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Optional projection under equivalent local martingale measures

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Abstract
We study optional projections of $\mathbb{G}$-adapted strict local martingales on a smaller filtration $\mathbb{F}$ under changes of equivalent martingale measures. General results are provided as well as a detailed analysis of two specific examples given by the inverse Bessel process and a class of stochastic volatility models. This analysis contributes to clarify the absence of arbitrage opportunities of market models under restricted information.

Keywords Local martingale · Optional projection · Local martingale measure · Filtration shrinkage

Mathematics Subject Classification (2020) 60G07 · 60H30 · 60G44

JEL Classification C650 · D82

1 Introduction

In this paper, we study optional projections of $\mathbb{G}$-adapted strict local martingales on a smaller filtration $\mathbb{F}$ under changes of equivalent local martingale measures.

It is a well-known fact that when projecting a stochastic process on a filtration with respect to which it is not adapted, some attributes of its dynamics may change; see for example Föllmer and Protter [11] and Bielecki et al. [3], where the authors study the semimartingale characteristics of projections of special semimartingales. Moreover, some basic properties of the process can be lost. Most notably, the optional projection of a local martingale may fail to be a local martingale; see for example [11, Theorem 3.7] and Larsson [25, Corollary 1], where conditions are stated under which this happens, and Kardaras and Ruf [23] for a study of optional projections of local

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martingale deflators. Optional projections of general semimartingales and stochastic integrals have been studied in Brémaud and Yor [4].

In the literature, it is a classical problem to investigate financial market models under restricted information, where full information on the asset prices is not available and agents' decisions are based on a restricted information flow. This could be due to a delay in the diffusion of information or to an incomplete data flow, as it may happen for example when investors only see an asset value when it crosses certain levels; see Jarrow et al. [19]. In general, this is represented by assuming that trading strategies are predictable for a filtration which can be smaller than the filtration $\mathbb{G}$ generated by the asset prices. Many of the classical problems in financial mathematics have been translated to this setting from the classical framework with full information; see for example Schweizer [29], Frey and Runggaldier [12], Runggaldier and Zaccaria [28], Callegaro et al. [5] and Fujimoto et al. [13] for studies on risk-minimisation and expected utility maximisation, respectively, under restricted information.

In Cuchiero et al. [7] and Kabanov and Stricker [21], it is shown that the analysis of characteristics and properties of financial markets under restricted or delayed information boils down to studying optional projections of the asset prices on the smaller filtration. In particular, it is shown in both papers how a certain "no-arbitrage" condition is equivalent to the existence of an equivalent measure Q such that the Q-optional projection of the asset price on $\mathbb{F}$ is a martingale. This setting can be applied to trading with delay, trading under restricted information, semistatic hedging and trading under transaction costs.

Our study of optional projections is related to [7] if we consider a more general definition of trading strategies. More precisely, in [7], prices are not fully revealed to observers in a large platonic financial market, and trading is possible only with $\mathbb{F}$-predictable simple strategies such that the resulting wealth or value process is L^p-integrable for some measure $\tilde{P}$ equivalent to P. We deviate from this definition and assume more classically that the set of value processes of admissible strategies is defined as those $\mathbb{F}$-adapted processes for which there exist a local martingale measure Q for the price process and a $\mathbb{G}$-adapted Q-martingale such that the value process is bounded from below by that martingale. If the optional projection of the price process X is a local martingale, it turns out, omitting the details here, that the superhedging price of a claim c over $\mathbb{F}$-adapted admissible strategies is bounded from below by

$$\sup_{Q\in\mathcal{Q}} E^Q[c],$$

where $\mathcal{Q}$ is the set of P-absolutely continuous probability measures under which the Q-optional projection of X is a Q-local martingale. Under full information, the message of the fundamental hedging duality is that the bound is tight so that "strong duality" holds. Under restricted information, it is an interesting point of further research to give conditions in this setup when the above "weak duality" holds as a strong one. This and the related more general indifference pricing formulas go beyond the scope of the present paper. Nonetheless, the indicated weak duality shows that the question whether the projection is a local martingale is relevant in the duality theory in pricing under restricted information. Further relevant applications of optional projections are provided by the works of Çetin et al. [6], Jarrow et al. [19] and Sezer [30] in the field of credit risk modelling: taking inspiration from Jarrow and Protter [16], the authors characterise reduced form models as optional projections of structural models on a smaller filtration. In particular, the cash balance of a firm, represented by a process $X = (X_t)_{t\geq 0}$, is adapted to the filtration $\mathbb{G}$ of the firm's management, but not necessarily to the filtration $\mathbb{F}$ representing the information available to the market. In this setting, the price of a zero-coupon bond issued by the firm is the optional projection of X.

An interesting question to analyse is then if traders with partial information $\mathbb{F} \subseteq \mathbb{G}$ may perceive different characteristics of the market they observe than individuals with access to the complete information $\mathbb{G}$, for example for what concerns the existence of perceived arbitrages.

As already noted in Jarrow and Protter [17], traders with limited information may interpret the presence of a bubble on the price process in the larger filtration as an arbitrage opportunity. This happens if X is a strict $(P, \mathbb{G})$-local martingale, ${}^{o}X$ fails to be a strict $(P, \mathbb{F})$-local martingale and in addition, there exists no measure $Q \approx P$ under which ${}^{o}X$ is a local martingale.

The above overview shows that it is important to assess the properties of optional projections under equivalent local martingale measures. In particular, we study the relation among the set $\mathcal{M}_{\mathrm{loc}}$ of equivalent local martingale measures (ELMMs) for a process X and the set $\mathcal{M}^{o}_{\mathrm{loc}}$ of measures $Q \approx P$ such that the optional projection under Q is a Q-local martingale. We treat in detail two main cases: the inverse Bessel process and an extension of the stochastic volatility model of Sin [31]. We also give general results for the optional projection of a $\mathbb{G}$-adapted process on the delayed filtration $(\mathcal{G}_{t-\epsilon})_{t\geq 0}$ with $\epsilon > 0$. In particular, we prove that the delayed projection of a strict local martingale is never a local martingale, and that in this setting, the equivalent measure extension problem studied in Larsson [25] admits no solution. This provides a family of examples showing that the sufficient condition of [25, Corollary 1] is not necessary. In financial applications, delayed information represents the scenario where investors in the market have access to the information with a given positive time delay. This setting has been extensively studied in the literature; see e.g. Guo et al. [14], Hillairet and Jiao [15], Jeanblanc and Lecam [20], Xing and Yiyun [33] in the setting of credit risk models, Dolinsky and Zouari [10] under model uncertainty and Bank and Dolinsky [1] in the context of option pricing.

We also provide an invariance theorem about local martingales which are solutions of a one-dimensional SDE in the natural filtration of an n-dimensional Brownian motion; see Proposition 4.4. Specifically, we show that under mild conditions, such a local martingale X has the same law under P as under every $Q \in \mathcal{M}_{\mathrm{loc}}(X)$. This result is closely related to the study in Jarrow et al. [18] where the authors provide invariance theorems for detecting asset price bubbles. Proposition 4.4 leads to Theorem 4.11 on optional projections on a filtration $\mathbb{F}$ that is smaller than the natural filtration $\mathbb{F}^X$ of X. Important applications of Theorem 4.11 are given by delayed information and by the model of [6], where the market does not see the value of a firm but only knows when the firm has positive cash balances or when it has negative or zero cash balances.

The rest of the paper is organised as follows. After setting up the notation in Sect. 2, we formulate in Sect. 3 the aims of our study as five properties about optional projections of strict local martingales. We also provide a synthetic overview of the main results of the paper in the light of these problems and recap when these questions have positive and when negative answers in the examples we consider along the paper. In Sect. 4, we give general results about optional projections of local martingales under equivalent local martingale measures. Section 5 is devoted to the inverse Bessel process, projected on different filtrations, whereas in Sect. 6, we focus on a class of two-dimensional stochastic volatility models. The novelty here is that we consider not only a reference measure P, but the whole set of equivalent local martingale measures.

2 Preliminaries

Consider a probability space $(\Omega, \mathcal{F}, P)$ equipped with two filtrations $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$, $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$. We assume that $\mathbb{F} \subseteq \mathbb{G}$ and $\mathcal{F} = \mathcal{G}_\infty$. **All filtrations** considered in the paper are supposed to be complete and right-continuous (and are made so if necessary). Let X be a nonnegative càdlàg $(P, \mathbb{G})$-local martingale.

Notation 2.1 We denote by ${}^oX$ the P-optional projection of X on $\mathbb{F}$, i.e., the unique optional process satisfying

$$\mathbb{1}_{\{\tau<\infty\}}\, {}^oX_\tau = E[\mathbb{1}_{\{\tau<\infty\}} X_\tau | \mathcal{F}_\tau] \qquad \text{a.s.}$$

for every $\mathbb{F}$-stopping time τ. We call ${}^{Q,o}X$ the Q-optional projection of X. If we do not specify the measure, the optional projection is with respect to P.

We denote by $\mathbb{F}^X$ the natural filtration of X. Moreover, if Q is a probability measure equivalent to P, we define $Z_\infty := \frac{dQ}{dP}$ and denote by ${}^{\mathcal{F}}Z$, ${}^{\mathcal{F}^X}Z$, ${}^{\mathcal{G}}Z$ the càdlàg processes characterised by

$$ {}^{\mathcal{F}}Z_t = E[Z_\infty|\mathcal{F}_t], \quad {}^{\mathcal{F}^X}Z_t = E[Z_\infty|\mathcal{F}^X_t], \quad {}^{\mathcal{G}}Z_t = E[Z_\infty|\mathcal{G}_t], \qquad t \geq 0, \quad (2.1)$$

respectively. Moreover, for $\mathbb{H} = \mathbb{F}, \mathbb{F}^X, \mathbb{G}$, we define

$$\begin{aligned}
\mathcal{M}_{\mathrm{loc}}(X, \mathbb{H}) &:= \{Q \approx P : X \text{ is a } (Q, \mathbb{H})\text{-local martingale}\},\\
\mathcal{M}_{\mathrm{true}}(X, \mathbb{H}) &:= \{Q \approx P : X \text{ is a true } (Q, \mathbb{H})\text{-martingale}\},\\
\mathcal{M}_{\mathrm{strict}}(X, \mathbb{H}) &:= \{Q \approx P : X \text{ is a strict } (Q, \mathbb{H})\text{-local martingale}\}.
\end{aligned}$$

We also set

$$\begin{aligned}
\mathcal{M}^o_{\mathrm{loc}}(X, \mathbb{F}) &:= \{Q \approx P :\ {}^{Q,o}X \text{ is a } (Q, \mathbb{F})\text{-local martingale}\},\\
\mathcal{M}_{\mathrm{loc}}(X, \mathbb{G}, \mathbb{F}) &:= \{Q \approx P :\ X \text{ is a } (Q, \mathbb{G})\text{-local martingale, } {}^{\mathcal{G}}Z \text{ is } \mathbb{F}\text{-adapted}\},\\
\mathcal{M}^o_{\mathrm{loc}}(X, \mathbb{G}, \mathbb{F}) &:= \{Q \approx P :\ {}^{Q,o}X \text{ is a } (Q, \mathbb{F})\text{-local martingale, } {}^{\mathcal{G}}Z \text{ is } \mathbb{F}\text{-adapted}\}.
\end{aligned}$$

For a given $(Q, \mathbb{G})$-Brownian motion $B = (B_t)_{t\geq 0}$ and a suitably integrable $\mathbb{G}$-predictable process $\alpha = (\alpha_t)_{t\geq 0}$, we denote by $\mathcal{E}_t(\int \alpha_s dB_s)$ the stochastic exponential at time t of the process $(\int_0^u \alpha_s dB_s)_{u\geq 0}$.

