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On the blow-up criterion for the Hall-MHD problem with partial dissipation in $\mathbb{R}^3$

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Abstract

In this paper, we investigate the 3D incompressible Hall-magnetohydrodynamics with partial dissipation. Based on the results in (Du in Bound. Value Probl. 2022:6, 2022; Du and Liu in Acta Math. Sci. 42A:5, 2022; Fei and Xiang in J. Math. Phys. 56:051504, 2015), we establish an improved blow-up criterion for classical solutions. Furthermore, using the blow-up criterion, we also obtain the existence of the classical solutions only under the condition that the initial data $\|V_0\|_{H^1} + \|B_0\|_{H^2}$ are sufficiently small.

MSC: 35L60; 35K55; 35Q80

Keywords: Blow-up criterion; Hall-MHD equations; Partial dissipation; Small initial data

1 Introduction

We discuss the following Hall-magnetohydrodynamic problem in three-dimensions:

$$\begin{cases} V_t + (V\cdot\nabla)V + \nabla(p+\pi) = \rho_1 V_{x_1x_1} + \rho_2 V_{x_2x_2} + \rho_3 V_{x_3x_3} + (B\cdot\nabla)B, \\ B_t + (V\cdot\nabla)B = (B\cdot\nabla)V - \Delta B - \nabla\times((\nabla\times B)\times B), \\ DivV = 0, \qquad DivB = 0, \\ V(0,x) = V_0, \qquad B(0,x) = B_0, \end{cases} \tag{1}$$

where $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^3$, V and B denote the velocity and magnetic fields, respectively, and ρ_1, ρ_2, ρ_3 are the kinematic viscosities. The Hall-MHD equations include $\nabla\times((\nabla\times B)\times B)$ (Hall term), which is different from the MHD equations. The Hall term is used to describe magnetic reconnection, and it appears when the magnetic shear is large.

Many mathematical results for MHD equations have been obtained [11, 12, 16, 19, 22, 23, 25–30, 32]. Recently, there are some contributions on the Hall-MHD system. The paper [1] presented derivations of Hall-MHD system. Chae and Lee [5] established two optimal blow-up criterions. For more research on the Hall-MHD system, we refer to [6–9, 13, 18, 20, 21, 24]. The authors of the papers [2, 3, 15] investigated the Boussinesq (or MHD) system with partial viscosity. Fei and Xiang [21] obtained a blow-up criterion for (1) with $\rho_1 = \rho_2 = 1$ and $\rho_3 = 0$; they also proved the global small data solution. Du [13] obtained the large data existence to $2\frac{1}{2}$-dimensional Hall-MHD problem with partial dissipation, provided that the coefficients of dissipation and magnetic diffusion are sufficiently large. Paper [14] established an improved blow-up criterion in terms of BMO norm for 3D Hall-magnetohydrodynamics with partial dissipation. Du [15] obtained the global existence of the classical solutions, provided that $(\|u_0\|_{L^2}^2 + \|B_0\|_{L^2}^2)(\|\nabla u_0\|_{L^2}^2 + \|\nabla B_0\|_{L^2}^2 + \|\nabla^2 u_0\|_{L^2}^2 + \|\nabla^2 B_0\|_{L^2}^2)/Q^4$ is sufficiently small.

In this paper, we study the incompressible Hall-magnetohydrodynamics (1) with partial dissipation in three dimensions. Inspired by [2–5, 13–15, 17, 21], we establish an improved blow-up criterion for classical solutions. Furthermore, we also get the small data global well-posedness.

Theorem 1.1 *Let* $\rho_1, \rho_2 > 0$, $\rho_3 = 0$, *and* $(V_0, B_0) \in H^3(\mathbb{R}^3)$ *with* $DivV_0 = DivB_0 = 0$. *Then the following two equalities are equivalent*:

$$(a) \quad \limsup_{t \nearrow \widetilde{\mathbf{T}}} \big(\|V(t)\|_{H^3}^2 + \|B(t)\|_{H^3}^2 \big) = \infty,$$

$$(b) \quad \int_0^{\widetilde{\mathbf{T}}} \big(\|V\|_{L^s}^k + \|\nabla B\|_{L^\lambda}^\alpha \big) \, dt = \infty,$$

where $\widetilde{\mathbf{T}} < \infty$ *is the first blow-up time to* (1), *and* s, k, λ, α *satisfy*

$$\frac{3}{s} + \frac{2}{k} \le 1, \qquad \frac{3}{\lambda} + \frac{2}{\alpha} \le 1, \quad \textit{and} \quad s \in (12, \infty], \lambda \in (3, \infty].$$

Remark 1.1 Compared to [19], the blow-up criterion imposes the condition on $\int_0^{\widetilde{\mathbf{T}}} (\|V\|_{L^s}^k + \|\nabla B\|_{L^\lambda}^\alpha) \, dt < \infty$ instead of $\int_0^{\widetilde{\mathbf{T}}} (\|\nabla V\|_{L^p}^q + \|\Delta B\|_{L^\beta}^\gamma) \, dt < \infty$, where $p, \beta \in (3, \infty]$.

Based on the Theorem 1.1, we can get the following small data existence to system (1) with $\rho_1, \rho_2 > 0$ and $\rho_3 = 0$.

Theorem 1.2 *Suppose* $\rho_1, \rho_2 > 0$, $\rho_3 = 0$, $(V_0, B_0) \in H^3(\mathbb{R}^3)$, $DivV_0 = DivB_0 = 0$, *and there exists a constant* $L > 0$, *such that* $\|V_0\|_{H^1} + \|B_0\|_{H^2} < L$. *Then* (1) *has a unique classical solution* $(V, B) \in L^\infty(0, \infty; H^3(\mathbb{R}^3))$.

We can also obtain results similar to Theorems 1.1 and 1.2 for problem (1) with $\rho_2 = 0$, $\rho_1, \rho_3 > 0$ and $\rho_1 = 0$, $\rho_2, \rho_3 > 0$.

Remark 1.2 Compared to the previous results, the smallness conditions are given for $\|V_0\|_{H^1} + \|B_0\|_{H^2}$ instead of sufficiently small $\|V_0\|_{H^3} + \|B_0\|_{H^3}$ in [21] and $\|V_0\|_{H^2} + \|B_0\|_{H^2}$ in [14].

In the paper, ∂_j and V_j represent the jth components of ∇ and V, and C denotes a generic positive constant. We adopt the following simplified notation:

$$D := \nabla = (\partial_1, \partial_2, \partial_3); \qquad D_p := (\partial_1, \partial_2, 0); \qquad V_p := (V_1, V_2, 0);$$

$$\|\cdot\|_s \triangleq \|\cdot\|_{L^s}; \qquad \rho := \min\{\rho_1, \rho_2\}; \qquad \rho_0 := \min\{\rho, 1\}.$$

2 Some a priori estimates

To establish Theorems 1.1 and 1.2, we need the following lemmas.

Lemma 2.1 ([10]) *Suppose f, g, h, $D_p f$, $D_p g$, and $\partial_3 h$ are all in $L^2(\mathbb{R}^3)$. Then*

$$\int_{\mathbb{R}^3} |fgh|\,dx \le C\|f\|_2^{\frac{1}{2}}\|D_p f\|_2^{\frac{1}{2}}\|g\|_2^{\frac{1}{2}}\|D_p g\|_2^{\frac{1}{2}}\|h\|_2^{\frac{1}{2}}\|\partial_3 h\|_2^{\frac{1}{2}}.$$

Lemma 2.2 ([31]) *Let f, g, h, $\partial_1 g$, $\partial_2 g$, $\partial_2 h$, and $\partial_3 h$ be all in $L^2(\mathbb{R}^3)$. Then*

$$\int_{\mathbb{R}^3} |fgh|\,dx \le C\|f\|_2\|\partial_1 g\|_2^{\frac{1}{2}}\|\partial_3 h\|_2^{\frac{1}{2}}\|g\|_2^{\frac{1}{4}}\|h\|_2^{\frac{1}{4}}\|\partial_2 g\|_2^{\frac{1}{4}}\|\partial_2 h\|_2^{\frac{1}{4}}.$$

