we have the uniform lower bound

$$\alpha_\varepsilon \ge C\omega_0.$$

This follows because each member of the corresponding family of metrics $\alpha_\varepsilon(J(\cdot), \cdot)$ are isometric to $\omega(J_\varepsilon(\cdot), \cdot)$, and this is a smooth family of metrics, for a compact set of parameter values (the *metric* also makes sense when $\varepsilon = 0$). The result then follows by expanding the metric quantities involved in the expression of L_{α_ε}, using standard techniques e.g. as in [36, Lemma 8.13]. □

The above proposition reduces the bound on $S(\omega_\varepsilon)$ to understanding the mapping properties of L_{α_ε}. For now all we need is that L_{α_ε} is an operator that is uniformly bounded in ε on the fibres. However, we prove a more refined result that will be need later. Let $\bar{\mathfrak{h}}$ be the space of holomorphy potentials on the cscK central fibre $\mathcal{X}_0$, with respect to ω. Then $\bar{\mathfrak{h}} = \langle h, 1\rangle$, for some average zero holomorphy potential h on $\mathcal{X}_0$. We then have the following.

Proposition 6 *There is an expansion*

$$L_{\alpha_\varepsilon} = L_\omega + O(\varepsilon),$$

where L_ω is the linearisation of the scalar curvature of ω at J_0. In particular, L_{α_ε} is surjective modulo $\overline{\mathfrak{h}}$ to leading order.

Proof The scalar curvature of α_ε is constant to order ε, so it suffices to establish the above for the Lichnerowicz operator.

We have that the Lichnerowicz operator is $\mathcal{D}_\varepsilon^* \mathcal{D}_\varepsilon$, where

$$\mathcal{D}_\varepsilon = \bar{\partial}_{J_\varepsilon}(\nabla^{1,0}_{\alpha_\varepsilon}).$$

We therefore have that

$$\mathcal{D}_\varepsilon - \mathcal{D}_0 = O(\varepsilon).$$

This uses that the metrics g_ε produced from the α_ε satisfy

$$g_\varepsilon - g_0 = O(\varepsilon)$$

and hence the same holds for the inverses, and the gradient. Again using this property, it follows that we have the same property for the adjoint, and hence for the full Lichnerowicz operator. □

With this in place, we finally have the estimate on the fibrewise scalar curvature that we need.

Proposition 7 *Let $\hat{S}$ be the average scalar curvature of the fibres of $\mathcal{X} \to \mathbb{P}^1$. Then*

$$S(\omega_\varepsilon|_{\mathcal{X}_t}) - \hat{S} = O(\varepsilon).$$

for any $t \in \mathbb{P}^1$.

Proof From Proposition 3, we know that $\phi_\varepsilon - \phi_t$ is $O(\varepsilon)$. Since we are interested in the scalar curvature on the fibre only, the term $\chi(\frac{|t|}{\varepsilon})$ is a constant. We can then apply Proposition 5 to estimate $S(\omega_\varepsilon|_{X_t})$. As remarked above, the scalar curvature is given by

$$S(\omega_\varepsilon) = S(\alpha_\varepsilon) + L_r\left(\chi\left(\frac{|t|}{\varepsilon}\right)(\phi_t - \phi_\varepsilon)\right),$$

where L_r is the linearisation of the scalar curvature operator at the metric

$$\alpha_\varepsilon + r \cdot i\partial\bar{\partial}\left(\chi\left(\frac{|t|}{\varepsilon}\right)(\phi_t - \phi_\varepsilon)\right),$$

for some $r \in [0, 1]$. From Proposition 5, we therefore get that

$$\begin{aligned}
&\|S(\omega_\varepsilon|_{X_t}) - \hat{S}\| \\
&\quad \leq \left\| S(\alpha_\varepsilon) - \hat{S}\| + \|L_r \left(\chi\left(\frac{|t|}{\varepsilon}\right)(\phi_t - \phi_\varepsilon)\right)\right\| \\
&\quad \leq \left\| S(\alpha_\varepsilon) - \hat{S}\right\| + \|L_0 \\
&\qquad \left(\chi\left(\frac{|t|}{\varepsilon}\right)(\phi_t - \phi_\varepsilon)\right)\right\| + \left\|(L_r - L_0)\left(\chi\left(\frac{|t|}{\varepsilon}\right)(\phi_t - \phi_\varepsilon)\right)\right\| \\
&\quad \leq \|S(\alpha_\varepsilon) - \hat{S}\| + \left\|L_0\left(\chi\left(\frac{|t|}{\varepsilon}\right)(\phi_t - \phi_\varepsilon)\right)\right\| + C\|\phi_t - \phi_\varepsilon\|^2.
\end{aligned}$$

Since $L_0 = L_{\alpha_\varepsilon}$ is a bounded operator independently of ε, the above is an $O(\varepsilon)$ term, as required. □

4 Constructing the extremal metrics

In this section, we prove the main result of the paper, Theorem 18. We first relate the parameter ε for the relative Kähler metrics on the test configuration $\mathcal{X}$ to the parameter k describing the polarisation. We then produce Kähler metrics on $\mathcal{X}$, that moreover are approximately extremal. We continue to improve these approximate solutions, before finally perturbing to a genuine solution, when the approximate solutions are sufficiently good.

4.1 Kähler metrics on the test configuration

We now relate the parameters k and ε. We define a closed 2-form

$$\Omega_\varepsilon = \omega_\varepsilon + \lambda\varepsilon^{-\delta}\pi^*\omega_{FS}$$

on $\mathcal{X}$, where $\delta, \lambda > 0$ are constant parameters and ω_{FS} is the Fubini–Study metric on $\mathbb{P}^1$. So the relationship between the two parameters is $\varepsilon = (\lambda/k)^{\frac{1}{\delta}}$. A priori, this is just a closed 2-form, but we will prove that when ε is sufficiently small, it is Kähler, provided δ is sufficiently large so that it compensates for the horizontal contribution of ω_ε.

Now,

$$\begin{aligned}
\omega_\varepsilon = {} & \chi\left(\frac{|t|}{\varepsilon}\right)\alpha_t + (1 - \chi\left(\frac{|t|}{\varepsilon}\right))\alpha_\varepsilon \\
& + i\partial\chi\left(\frac{|t|}{\varepsilon}\right) \wedge \bar{\partial}(\phi_t - \phi_\varepsilon) \\
& + i\partial(\phi_\varepsilon - \phi_t) \wedge \bar{\partial}\chi\left(\frac{|t|}{\varepsilon}\right) \\
& + (\phi_t - \phi_\varepsilon)i\partial\bar{\partial}\chi\left(\frac{|t|}{\varepsilon}\right).
\end{aligned}$$

The first line is a convex combination of relative Kähler forms, hence it is relatively Kähler. Since $J_\varepsilon - J_0 = O(\varepsilon)$, we therefore have that the horizontal contribution is $O(\varepsilon)$. We want to understand the contribution from the remaining terms, first to able to construct Kähler metrics, and then to understand the scalar curvature.

We first consider the term

$$\partial\chi\left(\frac{|t|}{\varepsilon}\right) = \chi'\left(\frac{|t|}{\varepsilon}\right)\cdot\partial(|t|)/\varepsilon.$$

Note that this is zero if $|t| < \frac{\varepsilon}{2}$, since χ' vanishes then. Note also that χ is uniformly bounded in C^1, as it is a fixed function. Finally,

$$\partial(|t|^2) = \bar{t}\partial(t),$$

and so

$$\partial(|t|) = \frac{1}{2|t|}\partial(|t|^2) = \frac{\bar{t}}{2|t|}\partial(t),$$

which is $O(1)$ for $t \in (\frac{\varepsilon}{2}, \varepsilon)$. The term $\partial\chi\left(\frac{|t|}{\varepsilon}\right)$ is therefore an $O(\varepsilon^{-1})$-term. Similarly for $\bar{\partial}\chi(\frac{|t|}{\varepsilon})$.

Finally, we consider $i\partial\bar{\partial}\chi\left(\frac{|t|}{\varepsilon}\right)$. Continuing from the calculations above, we have

$$\begin{aligned}
i\partial\bar{\partial}\chi\left(\frac{|t|}{\varepsilon}\right) =& -i\bar{\partial}\left(\chi'\left(\frac{|t|}{\varepsilon}\right)\cdot\frac{\bar{t}}{2\varepsilon|t|}\partial(t)\right)\\
=& -i\chi''\left(\frac{|t|}{\varepsilon}\right)\cdot\frac{\bar{\partial}(|t|)}{\varepsilon}\wedge\frac{\bar{t}}{2\varepsilon|t|}\partial(t) - i\chi'\left(\frac{|t|}{\varepsilon}\right)\bar{\partial}\left(\frac{\bar{t}}{2\varepsilon|t|}\partial(t)\right)\\
=&\frac{1}{4\varepsilon^2}\chi''\left(\frac{|t|}{\varepsilon}\right)i\partial(t)\wedge\bar{\partial}(\bar{t}) + \frac{1}{4\varepsilon|t|}\chi'\left(\frac{|t|}{\varepsilon}\right)i\partial(t)\wedge\bar{\partial}(\bar{t})\\
=&\frac{1}{8}\left(\frac{1}{\varepsilon^2}\chi''\left(\frac{|t|}{\varepsilon}\right) + \frac{1}{\varepsilon|t|}\chi'\left(\frac{|t|}{\varepsilon}\right)\right)idt\wedge d(\bar{t}),
\end{aligned}$$

which is an $O(\varepsilon^{-2})$-term.