Remark 2.2 Since X is a nonnegative process, its optional projection always exists by Dellacherie and Meyer [9, Theorem VI.43].

Remark 2.3 Since we assume $\mathcal{F} = \mathcal{G}_\infty$, the density $\frac{dQ}{dP}$ of any equivalent probability measure Q with respect to the original measure P must be $\mathcal{G}_\infty$-measurable. This implies that

$$\mathcal{M}_{\text{loc}}(X, \mathbb{G}, \mathbb{G}) = \mathcal{M}_{\text{loc}}(X, \mathbb{G}).$$

3 Overview of the results

The main focus of the paper is to study the following properties:

$$\mathcal{M}_{\text{loc}}(X, \mathbb{G}) \cap \mathcal{M}^o_{\text{loc}}(X, \mathbb{F}) \neq \emptyset; \tag{P1}$$

$$\mathcal{M}_{\text{loc}}({}^o X, \mathbb{F}) \neq \emptyset; \tag{P2}$$

$$\mathcal{M}_{\text{strict}}(X, \mathbb{G}) \cap \mathcal{M}^o_{\text{loc}}(X, \mathbb{F}) \neq \emptyset; \tag{P3}$$

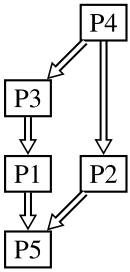
$$\mathcal{M}_{\text{loc}}(X, \mathbb{G}, \mathbb{F}) = \mathcal{M}^o_{\text{loc}}(X, \mathbb{G}, \mathbb{F}); \tag{P4}$$

$$\bigcup_{Q \in \mathcal{M}_{\text{loc}}(X, \mathbb{G})} \mathcal{M}_{\text{loc}}({}^{Q,o} X, \mathbb{F}) \neq \emptyset. \tag{P5}$$

These properties have the following financial interpretations. First, (P2) and (P5) are related to the study of perceived arbitrage opportunities. In particular, if (P2) holds, then the P-bubble modelled by X under full information is not perceived as an arbitrage opportunity under partial information, see also Jarrow and Protter [17], and more generally, when (P5) is satisfied, there exists at least one equivalent probability measure Q that defines an arbitrage-free market under restricted information. Moreover, if the optional projection is tradeable, (P1), (P3) and (P4) investigate the problem of a perceived bubble in the smaller filtration when there is a bubble in the bigger filtration. If it is not tradeable, the properties are related to pricing formulas under restricted information, as we explain in the introduction in detail. On the other hand, (P1) and (P3) address the classical mathematical question of when the optional projection of a (strict) local martingale on the smaller filtration $\mathbb{F}$ remains a (strict) local martingale; see for example Larsson [25].

It turns out that none of the above properties is equivalent to another, and that none of them holds for the whole class of nonnegative local martingales and arbitrary filtrations. In other words, their validity depends on both the process X and the filtrations $\mathbb{G}$ and $\mathbb{F}$. This is shown in Sects. 5 and 6. Section 4 provides general results on local martingales and their optional projections which are of independent interest.

In general, we note that $\mathcal{M}_{\text{strict}}(X, \mathbb{G}) \neq \emptyset$ and hence also $\mathcal{M}_{\text{loc}}(X, \mathbb{G}) \neq \emptyset$ whenever (as often assumed) $P \in \mathcal{M}_{\text{strict}}(X, \mathbb{G})$. Properties (P1)–(P3) and (P5) trivially hold if ${}^o X$ is an $\mathbb{F}$-local martingale; so the more interesting case is when ${}^o X$ is not a local martingale. If (P3) or (P4) holds, then (P1) holds as well. Moreover, if (P4) is satisfied, then $P \in \mathcal{M}^o_{\text{loc}}(X, \mathbb{G}, \mathbb{F})$ and so (P2) and (P3) are also satisfied. Finally, (P5) is implied by any of the other properties. This can be summarised in the following scheme:



As already mentioned, reverse implications do not hold in general.

We finish the section by summarising our main results. Note that (P1)–(P3) and (P5) trivially hold for the inverse Bessel process projected on the filtration generated by (B^1, B^2), as the optional projection is again a strict local martingale; see Sect. 5.1.

Property (P1)**.** In Sect. 6, we introduce a stochastic volatility model where X is a strict local martingale under suitable conditions on the coefficients of its SDE, but whose optional projection on a specific subfiltration is not a local martingale; see Theorem 6.11. Property (P1) holds because X admits a true martingale measure; see Proposition 6.2. On the other hand, (P1) is not true for the inverse Bessel process projected on a delayed filtration, i.e., on $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ with $\mathcal{F}_t = \mathcal{G}_{t-\epsilon}$ and $\epsilon > 0$; see Sect. 5.2.

Property (P2)**.** A particular case of the stochastic volatility model introduced in Sect. 6 gives a strict local martingale N such that ${}^oN$ is not a local martingale, but $\mathcal{M}_{\mathrm{loc}}({}^oN, \mathbb{F}) \neq \emptyset$; see Example 6.7. In this setting, (P2) holds. For the optional projection on the delayed filtration of the process introduced in Example 5.7, (P2) holds as well. On the other hand, the property does not hold in the case of the inverse Bessel process projected on the delayed filtration; see Theorem 5.5. It is not satisfied in the setting of Example 6.9 either.

Property (P3)**.** In Example 6.8, we consider the sum of X from the stochastic volatility model of Sect. 6 and a suitable strict local martingale adapted to a Brownian filtration $\mathbb{F}$; this is a strict local martingale whose optional projection on $\mathbb{F}$ is not a local martingale, but such that (P3) is satisfied. On the other hand, Property (P3) is never satisfied for a delayed filtration; see Proposition 4.12.

Property (P4)**.** This is satisfied by the inverse Bessel process projected on the filtration generated by (B^1, B^2); see Theorem 5.2. It does not hold for any of the examples where (P1) does not hold, e.g. in the case of the inverse Bessel process projected on a delayed filtration.

Property (P5)**.** This holds in all the considered examples except for the inverse Bessel process projected on the delayed filtration; see Theorem 5.5. In particular, Example 6.9 provides a case where (P5) holds and (P2) is not satisfied, whereas in Example 5.7, (P1) does not hold (and therefore also not (P3) and (P4)), whereas (P5) does.

We conclude by providing in Table 1 a schematic recap of the validity of properties (P1)–(P5). If it is not specified whether a property holds or not, it has not been investigated.

Table 1 Validity of properties (P1)–(P5) in the examples studies in Sects. 5 and 6

	(P1)	(P2)	(P3)	(P4)	(P5)
Inverse Bessel on $\sigma(B^1, B^2)$	YES	YES	YES	YES	YES
Inverse Bessel on delayed $\mathbb{F}^X$	NO	NO	NO	NO	NO
Inverse Bessel on delayed $\sigma(B^1, B^2, B^3)$	NO		NO	NO	
Example 5.7	NO	YES	NO	NO	YES
Example 6.7	YES	YES			YES
Example 6.8	YES		YES	NO	YES
Example 6.9	YES	NO	YES	NO	YES
Example 6.12	YES		NO	NO	YES

For any of the implications of the scheme above, we give a counterexample for the inverse implication:

- In Example 6.12, (P1) holds but (P3) does not.
- In Examples 6.8 and 6.12, (P1) holds but (P4) does not.
- In Example 5.7, (P2) holds but (P4) does not.
- In Examples 6.8 and 6.9, (P3) holds but (P4) does not.
- In Example 5.7, (P5) holds but (P1) does not.
- In Example 6.9, (P5) holds but (P2) does not.

Moreover, for every pair of properties which do not share any general implication, we have examples where the first is satisfied and the second is not, and vice versa. For instance, in Example 5.7, (P2) is satisfied whereas (P1) and (P3) are not, while the opposite holds in Example 6.9.

4 General results

We start by providing preliminary results which are used throughout the paper. They are also of independent interest.

The first result is due to Stricker [32] and is also stated in [11, Theorem 3.6].

Lemma 4.1 *Let $X = (X_t)_{t\geq 0}$ be a nonnegative $\mathbb{G}$-local martingale adapted to the filtration $\mathbb{F} \subseteq \mathbb{G}$. Then*

$$\mathcal{M}_{\text{loc}}(X, \mathbb{G}) \subseteq \mathcal{M}_{\text{loc}}(X, \mathbb{F}).$$

Lemma 4.2 *Let Q be a probability measure equivalent to P such that ${}^{\mathcal{G}}Z$ in (2.1) is $\mathbb{F}$-adapted. Then ${}^{Q,o}X = {}^{o}X$.*

Proof Fix an $\mathbb{F}$-stopping time τ. Recall that ${}^{\mathcal{F}}Z_\tau = E[\frac{dQ}{dP}|\mathcal{F}_\tau]$. Since ${}^{\mathcal{G}}Z$ is $\mathbb{F}$-adapted, we have

$${}^{\mathcal{G}}Z_\tau = \mathbb{E}[{}^{\mathcal{G}}Z_\tau|\mathcal{F}_\tau] = \mathbb{E}\left[\mathbb{E}\left[\frac{dQ}{dP}\middle|\mathcal{G}_\tau\right]\middle|\mathcal{F}_\tau\right] = \mathbb{E}\left[\frac{dQ}{dP}\middle|\mathcal{F}_\tau\right] = {}^{\mathcal{F}}Z_\tau,$$

and thus, by the Bayes formula,

$$\mathbb{1}_{\{\tau<\infty\}}\, {}^{Q,o}X_\tau = \mathbb{E}^Q[\mathbb{1}_{\{\tau<\infty\}}X_\tau|\mathcal{F}_\tau] = ({}^{\mathcal{F}}Z_\tau)^{-1}\mathbb{E}[{}^{\mathcal{G}}Z_\tau \mathbb{1}_{\{\tau<\infty\}}X_\tau|\mathcal{F}_\tau]$$
$$= \mathbb{E}[\mathbb{1}_{\{\tau<\infty\}}X_\tau|\mathcal{F}_\tau] = \mathbb{1}_{\{\tau<\infty\}}\, {}^{o}X_\tau. \qquad \square$$

The following theorem provides a condition under which the Q-optional projection of X is an $\mathbb{F}$-local martingale under any ELMM Q.

Theorem 4.3 *Suppose X admits an $\mathbb{F}$-localising sequence which makes it a bounded $(P,\mathbb{G})$-martingale. Then ${}^{Q,o}X$ is a $(Q,\mathbb{F})$-local martingale for any $Q \in \mathcal{M}_{\text{loc}}(X,\mathbb{G})$.*

Proof Let $Q \in \mathcal{M}_{\text{loc}}(X,\mathbb{G})$ and $(\tau_n)_{n\in\mathbb{N}}$ be the assumed localising sequence. Since X^{τ_n} is bounded for every $n \in \mathbb{N}$, $(\tau_n)_{n\in\mathbb{N}}$ localises X under Q as well, and the result follows by Föllmer and Protter [11, Theorem 3.7]. $\square$

We now give a result which provides a class of local martingales whose law under P is invariant under a change to any equivalent local martingale measure. Jarrow et al. [18, Theorem 3.1] provide a similar finding in the context of asset price bubbles detection.