Let $\rho_1 > 0$, $\rho_2 > 0$, and $\rho_3 = 0$, Taking the scalar products of $(1)_1$ and $(1)_2$ with V and B, we get

$$\frac{1}{2}\frac{d}{dt}\left(\|V(t)\|_2^2 + \|B(t)\|_2^2\right) + \rho_0\left(\|D_p V\|_2^2 + \|DB\|_2^2\right) = 0. \tag{2}$$

Integrating (2) over $(0,T)$, we get

$$\|V\|_2^2 + \|B\|_2^2 + 2\rho_0\int_0^T \left(\|D_p V(t)\|_2^2 + \|DB(t)\|_2^2\right)dt = \|V_0\|_2^2 + \|B_0\|_2^2. \tag{3}$$

Proposition 2.1 *Assume that $\rho_1 > 0$, $\rho_2 > 0$, $\rho_3 = 0$, and (V,B) is a solution to (1). Then*

$$\begin{aligned}&\frac{d}{dt}\left(\|DV\|_2^2 + \|DB\|_2^2\right) + \rho_0\left(\|DD_p V\|_2^2 + \|D^2 B\|_2^2\right)\\ &\quad\le C\left(\|DV\|_2^2 + \|DB\|_2^2\right)\left(\|V\|_s^{\frac{2s}{s-6}} + \|DB\|_\lambda^{\frac{2\lambda}{\lambda-3}}\right).\end{aligned} \tag{4}$$

Proof We operate D to $(1)_1$ and $(1)_2$. Then taking the inner product of them with DV and DB, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\left(\|DV(t)\|_2^2 + \|DB(t)\|_2^2\right) + \rho_1\|\partial_1 DV\|_2^2 + \rho_2\|\partial_2 DV\|_2^2 + \|D^2 B\|_2^2\\ &\quad= -\int_{\mathbb{R}^3} D(V\cdot DB)\cdot DB\,dx + \int_{\mathbb{R}^3} D(B\cdot DV)\cdot DB\,dx + \int_{\mathbb{R}^3} D(B\cdot DB)\cdot DV\,dx\\ &\qquad - \int_{\mathbb{R}^3} D(V\cdot DV)\cdot DV\,dx - \int_{\mathbb{R}^3} D\left[D\times\left((D\times B)\times B\right)\right]\cdot DB\,dx\\ &\quad= M_1 + M_2 + M_3 + M_4 + M_5.\end{aligned} \tag{5}$$

Firstly, we use $DivV = 0$, integration by parts, and interpolation to deduce that

$$\begin{aligned}|M_1| &= \left|\int_{\mathbb{R}^3} (V\cdot DB)\cdot D^2 B\,dx\right|\\ &\le C\|V\|_s\|DB\|_{\frac{2s}{s-2}}\|D^2 B\|_2\\ &\le C\|V\|_s\|DB\|_2^{\frac{s-3}{s}}\|D^2 B\|_2^{\frac{s+3}{s}}\\ &\le C\|V\|_s^{\frac{2s}{s-3}}\|DB\|_2^2 + \frac{1}{10}\|D^2 B\|_2^2.\end{aligned} \tag{6}$$

By cancelation property and integration by parts we rewrite $M_2 + M_3$ as follows:

$$M_2 + M_3 = -\sum_{j=1}^{3} \int_{\mathbb{R}^3} 2V(\partial_j B \cdot D)\partial_j B + V\big(\partial_j^2 B \cdot D\big)B + V(\partial_j B \cdot D)\partial_j B \, dx.$$

Hence, similarly to (6), we get

$$|M_2 + M_3| \le C\|V\|_s^{\frac{2s}{s-3}} \|DB\|_2^2 + \frac{3}{10}\big\|D^2 B\big\|_2^2. \tag{7}$$

We can decompose M_4 into three terms:

$$\begin{aligned} M_4 &= -\int_{\mathbb{R}^3} (D_p V \cdot D)V D_p V \, dx - \int_{\mathbb{R}^3} (\partial_3 V_p \cdot D_p)V \partial_3 V \, dx + \int_{\mathbb{R}^3} (D_p \cdot V_p)\partial_3 V \partial_3 V \, dx \\ &= M_{41} + M_{42} + M_{43}. \end{aligned} \tag{8}$$

Integrating by parts M_{41} and M_{42}, we have

$$\begin{aligned} |M_{41}| &= 2\left|\int_{\mathbb{R}^3} V D_p V D D_p V \, dx\right| \\ &\le C\|V\|_s \|D_p V\|_{\frac{2s}{s-2}} \|D D_p V\|_2 \\ &\le C\|V\|_s \|DV\|_2^{\frac{s-3}{s}} \|D D_p V\|_2^{\frac{s+3}{s}} \\ &\le C\|V\|_s^{\frac{2s}{s-3}} \|DV\|_2^2 + \frac{\rho}{6}\|D D_p V\|_2^2 \end{aligned}$$

and

$$\begin{aligned} |M_{42}| &= \left|\int_{\mathbb{R}^3} \partial_3 D_p V_p V \partial_3 V + \partial_3 V_p V \partial_3 D_p V \, dx\right| \\ &\le C\|V\|_s \|DV\|_{\frac{2s}{s-2}} \|D D_p V\|_2 \\ &\le C\|V\|_s^{\frac{s}{s-6}} \|DV\|_2^{\frac{s-12}{s-6}} \|D D_p V\|_2 \\ &\le C\|V\|_s^{\frac{2s}{s-6}} \|DV\|_2^2 + \frac{\rho}{6}\|D D_p V\|_2^2. \end{aligned}$$

Similarly, we have

$$|M_{43}| \le C\|V\|_s^{\frac{2s}{s-6}} \|DV\|_2^2 + \frac{\rho}{6}\|D D_p V\|_2^2.$$

Collecting the above estimates, we get

$$|M_4| \le C\|V\|_s^{\frac{2s}{s-6}} \|DV\|_2^2 + \frac{\rho}{2}\|D D_p V\|_2^2. \tag{9}$$

Applying cancelation property and interpolation inequality, we get

$$\begin{aligned} |M_5| &= \left|\int_{\mathbb{R}^3} \big[D\big((\nabla \times B) \times B\big) - D(\nabla \times B) \times B\big] \cdot D(\nabla \times B)\,dx\right| \\ &\leq C\|DB\|_\lambda \|DB\|_{\frac{2\lambda}{\lambda-2}} \big\|D^2B\big\|_2 \\ &\leq C\|DB\|_\lambda \|DB\|_2^{\frac{\lambda-3}{\lambda}} \big\|D^2B\big\|_2^{\frac{\lambda+3}{\lambda}} \\ &\leq C\|DB\|_\lambda^{\frac{2\lambda}{\lambda-3}} \|DB\|_2^2 + \frac{1}{10}\big\|D^2B\big\|_2^2. \end{aligned} \tag{10}$$

Combining (5)–(10) yields (4). □

Applying Gronwall's inequality to (4), we get the following inequality:

$$\begin{aligned} &\sup_{0<t<T}\big(\big\|DV(t)\big\|_2^2 + \big\|DB(t)\big\|_2^2\big) + \rho_0 \int_0^T \big(\big\|DD_pV(t)\big\|_2^2 + \big\|D^2B(t)\big\|_2^2\big)\,dt \\ &\quad\leq \big(\|DV_0\|_2^2 + \|DB_0\|_2^2\big)(1 + C\int_0^T \big(\big\|V(t)\big\|_s^{\frac{2s}{s-6}} + \big\|DB(t)\big\|_\lambda^{\frac{2\lambda}{\lambda-3}}\big)\,dt \\ &\quad\quad\times \exp\bigg(C\int_0^T \big(\big\|V(t)\big\|_k^{\frac{2s}{s-6}} + \big\|DB(t)\big\|_\lambda^{\frac{2\lambda}{\lambda-3}}\big)\,dt\bigg). \end{aligned} \tag{11}$$

Proposition 2.2 *Suppose the conditions in Proposition 2.1 hold. Then*

$$\begin{aligned} &\frac{d}{dt}\big(\big\|\Delta V(t)\big\|_2^2 + \big\|\Delta B(t)\big\|_2^2\big) + \rho_0\big(\|\Delta D_pV\|_2^2 + \big\|D^3B\big\|_2^2\big) \\ &\quad\leq C\big(\|\Delta V\|_2^2 + \|\Delta B\|_2^2\big)\big(\|V\|_s^{\frac{2s}{s-3}} + \|DB\|_\lambda^{\frac{2\lambda}{\lambda-3}} + \|DV\|_2^6\big). \end{aligned} \tag{12}$$