The above calculations allow us to prove the following.

Lemma 8 *For any $\delta \geq 1$, there exists $\varepsilon_0, \lambda > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, Ω_ε is Kähler on $\mathcal{X}$. When $\delta > 1$, the same conclusion holds for any λ, with ε_0 depending on λ.*

Proof The ω_ε are relatively Kähler, so all that needs to be checked is the horizontal component with respect to the fibration structure

$$\pi : \mathcal{X} \to \mathbb{P}^1.$$

It follows from Proposition 3 that the horizontal component of ω_ε is $O(\varepsilon^{-1})$. More precisely, it is bounded below by a constant multiple of $-\varepsilon^{-1}\pi^*\omega_{FS}$ when ε is sufficiently small. Thus if $\delta = 1$, we can pick $\lambda > 0$ sufficiently large to make Ω_ε Kähler. If $\delta > 1$, we can do so for any λ. □

4.2 Expansion of the scalar curvature

We now want to understand the scalar curvature of the above metric.

Proposition 9 *Let $\hat{S}$ denote the average scalar curvature of the fibres of $\mathcal{X} \to \mathbb{P}^1$. Suppose $\delta > 1$. Then*

$$S(\Omega_\varepsilon) - \hat{S} = O(\varepsilon^\tau)$$

for some $\tau > 0$, which can be taken to be 1 if $\delta \geq 2$. Moreover, the horizontal component of $S(\omega_\varepsilon)$ satisfies

$$(S(\Omega_\varepsilon) - \hat{S})_{\mathcal{H}} = \frac{\varepsilon^\delta}{\lambda} S(\omega_{FS}) + O(\varepsilon^{\delta+\tau'}),$$

for some $\tau' > 0$, which again can be taken to be 1 if $\delta \geq 2$.

Proof We follow the strategy of [23, Lemma 3.3] to compute the expansion of the scalar curvature. We begin with the Ricci curvature. The Ricci curvature of Ω_ε is the sum of the curvatures of the induced metrics on $\Lambda^n\mathcal{V}$ and $\mathcal{H}$, where $\mathcal{V} = \ker \pi_*$ is the vertical subbundle of the tangent bundle and $\mathcal{H} \cong \pi^*T\mathbb{P}^1$ is its orthogonal complement with respect to Ω_ε, due to the short exact sequence

$$0 \to \mathcal{V} \to T\mathcal{X} \to \pi^*T\mathbb{P}^1 \to 0.$$

We will denote by ρ_ε the curvature on $\Lambda^n\mathcal{V}$. Note that the vertical component $\rho_{\varepsilon,\mathcal{V}}$ of this is nothing but the Ricci curvature of the restriction of ω_ε to the fibres, so that $\Lambda_{\omega_\varepsilon}(\rho_\varepsilon) = S(\omega_\varepsilon)$ is the fibrewise scalar curvature. We will deal with its horizontal component $\rho_{\varepsilon,\mathcal{H}}$ at the end of the proof.

For the curvature $F_{\mathcal{H}}$ on $\mathcal{H}$, we also have another metric, namely $\pi^*\omega_{FS}$. The curvature of this metric is simply the pullback of the curvature on B, and the two curvatures differ by an exact term. We therefore have that the curvatures compare as

$$\begin{aligned} iF_{\mathcal{H}} - \mathrm{Ric}(\omega_{FS}) &= i\partial\bar{\partial}\log\left(\frac{\omega_\varepsilon + \lambda\varepsilon^{-\delta}\omega_{FS}}{\omega_{FS}}\right) \\ &= i\partial\bar{\partial}\log\left(1 + \frac{\varepsilon^\delta}{\lambda}\Lambda_{\omega_{FS}}(\omega_\varepsilon)\right) \\ &= \frac{\varepsilon^\delta}{\lambda} i\partial\bar{\partial}\Lambda_{\omega_{FS}}(\omega_\varepsilon) + O(\varepsilon^{2(\delta-1)}). \end{aligned}$$

Note that $\frac{\varepsilon^\delta}{\lambda} i\partial\bar{\partial}\Lambda_{\omega_{FS}}(\omega_\varepsilon)$ is an $O(\varepsilon^{\delta-1})$ term, since ω_ε is $O(\varepsilon^{-1})$. This is why the error term is $O(\varepsilon^{2(\delta-1)})$ above.

The upshot is that we have the expansion

$$\mathrm{Ric}(\Omega_\varepsilon) = \rho_\varepsilon + \mathrm{Ric}(\omega_{FS}) + \frac{\varepsilon^\delta}{\lambda} i\partial\bar{\partial}\Lambda_{\omega_{FS}}(\omega_\varepsilon) + O(\varepsilon^{2(\delta-1)}).$$

Next, we contract to obtain the scalar curvature. We have

$$\Lambda_{\Omega_\varepsilon} = \Lambda_{\mathcal{V}} + \frac{\varepsilon^\delta}{\lambda}\Lambda_{\omega_{FS}} + O(\varepsilon^{\delta+1}),$$

since the restriction of the metric to $\mathcal{V}$ is the restriction of ω_ε, and the metric on $\mathcal{H}$ is $\lambda\varepsilon^{-\delta}\omega_{FS}$ to leading order, the subleading order terms coming from the horizontal component of ω_ε. Therefore,

$$\begin{aligned} S(\Omega_\varepsilon) =& S(\omega_\varepsilon) + \varepsilon^\delta \Lambda_{\omega_{FS}}(\rho_{\varepsilon,\mathcal{H}}) \\ & + \frac{\varepsilon^\delta}{\lambda} S(\omega_{FS}) + \frac{\varepsilon^\delta}{\lambda}\Delta_{\mathcal{V}}(\Lambda_{\omega_{FS}}(\omega_\varepsilon)) + O(\varepsilon^{\min\{3\delta-2,\delta+1\}}). \end{aligned}$$

Note that $\frac{\varepsilon^\delta}{\lambda}\Delta_{\mathcal{V}}(\Lambda_{\omega_{FS}}(\omega_\varepsilon))$ is a vertical $O(\varepsilon^{\delta-1})$ term. Since $\delta > 1$, this is therefore a decaying vertical term, and is ε to an integer power if $\delta > 1$ is an integer.

The term $\rho_{\varepsilon,\mathcal{H}}$ is the horizontal component of the curvature of $\Lambda^n\mathcal{V}$ induced from ω_ε. We claim that $\Lambda_{\omega_{FS}}(\rho_{\varepsilon,\mathcal{H}})$ decays with ε, which gives the required expansion of the scalar curvature. Now, ρ_ε is determined by restricting ω_ε to a fibre $\mathcal{X}_t$ and then taking the Ricci curvature. When $|t| > \varepsilon$, ω_ε restricted to $\mathcal{X}_t$ is simply α_ε, which is constant in the horizontal direction, and so the horizontal contribution is entirely contained in the ball of radius ε in the base direction. In this region, ω_ε is a convex combination of α_t and α_ε, both of which are $O(\varepsilon)$ perturbations of ω. Thus, to leading order in ε, ρ_ε is simply the Ricci curvature of the central fibre, and so does not vary in the horizontal direction. Therefore, the term $\Lambda_{\omega_{FS}}(\rho_{\varepsilon,\mathcal{H}})$ decays with ε, and so the required expansion holds. □

Remark 1 The above explains why we picked the Fubini–Study metric on $\mathbb{P}^1$. This means that the leading order term in the horizontal direction is a constant.

4.3 Improving the approximate solution

Proposition 9 shows that Ω_ε is approximately extremal. We now want to show that we can perturb to a genuine solution of the extremal equation, for sufficiently small ε. The next step is to improve the approximate solution Ω_ε.

We first describe the holomorphy potential for the S^1-action on $\mathcal{X}$ with respect to our metrics. Recall that χ was the cut-off function used in the definition of the relative Kähler ω_ε.

Lemma 10 *Let h_ε be the average* 0 *holomorphy potential of the S^1-action on $\mathcal{X}$ with respect to Ω_ε. Then, after potentially changing the potentials ϕ_t by a function pulled back from $\mathbb{C}$, we have that*

$$h_\varepsilon = \chi(|t|/\varepsilon) \cdot h_0 + \lambda\varepsilon^{-\delta} h_{FS},$$

where

- *h_0 is the average* 0 *potential with respect to the central fibre, thought of as a vertical function on $\mathcal{X}$;*
- *h_{FS} is the average* 0 *potential of the corresponding action on $\mathbb{P}^1$ with respect to ω_{FS}, pulled back to $\mathcal{X}$.*

Proof Note that Ω_ε is an S^1-invariant Kähler form on $\mathcal{X}$, and as such we know the existence of h_ε. Note that because of the linearity of the holomorphy potential of a fixed vector field with respect to the symplectic form, we therefore have the existence of a potential with respect to the *relative* Kähler form ω_ε. Note that the notion of a hamiltonian makes sense even if a 2-form is not symplectic – it is just not guaranteed to exist. Now, if η_ε denotes this potential with respect to ω_ε, it suffices to show that $\eta_\varepsilon = \chi h_0$, again by the linearity of the potential ($\lambda\varepsilon^{-\delta} h_{FS}$ is the potential with respect to $\lambda\varepsilon^{-\delta}\pi^*\omega_{FS}$).