Proposition 4.4 *Let $X = (X_t)_{t\geq 0}$ be a $(P,\mathbb{G})$-local martingale given by*

$$dX_t = \sigma(t, X_t)dW_t, \qquad t \geq 0, \tag{4.1}$$

where W is a one-dimensional $(P,\mathbb{G})$-Brownian motion and $\sigma : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ is such that there exists a unique strong solution to (4.1). Suppose also that $\sigma(t, X_t) \neq 0$ a.s. for almost every $t \geq 0$. Then X has the same law under P as under any $Q \in \mathcal{M}_{\text{loc}}(X,\mathbb{G})$. In particular, if X is a strict $(P,\mathbb{G})$-local martingale, it is a strict $(Q,\mathbb{G})$-local martingale under any $Q \in \mathcal{M}_{\text{loc}}(X,\mathbb{G})$, and if it is a $(P,\mathbb{G})$-true martingale, it is a $(Q,\mathbb{G})$-true martingale under any $Q \in \mathcal{M}_{\text{loc}}(X,\mathbb{G})$.

Proof Fix a probability measure $Q \in \mathcal{M}_{\text{loc}}(X,\mathbb{G})$. Girsanov's theorem together with Lévy's characterisation of Brownian motion implies that W is a semimartingale under Q with decomposition

$$W = W^Q + A^Q, \tag{4.2}$$

where W^Q is a Q-Brownian motion and A^Q is a continuous adapted finite-variation process. Then the dynamics of X under Q are given by

$$X_t - X_0 = X_0 + \int_0^t \sigma(s, X_s)dW_s$$
$$= X_0 + \int_0^t \sigma(s, X_s)dA_s^Q + \int_0^t \sigma(s, X_s)dW_s^Q, \qquad t \geq 0. \tag{4.3}$$

Since X is a Q-local martingale by assumption, the finite-variation part on the right-hand side of (4.3) must be zero, and so the result follows. $\square$

The next results consider the case when there are no ELMMs defined by a non-trivial density adapted to $\mathbb{F}^X$.

Proposition 4.5 *Let X be a nonnegative $(P, \mathbb{G})$-local martingale, and suppose that*

$$\mathcal{M}_{\text{loc}}(X, \mathbb{F}^X, \mathbb{F}^X) = \{P\}.$$

Let Q be a probability measure with $Q \in \mathcal{M}_{\text{loc}}(X, \mathbb{G})$ and ${}^{\mathcal{G}}Z$ the density process defined in (2.1). Then for any $\mathbb{F}^X$-stopping time τ, we have

$$E[{}^{\mathcal{G}}Z_\tau | \mathcal{F}^X_\tau] = 1 \qquad a.s.$$

Proof If $Q \in \mathcal{M}_{\text{loc}}(X, \mathbb{G})$, then X is a $(Q, \mathbb{F}^X)$-local martingale by Lemma 4.1. By the assumption $\mathcal{M}_{\text{loc}}(X, \mathbb{F}^X, \mathbb{F}^X) = \{P\}$, we have for any $\mathbb{F}^X$-stopping time τ that

$$1 = {}^{\mathcal{F}^X}Z_\tau = E[Z_\infty | \mathcal{F}^X_\tau] = E\big[E[Z_\infty | \mathcal{G}_\tau] \big| \mathcal{F}^X_\tau\big] = E[{}^{\mathcal{G}}Z_\tau | \mathcal{F}^X_\tau] \qquad \text{a.s.} \qquad \square$$

Corollary 4.6 *Let X be a nonnegative $(P, \mathbb{G})$-local martingale, and suppose that*

$$\mathcal{M}_{\text{loc}}(X, \mathbb{F}^X, \mathbb{F}^X) = \{P\}.$$

Then either

$$\mathcal{M}_{\text{loc}}(X, \mathbb{G}) = \mathcal{M}_{\text{strict}}(X, \mathbb{G})$$

or

$$\mathcal{M}_{\text{loc}}(X, \mathbb{G}) = \mathcal{M}_{\text{true}}(X, \mathbb{G}).$$

Proof Let $Q \in \mathcal{M}_{\text{loc}}(X, \mathbb{G})$. Suppose that $P \in \mathcal{M}_{\text{strict}}(X, \mathbb{G})$. Then there exists $t \geq 0$ such that $\mathbb{E}^P[X_t] < X_0$. For the same t, we have

$$E^Q[X_t] = E^P[Z_\infty X_t] = E^P[{}^{\mathcal{G}}Z_t X_t] = E^P\big[X_t E^P[{}^{\mathcal{G}}Z_t | \mathcal{F}^X_t]\big] = E^P[X_t] < X_0,$$

where the last equality follows from Proposition 4.5. Therefore, $Q \in \mathcal{M}_{\text{strict}}(X, \mathbb{G})$. Analogously, it can be seen that if $P \in \mathcal{M}_{\text{true}}(X, \mathbb{G})$, then $Q \in \mathcal{M}_{\text{true}}(X, \mathbb{G})$. $\square$

Corollary 4.7 *Let X be a nonnegative $(P, \mathbb{G})$-local martingale and suppose that*

$$\mathcal{M}_{\text{loc}}(X, \mathbb{F}^X, \mathbb{F}^X) = \{P\}.$$

Then for every probability measure $Q \in \mathcal{M}_{\text{loc}}(X, \mathbb{G})$ and every subfiltration $\mathbb{F} \subseteq \mathbb{F}^X$, we have ${}^oX = {}^{Q,o}X$.

Proof Let $Q \in \mathcal{M}_{\text{loc}}(X, \mathbb{G})$. Then for any $\mathbb{F}$-stopping time τ, we have by conditioning Z_∞ on $\mathcal{G}_\tau$ that

$$
\begin{aligned}
E^Q[\mathbb{1}_{\{\tau<\infty\}}X_\tau|\mathcal{F}_\tau] &= (E^P[Z_\infty|\mathcal{F}_\tau])^{-1}E^P[{}^{\mathcal{G}}Z_\tau\mathbb{1}_{\{\tau<\infty\}}X_\tau|\mathcal{F}_\tau]\\
&= \big(E^P\big[E^P[Z_\infty|\mathcal{F}^X_\tau]\big|\mathcal{F}_\tau]\big)^{-1}E^P\big[E^P[{}^{\mathcal{G}}Z_\tau\mathbb{1}_{\{\tau<\infty\}}X_\tau|\mathcal{F}^X_\tau]\big|\mathcal{F}_\tau\big]\\
&= E^P\big[\mathbb{1}_{\{\tau<\infty\}}X_\tau E^P[{}^{\mathcal{G}}Z_\tau|\mathcal{F}^X_\tau]\big|\mathcal{F}_\tau\big]\\
&= E^P[\mathbb{1}_{\{\tau<\infty\}}X_\tau|\mathcal{F}_\tau] \qquad \text{a.s.,}
\end{aligned}
$$

where the second equality follows from $\mathbb{F}\subseteq\mathbb{F}^X$ and the last from Proposition 4.5. □

We now recall the equivalent measure extension problem of Larsson [25], together with the most important results relating this to the optional projection of strict local martingales. In the setting introduced in Sect. 2, define first the $\mathbb{G}$-stopping times

$$\tau_n := \inf\{t\geq 0 : X_t \geq n\}\wedge n, \qquad \tau := \lim_{n\to\infty}\tau_n, \tag{4.4}$$

and note that $\mathcal{G}_{\tau-}=\bigvee_{n\geq 1}\mathcal{G}_{\tau_n}$. The Föllmer measure Q_0 is defined on $\mathcal{G}_{\tau-}$ as the probability measure that coincides with Q_n on $\mathcal{G}_{\tau_n}$ for each $n\geq 1$, where $Q_n\approx P$ is defined on $\mathcal{G}_{\tau_n}$ by $dQ_n = X_{\tau_n}dP$. For more details, see [25, Sect. 2]. The measure Q_0 is only defined on $\mathcal{G}_{\tau-}$. Larsson [25, Problem 1], also known as the equivalent measure extension problem, deals with the issue of extending Q_0 to $\mathcal{G}_\infty$. We formulate this problem below.

Problem 4.8 *Consider the probability space $(\Omega,\mathcal{F},P)$ equipped with two filtrations $\mathbb{F}=(\mathcal{F}_t)_{t\geq 0}$, $\mathbb{G}=(\mathcal{G}_t)_{t\geq 0}$ with $\mathbb{F}\subseteq\mathbb{G}$, and let X be a strictly positive $(P,\mathbb{G})$-local martingale. Given the probability measure Q_0 introduced above, find a probability measure Q on $\mathcal{G}_\infty$ such that*

1. *$Q=Q_0$ on $\mathcal{G}_{\tau-}$;*
2. *the restrictions of P and Q to $\mathcal{F}_t$ are equivalent for each $t\geq 0$.*

The existence of a solution to this so-called *equivalent measure extension problem* is connected with the behaviour of the optional projection of X on $\mathbb{F}$ by the following result, which is [25, Corollary 1].

Corollary 4.9 *Let X be a strictly positive, strict $\mathbb{G}$-local martingale. If ${}^oX$ is an $\mathbb{F}$-local martingale, then the equivalent measure extension problem has no solution.*

We now provide a result about optional projections under equivalent local martingale measures on a filtration $\mathbb{F}\subseteq\mathbb{F}^X$. We start with a lemma.

Lemma 4.10 *Let X be a strictly positive $\mathbb{G}$-local martingale. Suppose that the equivalent measure extension problem admits a solution for P, $\mathbb{F}$ and $\mathbb{G}$ with $\mathbb{F}\subseteq\mathbb{F}^X$. Then it also admits a solution for P, $\mathbb{F}$ and $\mathbb{F}^X$.*

Proof Note first that X is an $\mathbb{F}^X$-local martingale by Lemma 4.1. Let Q be a solution of the equivalent measure extension problem for P, $\mathbb{F}$ and $\mathbb{G}$. Also let Q_0 and Q_0^X be the Föllmer measures on $\mathcal{G}_{\tau-}$ and $\mathcal{F}^X_{\tau-}$, respectively. By construction, we have that Q_0^X coincides with Q_0 on $\mathcal{F}^X_{\tau-}$. This implies that Q is also an extension of Q_0^X and equivalent to P on $\mathcal{F}_t$ for every $t \geq 0$. Then Q is a solution for the equivalent measure extension problem for P, $\mathbb{F}$ and $\mathbb{F}^X$. □

Theorem 4.11 *Let X be a strictly positive, strict $(P, \mathbb{G})$-local martingale. Consider a probability measure $\tilde{P} \in \mathcal{M}_{\text{strict}}(X, \mathbb{G})$ and suppose that X has the same law under P as under $\tilde{P}$. Also assume that the equivalent measure extension problem admits a solution for P, $\mathbb{F}$ and $\mathbb{G}$, and that $\mathbb{F} \subseteq \mathbb{F}^X$. Then the $\tilde{P}$-optional projection ${}^{\tilde{P},o}X$ of X on $\mathbb{F}$ is not a $(\tilde{P}, \mathbb{F})$-local martingale.*

Proof By Lemma 4.10, the equivalent measure extension problem admits a solution for P, $\mathbb{F}$ and $\mathbb{F}^X$. Consider now the construction of the Föllmer measure as above. The stopping times from (4.3) have by the assumption the same law under P as under $\tilde{P}$, and as $dQ_n = X_{\tau_n} dP$ and $d\tilde{Q}_n = X_{\tau_n} d\tilde{P}$, the measures Q_n and $\tilde{Q}_n$ coincide on $\mathcal{F}^X_{\tau_n}$. Therefore the equivalent measure extension problem also admits a solution for $\tilde{P}$, $\mathbb{F}$ and $\mathbb{F}^X$. By Theorem 4.9, it follows that the $\tilde{P}$-optional projection of X on $\mathbb{F}$ is not a $(\tilde{P}, \mathbb{F})$-local martingale. □

Note that Proposition 4.4 implies that Theorem 4.11 can be applied to all processes with dynamics as in (4.1). An important application when $\mathbb{F} \subseteq \mathbb{F}^X$ is the delayed information setting.