Proof Similarly to (5), we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big(\big\|D^2V(t)\big\|_2^2 + \big\|D^2B(t)\big\|_2^2\big) + \rho_1\big\|\partial_1D^2V\big\|_2^2 + \rho_2\big\|\partial_2D^2V\big\|_2^2 + \big\|D^3B\big\|_2^2 \\ &\quad= -\int_{\mathbb{R}^3} D^2(V\cdot DB)\cdot D^2B\,dx + \int_{\mathbb{R}^3} D^2(B\cdot DV)\cdot D^2B\,dx + \int_{\mathbb{R}^3} D^2(B\cdot DB)\cdot D^2V\,dx \\ &\quad\quad- \int_{\mathbb{R}^3} D^2(V\cdot DV)\cdot D^2V\,dx - \int_{\mathbb{R}^3} D^2\big[D\times\big((D\times B)\times B\big)\big]\cdot D^2B\,dx \\ &\quad= R_1 + R_2 + R_3 + R_4 + R_5. \end{aligned} \tag{13}$$

We decompose R_1 into two parts:

$$\begin{aligned} R_1 &= -\int_{\mathbb{R}^3} \big(D^2V\cdot D\big)B\cdot D^2B\,dx - 2\int_{\mathbb{R}^3} (DV\cdot D)DB\cdot D^2B\,dx \\ &= R_{11} + R_{12}. \end{aligned} \tag{14}$$

Further, we decompose R_{11} into two parts:

$$R_{11} = \int_{\mathbb{R}^3} (DV\cdot D)DB\cdot D^2B\,dx + \int_{\mathbb{R}^3} (DV\cdot D)B\cdot D^3B\,dx = R_{111} + R_{112}.$$

Applying integration by parts and interpolation, we have

$$\begin{aligned}
|R_{111}| &= \left| \int_{\mathbb{R}^3} (V\cdot D)DB\cdot D^3B + (V\cdot D)D^2B\cdot D^2B\,dx \right| \\
&\le C\|V\|_s \|\Delta B\|_{\frac{2s}{s-2}} \left\|D^3B\right\|_2 \\
&\le C\|V\|_s \|\Delta B\|_2^{\frac{s-3}{s}} \left\|D^3B\right\|_2^{\frac{s+3}{s}} \\
&\le C\|V\|_s^{\frac{2s}{s-3}} \|\Delta B\|_2^2 + \frac{1}{22}\left\|D^3B\right\|_2^2
\end{aligned}$$

and

$$|R_{12}| \le C\|V\|_s^{\frac{2s}{s-3}} \|\Delta B\|_2^2 + \frac{1}{22}\left\|D^3B\right\|_2^2.$$

We estimate R_{112} as

$$\begin{aligned}
|R_{112}| &\le C\|DV\|_2 \|DB\|_{L^\infty} \left\|D^3B\right\|_2 \\
&\le C\|DV\|_2 \|\Delta B\|_2^{\frac{1}{2}} \left\|D^3B\right\|_2^{\frac{3}{2}} \\
&\le C\|DV\|_2^4 \|\Delta B\|_2^2 + \frac{1}{22}\left\|D^3B\right\|_2^2.
\end{aligned}$$

Hence we have

$$|R_1| \le C\left(\|V\|_s^{\frac{2s}{s-3}} + \|DV\|_2^4\right)\|\Delta B\|_2^2 + \frac{3}{22}\left\|D^3B\right\|_2^2. \tag{15}$$

We can rewrite $R_2 + R_3$ as

$$\begin{aligned}
R_2 + R_3 &= \int_{\mathbb{R}^3} \left(D^2B\cdot D\right)B\cdot D^2V\,dx + 2\int_{\mathbb{R}^3} (DB\cdot D)DB\cdot D^2V\,dx \\
&\quad + \int_{\mathbb{R}^3} \left(D^2B\cdot DV\right)\cdot D^2B\,dx + 2\int_{\mathbb{R}^3} (DB\cdot D)DV\cdot D^2B\,dx \\
&= R_{231} + R_{232} + R_{233} + R_{234}.
\end{aligned} \tag{16}$$

The terms R_{231}, R_{232}, R_{234}, and R_{233} can be estimated as R_{11}, R_{12}. Hence we have

$$|R_2 + R_3| \le C\left(\|V\|_s^{\frac{2s}{s-3}} + \|DV\|_2^4\right)\|\Delta B\|_2^2 + \frac{7}{22}\left\|D^3B\right\|_2^2. \tag{17}$$

The term R_4 can be written as

$$R_4 = -\int_{\mathbb{R}^3} \left(D^2u\cdot D\right)u\cdot D^2u\,dx - 2\int_{\mathbb{R}^3} (Du\cdot D)Du\cdot D^2u\,dx = R_{41} + R_{42}. \tag{18}$$

We further decompose R_{41} and R_{42} into three parts:

$$\begin{aligned}
R_{41} &= -\int_{\mathbb{R}^3} (D_pDV\cdot D)V\cdot D_pDV\,dx - \int_{\mathbb{R}^3} \left(\partial_3^2V_p\cdot D_p\right)V\cdot \partial_3^2V\,dx \\
&\quad + \int_{\mathbb{R}^3} (\partial_3D_p\cdot V_p)\partial_3V\cdot \partial_3^2V\,dx \\
&= R_{411} + R_{412} + R_{413},
\end{aligned} \tag{19}$$

$$
\begin{aligned}
R_{42} &= -2\int_{\mathbb{R}^3}(D_pV\cdot D)DV\cdot D_pDV\,dx - 2\int_{\mathbb{R}^3}(\partial_3 V_p\cdot D_p)DV\cdot\partial_3^2 V\,dx \\
&\quad + 2\int_{\mathbb{R}^3}(D_p\cdot V_p)\partial_3 DV\cdot\partial_3^2 V\,dx \\
&= R_{421} + R_{422} + R_{423}.
\end{aligned}
\tag{20}
$$

By integration by parts and interpolation we obtain

$$
\begin{aligned}
|R_{411}| &= 2\left|\int_{\mathbb{R}^3} VDD_pVD^2D_pV\,dx\right| \\
&\le C\|V\|_s\|DD_pV\|_{\frac{2s}{s-2}}\|\Delta D_pV\|_2 \\
&\le C\|V\|_s\|DD_pV\|_2^{\frac{s-3}{s}}\|\Delta D_pV\|_2^{\frac{s+3}{s}} \\
&\le +C\|V\|_s^{\frac{2s}{s-3}}\|DD_pV\|_2^2 + \frac{\rho}{12}\|\Delta D_pV\|_2^2
\end{aligned}
$$

By integration by parts and Lemma 2.2 we get

$$
\begin{aligned}
|R_{412}| &= \left|\int_{\mathbb{R}^3}\partial_3^2 D_pV_pV\cdot\partial_3^2 V + \partial_3^2 V_pV\cdot\partial_3^2 D_pV\,dx\right| \\
&\le C\left\|\partial_3^2 D_pV\right\|_2\|\partial_3 V\|_2^{\frac{1}{2}}\left\|\partial_1\partial_3^2 V\right\|_2^{\frac{1}{2}}\|V\|_2^{\frac{1}{4}}\left\|\partial_3^2 V\right\|_2^{\frac{1}{4}}\|\partial_2 V\|_2^{\frac{1}{4}}\left\|\partial_2\partial_3^2 V\right\|_2^{\frac{1}{4}} \\
&\le C\|\Delta D_pV\|_2^{\frac{7}{4}}\|DV\|_2^{\frac{3}{4}}\|\Delta V\|_2^{\frac{1}{4}} \\
&\le C\|DV\|_2^6\|\Delta V\|_2^2 + \frac{\rho}{12}\|\Delta D_pV\|_2^2.
\end{aligned}
$$