We now prove that $\eta_\varepsilon = \chi(|t|/\varepsilon) \cdot h_0$. Since h_0 is a holomorphy potential for the generator ν of the S^1-action on the central fibre, we have that

$$d_{\mathcal{X}_0}(h_0) = \iota_\nu \omega.$$

Recall the description of ω_ε. We have a $\mathbb{C}^*$ equivariant biholomorphism $F : (M, J) \times \mathbb{C}^* \to \mathcal{X}_{|\pi^{-1}\mathbb{C}^*}$, giving diffeomorphisms $f_t = F_{|\mathcal{X}_t} : M \to M$ such that $f_t^* J_t = J$ and Kähler metrics $\alpha_t = f_t^*\omega = \omega + i\partial\bar{\partial}\phi_t$, for some background Kähler form ω on X. The cut-off function $\chi(s) : [0, \infty) \to [0, 1]$ is 1 when $s \leq \frac{1}{2}$ and 0 when $s \geq \frac{3}{4}$, and ω_ε is defined by

$$\omega_\varepsilon = \omega + i\partial\bar{\partial}\left(\chi\left(\frac{|t|}{\varepsilon}\right)\phi_t + (1 - \chi\left(\frac{|t|}{\varepsilon}\right))\phi_\varepsilon\right).$$

Note that this is exactly equal to α_ε for all fibres t with $|t| \geq \frac{3}{4}\varepsilon$. It follows that $(\iota_\nu\omega_\varepsilon)_{\pi^{-1}(\mathbb{P}^1 \setminus B_{\frac{3}{4}\varepsilon})} = 0$ and so we have $(h_\varepsilon)_{|\pi^{-1}(\mathbb{P}^1 \setminus B_{\frac{3}{4}\varepsilon})} = 0$. In other words, η_ε is supported on the preimage of the ball of radius $\frac{3}{4}\varepsilon$.

Using the expansion

$$\begin{aligned}\omega_\varepsilon =& \chi\left(\frac{|t|}{\varepsilon}\right)\alpha_t + (1 - \chi\left(\frac{|t|}{\varepsilon}\right))\alpha_\varepsilon \\ &+ i\partial\chi\left(\frac{|t|}{\varepsilon}\right) \wedge \bar{\partial}(\phi_t - \phi_\varepsilon) \\ &+ i\partial(\phi_\varepsilon - \phi_t) \wedge \bar{\partial}\chi\left(\frac{|t|}{\varepsilon}\right)\end{aligned}$$

$$+ (\phi_t - \phi_\varepsilon) i\partial\bar{\partial}\chi\left(\frac{|t|}{\varepsilon}\right).$$

we have that if we pull back to $\mathcal{X}$, then

$$\begin{aligned}(F^{-1})^*(\omega_\varepsilon) =& \chi\left(\frac{|t|}{\varepsilon}\right)\omega + (1-\chi\left(\frac{|t|}{\varepsilon}\right))\tau \\ &+ i(F^{-1})^*\partial\chi\left(\frac{|t|}{\varepsilon}\right)\wedge\bar{\partial}(\phi_t - \phi_\varepsilon) \\ &+ i(F^{-1})^*\partial(\phi_\varepsilon - \phi_t)\wedge\bar{\partial}\chi\left(\frac{|t|}{\varepsilon}\right) \\ &+ (F^{-1})^*(\phi_t - \phi_\varepsilon)i\partial\bar{\partial}\chi\left(\frac{|t|}{\varepsilon}\right).\end{aligned}$$

Here τ is the pullback of α_ε, extended to a form on the whole of $(M, J)\times\mathbb{C}^*$. It is not moving in the horizontal directions, so $\iota_\nu\tau = 0$. Similarly, $\iota_\nu i\partial\bar{\partial}\chi\left(\frac{|t|}{\varepsilon}\right) = 0$.

Next, we have that

$$\begin{aligned}\iota_\nu(F^{-1})^* i\partial\bar{\partial}(\chi\phi_t) =& -i\partial\chi\wedge\iota_\nu(F^{-1})^*\bar{\partial}\phi_t + i\bar{\partial}\chi\wedge\iota_\nu(F^{-1})^*\partial\phi_t \\ &+ \chi\iota_\nu(F^{-1})^* i\partial\bar{\partial}(\phi_t) \\ =& -i\partial\chi\wedge\iota_\nu(F^{-1})^*\bar{\partial}\phi_t + i\bar{\partial}\chi\wedge\iota_\nu(F^{-1})^*\partial\phi_t + \chi dh \\ =& -i\partial\chi\wedge\iota_\nu(F^{-1})^*\bar{\partial}\phi_t + i\bar{\partial}\chi\wedge\iota_\nu(F^{-1})^*\partial\phi_t + d(\chi h) - hd\chi \\ =& -\partial\chi\wedge(i\iota_\nu(F^{-1})^*\bar{\partial}\phi_t - h) + \bar{\partial}\chi\wedge(i\iota_\nu(F^{-1})^*\partial\phi_t + h) \\ &+ d(\chi h) - hd\chi.\end{aligned}$$

From

$$dh = \iota_\nu(F^{-1})^* i\partial\bar{\partial}(\phi_t)$$

we know that

$$h - i\iota_\nu(F^{-1})^*\bar{\partial}\phi_t$$

and

$$h + i\iota_\nu(F^{-1})^*\partial\phi_t$$

are closed. Now, ν generates an S^1 action, and if we restrict to the set $|t| = r$, for $r > 0$, then this is compact (real) manifold. In particular, the above two expressions have to be constant. In other words, there exists constants $c_{|t|}$ such that

$$h + i\iota_\nu(F^{-1})^*\partial\phi_t = c_{|t|},$$

and similarly

$$h - i\iota_v (F^{-1})^* \bar{\partial}\phi_t = c_{|t|}$$

since this is just the complex conjugate of the former.

We can then get rid of this constant by changing ϕ_t by a constant, fibrewise. This of course changes nothing for the metric on the fibre, but changes the global expression on the total space, by $i\partial\bar{\partial}$ of a function pulled back from $\mathbb{P}^1$, which moreover is S^1 invariant, as it depends on $|t|$ only. The result follows. □

Proposition 9 implies that the error is all vertical until order ε^δ. We therefore need to correct vertical errors until this order. To make sure we are not introducing horizontal errors interfering with the contribution from ω_{FS}, we need to understand the mapping properties of the linearised operator in more detail.

Proposition 11 *Suppose $\delta \geq 1$. The linearisation L_ε of the scalar curvature operator at Ω_ε admits an expansion*

$$L_\varepsilon = P_\varepsilon + \varepsilon^{2\delta} R_\varepsilon,$$

where the image of P_ε is vertical and R_ε is an operator bounded independently of ε. Further, P_ε and the horizontal component of R_ε admit expansions

$$P_\varepsilon = L_\omega + O(\varepsilon)$$

for any $f \in C^\infty(\mathcal{X})$, where $L_\omega = -\mathcal{D}^*_\omega \mathcal{D}_\omega$ is the linearisation of the scalar curvature at the central fibre, and

$$R_\varepsilon(\pi^* f) = -\frac{1}{\lambda^2} \cdot \pi^* \left(\mathcal{D}^*_{\omega_{FS}} \mathcal{D}_{\omega_{FS}}(f)\right) + O(\varepsilon)$$

for $f \in C^\infty(\mathbb{P}^1)$.

These expansions persist upon perturbations of Ω_ε by potentials ϕ_ε which are

- *$O(\varepsilon^\tau)$ for $\tau > 0$, if $\phi_\varepsilon \in C^\infty_0(X)$;*
- *$O(\varepsilon^{\tau-\delta})$ for $\tau > 0$, if $\phi_\varepsilon \in \pi^* C^\infty(\mathbb{P}^1)$.*

Proof We use the standard expression

$$L_\varepsilon(f) = -\mathcal{D}^*_{\Omega_\varepsilon} \mathcal{D}_{\Omega_\varepsilon}(f) + \frac{1}{2}\langle \nabla S(\Omega_\varepsilon), \nabla f\rangle$$

for the linearisation of the scalar curvature.

We begin with the leading order term of the expansion. First note that, since, by Proposition 9, $S(\Omega_\varepsilon) = \hat{S} + O(\varepsilon)$, the gradient term does not enter to leading order in ε. We are therefore left with expanding the Lichnerowicz operator. Now, the Lichnerowicz operator admits an expansion

$$L_{\Omega_\varepsilon}(f) = \Delta^2_{\Omega_\varepsilon}(f) + \langle \mathrm{Ric}(\Omega_\varepsilon), i\partial\bar{\partial}(f)\rangle_{\Omega_\varepsilon} + \frac{1}{2}\langle \nabla S(\Omega_\varepsilon), \nabla f\rangle_{\Omega_\varepsilon}.$$

So, we need to understand the expansion of the Laplacian, the Ricci curvature and the scalar curvature.