We conclude the section with a general result about optional projections on a delayed filtration. Such a finding also provides a counterexample to the converse implication of Theorem 4.9; see Proposition 4.14.

Proposition 4.12 *Let X be a nonnegative strict $(Q, \mathbb{G})$-local martingale and take $\mathcal{F}_t = \mathcal{G}_{t-h}$, $t \geq 0$, for a given $h > 0$. Then ${}^{Q,o}X$ is not a Q-local martingale.*

Proof We prove the claim by contradiction. Let $(\tau_n)_{n\in\mathbb{N}}$ be a localising sequence of $\mathbb{F}$-stopping times for ${}^{Q,o}X$. Then Föllmer and Protter [11, Theorem 3.7] show that such a sequence also localises X in $\mathbb{G}$. Since X is a strict $(Q, \mathbb{G})$-local martingale, there exists $t \in [h, \infty)$ such that $\delta_t := Q[B_t] > 0$ with

$$B_t := \{X_{t-h} > E^Q[X_t|\mathcal{G}_{t-h}]\}.$$

Denote now $A_t^n := \{\tau_n > t\} \cap B_t$. Note that $A_t^n \in \mathcal{F}_t = \mathcal{G}_{t-h}$ because τ_n is an $\mathbb{F}$-stopping time. Since $Q[\tau_n > t]$ increases to 1 as $n \to \infty$ by dominated convergence, there exists $N > 0$ such that for all $n > N$, we have

$$Q[A_t^n] \geq \frac{\delta_t}{2} > 0.$$

For any $\omega \in A_t^n$, we have

$$\begin{aligned} E^Q[X_t^{\tau_n}|\mathcal{G}_{t-h}](\omega) &= \mathbf{1}_{\{\tau_n>t\}}(\omega)E^Q[X_t^{\tau_n}|\mathcal{G}_{t-h}](\omega) = E^Q[\mathbf{1}_{\{\tau_n>t\}}X_t^{\tau_n}|\mathcal{G}_{t-h}](\omega) \\ &= E^Q[\mathbf{1}_{\{\tau_n>t\}}X_t|\mathcal{G}_{t-h}](\omega) = \mathbf{1}_{\{\tau_n>t\}}(\omega)E^Q[X_t|\mathcal{G}_{t-h}](\omega) \\ &= E^Q[X_t|\mathcal{G}_{t-h}](\omega) < X_{t-h}(\omega) = X_{t-h}^{\tau_n}(\omega), \end{aligned}$$

where the second and fourth equalities follow since $\mathbf{1}_{\{\tau_n>t\}}$ is $\mathcal{G}_{t-h}$-measurable and the strict inequality comes from the definitions of A^n_t and B_t. Thus $(\tau_n)_{n\in\mathbb{N}}$ is not a localising sequence for X, and ${}^{Q,o}X$ cannot be a Q-local martingale. □

Proposition 4.12 immediately implies the following corollary.

Corollary 4.13 *Let X be a nonnegative $\mathbb{G}$-strict local martingale and $\mathcal{F}_t = \mathcal{G}_{t-h}$, $t \geq 0$, for a given $h > 0$. Then*

$$\mathcal{M}_{\text{strict}}(X,\mathbb{G}) \cap \mathcal{M}^o_{\text{loc}}(X,\mathbb{F}) = \emptyset,$$

i.e., (P3) *does not hold.*

The next result, together with Proposition 4.12, shows that optional projections of strict local martingales on delayed filtrations provide a family of examples showing that the sufficient condition of Larsson [25, Corollary 1] is not necessary.

Proposition 4.14 *Take a strictly positive strict $\mathbb{G}$-local martingale X and $\mathcal{F}_t = \mathcal{G}_{t-h}$, $t \geq 0$, for a given $h > 0$. Then the equivalent measure extension problem admits no solution for P, $\mathbb{F}$ and $\mathbb{G}$.*

Proof Let Q_0 be the Föllmer measure introduced above and Q an extension of Q_0 to $\mathcal{G}_\infty$. We show that there exists a time $t \geq 0$ such that the restrictions of P and Q to $\mathcal{F}_t$ are not equivalent. Since X is a strict local martingale, there exists some $r > 0$ such that $E[X_s] < X_0$ for all $s \geq r$. Take $t \geq r + h$ and let τ be the stopping time introduced in (4.4). Then we have

$$P[\tau \leq t-h] = 0, \qquad Q[\tau \leq t-h] = 1 - E[X_{t-h}]/X_0 > 0; \tag{4.5}$$

see Larsson [25]. Since $\{\tau \leq t-h\} \in \mathcal{F}_t$, (4.5) implies that the restrictions of Q and P to $\mathcal{F}_t$ are not equivalent. □

5 The inverse Bessel process

In this section, we study (P1)–(P5) in the case of the inverse Bessel process. Let $B^1 = (B^1_t)_{t\geq 0}$, $B^2 = (B^2_t)_{t\geq 0}$, $B^3 = (B^3_t)_{t\geq 0}$ be standard, independent Brownian motions, starting at $(B^1_0, B^2_0, B^3_0) = (1,0,0)$, on $(\Omega, \mathcal{F}, P)$. We specify the filtration later. In the notation of Sect. 2, we now assume that the nonnegative local martingale X is given by the inverse Bessel process

$$X_t := \big((B^1_t)^2 + (B^2_t)^2 + (B^3_t)^2\big)^{-1/2}, \qquad t \geq 0.$$

Itô's formula implies that under the original probability measure P, X has the dynamics

$$dX_t = -X^3_t(B^1_t dB^1_t + B^2_t dB^2_t + B^3_t dB^3_t), \qquad t \geq 0, \tag{5.1}$$

with $X_0 = 1$. We note that X also solves the SDE

$$dX_t = -X_t^2 dW_t, \qquad t \geq 0, \tag{5.2}$$

where the process W with

$$W_t = \int_0^t X_s (B_s^1 dB_s^1 + B_s^2 dB_s^2 + B_s^3 dB_s^3), \qquad t \geq 0, \tag{5.3}$$

is a one-dimensional Brownian motion by Lévy's characterisation theorem.

In the notation of Sect. 2, we now consider two different choices for the filtration $\mathbb{G}$. In Sect. 5.1, we let $\mathbb{G}$ be the filtration generated by (B^1, B^2, B^3), whereas in Sect. 5.2, it is generated by the Brownian motion W in (5.3). In both cases, X is a strict $\mathbb{G}$-local martingale, and it is therefore interesting to investigate (P1)–(P5) when X is projected on a smaller filtration $\mathbb{F}$. It is well known that the projection of X on the natural filtration of B^1 is not a local martingale; see Föllmer and Protter [11, Theorem 5.1]. See also Kardaras and Ruf [24] for a more general study of projections of sums of Bessel processes. In Sect. 5.1, we consider the case when $\mathbb{F}$ is generated by B^1 and B^2, while in Sect. 5.2, we study delayed information.

Remark 5.1 In the case of the inverse Bessel process, whether $\mathcal{M}_{\mathrm{loc}}(X, \mathbb{G})$ is a singleton or not depends on the choice of $\mathbb{G}$. If $\mathbb{G}$ is generated by one Brownian motion, as in Sect. 5.2 below, P is the only ELMM; see also Delbaen and Schachermayer [8].

In contrast, if $\mathbb{G}$ is the natural filtration of (B^1, B^2, B^3), as in Sect. 5.1 below, we have $\mathcal{M}_{\mathrm{loc}}(X, \mathbb{G}) \neq \{P\}$. Consider for example the $\mathbb{G}$-predictable processes

$$\alpha_t^1 = -\frac{B_t^2}{(B_t^1)^2 + (B_t^2)^2 + 1}, \qquad \alpha_t^2 = \frac{B_t^1}{(B_t^1)^2 + (B_t^2)^2 + 1}, \qquad t \geq 0,$$

and $Z = (Z_t)_{t \geq 0}$ defined by

$$Z_t = \mathcal{E}_t(L), \qquad t \geq 0, \tag{5.4}$$

with

$$L_t := \int_0^t e^{-\delta s} \alpha_s^1 dB_s^1 + \int_0^t e^{-\delta s} \alpha_s^2 dB_s^2, \qquad t \geq 0,$$

for a given $\delta > 0$. With this choice of α^i, $i = 1, 2$, the stochastic exponential Z in (5.4) is a $\mathbb{G}$-adapted process such that $[X, Z] = 0$ a.s. Moreover, the Novikov condition for Z is fulfilled since

$$\exp\left(\frac{1}{2}[Z, Z]_\infty\right) = \exp\left(\frac{1}{2}\int_0^\infty e^{-2\delta s} |\alpha_s|^2 ds\right) \leq \exp\left(\frac{1}{4\delta}\right),$$

and so Z is a uniformly integrable martingale. Defining Q by

$$\left.\frac{dQ}{dP}\right|_{\mathcal{G}_t} = Z_t, \qquad t \geq 0,$$

we thus have $Q \in \mathcal{M}_{\mathrm{loc}}(X, \mathbb{G})$ and $Q \neq P$.

However, Proposition 4.4 together with (5.2) implies that $\mathcal{M}_{\mathrm{true}}(X, \mathbb{H}) = \emptyset$ for every filtration $\mathbb{H}$ to which X is adapted, i.e., there does not exist any measure $Q \approx P$ such that X is a true martingale under Q in $\mathbb{H}$. This also means that for the examples of Sects. 5.1 and 5.2, (P1) holds if and only if (P3) holds.