We can use Lemma 2.1 to estimate R_{313} as follows:

$$
\begin{aligned}
|R_{413}| &\le C\|\partial_3 D_pV\|_2^{\frac{1}{2}}\|\partial_3 V\|_2^{\frac{1}{2}}\left\|\partial_3^2 V\right\|_2^{\frac{1}{2}}\left\|\partial_3^2 D_pV\right\|_2^{\frac{1}{2}}\|\partial_3 D_pV\|_2^{\frac{1}{2}}\left\|\partial_3^2 D_pV\right\|_2^{\frac{1}{2}} \\
&\le C\left\|D^2D_pV\right\|_2\|DD_pV\|_2\|DV\|_{L^2}^{\frac{1}{2}}\left\|D^2V\right\|_2^{\frac{1}{2}} \\
&\le C\left\|D^2D_pV\right\|_2^{\frac{3}{2}}\|DV\|_2\left\|D^2V\right\|_2^{\frac{1}{2}} \\
&\le C\|DV\|_2^4\|\Delta V\|_2^2 + \frac{\rho}{12}\left\|D^2D_pV\right\|_2^2.
\end{aligned}
$$

Therefore we have

$$
|R_{41}| \le C\left(\|DV\|_2^6 + \|V\|_s^{\frac{2s}{s-3}}\right)\|\Delta V\|_2^2 + \frac{3\rho}{12}\|\Delta D_pV\|_2^2.
$$

Clearly, R_{421}, R_{422}, R_{423} can be estimated as R_{411}, R_{413}, R_{412}. Hence we have

$$
|R_{42}| \le C\left(\|DV\|_2^6 + \|V\|_s^{\frac{2s}{s-3}}\right)\|\Delta V\|_2^2 + \frac{3\rho}{12}\|\Delta D_pV\|_2^2.
$$

Therefore we get

$$
|R_4| \le C\left(\|DV\|_2^6 + \|V\|_s^{\frac{2s}{s-3}}\right)\|\Delta V\|_2^2 + \frac{\rho}{2}\|\Delta D_pV\|_2^2. \tag{21}
$$

Using the cancelation property, we obtain

$$\begin{aligned}|R_5| &= \left|\int_{\mathbb{R}^3}\left[D^2\big((\nabla\times B)\times B\big) - D^2(\nabla\times B)\times B\right]\cdot D^2(\nabla\times B)\,dx\right| \\ &\le C\left\|D^3B\right\|_2\|DB\|_\lambda\left\|D^2B\right\|_{\frac{2\lambda}{\lambda-2}} \\ &\le C\left\|D^3B\right\|_2^{\frac{\lambda+3}{\lambda}}\|DB\|_\lambda\left\|D^2B\right\|_2^{\frac{\lambda-3}{\lambda}} \\ &\le C\|DB\|_\lambda^{\frac{2\lambda}{\lambda-3}}\left\|D^2B\right\|_2^2 + \frac{1}{22}\left\|D^3B\right\|_2^2.\end{aligned} \tag{22}$$

Combining (13)–(22), we get (12). □

Proposition 2.3 *Let* (V,B) *solve system* (1) *with* $\rho_1 > 0$, $\rho_2 > 0$, *and* $\rho_3 = 0$. *Then*

$$\begin{aligned}&\frac{d}{dt}\big(\left\|D^3V(t)\right\|_2^2 + \left\|D^3B(t)\right\|_2^2\big) + \rho_0\big(\left\|D^3D_pV\right\|_2^2 + \left\|D^4B\right\|_2^2\big) \\ &\quad\le C\big(\left\|D^3V\right\|_2^2 + \left\|D^3B\right\|_2^2\big)\big(\|V\|_s^{\frac{2s}{s-3}} + \|DB\|_\lambda^{\frac{2\lambda}{\lambda-3}} + \|DV\|_2^6 + \|\Delta B\|_2^2\big).\end{aligned} \tag{23}$$

Proof Similarly to the derivation of (5), we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\left\|D^3V(t)\right\|_2^2 + \left\|D^3B(t)\right\|_2^2\big) + \rho_1\left\|\partial_1D^3V\right\|_2^2 + \rho_2\left\|\partial_2D^3V\right\|_2^2 + \left\|D^4B\right\|_2^2 \\ &\quad= -\int_{\mathbb{R}^3}D^3(V\cdot DB)\cdot D^3B\,dx + \int_{\mathbb{R}^3}D^3(B\cdot DV)\cdot D^3B\,dx + \int_{\mathbb{R}^3}D^3(B\cdot DB)\cdot D^3V\,dx \\ &\qquad -\int_{\mathbb{R}^3}D^3(V\cdot DV)\cdot D^3V\,dx - \int_{\mathbb{R}^3}D^3\big[D\times\big((D\times B)\times B\big)\big]\cdot D^3B\,dx \\ &\quad= T_1 + T_2 + T_3 + T_4 + T_5.\end{aligned} \tag{24}$$

We decompose T_1 into three parts:

$$\begin{aligned}T_1 &= -\int_{\mathbb{R}^3}\big(D^3V\cdot D\big)B\cdot D^3B\,dx - 3\int_{\mathbb{R}^3}\big(D^2V\cdot D\big)DB\cdot D^3B\,dx \\ &\quad - 3\int_{\mathbb{R}^3}(DV\cdot D)D^2B\cdot D^3B\,dx \\ &= T_{11} + T_{12} + T_{13}.\end{aligned}$$

We apply the Hölder inequality and interpolation to estimate T_{11}:

$$\begin{aligned}|T_{11}| &\le C\left\|D^3V\right\|_2\|DB\|_3\left\|D^3B\right\|_6 \\ &\le C\left\|D^3V\right\|_2\|\Delta B\|_2^{\frac{3}{4}}\|B\|_2^{\frac{1}{4}}\left\|D^4B\right\|_2 \\ &\le C\|\Delta B\|_2^2\left\|D^3V\right\|_2^2 + \frac{1}{20}\left\|D^4B\right\|_2^2.\end{aligned}$$

Using integration by parts and $DivV = 0$, we get

$$\begin{aligned}|T_{12}| &= 3\left|\int_{\mathbb{R}^3} D^2VDB\cdot D^4B\,dx\right|\\ &\le C\|D^4B\|_2\|\Delta V\|_6\|DB\|_3\\ &\le C\|D^4B\|_2\|D^3V\|_2\|\Delta B\|_2^{\frac{3}{4}}\|B\|_2^{\frac{1}{4}}\\ &\le C\|\Delta B\|_2^2\|D^3V\|_2^2+\frac{1}{20}\|D^4B\|_2^2.\end{aligned}$$

We used the boundedness of $\|B\|_{L^2}$ in the above two estimates. Similarly, we estimate T_{13} as follows:

$$\begin{aligned}|T_{13}| &= 3\left|\int_{\mathbb{R}^3}(V\cdot D)D^2B\cdot D^4B+(V\cdot D)D^3B\cdot D^3B\,dx\right|\\ &\le C\|V\|_s\|D^3B\|_{\frac{2s}{s-2}}\|D^4B\|_2\\ &\le C\|V\|_s\|D^3B\|_2^{\frac{s-3}{s}}\|D^4B\|_2^{\frac{s+3}{s}}\\ &\le C\|V\|_s^{\frac{2s}{s-3}}\|D^3B\|_2^2+\frac{1}{20}\|D^4B\|_2^2.\end{aligned}$$

Hence we get

$$|T_1|\le C\big(\|V\|_s^{\frac{2s}{s-3}}+\|\Delta B\|_2^2\big)\big(\|D^3V\|_2^2+\|D^3B\|_2^2\big)+\frac{3}{20}\|D^4B\|_2^2. \tag{25}$$

We decompose $T_2 + T_3$ into six terms:

$$\begin{aligned}T_2+T_3 &= \int_{\mathbb{R}^3}\big(D^3B\cdot D\big)B\cdot D^3V\,dx+3\int_{\mathbb{R}^3}\big(D^2B\cdot D\big)DB\cdot D^3V\,dx\\ &\quad+3\int_{\mathbb{R}^3}(DB\cdot D)D^2B\cdot D^3V\,dx+\int_{\mathbb{R}^3}\big(D^3B\cdot D\big)V\cdot D^3B\,dx\\ &\quad+3\int_{\mathbb{R}^3}\big(D^2B\cdot D\big)DV\cdot D^3B\,dx+3\int_{\mathbb{R}^3}(DB\cdot D)D^2V\cdot D^3B\,dx\\ &= T_{231}+T_{232}+T_{233}+T_{234}+T_{235}+T_{236}.\end{aligned}$$