We begin with the Laplacian. The Laplacian operator can be written

$$\Delta_{\Omega_\varepsilon}(f)\Omega_\varepsilon^{n+1} = (n+1)i\partial\bar{\partial}(f) \wedge \Omega_\varepsilon^n.$$

since the base $\mathbb{P}^1$ is one-dimensional, we have expansions

$$\begin{aligned}\Omega_\varepsilon^{n+1} =& \omega_\varepsilon^{n+1} + (n+1)\lambda\varepsilon^{-\delta}\pi^*\omega_{FS} \wedge \omega_\varepsilon^n \\ \Omega_\varepsilon^n =& \omega_\varepsilon^n + n\lambda\varepsilon^{-\delta}\pi^*\omega_{FS} \wedge \omega_\varepsilon^{n-1}\end{aligned}$$

Thus the leading order term of the Laplacian equation is

$$\Delta_{\Omega_\varepsilon}(f)\pi^*\omega_{FS} \wedge \omega_\varepsilon^n = ni\partial\bar{\partial}(f) \wedge \pi^*\omega_{FS} \wedge \omega_\varepsilon^{n-1} + O(\varepsilon^{-1}).$$

In the leading order term, we therefore only get a vertical contribution from $i\partial\bar{\partial}(f)$. This gives that

$$\begin{aligned}\Delta_{\Omega_\varepsilon}(f) =& \Lambda_{\omega_\varepsilon}(i\partial\bar{\partial}_{\mathcal{V}}(f)) + O(\varepsilon) \\ =& \Delta_{\mathcal{X}_0}(f) + O(\varepsilon).\end{aligned}$$

Here we are also using that $J_\varepsilon - J_0 = O(\varepsilon)$ so that we get a similar bound for the difference of the $i\partial\bar{\partial}$-operators near the central fibre, and hence everywhere (since the family of Kähler manifolds is constant for a given ε away from a small neighbourhood of the central fibre).

For the Ricci term, we have from the proof of Proposition 9 that

$$\mathrm{Ric}(\Omega_\varepsilon) = \rho_\varepsilon + \mathrm{Ric}(\omega_{FS}) + O(\varepsilon).$$

Since the inner product $\langle \mathrm{Ric}(\Omega_\varepsilon), i\partial\bar{\partial}(f)\rangle_{\Omega_\varepsilon}$ equals the inner product with respect to the vertical metric from the central fibre to leading order, we therefore have

$$\langle \mathrm{Ric}(\Omega_\varepsilon), i\partial\bar{\partial}(f)\rangle_{\Omega_\varepsilon} = \langle \mathrm{Ric}(\omega), i\partial\bar{\partial}(f)\rangle_{\omega} + O(\varepsilon).$$

Finally, the scalar curvature term vanishes to leading order, since $S(\Omega_\varepsilon)$ is constant to leading order. This term also does not enter in the Lichnerowicz operator on the central fibre, since the central fibre has constant scalar curvature.

Together with the above, this shows that the leading order term of the expansion above is the negative of the Lichnerowicz operator of the metric on the central fibre, which gives the expansion of P_ε (since $\delta \geq 1$, and so all the errors are $O(\varepsilon)$).

Next, we have a refined expansion when the function is of the form $\pi^*(f)$, i.e. the function is pulled back from $\mathbb{P}^1$. In this case, the expansion of the Laplacian becomes

$$\lambda\varepsilon^{-\delta}\Delta_{\Omega_\varepsilon}(f)\pi^*\omega_{FS} \wedge \omega_\varepsilon = ni\partial\bar{\partial}(f) \wedge \omega_\varepsilon^n + O(\varepsilon)$$

where we are omitting pullback from the notation. The above equation follows because $ni\partial\bar{\partial}(f) \wedge \pi^*\omega_{FS} \wedge \omega_\varepsilon^{n-1} = 0$, as f is pulled back from $\mathbb{P}^1$. It follows that

$$\Delta_{\Omega_\varepsilon}(\pi^*(f)) = \frac{\varepsilon^\delta}{\lambda} \cdot \pi^*(\Delta_{\omega_{FS}}(f)) + O(\varepsilon^{\delta+1}).$$

and so

$$\Delta^2_{\Omega_\varepsilon}(\pi^*(f)) = \frac{\varepsilon^{2\delta}}{\lambda^2} \cdot \pi^*(\Delta^2_{\omega_{FS}}(f)) + O(\varepsilon^{2\delta+1}).$$

For the Ricci curvature, we now have

$$\begin{aligned}\langle \mathrm{Ric}(\Omega_\varepsilon), i\partial\bar{\partial}(f)\rangle_{\Omega_\varepsilon} =& \langle \mathrm{Ric}(\omega_{FS}), i\partial\bar{\partial}(f)\rangle_{\lambda\varepsilon^{-\delta}\omega_{FS}+(\omega_\varepsilon)_{\mathcal{H}}} + O(\varepsilon^{3\delta}) \\ =& \frac{\varepsilon^{2\delta}}{\lambda^2}\langle \mathrm{Ric}(\omega_{FS}), i\partial\bar{\partial}(f)\rangle_{\omega_{FS}} + O(\varepsilon^{3\delta})\end{aligned}$$

Note that here we are using the induced inner product on 2-forms, which scales like the scaling factor to the power -2, hence the equation in the second line and that the error is $O(\varepsilon^{3\delta})$.

Again, the gradient of the scalar curvature term does not enter either in the expansion or in the model on $\mathbb{P}^1$ to leading order, as ω_{FS} is of constant curvature (though, if we had pulled back another metric, we would have a matching of these terms anyway, so this does not use that we have the Fubini–Study metric in any essential way). Thus the claimed expansion on pulled back functions also holds.

Finally, we need to check that the expansions persist under certain perturbations of Ω_ε. It is immediate that the $O(1)$-term of the expansion will be unchanged upon perturbing Ω_ε by $\varepsilon^\tau i\partial\bar{\partial}\phi$, for any ϕ. We can also allow a negative power of ε and keep the same expansion, when $\phi = \pi^* f$ is pulled back from $\mathbb{P}^1$. The vertical component of the metric is then unaffected. In the horizontal direction, we then need to be perturbing ω_{FS}, which we do provided the exponent is strictly larger than $-\delta$. This completes the proof. □

As used at the end of the proof above, we can also perturb ω_{FS} before pulling back. This perturbs Ω_ε by terms that at first glance look like they blow up.

Lemma 12 *Let $f \in C^\infty(\mathbb{P}^1)$. Then for any $\tau > 0$, we have that*

$$S\left(\Omega_\varepsilon + \varepsilon^{\tau-\delta} i\partial\bar{\partial}\pi^*(f)\right) = S(\Omega_\varepsilon) - \frac{\varepsilon^{\delta+\tau}}{\lambda^2}\pi^*\left(\mathcal{D}^*_{\omega_{FS}}\mathcal{D}_{\omega_{FS}}(f)\right) + O(\varepsilon^{\delta+\tau+1}).$$

Proof This is exactly the same as the computations for the horizontal terms in Proposition 11. □

Remark 2 Of course, the above lemma can be applied iteratively. If we define $\Omega'_\varepsilon = \Omega_\varepsilon + \varepsilon^{\tau-\delta} i\partial\bar{\partial}\pi^*(f)$ and then perturb Ω'_ε to $\Omega'_\varepsilon + \varepsilon^{\tau'-\delta} i\partial\bar{\partial}(\pi^*\phi)$ for some $\tau' > \tau$, then

$$S\left(\Omega'_\varepsilon + \varepsilon^{\tau'-\delta} i\partial\bar{\partial}\pi^*(\phi)\right) = S\left(\Omega'_\varepsilon\right) - \frac{\varepsilon^{\delta+\tau'}}{\lambda^2}\pi^*\left(\mathcal{D}^*_{\omega_{FS}}\mathcal{D}_{\omega_{FS}}(\phi)\right) + O(\varepsilon^{\delta+\tau'+1}).$$

Similar statements hold when we iterate multiple times, and also when we further perturb Ω'_ε by functions not necessarily pulled back from $\mathbb{P}^1$, but which are decaying with ε.

Having this understanding of the linearised operator, we can now improve the approximate solutions to be extremal to as high order in ε as we would like.

Proposition 13 *For any $\kappa \geq 0$, there exists a Kähler form $\Omega_{\varepsilon,\kappa}$ on $\mathcal{X}$ and a holomorphy potential $h_{\varepsilon,\kappa}$ with respect to $\Omega_{\varepsilon,\kappa}$ such that*

$$S\left(\Omega_{\varepsilon,\kappa}\right) = h_{\varepsilon,\kappa} + O(\varepsilon^{\kappa+1}).$$

Proof For $\kappa = 0$, we take $\Omega_{\varepsilon,0} = \Omega_\varepsilon$. Proposition 9 then shows that we have the required expansion, with $h_{\varepsilon,0} = \hat{S}$ a constant, which is a potential for the trivial holomorphic vector field on $\mathcal{X}$.