5.1 Optional projection on the filtration generated by (B^1, B^2)

In this section, we provide an example for which (P4) is satisfied by introducing two filtrations $\mathbb{F} \subseteq \mathbb{G}$ such that for the inverse Bessel process X, we have

$$\mathcal{M}_{\mathrm{loc}}(X, \mathbb{G}, \mathbb{F}) = \mathcal{M}^o_{\mathrm{loc}}(X, \mathbb{G}, \mathbb{F}).$$

Let $\mathbb{G}$ be the natural filtration of (B^1, B^2, B^3) and $\mathbb{F}$ be generated by (B^1, B^2). As usual, we denote the optional projection of X on $\mathbb{F}$ by ${}^oX$. Föllmer and Protter [11, Theorem 5.2] state that ${}^oX$ is an $\mathbb{F}$-local martingale and has the form

$${}^oX_t = u(B^1_t, B^2_t, t), \qquad t \geq 0,$$

with

$$u(x, y, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{x^2 + y^2}{4t}\right) K_0\left(\frac{x^2 + y^2}{4t}\right).$$

Here and below, K_n, $n \geq 0$, are the modified Bessel functions of the second kind. In particular, we have

$$\partial_x u(x, y, t) = x\psi(x, y, t), \qquad \partial_y u(x, y, t) = y\psi(x, y, t), \tag{5.5}$$

where

$$\psi(x, y, t) = \frac{1}{\sqrt{2\pi t}} \exp\left(\frac{x^2 + y^2}{4t}\right)\left(K_0\left(\frac{x^2 + y^2}{4t}\right) - K_1\left(\frac{x^2 + y^2}{4t}\right)\right). \tag{5.6}$$

Since ${}^oX$ is an $\mathbb{F}$-local martingale, we focus here on (P4). The following theorem shows that it holds in this example.

Theorem 5.2 *Let $\mathbb{F}$ be the natural filtration of (B^1, B^2). Then*

$$\mathcal{M}_{\mathrm{loc}}(X, \mathbb{G}, \mathbb{F}) = \mathcal{M}^o_{\mathrm{loc}}(X, \mathbb{G}, \mathbb{F}).$$

Proof We introduce the sequence of $\mathbb{F}$-stopping times $(\tau_n)_{n\in\mathbb{N}}$ with

$$\tau_n := \inf\left\{t \geq 0 : (B^1_t)^2 + (B^2_t)^2 \leq \frac{1}{n}\right\}, \qquad n \geq 1.$$

We have $\lim_{n\to\infty} \tau_n = \infty$ because the origin $(0, 0)$ is polar for a two-dimensional Brownian motion; so the sequence localises X to a bounded martingale. Given a probability measure $Q \in \mathcal{M}_{\mathrm{loc}}(X, \mathbb{G}, \mathbb{F})$, Theorem 4.3 implies that ${}^{Q,o}X$ is a $(Q, \mathbb{F})$-local martingale, and so $\mathcal{M}_{\mathrm{loc}}(X, \mathbb{G}, \mathbb{F}) \subseteq \mathcal{M}^o_{\mathrm{loc}}(X, \mathbb{G}, \mathbb{F})$.

To prove the converse inclusion, take $Q \in \mathcal{M}^o_{\text{loc}}(X, \mathbb{G}, \mathbb{F})$, which means that ${}^{Q,o}X$ is a $(Q, \mathbb{F})$-local martingale and ${}^{\mathcal{G}}Z$ is $\mathbb{F}$-adapted. By Lemma 4.2, ${}^{Q,o}X = {}^oX$; so ${}^oX$ is both a $(P, \mathbb{F})$- and a $(Q, \mathbb{F})$-local martingale, and thus $[{}^{\mathcal{F}}Z, {}^oX]$ is zero by Girsanov's theorem. Now

$$ {}^{\mathcal{F}}Z_t = \frac{dQ}{dP}\bigg|_{\mathcal{F}_t} = \mathcal{E}_t\bigg(\int \alpha^1_s dB^1_s + \int \alpha^2_s dB^2_s\bigg), \qquad t \geq 0, \tag{5.7} $$

where α^1 and α^2 are $\mathbb{F}$-predictable processes such that the Doléans-Dade exponential in (5.7) is well defined and a uniformly integrable martingale. By (5.5)–(5.7), we have

$$ [{}^{\mathcal{F}}Z, {}^oX]_t = \int_0^t {}^{\mathcal{F}}Z_s \psi(B^1_s, B^2_s, s)(\alpha^1_s B^1_s + \alpha^2_s B^2_s) ds, \qquad t \geq 0. \tag{5.8} $$

Since the left-hand side of (5.8) is zero and $\psi(x, y, t) < 0$ for $x, y < \infty$ and $t > 0$ (see for example Yang and Chu [34]), we get

$$ \alpha^1_t B^1_t + \alpha^2_t B^2_t = 0, \qquad t \geq 0, \tag{5.9} $$

P-almost surely, as P is equivalent to Q. On the other hand, from (5.1) and (5.7) and since ${}^{\mathcal{F}}Z = {}^{\mathcal{G}}Z$, it follows that

$$ [{}^{\mathcal{G}}Z, X]_t = -\int_0^t {}^{\mathcal{G}}Z_s X^3_s (\alpha^1_s B^1_s + \alpha^2_s B^2_s) ds, \qquad t \geq 0, $$

and this is zero P-a.s. by (5.9). Since X is a $(P, \mathbb{G})$-local martingale, this implies that X is also a $(Q, \mathbb{G})$-local martingale. Hence $Q \in \mathcal{M}_{\text{loc}}(X, \mathbb{G}, \mathbb{F})$.

□

5.2 Delayed information

We now consider a market model where $\mathbb{G}$ is the filtration generated by the Brownian motion W in (5.3) and $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ is given by $\mathcal{F}_t = \mathcal{G}_{t-\epsilon}$ with $\epsilon > 0$. As explained above, this means that investors have access to the information about W, with respect to which X is adapted by (5.2), only with a positive delay ϵ.

In this setting, Proposition 4.12 implies that the optional projection of X on $\mathbb{F}$ is not a local martingale so that (P1), (P3) and (P4) are not satisfied. We show that (P2) and (P5) (which are equivalent since $\mathcal{M}_{\text{loc}}(X, \mathbb{G}) = \{P\}$) do not hold either. However, in Example 5.7, we introduce a modification M of the inverse Bessel process X such that (P2) and (P5) are satisfied, but (P1), (P3) and (P4) are not.

We start our analysis with the following result.

Lemma 5.3 *For every* $\epsilon > 0$, *we have*

$$ E[X_{t+\epsilon} | \sigma(B^1_t, B^2_t, B^3_t)] = X_t \operatorname{erf}\bigg(\frac{1}{X_t\sqrt{2\epsilon}}\bigg), $$

where $\operatorname{erf}(x) := \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt$.

Proof We have

$$E[X_{t+\epsilon}|\sigma(B_t^1, B_t^2, B_t^3)] = u(\epsilon, B_t^1, B_t^2, B_t^3)$$

with

$$u(t,a,b,c) = (2\pi t)^{-3/2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{e^{-\frac{1}{2t}((x-a)^2+(y-b)^2+(z-c)^2)}}{\sqrt{x^2+y^2+z^2}}dzdydx =: I.$$

We set $R = \sqrt{a^2+b^2+c^2}$, $r = \sqrt{x^2+y^2+z^2}$. The above integral can be written in spherical coordinates as

$$\begin{aligned} I &= (2\pi t)^{-3/2}\int_0^{2\pi}\int_0^{\infty} r^2\frac{1}{r}\int_0^{\pi}\sin(\theta)e^{-\frac{1}{2t}(r^2-2rR\cos(\theta)+R^2)}d\theta dr d\phi \\ &= \frac{2}{R\sqrt{\pi}}\int_0^{\frac{R}{\sqrt{2t}}} e^{-r^2}dr = \frac{1}{\sqrt{a^2+b^2+c^2}}\mathrm{erf}\left(\sqrt{\frac{a^2+b^2+c^2}{2t}}\right). \end{aligned}$$

Thus

$$\begin{aligned} &E[X_{t+\epsilon}|\sigma(B_t^1, B_t^2, B_t^3)] \\ &= \frac{1}{\sqrt{(B_t^1)^2+(B_t^2)^2+(B_t^3)^2}}\mathrm{erf}\left(\sqrt{\frac{(B_t^1)^2+(B_t^2)^2+(B_t^3)^2}{2\epsilon}}\right) \\ &= X_t\mathrm{erf}\left(\frac{1}{X_t\sqrt{2\epsilon}}\right). \end{aligned}$$

□

Proposition 5.4 *Let $\mathbb{G} = (\mathcal{G}_t)_{t\ge 0}$ be the filtration generated by the Brownian motion W in (5.3) and $\mathbb{F} = (\mathcal{F}_t)_{t\ge 0}$ be given by $\mathcal{F}_t := \mathcal{G}_{t-\epsilon}$ with $\epsilon > 0$. Then*

$$^oX_{t+\epsilon} = E[X_{t+\epsilon}|\mathcal{F}_{t+\epsilon}] = X_t\mathrm{erf}\left(\frac{1}{X_t\sqrt{2\epsilon}}\right), \qquad t \ge 0.$$

Proof By (5.2) and (5.3), we have $\sigma(W_t) \subseteq \sigma(B_t^1, B_t^2, B_t^3)$. Moreover, X is like its reciprocal, the Bessel process, a Markov process with respect to $\mathbb{G}$. Combining this with Lemma 5.3 gives

$$\begin{aligned} E[X_{t+\epsilon}|\mathcal{F}_{t+\epsilon}] &= E[X_{t+\epsilon}|\sigma(W_t)] = E\big[E[X_{t+\epsilon}|\sigma(B_t^1, B_t^2, B_t^3)]\big|\sigma(W_t)\big] \\ &= E\left[X_t\mathrm{erf}\left(\frac{1}{X_t\sqrt{2\epsilon}}\right)\middle|\sigma(W_t)\right] = X_t\mathrm{erf}\left(\frac{1}{X_t\sqrt{2\epsilon}}\right), \qquad t\ge 0, \end{aligned}$$

as X_t is $\sigma(W_t)$-measurable by the Markov property. □

By Proposition 5.4, we have

$$^oX_{t+\epsilon} = f(X_t), \qquad t \ge 0,$$

where $f(x) = x\mathrm{erf}(\frac{1}{x\sqrt{2\epsilon}})$. Since

$$f'(x) = -\frac{\sqrt{2}e^{-\frac{1}{2\epsilon x^2}}}{x\sqrt{\pi\epsilon}} + \mathrm{erf}\Big(\frac{1}{x\sqrt{2\epsilon}}\Big), \qquad f''(x) = -\frac{\sqrt{2}\epsilon^{-\frac{3}{2}}e^{-\frac{1}{2\epsilon x^2}}}{x^4\sqrt{\pi}},$$

Itô's formula gives

$$\begin{aligned} d^oX_{t+\epsilon} &= \Big(-\frac{\sqrt{2}e^{-\frac{1}{2\epsilon X_t^2}}}{X_t\sqrt{\pi\epsilon}} + \mathrm{erf}\Big(\frac{1}{X_t\sqrt{2\epsilon}}\Big)\Big)dX_t - \frac{\sqrt{2}\epsilon^{-\frac{3}{2}}e^{-\frac{1}{2\epsilon X_t^2}}}{X_t^4\sqrt{\pi}}d[X,X]_t \\ &= \Big(\frac{\sqrt{2}e^{-\frac{1}{2\epsilon X_t^2}}}{\sqrt{\pi\epsilon}}X_t - \mathrm{erf}\Big(\frac{1}{X_t\sqrt{2\epsilon}}\Big)X_t^2\Big)dW_t - \sqrt{\frac{2}{\pi}}\epsilon^{-\frac{3}{2}}e^{-\frac{1}{2\epsilon X_t^2}}dt. \end{aligned} \tag{5.10}$$

The above expression shows that the optional projection is a strict $\mathbb{F}$-supermartingale as the drift is strictly negative. Note that this is stated for the projection on the delayed filtration for a general nonnegative strict local martingale in Proposition 4.12. Since we have $\mathcal{M}_{\mathrm{loc}}(X,\mathbb{G}) = \{P\}$ by Remark 5.1, this implies that

$$\mathcal{M}_{\mathrm{loc}}(X,\mathbb{G}) \cap \mathcal{M}^o_{\mathrm{loc}}(X,\mathbb{F}) = \emptyset, \tag{5.11}$$

i.e., (P1), (P3) and (P4) do not hold.