Integrating by parts T_{232}, we get

$$T_{232}=-3\int_{\mathbb{R}^3}\big(D^3B\cdot D\big)DB\cdot D^2V\,dx-3\int_{\mathbb{R}^3}\big(D^2B\cdot D\big)D^2B\cdot D^2V\,dx.$$

Therefore $(T_{231}, T_{233}, T_{236})$, (T_{232}, T_{235}), and T_{234} can be estimated as T_{11}, T_{12}, T_{13}. Hence we get

$$|T_2+T_3|\le C\big(\|V\|_s^{\frac{2s}{s-3}}+\|\Delta B\|_2^2\big)\big(\|D^3V\|_2^2+\|D^3B\|_2^2\big)+\frac{6}{20}\|D^4B\|_2^2. \tag{26}$$

We split T_4 into three terms:

$$\begin{aligned} T_4 &= -\int_{\mathbb{R}^3} \big(D^3V\cdot D\big)VD^3V\,dx - 3\int_{\mathbb{R}^3}\big(D^2V\cdot D\big)DVD^3V\,dx \\ &\quad - 3\int_{\mathbb{R}^3}(DV\cdot D)D^2VD^3V\,dx \\ &= T_{41}+T_{42}+T_{43}, \end{aligned}$$

and T_{41} and T_{42} can be further decomposed into three parts as follows:

$$\begin{aligned} T_{41} &= -\int_{\mathbb{R}^3}\big(D^2D_pV\cdot D\big)VD^2D_pV\,dx - \int_{\mathbb{R}^3}\big(\partial_3^3V_p\cdot D_p\big)V\partial_3^3V\,dx \\ &\quad + \int_{\mathbb{R}^3}\big(\partial_3^2D_p\cdot V_p\big)\partial_3V\partial_3^3V\,dx \\ &= T_{411}+T_{412}+T_{413}, \\ T_{42} &= -3\int_{\mathbb{R}^3}(DD_pV\cdot D)DVD^2D_pV\,dx - 3\int_{\mathbb{R}^3}\big(\partial_3^2V_p\cdot D_p\big)\partial_3V\partial_3^3V\,dx \\ &\quad + 3\int_{\mathbb{R}^3}(\partial_3D_p\cdot V_p)\partial_3^2V\partial_3^3V\,dx \\ &= T_{421}+T_{422}+T_{423}. \end{aligned}$$

Integrating by parts in T_{411} and applying the Hölder inequality, we get

$$\begin{aligned} |T_{411}| &= 2\left|\int_{\mathbb{R}^3}(VD^2D_pVD^3D_pV\,dx\right| \\ &\le C\|V\|_s\big\|D^2D_pV\big\|_{\frac{2s}{s-2}}\big\|D^3D_pV\big\|_2 \\ &\le C\|V\|_s\big\|D^2D_pV\big\|_2^{\frac{s-3}{s}}\big\|D^3D_pV\big\|_2^{\frac{s+3}{s}} \\ &\le C\|V\|_s^{\frac{2s}{s-3}}\big\|D^3V\big\|_2^2+\frac{\rho}{22}\big\|D^3D_pV\big\|_2^2. \end{aligned}$$

Integrating by parts T_{412}, we have

$$\begin{aligned} |T_{412}| &= \left|\int_{\mathbb{R}^3}V\partial_3^3D_pV_p\partial_3^3V+V\partial_3^3V_p\partial_3^3D_pV\,dx\right| \\ &\le C\big\|\partial_3^3D_pV\big\|_2\|\partial_3V\|_2^{\frac12}\big\|\partial_1\partial_3^3V\big\|_2^{\frac12}\|V\|_2^{\frac14}\big\|\partial_3^3V\big\|_2^{\frac14}\|\partial_2V\|_2^{\frac14}\big\|\partial_2\partial_3^3V\big\|_2^{\frac14} \\ &\le C\big\|D^3D_pV\big\|_2^{\frac74}\|DV\|_2^{\frac34}\big\|D^3V\big\|_2^{\frac14} \\ &\le C\|DV\|_2^6\big\|D^3V\big\|_2^2+\frac{\rho}{22}\big\|D^3D_pV\big\|_2^2. \end{aligned}$$

By Lemma 2.1 we have

$$\begin{aligned} |T_{413}| &\le C\big\|\partial_3^2D_pV\big\|_2^{\frac12}\|\partial_3V\|_2^{\frac12}\big\|\partial_3^3V\big\|_2^{\frac12}\big\|\partial_3^3D_pV\big\|_2^{\frac12}\|\partial_3D_pV\|_2^{\frac12}\big\|\partial_3^3D_pV\big\|_2^{\frac12} \\ &\le C\big\|D^3D_pV\big\|_2\big\|D^2D_pV\big\|_{L^2}^{\frac12}\|DD_pV\|_2^{\frac12}\|DV\|_2^{\frac12}\big\|D^3V\big\|_2^{\frac12} \end{aligned}$$

$$
\begin{aligned}
&\le C\|D^3D_pV\|_2^{\frac{3}{2}}\|DV\|_2\|D^3V\|_2^{\frac{1}{2}}\\
&\le C\|DV\|_2^4\|D^3V\|_2^2+\frac{\rho}{22}\|D^3D_pV\|_2^2.
\end{aligned}
$$

Therefore

$$
|T_{41}|\le C\big(\|DV\|_2^6+\|V\|_s^{\frac{2s}{s-3}}\big)\|D^3V\|_2^2+\frac{3\rho}{22}\|D^3D_pV\|_2^2.
$$

We can use Lemma 2.1 to estimate T_{421} as follows:

$$
\begin{aligned}
|T_{421}|&\le C\|DD_pV\|_2^{\frac{1}{2}}\|D^2V\|_2^{\frac{1}{2}}\|D^2D_pV\|_2^{\frac{1}{2}}\|\partial_3DD_pV\|_2^{\frac{1}{2}}\|D^2D_pV\|_2^{\frac{1}{2}}\|D^2D_p^2V\|_2^{\frac{1}{2}}\\
&\le C\|D^3D_pV\|_2^{\frac{1}{2}}\|D^2D_pV\|_2^{\frac{3}{2}}\|DD_pV\|_2^{\frac{1}{2}}\|D^2V\|_2^{\frac{1}{2}}\\
&\le C\|D^3D_pV\|_2^{\frac{5}{3}}\|D_pV\|_2^{\frac{5}{6}}\|D^3V\|_2^{\frac{1}{3}}\|V\|_2^{\frac{1}{6}}\\
&\le C\|DV\|_2^5\|D^3V\|_2^2+\frac{\rho}{22}\|D^3D_pV\|_2^2,
\end{aligned}
$$

where we used the boundedness of $\|u\|_2$. We further divide T_{422} into two terms:

$$
\begin{aligned}
T_{422}&=-3\int_{\mathbb{R}^3}\big(\partial_3^2V_p\cdot D_p\big)\partial_3V_p\partial_3^3V_p\,dx-3\int_{\mathbb{R}^3}\big(\partial_3^2V_p\cdot D_p\big)\partial_3V_3\partial_3^3V_3\,dx\\
&=\frac{3}{2}\int_{\mathbb{R}^3}\big(\partial_3^2V_p\cdot D_p\big)\partial_3^2V_p\partial_3^2V_p\,dx+3\int_{\mathbb{R}^3}\big(\partial_3^2V\cdot D_p\big)\partial_3V_3\big(\partial_3^2D_p\cdot V_p\big)\,dx\\
&=T_{4221}+T_{4222}.
\end{aligned}
$$

We estimate T_{4221} as

$$
\begin{aligned}
|T_{4221}|&\le C\|\partial_3^2V\|_2^{\frac{1}{2}}\|\partial_3^2D_pV\|_2^{\frac{1}{2}}\|\partial_3^2V\|_2^{\frac{1}{2}}\|\partial_3^2D_pV\|_2^{\frac{1}{2}}\|\partial_3^3D_pV\|_2^{\frac{1}{2}}\|\partial_3^2D_pV\|_2^{\frac{1}{2}}\\
&\le C\|D^3D_pV\|_2^{\frac{1}{2}}\|D^2D_pV\|_2^{\frac{3}{2}}\|D^2V\|_2\\
&\le C\|D^3D_pV\|_2^{\frac{3}{2}}\|DV\|_2\|D^3V\|_2^{\frac{1}{2}}\\
&\le C\|DV\|_2^4\|D^3V\|_2^2+\frac{\rho}{22}\|D^3D_pV\|_2^2.
\end{aligned}
$$