We proceed to construct better approximate solutions inductively. For simplicity we will assume that δ is an integer. This is not essential, and is simply for notational convenience. We can then do induction over $\kappa \in \mathbb{Z}_{\geq 0}$ and assume that all terms appear at integer values of ε. Ultimately we can choose $\delta = 2$ to make the construction work, so we lose nothing in the proof of the main result in assuming this.

We will inductively show that we can find

- functions $f_0, \ldots, f_\kappa \in C^\infty_0(\mathcal{X})$;
- functions $\phi_0, \ldots, \phi_\kappa \in C^\infty(\mathbb{P}^1)$;
- holomorphy potentials $h_0, \ldots, h_\kappa$ with respect to Ω_ε;

such that, if we put

$$\Omega_{\varepsilon,\kappa} = \Omega_\varepsilon + i\partial\bar{\partial}\left(\sum \varepsilon^j f_j + \sum_j \varepsilon^{j-\delta}\pi^*\phi_j\right)$$

then $\Omega_{\varepsilon,\kappa}$ satisfies

$$S\left(\Omega_{\varepsilon,\kappa}\right) = h_{\varepsilon,\kappa} + O(\varepsilon^{\kappa+1})$$

where

$$h_{\varepsilon,\kappa} = \sum_{j=0}^{\kappa} \varepsilon^j h_j + \left\langle \nabla\left(\sum_{j=0}^{\kappa}\varepsilon^j h_j\right), \nabla\left(\sum_{j=0}^{\kappa}\varepsilon^j f_j + \sum_{j=0}^{\kappa}\varepsilon^{j-\delta}\pi^*\phi_j\right)\right\rangle,$$

which is a holomorphy potential with respect to $\Omega_{\varepsilon,\kappa}$. As remarked above, for $\kappa = 0$, we have the required statement with $f_0 = 0 = \phi_0$ and $h_0 = \hat{S}$, the average scalar curvature of the fibres.

The argument differs depending on whether or not $\kappa \geq \delta$. So we begin with $\kappa < \delta$. In this case, we will additionally show inductively that there is no horizontal term up to order ε^{δ} and that the horizontal term at order ε^{δ} is precisely given by the horizontal term in the expansion of Proposition 9.

By the induction assumption,

$$S\left(\Omega_{\varepsilon,\kappa}\right) - h_{\varepsilon,\kappa} = l_{\kappa+1}\varepsilon^{\kappa+1} + O(\varepsilon^{\kappa+2})$$

for some function $l_{\kappa+1} \in C_0^{\infty}(\mathcal{X})$. By the mapping properties of L_{ω}, there exists $f_{\kappa+1} \in C_0^{\infty}(\mathcal{X})$ and a holomorphy potential $h_{\kappa+1}$ such that

$$L_{\omega}(f_{\kappa+1}) = -l_{\kappa+1} + h_{\kappa+1}.$$

By Proposition 11 applied to $\Omega_{\varepsilon,\kappa}$, we therefore have that

$$\begin{aligned} S&\left(\Omega_{\varepsilon,\kappa} + i\partial\bar{\partial}(\varepsilon^{\kappa+1} f_{\kappa+1})\right) \\ &= S(\Omega_{\varepsilon,\kappa}) - \varepsilon^{\kappa+1} l_{\kappa+1} + \varepsilon^{\kappa+1} h_{\kappa+1} + O(\varepsilon^{\kappa+2}) \\ &= \sum_{j=0}^{\kappa+1} \varepsilon^j h_j + \left\langle \nabla\left(\sum_{j=0}^{\kappa} \varepsilon^j h_j\right), \nabla\left(\sum_{j=0}^{\kappa} \varepsilon^j f_j + \sum_{j=0}^{\kappa} \varepsilon^{j-\delta}\pi^*\phi_j\right)\right\rangle + O(\varepsilon^{\kappa+2}). \end{aligned}$$

Moreover, since the terms

$$\left\langle \nabla\left(\varepsilon^{\kappa+1} h_j\right), \nabla\left(\sum_{j=0}^{\kappa+1} \varepsilon^j f_j + \sum_{j=0}^{\kappa+1} \varepsilon^{\delta-j}\pi^*\phi_j\right)\right\rangle$$

and

$$\left\langle \nabla\left(\sum_{j=0}^{\kappa+1} \varepsilon^j h_j\right), \nabla\left(\varepsilon^{\kappa+1} f_{\kappa+1}\right)\right\rangle$$

are $O(\varepsilon^{\kappa+2})$ (since the $\kappa = 0$ terms are all constant), we have that the above agrees with $h_{\varepsilon,\kappa+1}$ at order $\varepsilon^{\kappa+1}$, i.e.

$$S\left(\Omega_{\varepsilon,\kappa} + i\partial\bar{\partial}(\varepsilon^{\kappa+1} f_{\kappa+1})\right) = h_{\varepsilon,\kappa+1} + O(\varepsilon^{\kappa+2}),$$

where

$$h_{\varepsilon,\kappa+1} = \sum_{j=0}^{\kappa+1} \varepsilon^j h_j + \left\langle \nabla \left(\sum_{j=0}^{\kappa+1} \varepsilon^j h_j \right), \nabla \left(\sum_{j=0}^{\kappa+1} \varepsilon^j f_j + \sum_{j=0}^{\kappa+1} \varepsilon^{\delta-j} \pi^* \phi_j \right) \right\rangle,$$

Thus we have the required expansion with $f_{\kappa+1}$ and $h_{\kappa+1}$ as above and $\phi_{\kappa+1} = 0$.

It remains to show that there are no horizontal terms up to order ε^δ, apart from our original one at order exactly ε^δ. This is again a consequence of Proposition 11, since the linearisation only hits horizontal terms at order $\varepsilon^{2\delta}$.

We now continue for $\kappa \geq \delta$. In this case, there will be horizontal error terms, too, and we begin by correcting these. Suppose

$$S\left(\Omega_{\varepsilon,\kappa}\right) - h_{\varepsilon,\kappa} = l_{\kappa+1}\varepsilon^{\kappa+1} + \psi_{\kappa+1}\varepsilon^{\kappa+1} + O(\varepsilon^{\kappa+2}),$$

where $l_{\kappa+1} \in C_0^\infty(\mathcal{X})$ and $\psi_{\kappa+1} \in C^\infty(\mathbb{P}^1)$. By the mapping properties of the Lichnerowicz operator on $\mathbb{P}^1$, there exists $\phi_{\kappa+1} \in C^\infty(\mathbb{P}^1)$ and a holomorphy potential $h^1_{\kappa+1}$ on $\mathbb{P}^1$, with respect ω_{FS}, such that

$$\mathcal{D}^*_{\omega_{FS}}\mathcal{D}_{\omega_{FS}}(\phi_{\kappa+1}) = \psi_{\kappa+1} - h^1_{\kappa+1}.$$

By Proposition 11 and Lemma 12, we therefore have that $\Omega_{\varepsilon,\kappa} + i\partial\bar{\partial}\left(\varepsilon^{\kappa+1-2\delta}\right)$ is Kähler and

$$S\left(\Omega_{\varepsilon,\kappa} + i\partial\bar{\partial}\left(\varepsilon^{\kappa+1-2\delta}\phi_{\kappa+1}\right)\right) = S\left(\Omega_{\varepsilon,\kappa}\right) - \varepsilon^{\kappa+1}\psi_{\kappa+1} + \varepsilon^{\kappa+1}h^1_{\kappa+1}.$$

Note that Lemma 12 applies because $\kappa \geq \delta$, so $\kappa + 1 - 2\delta > -\delta$.

We now invoke Lemma 10, which allows us to compare $h^1_{\kappa+1}$ to an actual holomorphy potential for Ω_ε on $\mathcal{X}$. We have that $h^1_{\kappa+1} = \upsilon h_{FS} + c$ for some υ constants υ and c. If we then write

$$h^2_{\kappa+1} = \upsilon h_\varepsilon = \upsilon\left(\chi(t/\varepsilon) \cdot h_0 + \lambda\varepsilon^{-\delta}h_{FS}\right)$$

for the average zero potential for the corresponding multiple of the generator of the circle action on $\mathcal{X}$, we then have that

$$\begin{aligned} S\left(\Omega_{\varepsilon,\kappa} + i\partial\bar{\partial}\left(\varepsilon^{\kappa+1-2\delta}\phi_{\kappa+1}\right)\right) =& \varepsilon^{\kappa+1-\delta}h^2_{\kappa+1} + \varepsilon^{\kappa+1}c + h_{\varepsilon,\kappa} \\ &+ \left\langle \nabla h_{\varepsilon,\kappa}, \nabla\left(\sum_{j=0}^{\kappa}\varepsilon^j f_j + \sum_{j=1}^{\kappa+1}\varepsilon^{j-\delta}\pi^*\phi_j\right)\right\rangle \\ &+ \varepsilon^{\kappa+1}l_{\kappa+1} + O(\varepsilon^{\kappa+2}). \end{aligned}$$