Moreover, the following result implies that (P2) and (P5) are not satisfied.

Theorem 5.5 *Let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be the filtration generated by the Brownian motion W in (5.3) and $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ with $\mathcal{F}_t := \mathcal{G}_{t-\epsilon}$ with $\epsilon > 0$. Then*

$$\mathcal{M}_{\mathrm{loc}}({}^oX,\mathbb{F}) = \emptyset.$$

To prove Theorem 5.5, we rely on some results provided by Mijatović and Urusov [27] which we now recall. Consider the state space $J = (\ell, r)$, $-\infty \leq \ell < r \leq \infty$, and a J-valued diffusion $Y = (Y_t)_{t\geq 0}$ on some filtered probability space, governed by the SDE

$$dY_t = \mu_Y(Y_t)dt + \sigma_Y(Y_t)dB_t, \qquad t \geq 0, \tag{5.12}$$

where $Y_0 = y_0 \in J$, B is a one-dimensional Brownian motion and $\mu_Y, \sigma_Y : \mathbb{R} \to \mathbb{R}$ satisfy

$$\sigma_Y(x) \neq 0, \qquad \forall x \in J \tag{5.13}$$

and

$$\frac{1}{\sigma_Y^2}, \frac{\mu_Y}{\sigma_Y^2} \in L^1_{\mathrm{loc}}(J). \tag{5.14}$$

Here $L^1_{\text{loc}}(J)$ denotes the class of locally integrable functions ψ on J, i.e., the measurable functions $\psi : (J, \mathcal{B}(J)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ that are Lebesgue-integrable on compact subsets of J. Consider the stochastic exponential

$$\mathcal{E}_t\left(\int g(Y_u)dB_u\right), \qquad t \geq 0, \tag{5.15}$$

with $g : \mathbb{R} \to \mathbb{R}$ such that

$$\frac{g^2}{\sigma_Y^2} \in L^1_{\text{loc}}(J). \tag{5.16}$$

Put $\bar{J} = [\ell, r]$ and, fixing an arbitrary $c \in J$, define

$$\rho(x) := \exp\left(-\int_c^x \frac{2\mu_Y}{\sigma_Y^2}(y)dy\right), \qquad x \in J, \tag{5.17}$$

$$\tilde{\rho}(x) := \rho(x)\exp\left(-\int_c^x \frac{2g}{\sigma_Y}(y)dy\right), \qquad x \in J, \tag{5.18}$$

$$s(x) := \int_c^x \rho(y)dy, \qquad x \in \bar{J}, \tag{5.19}$$

$$\tilde{s}(x) := \int_c^x \tilde{\rho}(y)dy, \qquad x \in \bar{J}. \tag{5.20}$$

Define $s(r) = \lim_{x\to r-} s(x)$, $s(\ell) = \lim_{x\to \ell+} s(x)$ and analogously for $\tilde{s}$ and $\tilde{\rho}$. Define

$$L^1_{\text{loc}}(r-) := \left\{\psi : \big(J, \mathcal{B}(J)\big) \to \big(\mathbb{R}, \mathcal{B}(\mathbb{R})\big) : \int_x^r |\psi(y)|dy < \infty \text{ for some } x \in J\right\},$$

and $L^1_{\text{loc}}(\ell+)$ analogously. We report here [27, Theorem 2.1].

Theorem 5.6 *Let the functions μ_Y, σ_Y and g satisfy conditions* (5.13), (5.14) *and* (5.16) *and let Y be a solution of the SDE* (5.12). *Then the Doléans-Dade exponential given by* (5.15) *is a true martingale if and only if both of the following requirements are satisfied*:

1) *It does not hold that*

$$\tilde{s}(r) < \infty \quad \textit{and} \quad \frac{\tilde{s}(r) - \tilde{s}}{\tilde{\rho}\sigma_Y^2} \in L^1_{\text{loc}}(r-), \tag{5.21}$$

or it holds that

$$s(r) < \infty \quad \textit{and} \quad \frac{(s(r) - s)g^2}{\rho\sigma_Y^2} \in L^1_{\text{loc}}(r-). \tag{5.22}$$

2) *It does not hold that*

$$\tilde{s}(\ell) > -\infty \quad \textit{and} \quad \frac{\tilde{s} - \tilde{s}(\ell)}{\tilde{\rho}\sigma_Y^2} \in L^1_{\mathrm{loc}}(\ell+),$$

or it holds that

$$s(\ell) > -\infty \quad \textit{and} \quad \frac{(s - s(\ell))g^2}{\rho\sigma_Y^2} \in L^1_{\mathrm{loc}}(\ell+).$$

We now use Theorem 5.6 in order to prove Theorem 5.5.

Proof of Theorem 5.5 By (5.10), we have

$$d\,{}^oX_{t+\epsilon} = \mu(X_t)dt + \sigma(X_t)dW_t, \qquad t \geq 0,$$

with

$$\mu(x) = -\sqrt{\frac{2}{\pi}}\epsilon^{-\frac{3}{2}}e^{-\frac{1}{2\epsilon x^2}}, \qquad \sigma(x) = x\sqrt{\frac{2}{\pi\epsilon}}e^{-\frac{1}{2\epsilon x^2}} - x^2\mathrm{erf}\left(\frac{1}{x\sqrt{2\epsilon}}\right). \tag{5.23}$$

By Girsanov's theorem, there exists a probability measure $Q \in \mathcal{M}_{\mathrm{loc}}({}^oX, \mathbb{F})$ only if the Doléans-Dade exponential

$$\left.\frac{dQ}{dP}\right|_{\mathcal{G}_t} = Z_t = \mathcal{E}_t\left(\int \alpha_s dW_s\right), \qquad t \geq 0, \tag{5.24}$$

with

$$\alpha_t = -\frac{\mu(X_t)}{\sigma(X_t)}, \qquad t \geq 0, \tag{5.25}$$

is a true martingale. In order to prove that this is not the case, we apply Theorem 5.6. In our case, by (5.2), (5.23) and (5.25), we have $Y = X$, $J = (0, \infty)$, $\mu_Y \equiv 0$, $\sigma_Y(x) = -x^2$ and

$$g(x) = \frac{\sqrt{2/\pi}\epsilon^{-\frac{3}{2}}e^{-\frac{1}{2\epsilon x^2}}}{x(\sqrt{\frac{2}{\pi\epsilon}}e^{-\frac{1}{2\epsilon x^2}} - x\mathrm{erf}(\frac{1}{x\sqrt{2\epsilon}}))}.$$

Note that conditions (5.13) and (5.14) are satisfied. In order to prove that (5.16) also holds, it is enough to check that

$$x\mathrm{erf}\left(\frac{1}{x\sqrt{2\epsilon}}\right) - \sqrt{\frac{2}{\pi\epsilon}}e^{-\frac{1}{2\epsilon x^2}} > 0 \qquad \text{for every } x \in (0, \infty). \tag{5.26}$$

This holds if and only if

$$\mathrm{erf}(y)\frac{1}{y\sqrt{2\epsilon}} - \sqrt{\frac{2}{\pi\epsilon}}e^{-y^2} > 0 \qquad \text{for every } y \in (0, \infty),$$

i.e., if and only if

$$F(y) := \operatorname{erf}(y) - \frac{2}{\sqrt{\pi}} y e^{-y^2} > 0 \qquad \text{for every } y \in (0, \infty).$$

The last condition if fulfilled since $F(0) = 0$ and $F'(y) = \frac{4}{\sqrt{\pi}} y^2 e^{-y^2} > 0$ for every $y > 0$. So we have (5.26) and the assumptions of Theorem 5.6 are thus satisfied.

We now show that condition (5.22) fails whereas (5.21) is satisfied, implying that the process Z introduced in (5.24) is not a martingale. Consider first ρ and s defined in (5.17) and (5.19), respectively. We have $\rho \equiv 1$ so that $s(x) = x - c$ for any $c > 0$. This implies that $s(\infty) = +\infty$ so that condition (5.22) fails. We now check condition (5.21). We have

$$\lim_{x\to\infty} \frac{e^{-\frac{1}{2\epsilon x^2}}}{x^2(\sqrt{\frac{2}{\pi\epsilon}} e^{-\frac{1}{2\epsilon x^2}} - x \operatorname{erf}(\frac{1}{x\sqrt{2\epsilon}}))} = -3\sqrt{\frac{\pi}{2}} \epsilon^{3/2},$$

and so

$$\lim_{x\to\infty} -x\frac{2g(x)}{\sigma_Y(x)} = \lim_{x\to\infty} 2\sqrt{2/\pi}\,\epsilon^{-\frac{3}{2}} \frac{e^{-\frac{1}{2\epsilon x^2}}}{x^2(\sqrt{\frac{2}{\pi\epsilon}} e^{-\frac{1}{2\epsilon x^2}} - x \operatorname{erf}(\frac{1}{x\sqrt{2\epsilon}}))} = -6.$$

Hence for every $\delta > 0$, there exists $\bar{x} > 0$ such that

$$\left| -x\frac{2g(x)}{\sigma_Y(x)} + 6 \right| \le \delta \qquad \text{for every } x \ge \bar{x}. \tag{5.27}$$

We fix $\delta < 1$ and choose $\bar{x} > 0$ such that (5.27) holds. For every $x > \bar{x}$, we get the estimate

$$\begin{aligned} \left| -\int_{\bar{x}}^{x} \frac{2g(y)}{\sigma_Y(y)} dy + \int_{\bar{x}}^{x} \frac{6}{y} dy \right| &\le \int_{\bar{x}}^{x} \left| -\frac{2g(y)}{\sigma_Y(y)} + \frac{6}{y} \right| dy \\ &\le \int_{\bar{x}}^{x} \frac{1}{y} \left| -y\frac{2g(y)}{\sigma_Y(y)} + 6 \right| dy \\ &\le \delta(\log x - \log \bar{x}). \end{aligned}$$

Thus for every $x > \bar{x}$, we have

$$(-6-\delta)(\log x - \log \bar{x}) \le -\int_{\bar{x}}^{x} \frac{2g(y)}{\sigma_Y(y)} dy \le (-6+\delta)(\log x - \log \bar{x}).$$

Taking $\tilde{\rho}$ as in (5.18) and choosing $c = \bar{x}$, this implies for every $x > \bar{x}$ that

$$\left(\frac{x}{\bar{x}}\right)^{-6-\delta} \le \tilde{\rho}(x) \le \left(\frac{x}{\bar{x}}\right)^{-6+\delta}. \tag{5.28}$$