We can estimate T_{4222} as T_{421}:

$$
|T_{4222}|\le C\|DV\|_2^5\|D^3V\|_2^2+\frac{\rho}{22}\|D^3D_pV\|_2^2.
$$

Similarly,

$$
|T_{423}|\le C\|DV\|_2^5\|D^3V\|_2^2+\frac{2\rho}{22}\|D^3D_pV\|_2^2.
$$

Hence we get

$$
|T_{42}|\le C\|DV\|_2^5\|D^3V\|_2^2+\frac{5\rho}{22}\|D^3D_pV\|_2^2.
$$

We can estimate T_{43} as T_{41}:

$$|T_{43}| \leq C\big(\|DV\|_2^6 + \|V\|_s^{\frac{2s}{s-3}}\big)\|D^3V\|_2^2 + \frac{3\rho}{22}\|D^3D_pV\|_2^2.$$

Putting the above estimates together, we obtain

$$|T_4| \leq C\big(\|DV\|_2^6 + \|V\|_s^{\frac{2s}{s-3}}\big)\|D^3V\|_2^2 + \frac{\rho}{2}\|D^3D_pV\|_2^2. \tag{27}$$

Similarly to (16), we get

$$\begin{aligned}
|T_5| &= \left|\int_{\mathbb{R}^3}\big[D^3\big((\nabla\times B)\times B\big) - D^3(\nabla\times B)\times B\big]\cdot D^3(\nabla\times B)\,dx\right| \\
&\leq C\|DB\|_\lambda\|D^3B\|_{\frac{2\lambda}{\lambda-2}}\|D^4B\|_2 \\
&\leq C\|DB\|_\lambda\|D^3B\|_2^{\frac{\lambda-3}{\beta}}\|D^4B\|_2^{\frac{\lambda+3}{\lambda}} \\
&\leq C\|DB\|_\lambda^{\frac{2\lambda}{\lambda-3}}\|D^3B\|_2^2 + \frac{1}{20}\|D^4B\|_2^2.
\end{aligned} \tag{28}$$

Combining (24)–(28) yields (23). □

3 Proof of Theorem 1.1

Putting (2), (4), (12), and (23) together, we get

$$\begin{aligned}
&\frac{d}{dt}\big(\|V(t)\|_{H^3}^2 + \|B(t)\|_{H^3}^2\big) + \rho_0\big(\|D_pV\|_{H^3}^2 + \|DB\|_{H^3}^2\big) \\
&\quad\leq C\big(\|V\|_s^{\frac{2s}{s-3}} + \|DB\|_\lambda^{\frac{2\lambda}{\lambda-3}} + \|DV\|_2^6 + \|D^2B\|_2^2\big)\big(\|V\|_{H^3}^2 + \|B\|_{H^3}^2\big).
\end{aligned}$$

Applying Gronwall's inequality to this inequality, we obtain

$$\begin{aligned}
&\sup_{0<t<T}\big(\|V(t)\|_{H^3}^2 + \|B(t)\|_{H^3}^2\big) + \rho_0\int_0^T\big(\|D_pV(t)\|_{H^3}^2 + \|DB(t)\|_{H^3}^2\big)\,dt \\
&\quad\leq \big(\|V_0\|_{H^3}^2 + \|B_0\|_{H^3}^2\big) \\
&\qquad\times\Big(1 + C\int_0^T\big(\|V(t)\|_s^{\frac{2s}{s-6}} + \|DB(t)\|_\lambda^{\frac{2\lambda}{\lambda-3}} + \|DV(t)\|_2^6 + \|D^2B(t)\|_2^2\big)\,dt\Big) \\
&\qquad\times\exp\Big[C\int_0^T\big(\|V(t)\|_s^{\frac{2s}{s-6}} + \|DB(t)\|_\lambda^{\frac{2\lambda}{\lambda-3}} + \|DV(t)\|_2^6 + \|D^2B(t)\|_2^2\big)\,dt\Big].
\end{aligned}$$

which, together with (11), gives that if

$$\int_0^{\widetilde{\mathrm{T}}}\big(\|V(t)\|_s^{\frac{2s}{s-6}} + \|DB(t)\|_\lambda^{\frac{2\lambda}{\lambda-3}}\big)\,dt < \infty,$$

then

$$\sup_{0<t<\widetilde{\mathrm{T}}}\big(\|V(t)\|_{H^3}^2 + \|B(t)\|_{H^3}^2\big) + \rho_0\int_0^{\widetilde{\mathrm{T}}}\big(\|D_pV(t)\|_{H^3}^2 + \|DB(t)\|_{H^3}^2\big)\,dt < \infty.$$

Notice that $k \geq \frac{2s}{s-3}$ and $\alpha \geq \frac{2\lambda}{\lambda-3}$. Hence Theorem 1.1 holds.

4 Proof of Theorem 1.2

We begin with estimating the terms M_1–M_5 in (5), First, we estimate M_1:

$$\begin{aligned} |M_1| &= \left| \int_{\mathbb{R}^3} (DV \cdot D)B \cdot DB \, dx \right| \\ &\le C\|DV\|_2 \|DB\|_4^2 \\ &\le C\|DV\|_2 \|DB\|_2^{\frac{1}{2}} \left\|D^2 B\right\|_2^{\frac{3}{2}} \\ &\le C\left(\|DV\|_2^2 + \|DB\|_2\right) \left\|D^2 B\right\|_2^{\frac{3}{2}}. \end{aligned} \tag{29}$$

The sum $M_2 + M_3$ can be rewritten as

$$M_2 + M_3 = \sum_{j=1}^{3} \int_{\mathbb{R}^3} (\partial_j B \cdot D)V \cdot \partial_j B + (\partial_j B \cdot D)B \cdot \partial_j V \, dx,$$

and hence we can estimate $M_2 + M_3$ as M_1 to obtain

$$|M_2 + M_3| \le C\left(\|DV\|_2^2 + \|DB\|_2\right) \left\|D^2 B\right\|_2^{\frac{3}{2}}. \tag{30}$$

For M_4, we estimate each term M_{41}–M_{43} in (8). By Lemma 2.1 and interpolation we have

$$\begin{aligned} |M_{41}| &\le C\|D_p V\|_2^{\frac{1}{2}} \|DV\|_2^{\frac{1}{2}} \|D_p V\|_2^{\frac{1}{2}} \|\partial_3 D_p V\|_2^{\frac{1}{2}} \|DD_p V\|_2^{\frac{1}{2}} \left\|D_p^2 V\right\|_2^{\frac{1}{2}} \\ &\le C\|DV\|_2^{\frac{3}{2}} \|DD_p V\|_2^{\frac{3}{2}} \end{aligned}$$

and

$$\begin{aligned} |M_{42}| &\le C\|\partial_3 V\|_2^{\frac{1}{2}} \|D_p V\|_2^{\frac{1}{2}} \|\partial_3 V\|_2^{\frac{1}{2}} \|\partial_3 D_p V_p\|_2^{\frac{1}{2}} \|\partial_3 D_p V\|_2^{\frac{1}{2}} \|\partial_3 D_p V\|_2^{\frac{1}{2}} \\ &\le C\|DV\|_2^{\frac{3}{2}} \|DD_p V\|_2^{\frac{3}{2}}. \end{aligned}$$

In a similar manner, we have

$$|M_{43}| \le C\|DV\|_2^{\frac{3}{2}} \|DD_p V\|_2^{\frac{3}{2}}.$$

Therefore

$$|M_4| \le C\|DV\|_2^{\frac{3}{2}} \|DD_p V\|_2^{\frac{3}{2}}. \tag{31}$$

By the boundedness of $\|B\|_2$ we get

$$\begin{aligned} |M_5| &= \left| \int_{\mathbb{R}^3} \big((\nabla \times B) \times B\big) \cdot D^2 (\nabla \times B) \, dx \right| \\ &\le C\|DB\|_6 \|B\|_3 \left\|D^3 B\right\|_2 \\ &\le C\left\|D^2 B\right\|_2 \left\|D^2 B\right\|_2^{\frac{1}{4}} \|B\|_2^{\frac{3}{4}} \left\|D^3 B\right\|_2 \\ &\le C\left\|D^2 B\right\|_2^{\frac{5}{4}} \left\|D^3 B\right\|_2. \end{aligned} \tag{32}$$