We now define

$$\widetilde{h}_{\varepsilon,\kappa+1} = h_{\varepsilon,\kappa} + \varepsilon^{\kappa+1}c + \varepsilon^{\kappa+1-\delta}h^2_{\kappa+1},$$

so we have added an $O(\varepsilon^{\kappa+1})$ constant term, but also *altered* the $\varepsilon^{\kappa+1-\delta}$-term, compared to $h_{\varepsilon,\kappa}$. If we then compare

$$S\left(\Omega_{\varepsilon,\kappa}+i\partial\bar{\partial}\left(\varepsilon^{\kappa+1-2\delta}\phi_{\kappa+1}\right)\right)$$

to

$$\widetilde{h}_{\varepsilon,\kappa+1}+\left\langle\nabla\widetilde{h}_{\varepsilon,\kappa+1},\nabla\left(\sum_{j=0}^{\kappa}\varepsilon^{j}f_{j}+\sum_{j=1}^{\kappa+1}\varepsilon^{j-\delta}\pi^{*}\phi_{j}\right)\right\rangle$$

we have no horizontal term to up and including order $\varepsilon^{\kappa+1}$: the only horizontal term that could appear comes from

$$\left\langle\varepsilon^{\kappa+1-\delta}h^{2}_{\kappa+1},\nabla\left(\sum_{j=1}^{\kappa+1}\varepsilon^{j-\delta}\pi^{*}\phi_{j}\right)\right\rangle$$

whose horizontal component to leading order is

$$\begin{aligned}\left\langle\varepsilon^{\kappa+1}h^{1}_{\kappa+1},\nabla\left(\sum_{j=1}^{\kappa+1}\varepsilon^{j-\delta}\pi^{*}\phi_{j}\right)\right\rangle_{\varepsilon^{-\delta}\omega_{FS}}&=\varepsilon^{(\kappa+1)+(1-\delta)+\delta}\left\langle h^{1}_{\kappa+1},\phi_{1}\right\rangle_{\omega_{FS}}\\&=\varepsilon^{\kappa+2}\left\langle h^{1}_{\kappa+1},\phi_{1}\right\rangle_{\omega_{FS}}.\end{aligned}$$

So the horizontal terms appearing in the difference of the two terms are at least $O(\varepsilon^{\kappa+2})$, as claimed.

On the other hand, we have introduced new vertical terms at smaller orders. These come from accounting for the inner product of

$$\nabla\left(\sum_{j=0}^{\kappa}\varepsilon^{j}f_{j}+\sum_{j=1}^{\kappa+1}\varepsilon^{j-\delta}\pi^{*}\phi_{j}\right)$$

with the gradient of $\varepsilon^{\kappa+1-\delta}h^{2}_{\kappa+1}$. The leading order contribution is therefore the inner product with the gradient of $\varepsilon^{\kappa+1-\delta}\cdot\upsilon\cdot\chi(t/\varepsilon)\cdot h_{0}$. Since $f_{0}=0$, this means we get new vertical contributions to all orders starting from order $\varepsilon^{\kappa+2-\delta}$. The upshot is that we have an expansion

$$\begin{aligned}&S\left(\Omega_{\varepsilon,\kappa}+i\partial\bar{\partial}\left(\varepsilon^{\kappa+1-2\delta}\phi_{\kappa+1}\right)\right)\\=&\widetilde{h}_{\varepsilon,\kappa+1}+\left\langle\nabla\widetilde{h}_{\varepsilon,\kappa+1},\nabla\left(\sum_{j=0}^{\kappa}\varepsilon^{j}f_{j}+\sum_{j=1}^{\kappa+1}\varepsilon^{j-\delta}\pi^{*}\phi_{j}\right)\right\rangle\end{aligned}$$

$$+\sum_{j=0}^{\delta} \varepsilon^{\kappa+2+j-\delta} l_j^{\kappa+1} + O(\varepsilon^{\kappa+2}),$$

where all the $l_j^{\kappa+1} \in C_0^{\infty}(\mathcal{X})$ are vertical.

Following exactly the same argument as for the case when $\kappa < \delta$, we can remove these errors *without reintroducing horizontal terms* until order $\varepsilon^{\kappa+2}$. This relies on the fact that when using vertical functions added at order ε^i, we are not introducing any horizontal terms until order $\varepsilon^{i+\delta}$. The upshot is that by altering the f_j's in the previous steps and adding a suitable function $f_{\kappa+1}$ at order $\varepsilon^{\kappa+1}$, we get an expansion of the required type. This completes the proof. □

4.4 Perturbing to a solution

We are now ready to perturb the approximate solution constructed above to a genuine solution.

Proposition 14 *Suppose* $\delta > 1$. *Let* $\Psi_{\varepsilon,\kappa} : C^{k+4,\alpha} \times \overline{\mathfrak{h}} \to C^{k,\alpha}$ *denote the operator*

$$\Psi_{\varepsilon,\kappa}(\phi, h) = L_{\varepsilon,\kappa}(\phi) - h_{\varepsilon,\kappa} - \frac{1}{2}\langle \nabla_{\Omega_{\varepsilon,\kappa}}(h_{\varepsilon,\kappa}), \nabla\phi\rangle,$$

where $L_{\varepsilon,\kappa}$ *is the linearisation of the scalar curvature operator at* $\Omega_{\varepsilon,\kappa}$. *Then* $\Psi_{\varepsilon,\kappa}$ *admits a right-inverse* $Q_{\varepsilon,\kappa}$ *satisfying*

$$\| Q_{\varepsilon,\kappa} \| \leq C\varepsilon^{-2\delta}.$$

Proof The proof is very similar to analogous results for other fibrations, see [23, Lemma 6.5, 6.6, 6.7] or [18, Lemma 6.5], and we omit the details. The precise rate $\varepsilon^{-2\delta}$ of the bound comes from Proposition 11, which shows that $\Psi_{\varepsilon,\kappa}$ is surjective modulo $\overline{\mathfrak{h}}$ at order $\varepsilon^{2\delta}$. □

We are now ready to perturb to a genuine solution of the extremal equation. The key will be the following Quantitative Inverse Function Theorem.

Theorem 15 *Suppose* $\Phi : V \to W$ *is a differentiable map of Banach spaces, with surjective differential at* $0 \in V$. *Let* Ψ *be a right inverse for* $D\Phi_0$. *Let*

- r' *be the radius of the closed ball in* V *where* $\Phi - d\Phi$ *is Lipschitz, with Lipschitz constant* $\frac{1}{2\|\Psi\|}$;
- $r = \frac{1}{2\|\Psi\|}$.

Then for all $w \in W$ *such that* $\|w - \Phi(0)\| < r$, *there exists a* $v \in V$ *with* $\|v\| < r'$ *such that* $\Phi(v) = w$.

We will apply this to the operator

$$\Phi_{\varepsilon,\kappa} : C^{k+4,\alpha}(\mathcal{X}) \times \overline{\mathfrak{h}} \to C^{k,\alpha}(\mathcal{X})$$

given by

$$(f,h) \mapsto S(\Omega_{\varepsilon,\kappa} + i\partial\bar{\partial} f) - h_{\varepsilon,\kappa} - \frac{1}{2}\langle \nabla_{\Omega_{\varepsilon,\kappa}}(h_{\varepsilon,\kappa}), \nabla\phi\rangle. \tag{2}$$

Notice that the linearisation of $\Phi_{\varepsilon,\kappa}$ is the operator $\Psi_{\varepsilon,\kappa}$ in Proposition 14. In order to apply Theorem 15, we need to a definite region on which $\Phi = \Phi_{\varepsilon,\kappa}$ is Lipschitz of Lipschitz constant $\frac{1}{2\|\Psi\|}$. This is provided by the following lemma.

Lemma 16 *Let $\mathcal{N}_{\varepsilon,\kappa} = \Phi_{\varepsilon,\kappa} - d\Phi_{\varepsilon,\kappa}$, where $\Phi_{\varepsilon,\kappa} : C^{k+4,\alpha} \times \mathfrak{h} \to C^{k,\alpha}$ is the operator given by Equation (2). There are constant $c, C > 0$, such that for all $\varepsilon > 0$ sufficiently small, if $f_i \in C^{k+4,\alpha} \times \mathfrak{h}$ for $i = 1, 2$ satisfy $\|f_i\| \le c$, then*

$$\|\mathcal{N}_{\varepsilon,\kappa}(f_1) - \mathcal{N}_{\varepsilon,\kappa}(f_2)\| \le C\,(\|f_1\| + \|f_2\|)\,\|f_1 - f_2\|.$$

The proof of Lemma 16 is a direct consequence of the Mean Value Theorem, using that the $\Psi_{\varepsilon,\kappa}$ are bounded, independently of ε, and the following result, which is a global version of Proposition 5, and whose proof is similar.

Proposition 17 *For each k, α, there exists $c, C > 0$ such that if $\|\phi\|_{C^{k+4,\alpha}} \le c$, then*

$$\|L_{\Omega_\varepsilon + i\partial\bar{\partial}\phi}(f) - L_{\Omega_\varepsilon}(f)\|_{C^{k,\alpha}} \le C\|\phi\|_{C^{k+4,\alpha}}\|f\|_{C^{k+4,\alpha}}.$$

We are now ready to prove our main result. The remaining argument follows exactly similar arguments in e.g. [21, 23]. The statement below is just rephrasing Theorem 1, using the parameter ε instead of k.