Hence, taking $\tilde{s}$ as in (5.20) and choosing again $c = \bar{x}$, we have for every $x > \bar{x}$ that

$$\int_{\bar{x}}^{x} \left(\frac{y}{\bar{x}}\right)^{-6-\delta} dy \leq \tilde{s}(x) = \int_{\bar{x}}^{x} \tilde{\rho}(y) dy \leq \int_{\bar{x}}^{x} \left(\frac{y}{\bar{x}}\right)^{-6+\delta} dy$$

so that $\tilde{s}(\infty) < \infty$, and in particular,

$$\bar{x}^{6+\delta} \frac{x^{-5-\delta}}{5+\delta} \leq \tilde{s}(\infty) - \tilde{s}(x) = \int_{x}^{\infty} \tilde{\rho}(y) dy \leq \bar{x}^{6-\delta} \frac{x^{-5+\delta}}{5-\delta}.$$

Together with (5.28), this implies

$$\frac{\bar{x}^{2\delta}}{5+\delta} x^{1-2\delta} \leq \frac{\tilde{s}(\infty) - \tilde{s}(x)}{\tilde{\rho}(x)} \leq \frac{\bar{x}^{-2\delta}}{5-\delta} x^{1+2\delta}$$

for every $x > \bar{x}$, so that

$$\frac{\bar{x}^{2\delta}}{5+\delta} x^{-3-2\delta} \leq \frac{\tilde{s}(\infty) - \tilde{s}(x)}{\sigma_Y^2(x)\tilde{\rho}(x)} \leq \frac{\bar{x}^{-2\delta}}{5-\delta} x^{-3+2\delta}.$$

Therefore, as $\delta < 1$ by the choice of $\bar{x}$, we have $\frac{\tilde{s}(\infty) - \tilde{s}}{\tilde{\rho}\sigma_Y^2} \in L^1_{\text{loc}}(\infty-)$ and (5.21) is satisfied. By Theorem 5.6, it follows that Z in (5.24) is not a martingale. □

We now give an example of a process whose optional projection on a delayed filtration is not a local martingale, but admits an equivalent local martingale measure.

Example 5.7 Consider again the filtration $\mathbb{G} = (\mathcal{G}_t)_{t \geq 0}$ generated by the Brownian motion W in (5.3), and define $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ by $\mathcal{F}_t := \mathcal{G}_{t-\epsilon}$ with $\epsilon > 0$. Introduce the process $M = (M_t)_{t \geq 0}$ with $M_t = X_t - \int_0^T (1+s) dW_s$, where X is the inverse Bessel process. Thus

$$\begin{aligned} {}^o M_{t+\epsilon} &= E[M_{t+\epsilon} | \mathcal{G}_t] = E\left[X_{t+\epsilon} - \int_0^{t+\epsilon} (1+s) dW_s \middle| \mathcal{G}_t \right] \\ &= X_t \operatorname{erf}\left(\frac{1}{X_t \sqrt{2\epsilon}} \right) - \int_0^t (1+s) dW_s, \qquad t \geq 0, \end{aligned}$$

where the last equality comes from Proposition 5.4 and the martingale property of $\int (1+s) dW_s$. From (5.10), we therefore have

$$\begin{aligned} d\,{}^o M_{t+\epsilon} &= \left(\sqrt{\frac{2}{\pi\epsilon}} e^{-\frac{1}{2\epsilon X_t^2}} X_t - \operatorname{erf}\left(\frac{1}{X_t \sqrt{2\epsilon}} \right) X_t^2 - (1+t) \right) dW_t \\ &\quad - \sqrt{\frac{2}{\pi}} \epsilon^{-\frac{3}{2}} e^{-\frac{1}{2\epsilon X_t^2}} dt, \qquad t \geq 0. \end{aligned}$$

It is then clear that ${}^o M$ is not an $\mathbb{F}$-local martingale. This implies that

$$\mathcal{M}_{\text{loc}}(M, \mathbb{G}) \cap \mathcal{M}^o_{\text{loc}}(M, \mathbb{F}) = \emptyset,$$

since $\mathcal{M}_{\text{loc}}(M, \mathbb{G}) = \{P\}$ by Remark 5.1. So (P1), (P3) and (P4) do not hold here.

We now introduce the Doléans-Dade exponential

$$\bar{Z}_t = \mathcal{E}_t\left(\int \bar{\alpha}_s dW_s\right), \qquad t \geq 0,$$

with

$$\bar{\alpha}_t = \frac{\sqrt{\frac{2}{\pi}}\epsilon^{-\frac{3}{2}} e^{-\frac{1}{2\epsilon X_t^2}}}{\sqrt{\frac{2}{\pi\epsilon}} e^{-\frac{1}{2\epsilon X_t^2}} X_t - \mathrm{erf}(\frac{1}{X_t\sqrt{2\epsilon}}) X_t^2 - (1+t)}, \qquad t \geq 0,$$

and define $\bar{Q}$ by $\frac{d\bar{Q}}{dP}|_{\mathcal{G}_t} = \bar{Z}_t$, $t \geq 0$. By (5.26), we have $|\bar{\alpha}_t| \leq \sqrt{\frac{2}{\pi}}\epsilon^{-\frac{3}{2}}(1+t)^{-1}$ for all $t \geq 0$. Thus

$$\exp\left(\frac{1}{2}\int_0^\infty |\bar{\alpha}_s|^2 ds\right) \leq \exp\left(\frac{1}{\pi}\epsilon^{-3}\int_0^\infty (1+s)^{-2} ds\right) < \infty,$$

and so Novikov's condition is satisfied and $\bar{Z}$ is a uniformly integrable martingale. By Girsanov's theorem, $\bar{Q} \in \mathcal{M}_{\mathrm{loc}}({}^o M, \mathbb{G})$. Hence (P2) and (P5) are satisfied.

Remark 5.8 Example 5.7 provides a case where we can explicitly compute the optional projection of the stochastic process M. In particular, in contrast to the general setting, the nonnegativity of the process is not needed here to ensure the existence of the optional projection. Furthermore, M retains the supermartingale property since it is the sum of $-\int_0^t (1+s) dW_s$, which is a martingale, and X, which is a local martingale bounded from below. For this reason, the optional projection is also a supermartingale.

Remark 5.9 In Sect. 5.2, $\mathbb{G}$ is the natural filtration of W and hence coincides with the natural filtration of X by (5.2). From (5.11), we obtain that (P1) is not satisfied in this setting, and also not if we project from the natural filtration $\bar{\mathbb{G}}$ of (B^1, B^2, B^3). In fact, from Corollary 4.6, we have $\mathcal{M}_{\mathrm{loc}}(X, \bar{\mathbb{G}}) = \mathcal{M}_{\mathrm{strict}}(X, \bar{\mathbb{G}})$, and Proposition 4.12 implies that $\mathcal{M}_{\mathrm{strict}}(X, \bar{\mathbb{G}}) \cap \mathcal{M}^o_{\mathrm{loc}}(X, \mathbb{F}) = \emptyset$ if $\mathcal{F}_t = \bar{\mathcal{G}}_{t-h}$ or $\mathcal{F}_t = \mathcal{F}^X_{t-h}$, $t \geq 0$, with $h > 0$. Note that when $\mathbb{F}$ is the delayed filtration of $\mathbb{F}^X$, one can get the same result from Corollary 4.7, which implies that for every $Q \in \mathcal{M}_{\mathrm{loc}}(X, \bar{\mathbb{G}})$,

$$E^Q[X_t|\mathcal{F}_t] = E^P[X_t|\mathcal{F}_t], \qquad \text{a.s.}, t \geq 0.$$

In particular, for that choice of $\mathbb{F}$, the Q-optional projection ${}^{Q,o}X$ is a $(Q, \mathbb{F})$-local martingale if and only if Q is an equivalent local martingale measure for ${}^o X$, which has dynamics given in (5.10). However, this is never the case because

$$\mathcal{M}_{\mathrm{loc}}({}^o X, \mathbb{F}) = \emptyset$$

by the same arguments as in the proof of Theorem 5.5.

6 A stochastic volatility model

In this section, we assume that X comes from a stochastic volatility model and is a local martingale with respect to a filtration $\mathbb{G}$. We then consider a subfiltration $\hat{\mathbb{F}}$ of $\mathbb{G}$ such that (P1) is satisfied even if the optional projection of X on $\hat{\mathbb{F}}$ is not a local martingale; see Theorem 6.11 below. We give Examples 6.7 and 6.8 to show when (P2) or (P3) hold, respectively. Example 6.9 provides instead a setting where (P5) is satisfied whereas (P2) is not. Finally, Example 6.12 shows a case where (P3) does not hold whereas (P1) does.

We introduce a three-dimensional Brownian motion $B = (B^1, B^2, B^3)$ on a filtered probability space $(\Omega, \mathcal{F}, P, \mathbb{G})$, and consider a stochastic volatility model of the form

$$dX_t = \sigma_1 v_t^\alpha X_t dB_t^1 + \sigma_2 v_t^\alpha X_t dB_t^2, \quad t \geq 0, \qquad X_0 = x > 0, \tag{6.1}$$

$$dv_t = a_1 v_t dB_t^1 + a_2 v_t dB_t^2 + a_3 v_t dB_t^3 + \rho(L - v_t)dt, \quad t \geq 0, \qquad v_0 = 1, \tag{6.2}$$

where $\alpha, \rho, L \in \mathbb{R}_+$ and $\sigma_1, \sigma_2, a_1, a_2, a_3 \in \mathbb{R}$.

Remark 6.1 The class of stochastic volatility models (6.1) and (6.2) reduces to the class considered in Sin [31] for $a_3 = 0$ and to the class presented in Biagini et al. [2] for $\rho = 0$ and $\alpha = 1$. Therefore, all the results of this section can be applied to these particular cases. For analogous studies of a similar class of stochastic volatility models, see also Lions and Musiela [26].

The next result states that under a condition on the coefficients of (6.1) and (6.2), X is a strict $(P, \mathbb{G})$-local martingale, but $\mathcal{M}_{\text{true}}(X, \mathbb{G}) \neq \emptyset$.

Proposition 6.2 *The system of SDEs* (6.1) *and* (6.2) *admits a unique strong solution* (X, v). *The process* X *satisfies the following properties*:

1) X *is a local martingale, and it is a true martingale if and only if*

$$a_1\sigma_1 + a_2\sigma_2 \leq 0.$$

2) *For every* $T > 0$, *there exists a probability measure* Q *equivalent to* P *on* $\mathcal{G}_T$ *such that* X *is a true* Q*-martingale on* $[0, T]$.

Proof Existence and uniqueness of a strong solution to (6.1) and (6.2) can be proved as an extension of [31, Remark 2.2]. The proofs of the two claims are easy extensions of the proofs of [31, Theorem 3.2] and [2, Theorem 5.1], respectively. □

We now give two results that provide a relation between the expectation of X and the explosion time of a process associated to the volatility v.