Putting (5) and (29)–(32) together, we have

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|DV\|_2^2 + \|DB\|_2^2\right) + \rho_0\left(\|DD_pV\|_2^2 + \left\|D^2B\right\|_2^2\right) \\ &\quad\le C\left(\|DV\|_2^2 + \|DB\|_2^2\right)\left(\|DD_pV\|_2^2 + \left\|D^2B\right\|_2^2\right) + \|\Delta B\|_2^2\left\|D^3B\right\|_2^2. \end{aligned} \tag{33}$$

Next, we estimate each term R_1–R_5 in (13). To estimate R_1, we need to estimate R_{11} and R_2 in (14). Applying the Hölder inequality and interpolation inequality, we have

$$\begin{aligned} |R_{11}| &\le C\left\|D^2V\right\|_2\|DB\|_3\left\|D^2B\right\|_6 \\ &\le C\left\|D^2V\right\|_2\left\|D^2B\right\|_2^{\frac{1}{2}}\|DB\|_2^{\frac{1}{2}}\left\|D^3B\right\|_2 \\ &\le C\|\Delta B\|_2\left\|D^3B\right\|_2^2 + C\|DB\|_2\|\Delta V\|_2^2 \end{aligned}$$

and

$$\begin{aligned} |R_{12}| &\le C\left\|D^2B\right\|_6\left\|D^2B\right\|_2\|DV\|_3 \\ &\le C\left\|D^3B\right\|_2\left\|D^2B\right\|_{L^2}\left\|D^2V\right\|_2^{\frac{3}{4}}\|V\|_2^{\frac{1}{4}} \\ &\le C\|\Delta B\|_2^2\left\|D^3B\right\|_2^2 + C\|\Delta V\|_2^{\frac{3}{2}}, \end{aligned}$$

where we used the boundedness of $\|u\|_2$. Hence we get

$$|R_1| \le C\|\Delta B\|_2^2\left\|D^3B\right\|_2^2 + C\left(1 + \|DB\|_2\right)\|\Delta V\|_2^2. \tag{34}$$

Based on (16), we can, similarly to R_{11} and R_{12}, estimate $(R_{231}, R_{232}, R_{234})$, R_{233}. Hence, we have

$$|R_2 + R_3| \le C\|\Delta B\|_2^2\left\|D^3B\right\|_2^2 + C\left(1 + \|DB\|_2\right)\|\Delta V\|_2^2. \tag{35}$$

For R_4, we estimate each term $R_{411} - R_{413}$ in (19). Applying the Hölder inequality and interpolation inequality, we have

$$\begin{aligned} |R_{411}| &\le C\|DV\|_2\|DD_pV\|_4^2 \\ &\le C\|DV\|_2\|DD_pV\|_2^{\frac{7}{4}}\|D_pV\|_2^{\frac{1}{4}} \\ &\le C\|DV\|_2^{\frac{5}{4}}\|DD_pV\|_2^{\frac{7}{4}}. \end{aligned}$$

We can further divide R_{412} of (19) into two terms:

$$R_{412} = -\int_{\mathbb{R}^3}\left(\partial_3^2V_p\cdot D_p\right)V_p\cdot\partial_3^2V_p\,dx + \int_{\mathbb{R}^3}\left(\partial_3^2V_p\cdot D_p\right)V_3(\partial_3D_p\cdot V_p)\,dx = R_{4121} + R_{4122}.$$

Applying integration by parts and Lemma 2.2, we have

$$\begin{aligned}|R_{4121}| &= 2\left|\int_{\mathbb{R}^3} V_p\partial_3^2 V_p\partial_3^2 D_p V_p\,dx\right|\\ &\le C\left\|\partial_3^2 D_p V_p\right\|_2\|\partial_3 V_p\|_2^{\frac{1}{2}}\left\|\partial_1\partial_3^2 V_p\right\|_2^{\frac{1}{2}}\|V_p\|_2^{\frac{1}{4}}\left\|\partial_3^2 V_p\right\|_2^{\frac{1}{4}}\|\partial_2 V_p\|_2^{\frac{1}{4}}\left\|\partial_2\partial_3^2 V_p\right\|_2^{\frac{1}{4}}\\ &\le C\|\Delta D_p V\|_2^{\frac{7}{4}}\|DV\|_2^{\frac{1}{2}}\|\Delta V\|_2^{\frac{1}{4}}\|D_p V\|_2^{\frac{1}{4}}\\ &\le C\|DV\|_2^{\frac{4}{7}}\|\Delta D_p V\|_2^2 + C\|D_p V\|_2^2\|\Delta V\|_2^2.\end{aligned}$$

We estimate R_{4122} as follows:

$$\begin{aligned}|R_{4122}| &\le C\left\|\partial_3^2 V_p\right\|_2^{\frac{1}{2}}\|D_p V_3\|_2^{\frac{1}{2}}\|\partial_3 D_p V_p\|_2^{\frac{1}{2}}\left\|\partial_3^2 D_p V_p\right\|_2^{\frac{1}{2}}\|\partial_3 D_p V_3\|_2^{\frac{1}{2}}\left\|\partial_3 D_p^2 V_p\right\|_2^{\frac{1}{2}}\\ &\le C\|\Delta D_p V\|_2\|DD_p V\|_2\|D_p V\|_2^{\frac{1}{2}}\|\Delta V\|_2^{\frac{1}{2}}\\ &\le C\|\Delta D_p V\|_2^{\frac{3}{2}}\|D_p V\|_2^{\frac{1}{2}}\|D_p V\|_2^{\frac{1}{2}}\|\Delta V\|_2^{\frac{1}{2}}\\ &\le C\|DV\|_2^{\frac{2}{3}}\|\Delta D_p V\|_2^2 + C\|D_p V\|_2^2\|\Delta V\|_2^2.\end{aligned}$$

In a similar manner, we obtain

$$|R_{413}| \le C\|DV\|_2^{\frac{2}{3}}\|\Delta D_p V\|_2^2 + C\|D_p V\|_2^2\|\Delta V\|_2^2.$$

Clearly, R_{421}, R_{422}, and R_{423} in (20) can be estimated as R_{411}, R_{413}, and R_{412}. Hence we have

$$|R_4| \le C\|DV\|_2^{\frac{5}{4}}\|\Delta D_p V\|_2^2 + C\|D_p V\|_2^2\|\Delta V\|_2^2. \tag{36}$$

Similarly to (29), we have

$$\begin{aligned}|R_5| &= \left|\int_{\mathbb{R}^3}\left[D^2\big((\nabla\times B)\times B\big) - D^2(\nabla\times B)\times B\right]\cdot D^2(\nabla\times B)\,dx\right|\\ &\le C\|DB\|_\infty\left\|D^2 B\right\|_2\left\|D^3 B\right\|_2\\ &\le C\|\Delta B\|_2^{\frac{3}{2}}\left\|D^3 B\right\|_2^{\frac{3}{2}}.\end{aligned}\tag{37}$$

Combining (13) and (34)–(37), we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|\Delta V\|_2^2 + \|\Delta B\|_2^2\big) + \rho_0\big(\|\Delta D_p V\|_2^2 + \left\|D^3 B\right\|_2^2\big)\\ &\quad\le C\big(\|DV\|_2^2 + \|\Delta B\|_2^2\big)\big(\|\Delta D_p V\|_2^2 + \left\|D^3 B\right\|_2^2\big)\\ &\qquad + C\big(\|D_p V\|_2^2 + \|DB\|_2^2\big)\big(\|\Delta V\|_2^2 + \|\Delta B\|_2^2\big).\end{aligned}\tag{38}$$

Adding (2), (33), and (38) together, we have

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\big(\|V\|_{H^2}^2 + \|B\|_{H^2}^2\big) + \left[\rho_0 - C\big(\|DV\|_2^2 + \|DB\|_2^2 + \|\Delta B\|_2^2\big)\right]\\ &\quad\times\big(\|D_p V\|_{H^2}^2 + \|DB\|_{H^2}^2\big) \le C\big(\|D_p V\|_2^2 + \|DB\|_2^2\big)\big(\|V\|_{H^2}^2 + \|B\|_{H^2}^2\big),\end{aligned}\tag{39}$$