Theorem 18 *For all $\varepsilon > 0$ sufficiently small, there exists an extremal metric $\widetilde{\Omega}_\varepsilon \in [\Omega_\varepsilon]$.*

Proof Lemma 16 implies that $\mathcal{N}_{\varepsilon,\kappa}$ is Lipschitz on the ball of radius ϱ, with Lipschitz constant $c\varrho$, for all ϱ sufficiently small. Letting r' be as in Theorem 15 for the extremal operator, we therefore have that $r' \ge c\varepsilon^{2\delta}$, for some constant $c > 0$, by Proposition 14. Letting $r = \frac{r'}{2\|Q_{\varepsilon,\kappa}\|}$, we then have that

$$r \ge C\varepsilon^{4\delta},$$

for some constant $C > 0$. In particular, Theorem 15 implies that we can perturb the scalar curvature, up to a holomorphy potential, to anything in the ball of radius $C\varepsilon^{4\delta}$ about $S(\Omega_{\varepsilon,\kappa})$. Since C depends only on κ, not ε, we see that if κ is chosen sufficiently large (we need $\kappa > 4\delta$), then we can solve the extremal equation, since $S(\Omega_{\varepsilon,\kappa})$ is extremal to order $\varepsilon^{\kappa+1}$. □

Remark 3 In fact, as used e.g. in [21, Remark 3.8], the actual solutions can be ensured to be an $O(\varepsilon^{\kappa-2\delta})$ perturbation of $\Omega_{\varepsilon,\kappa}$. In particular, we can, for any κ, choose a $\kappa' > \kappa$ such that the actual solutions produced from $\Omega_{\varepsilon,\kappa'}$ agree with $\Omega_{\varepsilon,\kappa'}$ to order

ε^κ. In particular, for any desired k and α, and for all ε sufficiently small, the solution will agree with

$$\Omega_\varepsilon = \omega_\varepsilon + \lambda \varepsilon^{-\delta} \pi^* \omega_{FS}.$$

in $C^{k,\alpha}$, up to terms that are decaying in ε in both the horizontal and vertical direction.

Remark 4 Treating δ as a parameter is not strictly speaking necessary in our construction. One could e.g. choose $\delta = 2$. On the other hand, under more general assumptions on the relationship between the automorphism group of the central fibre and the general fibre, constructions like ours should be possible. This is the main reason we have treated δ as a parameter.

These constructions should be obstructed, as there is no longer an action of the full maximal torus T_0 of the central fibre, on the total space of the test configuration. It is likely that there one needs a more specific choice of δ to make the construction work. For example, if one would need to pick δ to be minimal, i.e. $\delta = 1$, in the construction, one would potentially get a different equation than the cscK equation for the metric on $\mathbb{P}^1$ by looking at the horizontal term at this order – this would likely be a twisted cscK equation, with twisting term coming from the pushforward of certain terms from the total space of the test configuration. This would also lead to obstructions at order ε in the vertical direction, that one cannot deal with using the linearised operator.

In fact, at the time of writing, a general framework for attacking these questions on fibrations with only semistable fibres is being developed by Annamaria Ortu ([29]). Her work studies the analogue of the optimal symplectic connection equation in this setting, and constructs extremal metrics on the total space of such fibrations, when this equation and a suitable equation on the base can be solved. This requires one to see the total space of the fibration as a deformation of a cscK fibration, which is very different from the approach taken in our specific case.

It seems possible that our work could fit into her framework. The key to this would be to show that the test configuration we consider can be seen as a deformation of a product test configuration over $\mathbb{P}^1$ for $(\mathcal{X}_0, \mathcal{L}_0)$. The likely choice is to use the product test configuration produced from the $\mathbb{C}^*$ action on $\mathcal{X}_0$, that we used to define the $\mathbb{C}^*$ action on the initial local test configuration we have considered. This means that we would *not* see this as a deformation of $\mathcal{X}_0 \times \mathbb{P}^1$, in general. Showing that indeed this is the case may be the better avenue to pursue in order to extend our results to cases with more general automorphism groups.

5 Metric limits of the adiabatic extremal metrics

It is interesting to analyze from the metric viewpoint what happens to the extremal metrics $\widetilde{\Omega}_\varepsilon$ when $\varepsilon \to 0$.

Proposition 19 *For any point $p \in \mathcal{X}$, the metrics $\widetilde{\Omega}_\varepsilon$ converge as $\varepsilon \to 0$ in the pointed Gromov-Hausdorff sense to the product $X_0 \times \mathbb{R}^2$, with X_0 equipped with its constant scalar curvature metric and flat $\mathbb{R}^2$.*

Remark 5 This shows that a phenomenon of jumping of complex structures in metric limits happens in the strictly K-semistable fibres X_t as $t \neq 0$. Indeed, instead of converging to some Kähler metric on the product $X_t \times \mathbb{R}^2$, as ε goes to 0, these extremal metrics $\widetilde{\Omega}_\varepsilon$ are resembling in all the vertical directions more and more the constant scalar curvature Kähler metric at the central fibre of the test configuration.

Proof The result follows by observing that this limit behavior happens for the background model metric Ω_ε. At any fixed fibre as $\varepsilon \to 0$, the vertical part of Ω_ε, namely ω_ε, gets metrically closer and closer to the cscK manifold (X_0, ω_0). Indeed, the vertical background metrics on the fibres for $|t| \geq \varepsilon$ get smoothly closer to the cscK metric, thanks to their construction as smooth equivariant deformations in [34]. Moreover, the horizontal part gets more and more dominated by a large multiple of the Fubini-Study metric. Then the analytic estimates in the perturbative analysis, see Remark 3, show that the two metrics tensors corresponding to Ω_ε and $\widetilde{\Omega}_\varepsilon$ are point-wise getting closer as $\varepsilon \to 0$, so the difference of the distances measured with the two metrics goes to zero. Hence, passing to the limit, both the model metrics Ω_ε and the extremal metrics $\widetilde{\Omega}_\varepsilon$ converge in the pointed Gromov-Hausdorff sense to the same space $X_0 \times \mathbb{R}^2$, with X_0 equipped with its constant scalar curvature metric and flat $\mathbb{R}^2$, as claimed. □

Finally, note that if instead we would have rescaled the metrics so that the horizontal direction remains of fixed diameter (while the fibres shrink to zero) the metrics would converge to the Fubini-Study metric on the base $\mathbb{P}^1$.

6 Examples

In this final section we show that our construction can be used to produce many new extremal metrics. In order to apply the construction, we need to find a destabilising test configuration to a smooth cscK central fibre, for a strictly semistable manifold. Moreover, we need the $\mathbb{C}^*$ discrepancy condition to hold. All the examples we give here will come from explicit such families of Fano manifolds.

We begin with a useful lemma for providing examples where the reduced automorphism group of the central fibre is $\mathbb{C}^*$ and the strictly K-semistable manifold has discrete automorphism group. This will be used in a few different specific instances below.

Lemma 20 *Let X_0 be a Fano manifold with connected component of the identity of the automorphisms group $\mathrm{Aut}_0(X_0) \cong \mathbb{C}^*$. Assume $\mathrm{Aut}_0(X_0)$ acts non-trivially on $H^1(TX_0)$. Then there exist a smooth compactified test configuration $\pi : \mathcal{X} \to \mathbb{P}^1$ with X_0 as central fibre and $\mathrm{Aut}_0(X_t)$ trivial for $t \neq 0$.*

Proof It follows by Kuranishi theory that the versal space of deformation is smooth since $H^2(TX_0)$ vanishes, as X_0 is Fano, and so it can be identified with an open neighbourhood of the origin in $H^1(TX_0)$. Since the natural linear action of $\mathrm{Aut}_0(X_0)$ is by hypothesis non-trivial, by taking a one dimensional positive weight eigenspaces (if only negative exist just take the opposite action), eventually base changing and completing that family over $\mathbb{P}^1$, we obtain a smooth compactified test configuration.

Finally, let us show that there are no holomorphic vector fields on X_t, $t \neq 0$. Suppose for a contradiction that there are holomorphic vector fields on X_t. Since $\chi(TX_t)$ is constant, $h^1(TX_t)$ must be equal to $h^1(TX_0)$, since by Kodaira–Nakano vanishing, $h^i(TX_t) = 0$ for all $i > 0$. By openness of versality (see, e.g., [9]), the Kodaira–Spencer map at $t \neq 0$ will still be surjective. On the other hand, the non-zero fibres of the test configuration are all isomorphic, and so there is some kernel in such a map at $t \neq 0$. But this contradicts versality, because then, under the assumption that $h^1(TX_0) = h^1(TX_t)$, the Kodaira–Spencer map cannot be surjective either. ◻

Remark 6 It should be possible to easily generalise the above lemma to other situations, under certain assumptions. However, we will only require the above for the examples we consider.