Choose L so small that

$$C\big(\|DV_0\|_2^2 + \|DB_0\|_2^2 + \|\Delta B_0\|_2^2\big) \le \frac{\rho_0}{2}.$$

Substituting this inequality into (39), we get

$$\begin{aligned} &\frac{d}{dt}\big(\|V\|_{H^2}^2 + \|B\|_{H^2}^2\big) + \rho_0\big(\|D_pV\|_{H^2}^2 + \|DB\|_{H^2}^2\big) \\ &\quad \le C\big(\|D_pV\|_2^2 + \|DB\|_2^2\big)\big(\|V\|_{H^2}^2 + \|B\|_{H^2}^2\big). \end{aligned} \tag{40}$$

Applying Gronwall's inequality to (40), we have

$$\begin{aligned} &\sup_{0<t<T}\big(\|V(t)\|_{H^2}^2 + \|B(t)\|_{H^2}^2\big) + \rho_0\int_0^T\big(\|D_pV(t)\|_{H^2}^2 + \|DB(t)\|_{H^2}^2\big)\,dt \\ &\quad \le \big(\|V_0\|_{H^2}^2 + \|B_0\|_{H^2}^2\big)\bigg[1 + C\int_0^T\big(\|D_pV(t)\|_2^2 + \|DB(t)\|_2^2\big)\,dt\bigg] \\ &\qquad \times \exp\bigg[C\int_0^T\big(\|D_pV(t)\|_2^2 + \|DB(t)\|_2^2\big)\,dt\bigg]. \end{aligned}$$

which, together with (3), yields that for any $T \in (0, \widetilde{\mathbf{T}})$,

$$(V, B) \in L^\infty\big(0, T; H^2\big), \qquad (D_pV, DB) \in L^2\big(0, T; H^2\big).$$

Noticing that

$$H^2\big(\mathbb{R}^3\big) \hookrightarrow L^\infty\big(\mathbb{R}^3\big),$$

we get

$$(V, DB) \in L^2\big(0, T; L^\infty\big(\mathbb{R}^3\big)\big), \quad \forall T \in (0, \widetilde{\mathbf{T}}).$$

Based on Theorem 1.1 ($p = \beta = \infty$ and $q = \gamma = 2$), we have $\widetilde{\mathbf{T}} = \infty$, which yields Theorem 1.2.

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Baoying Du, contributed to the conception of the paper; contributed significantly to analysis and manuscript preparation; wrote the manuscript.

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References

1. Acheritogaray, M., Degond, P., Frouvelle, A., Liu, J.-G.: Kinetic formulation and global existence for the Hall-magnetohydrodynamic system. Kinet. Relat. Models **4**, 908–918 (2011)
2. Cao, C., Wu, J.: Global regularity for the 2D MHD equations with mixed partial dissipation and magnetic diffusion. Adv. Math. **226**, 1803–1822 (2011)
3. Chae, D.: Global regularity for the 2D Boussinesq equations with partial viscosity terms. Adv. Math. **203**, 497–513 (2006)
4. Chae, D., Degond, P., Liu, J.-G.: Well-posedness for Hall-magnetohydrodynamics. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **31**, 555–565 (2014)
5. Chae, D., Lee, J.: On the blow-up criterion and small data global existence for the Hall-magnetohydrodynamics. J. Differ. Equ. **256**, 3835–3858 (2014)
6. Chae, D., Schonbek, M.: On the temporal decay for the Hall-magnetohydrodynamic equations. J. Differ. Equ. **255**, 3871–3882 (2013)
7. Chae, D., Weng, S.: Singularity formation for the incompressible Hall-magnetohydrodynamic equations without resistivity. Ann. Inst. Henri Poincaré, Anal. Non Linéaire **33**, 1009–1022 (2016)
8. Chae, D., Wolf, J.: On partial regularity for the steady Hall magnetohydrodynamics system. Commun. Math. Phys. **339**, 1147–1166 (2015)
9. Chae, D., Wolf, J.: On partial regularity for the 3D non-stationary Hall magnetohydrodynamics equations on the plane. SIAM J. Math. Anal. **48**, 443–469 (2016)
10. Chemin, J.Y., Desjardins, B., Gallagher, I., Grenier, E.: Fluids with anisotropic viscosity. Modél. Math. Anal. Numér. **34**, 315–335 (2000)
11. Cheng, J., Du, L.: On two-dimensional magnetic Bénard problem with mixed partial viscosity. J. Math. Fluid Mech. **17**, 769–797 (2015)
12. Cheng, J., Liu, Y.: Global regularity of the 2D magnetic-micropolar fluid flows with mixed partial viscosity. Comput. Math. Appl. **70**, 66–72 (2015)
13. Du, B.: Global regularity for the $2\frac{1}{2}$D incompressible Hall-MHD system with partial dissipation. J. Math. Anal. Appl. **484**, 123701 (2020)
14. Du, B.: On three-dimensional Hall-magnetohydrodynamic equations with partial dissipation. Bound. Value Probl. **2022**, 6 (2022). https://doi.org/10.1186/s13661-022-01587-0
15. Du, B., Liu, J.: Global regularity for the incompressible 3D Hall-magnetohydrodynamics with partial dissipation. Acta Math. Sci. **42A**, 5 (2022)
16. Du, L., Lin, H.: Regularity criteria for incompressible magnetohydrodynamics equations in three dimensions. Nonlinearity **26**, 219–239 (2013)
17. Du, L., Zhou, D.: Global well-posedness of two-dimensional magnetohydrodynamic flows with partial dissipation and magnetic diffusion. SIAM J. Math. Anal. **47**, 1562–1589 (2015)
18. Dumas, E., Sueur, F.: On the weak solutions to the Maxwell–Landau–Lifshitz equations and to Hall-magnetohydrodynamic equations. Commun. Math. Phys. **330**, 1179–1225 (2014)
19. Duvaut, G., Lions, J.: Inéquations en thermoélasticité et magnétohydrodynamique. Arch. Ration. Mech. Anal. **46**, 241–279 (1972)
20. Fan, J., Huang, S., Nakamura, G.: Well-posedness for the axisymmetric incompressible viscous Hall-magnetohydrodynamic equations. Appl. Math. Lett. **26**, 963–967 (2013)
21. Fei, M., Xiang, Z.: On the blow-up criterion and small data global existence for the Hall-magnetohydrodynamics with horizontal dissipation. J. Math. Phys. **56**, 051504 (2015)
22. He, C., Xin, Z.: On the regularity of weak solutions to the magnetohydrodynamic equations. J. Differ. Equ. **213**, 235–252 (2005)
23. He, C., Xin, Z.: Partial regularity of suitable weak solutions to the incompressible magnetohydrodynamic equations. J. Funct. Anal. **227**, 113–152 (2005)
24. Homann, H., Grauer, R.: Bifurcation analysis of magnetic reconnection in Hall-MHD systems. Physica D **208**, 59–72 (2005)
25. Lin, F., Xu, L., Zhang, P.: Global small solutions to 2-D incompressible MHD system. J. Differ. Equ. **259**, 5440–5485 (2015)
26. Ma, L.: Global existence of three-dimensional incompressible magneto-micropolar system with mixed partial dissipation, magnetic diffusion and angular viscosity. Comput. Math. Appl. **75**, 170–186 (2018)
27. Ma, L.: On two-dimensional incompressible magneto-micropolar system with mixed partial viscosity. Nonlinear Anal., Real World Appl. **40**, 95–129 (2018)
28. Mdjda, A.J., Bertozzi, A.L.: Voriticity and Incompressible Flow. Cambridge Uiversity Press, Cambridge (2001)
29. Sermange, M., Teman, R.: Some mathematical questions related to the MHD equations. Commun. Pure Appl. Math. **36**, 635–664 (1983)
30. Wang, F., Wang, K.: Global existence of 3D MHD equations with mixed partial dissipation and magnetic diffusion. Nonlinear Anal., Real World Appl. **14**(1), 526–535 (2013)
31. Xu, L., Zhang, P.: Global small solutions to three-dimensional incompressible MHD system. SIAM J. Math. Anal. **47**(1), 26–65 (2015)
32. Ye, X., Zhu, M.: Global regularity for 3D MHD system with partial viscosity and magnetic diffusion terms. J. Math. Anal. Appl. **458**, 980–991 (2018)