We now consider some different families of the Mori–Mukai classification of Fano threefolds in turn. This relies heavily on recent work on Fano threefolds by many authors. We will use the book [5] as our chief reference for the results relevant to our construction.

6.1 Single members of families

In this section, we will consider several examples of a similar type. In each case, we will consider one special member of the family and verify that it admits a degeneration to another smooth K-polystable member of the same family, with the $\mathbb{C}^*$ discrepancy condition being satisfied.

We illustrate the method in most detail with a member of the family 3.5. We follow [5, Sect. 7.5].

Lemma 21 *Let a, b, c be complex numbers, not all 0. For $t \in \mathbb{C}$, let C be the curve in $\mathbb{P}^1 \times \mathbb{P}^2$ given as the image of*

$$xv^5 + y(w^5 + avw^4 + bv^2w^3 + cv^3w^2) = 0$$

via the embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^2$ given by

$$([x, y], [v, w]) \mapsto ([x, y], [v^2, vw, w^2])$$

in homogeneous coordinates. Let $X = \mathrm{Bl}_C\, \mathbb{P}^1 \times \mathbb{P}^2$ be the Fano threefold in the family 3.5 of the Mori–Mukai list given by blowing up $\mathbb{P}^1 \times \mathbb{P}^2$ in C. Then there exists a test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, -K_X)$ for which the construction of Theorem 1 applies.

Proof Consider the family X_s with $s \in \mathbb{C}$ given by blowing up $\mathbb{P}^1 \times \mathbb{P}^2$ in the curve C_s which is the image of

$$xv^5 + y(w^5 + savw^4 + s^2bv^2w^3 + s^3cv^3w^2) = 0$$

via the embedding $\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^1 \times \mathbb{P}^2$ above. One can show that X_s is isomorphic to $X = X_1$ when $s \neq 0$ ([5, Lemma 7.5.1]). On the other hand, when $s = 0$, we obtain a

Fano manifold X_0 which by [5, Lemma 5.14.8] is K-polystable. Thus the family X_s is a destabilising test configuration, showing that X is strictly K-semistable. Moreover, by [11, Lemma 8.7, Corollary 2.7], X_0 has reduced automorphism group $\mathbb{C}^*$, while X has discrete automorphism group.

Versality in the Kuranishi theory then implies that there exists a test configuration $(\mathcal{X}, \mathcal{L})$ for $(X, -K_X)$ in the Kuranishi family, whose central fibre is $(X_0, -K_{X_0})$. The fact that central fibre of this potentially different test configuration to the one we have explicitly produced is $(X_0, -K_{X_0})$ follows by uniqueness of cscK degenerations [13, Theorem 1.3]. Since the condition on the automorphism group is satisfied, Theorem 1 applies to construct extremal metrics on $(\mathcal{X}, \mathcal{L} + \mathcal{O}(d))$ for all sufficiently large d. □

One can follow the same strategy to produce examples in the remaining families under consideration in this section. The details in each case are the same as the proof of Lemma 21 and rely on explicit examples of destabilising test configurations given in [5]. The only difference is that the strictly K-semistable member in each case will have discrete automorphism group, and the central fibre has reduced automorphism group given by $\mathbb{C}^*$.

Lemma 22 *In each of the families 2.20, 2.22, 3.8, 3.12 and 4.13 of the Mori–Mukai list of Fano threefolds, there exists at least one member that admits a test configuration to which Theorem 1 applies.*

Proof For each of these families, there is a member X with discrete automorphism group that admits an explicit test configuration for $(X, -K_X)$ whose central fibre is a smooth K-polystable Fano with reduced automorphism group given by $\mathbb{C}^*$. All of these are described in [5]. For the member of family number 2.20 this is given in [5, Lemma 7.2.5], for number 2.22 in [5, Lemma 7.4.2], for number 3.5 in [5, Lemma 7.5.1], for number 3.8 in [5, Lemma 7.6.1], for number 3.12 in [5, Sect. 7.7], and finally, for 4.13, the test configuration is described in [5, Corollary 5.22.3]. The statement regarding the automorphism group of X and of the central fibre is described in [5] and follows from [11]. □

6.2 One parameter families

We can also do the construction in families. We will begin by applying it to a one-parameter family of strictly K-semistable manifolds in the family 3.10. Let X_c for $c \neq \pm 1$ be the Fano threefold in the family 3.10 of the Mori–Mukai list given as follows. Endow $\mathbb{P}^4$ with homogeneous coordinates $[v, w, x, y, z]$. Let C_1 and C_2 be the curves in $\mathbb{P}^4$ given by

$$\begin{aligned} C_1 =& \{0 = w^2 + zv = y = x\}, \\ C_2 =& \{0 = w^2 + xy = v = z\}. \end{aligned}$$

These curves are disjoint and are both contained in the quadric surfaces

$$Q_c = \{w^2 + xy + zv + c(xv + yz) + xz = 0\}$$

for any c. Let $X_c = \mathrm{Bl}_{C_1,C_2} Q_c$ be the blowup of Q_c in the two curves C_1 and C_2. For all c, the reduced automorphism group of X_c is trivial [11, Lemma 5.9].

Proposition 23 *For all $c \neq 0, \pm 1$, there exists a test configuration $(\mathcal{X}_c, \mathcal{L}_c)$ for the member $(X_c, -K_{X_c})$ of the family 3.10 for which the construction of Theorem 1 applies.*

Note that the condition $c \neq \pm 1$ is made just to ensure that we have a smooth threefold. The case $c = 0$ is excluded because there is then a rank two maximal torus.

Proof This follows [5, Corollary 5.17.7], which gives an explicit test configuration for X_c, given as follows. Let

$$Q_{c,s} = \{w^2 + xy + zv + c(xv + yz) + sxz = 0\}.$$

All the $Q_{c,s}$ contain C_1 and C_2, and so we can define $X_{c,s} = \mathrm{Bl}_{C_1,C_2} Q_{c,s}$. Then for all non-zero s, $X_{c,s}$ is isomorphic to X_c. On the other hand, when $s = 0$, $Y_c = X_{c,0}$ is K-polystable [5, Lemma 5.17.6]. This gives a destabilising test configuration for $(X_s, -K_{X_s})$. Moreover, provided $c \neq 0$, by [11, Lemma 5.9], the automorphism group of Y_c is $\mathbb{C}^*$, showing the construction can be applied to all of these X_c. □

For the remaining case $c = 0$, the reduced automorphism group of Y_c is $(\mathbb{C}^*)^2$, and so the condition on the automorphism group required for the result to apply is not satisfied in this case.

For our final examples whose fibres are Fano threefolds, we will invoke Lemma 20. In the family 1.10 of Fano threefolds, there exists a one-parameter family of distinct Fano manifolds with $\mathrm{Aut}_0(X_0) \cong \mathbb{C}^*$ acting non trivially on their deformation space (see for example [22] for the explicit description of their deformation space near the Mukai–Umemura threefold). All of these admit Kähler–Einstein metrics, by works of Cheltsov and Shramov [10, Corollary 1.7] and Fujita [24, Theorem 1.2]. Similarly, we have a completely analogous picture in the family 2.21, where there also is a one-parameter family of distinct Fano manifolds with $\mathrm{Aut}_0(X_0) \cong \mathbb{C}^*$. All of these admit Kähler–Einstein metrics [5, Proposition 5.9.11]. By Lemma 20, all of the members of these two one-parameter families appear as the central fibre in some test configuration to which the construction of Theorem 1 applies. We have thus shown the following.

Lemma 24 *Each of the families 1.10 and 2.21 of Fano threefolds induce a one-parameter family of distinct manifolds to which the construction of Theorem 1 applies.*

6.3 Summary of examples arising from Fano threefolds

We summarise all the special cases we have considered in the following theorem.

Theorem 25 *The following families from the Mori–Mukai list of Fano threefolds produce at least one test configuration for a strictly K-semistable manifold to which the construction of Theorem 1 applies:*

- *1.10*
- *2.20*
- *2.21*
- *2.22*
- *3.5*
- *3.8*
- *3.10*
- *3.12*
- *4.13*

Together, they give infinitely many projective manifolds that admit extremal Kähler metrics in some classes.

Remark 7 The above result says that in each of the cases mentioned, we have found at least *one* member, to which we can apply our construction. In some cases, we have checked families rather than a single member, but we have not checked every possible member of the family in order to get a complete classification. There could very well be other members of these families from which one can obtain further examples.

Remark 8 Not all the families will give test configurations to which we can apply the construction, even if there are strictly K-semistable members of these families. For example, there are some strictly K-semistable members of the family 2.26, but they degenerate to a *singular* K-stable Fano variety. There are no K-polystable members of this family [5, Sect. 5.10], and so our construction cannot be applied to any members of this family. In the family 3.13 there is a unique strictly K-semistable member [5, Lemma 5.19.8], but the connected component of its automorphism group is the non-reductive group $(\mathbb{C}, +)$, and the central fibre has automorphism group $\mathrm{PGL}_2(\mathbb{C})$. Thus the condition on the automorphism group is not satisfied in this case.

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Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest. All data generated or analysed during this study are included in this published article (and its supplementary information files).

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