Lemma 6.3 *Let* (X, v) *satisfy the system of SDEs* (6.1) *and* (6.2). *Then*

$$E[X_t] = X_0 P[\hat{v} \textit{ does not explode to } +\infty \textit{ up to time } t], \qquad t \geq 0,$$

where $\hat{v} = (\hat{v}_t)_{t \geq 0}$ *is given by*

$$d\hat{v}_t = a_1 \hat{v}_t dB_t^1 + a_2 \hat{v}_t dB_t^2 + a_3 \hat{v}_t dB_t^3 + \rho(L - \hat{v}_t)dt + (a_1\sigma_1 + a_2\sigma_2)\hat{v}_t^{\alpha+1} dt, \quad t \geq 0, \qquad \hat{v}_0 = 1. \tag{6.3}$$

Proof This result is a particular case of Proposition 6.10 below. □

Lemma 6.4 *The (unique) solution to (6.3) explodes to* $+\infty$ *in finite time with positive probability if and only if* $a_1\sigma_1 + a_2\sigma_2 > 0$. *Moreover, if* $a_1\sigma_1 + a_2\sigma_2 > 0$, *it does not reach zero in finite time.*

Proof The result is given in Sin [31, Lemma 4.3] when $a_3 = 0$, and proved by using Feller's test of explosions. In that framework, the test is applicable because $\hat{v}$ is a one-dimensional Itô diffusion with respect to the Brownian motion $(a \cdot B)/|a|$, with $a = (a_1, a_2)$ and $B = (B^1, B^2)$. The author introduces $\sigma = (\sigma_1, \sigma_2)$ and proves that $\hat{v}$ explodes to $+\infty$ with positive probability in finite time and does not reach the origin in finite time if $a \cdot \sigma > 0$. In our case, the proof comes as an easy extension by considering now $a = (a_1, a_2, a_3)$ and $\sigma = (\sigma_1, \sigma_2, 0)$. □

The following is from Karatzas and Ruf [22, Proposition 5.2].

Lemma 6.5 *Fix an open interval* $I = (\ell, r)$ *with* $-\infty \leq \ell < r \leq \infty$ *and consider the stochastic differential equation*

$$dY_t = s(Y_t)\big(dW_t + b(Y_t)dt\big), \quad t \geq 0, \qquad Y_0 = \xi, \tag{6.4}$$

where $\xi \in I$ *and* W *denotes a standard Brownian motion. Suppose that the functions* $b : (I, \mathcal{B}(I)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$ *and* $s : (I, \mathcal{B}(I)) \to (\mathbb{R} \setminus \{0\}, \mathcal{B}(\mathbb{R} \setminus \{0\}))$ *are measurable and satisfy*

$$\int_K \left(\frac{1}{s^2(y)} + \left| \frac{b(y)}{s(y)} \right| \right) dy < \infty \qquad \text{for every compact set } K \subseteq I. \tag{6.5}$$

Call τ^ξ *the first time when the weak solution* Y *to (6.4), unique in the sense of distribution, exits the open interval* I. *Introduce the function* $U : (0, \infty) \times I \to \mathbb{R}_+$ *by*

$$U(t, \xi) := P[\tau^\xi > t].$$

If the functions s *and* b *are locally Hölder-continuous on* I, *the function* U *is of class* $C([0, \infty) \times I) \cap C^{1,2}((0, \infty) \times I)$.

Applying Lemma 6.5 to our setting, we get the following result.

Lemma 6.6 *Consider the solution* $\hat{v}$ *to (6.3), supposing* $a_1\sigma_1 + a_2\sigma_2 > 0$ *and* $\rho = 0$. *Define the function* $m : (0, \infty) \to \mathbb{R}_+$ *by*

$$m_t = P[\hat{v} \text{ does not explode to } +\infty \text{ up to time } t]. \tag{6.6}$$

Then $m \in C^1((0, \infty))$.

Proof Note that $\hat{v}$ is a one-dimensional Itô diffusion with respect to the Brownian motion $W = (a \cdot B)/|a|$, with $a = (a_1, a_2, a_3)$ and $B = (B^1, B^2, B^3)$. In particular, we have

$$d\hat{v}_t = |a|\hat{v}_t dW_t + (a_1\sigma_1 + a_2\sigma_2)\hat{v}^{\alpha+1}dt, \qquad t \geq 0.$$

We are thus in the setting of Lemma 6.5 with $I = (0, \infty)$ and

$$s(x) = |a|x, \qquad b(x) = \frac{a_1\sigma_1 + a_2\sigma_2}{|a|}x^\alpha.$$

Condition (6.5) is satisfied because for every compact interval $K \subseteq (0, \infty)$, we have

$$\int_K \left(\frac{1}{s^2(y)} + \left|\frac{b(y)}{s(y)}\right|\right)dy = \int_K \left(\frac{1}{|a|y^2} + \left|\frac{a_1\sigma_1 + a_2\sigma_2}{|a|^2}y^{\alpha-1}\right|\right)dy < \infty.$$

Moreover, s and b are locally Hölder-continuous on $(0, \infty)$. The result follows from Lemma 6.5 since $\hat{v}$ does not reach zero in finite time by Lemma 6.4. □

We are now ready to state our first result in this setting.

Example 6.7 Consider the solution (X, v) to the system of SDEs (6.1) and (6.2), supposing $\rho = 0$ and $a_1\sigma_1 + a_2\sigma_2 > 0$. Let $\mathbb{H}$ be the filtration generated by (B^1, B^2, B^3) and $\mathbb{F}$ the filtration generated by a fourth Brownian motion B^4, independent of (B^1, B^2, B^3). Introduce the $\mathbb{F}$-local martingale $N = (N_t)_{t\geq 0}$ with dynamics given by

$$dN_t = (1 + m'_t)(1 + t)dB^4_t, \qquad t \geq 0,$$

where m' is the first derivative of the function m from (6.6). Then the stochastic process $R := N + X$ is a $\mathbb{G}$-local martingale for $\mathbb{G} := \mathbb{H} \vee \mathbb{F}$, and its optional projection on $\mathbb{F}$ is ${}^oR = N + m$ by Lemma 6.3. Thus ${}^oR$ is not an $\mathbb{F}$-local martingale because m is not constant by Lemma 6.4. We introduce the stochastic exponential

$$Z_t = \mathcal{E}_t\left(\int \alpha_s dB^4_s\right), \qquad t \geq 0,$$

with

$$\alpha_t = \frac{m'_t}{(1 + m'_t)(1 + t)}, \qquad t \geq 0.$$

As $m \in C^1((0, \infty))$ from (6.6) is increasing, we get $\int_0^\infty \alpha_s^2 ds \leq \int_0^\infty \frac{1}{(1+s)^2}ds < \infty$. So Novikov's condition is satisfied, Z is an $\mathbb{F}$-uniformly integrable martingale and we can introduce the probability measure $Q \approx P$ defined by Z. Then ${}^oR = N + m$ is a $(Q, \mathbb{F})$-local martingale. Hence (P2) is satisfied.

Example 6.8 In the setting of Example 6.7, introduce a strict $(P, \mathbb{F})$-local martingale Y independent of $\mathbb{H}$. Let Q be the probability measure from Proposition 6.2 under which X is a true martingale. Since the density of Q with respect to P only depends on (B^1, B^2, B^3), Y is a strict local martingale also with respect to Q, and so is $U := Y + X$. The Q-optional projection of U on $\mathbb{F}$ is

$$^{Q,o}U_t = Y_t + \mathbb{E}[X_t] = Y_t + X_0, \qquad t \geq 0,$$

which is a $(Q, \mathbb{F})$-local martingale. Thus $P \notin \mathcal{M}^o_{\text{loc}}(U, \mathbb{F})$, but

$$\mathcal{M}_{\text{strict}}(U, \mathbb{H} \vee \mathbb{F}) \cap \mathcal{M}^o_{\text{loc}}(U, \mathbb{F}) \neq \emptyset,$$

i.e., (P3) holds for U.

The next example provides a case where (P2) does not hold whereas (P5) does.

Example 6.9 Consider again the solution (X, v) to (6.1) and (6.2), supposing $\rho = 0$ and $a_1\sigma_1 + a_2\sigma_2 > 0$. Let $\mathbb{H}$ be the filtration generated by (B^1, B^2, B^3) and $\mathbb{F}$ a filtration independent of $\mathbb{H}$. Consider a $(P, \mathbb{F})$-local martingale $\bar{N}$ such that the measure $d[\bar{N}]_t$ is singular with respect to Lebesgue measure[1]. Then the process $\bar{R} := \bar{N} + X$ is a local martingale for the filtration $\mathbb{G} := \mathbb{H} \vee \mathbb{F}$. As in Example 6.7, we obtain that the optional projection of $\bar{R}$ on $\mathbb{F}$ is given by $^o\bar{R} = \bar{N} + m$, where m is defined in (6.6). Thus $^o\bar{R}$ is not an $\mathbb{F}$-local martingale as m is not a constant function.

Since for any $\mathbb{F}$-adapted density Z, the measure $d[\bar{N}, Z]_t$ is absolutely continuous with respect to $d[\bar{N}]_t$ and hence singular with respect to Lebesgue measure, we have $\mathcal{M}_{\text{loc}}(^o\bar{R}, \mathbb{F}) = \emptyset$. Hence (P2) does not hold. Consider now the optional projection $^{Q,o}\bar{R}$ of $\bar{R}$ on $\mathbb{F}$ under the martingale measure Q from Proposition 6.2. Since X is a true martingale under Q, the process $^{Q,o}\bar{R}$ is a local martingale under Q itself, and so (P5) is satisfied.

The next result is a generalisation of Sin [31, Lemma 4.2].

Proposition 6.10 *Suppose the two-dimensional process (X, v) satisfies the system of SDEs (6.1) and (6.2) and call $\mathbb{F}$ the natural filtration of B^1. Introduce the process $(\hat{X}_t)_{t\geq 0}$ defined by*

$$\hat{X}_t = B^1_t - \sigma_1 \int_0^t v^\alpha_s ds, \qquad t \geq 0, \tag{6.7}$$

and call $\hat{\mathbb{F}}$ the natural filtration of $\hat{X}$. Then for every $\hat{\mathbb{F}}$-stopping time $\hat{\tau}$, there exists an $\mathbb{F}$-stopping time τ such that

$$E[X_{T\wedge\hat{\tau}}] = X_0 P[\hat{v} \textit{ does not explode to } +\infty \textit{ up to time } T \wedge \tau],$$

where $\hat{v}$ is defined in (6.3).

[1] One can easily provide an example of such a pair $(\mathbb{F}, \bar{N})$: consider a Brownian motion B^4 independent of (B^1, B^2, B^3) and an increasing process $f = (f_t)_{t\geq 0}$ such that df_t is singular with respect to Lebesgue measure. Let then $\mathbb{F}$ be the filtration generated by (B^4, f) and set $\bar{N}_t := B^4_{f_t}$, $t \geq 0$.

Proof By (6.1), X is a positive $(P, \mathbb{G})$-local martingale. Define the sequence of $\mathbb{G}$-stopping times $(\tau_n)_{n\in\mathbb{N}}$ by

$$\tau_n = \inf\left\{t \geq 0 : |\sigma_1 + \sigma_2|^2 \int_0^t v_s^{2\alpha} ds \geq n\right\} \wedge T,$$

with $v = (v_t)_{t\geq 0}$ as in (6.2). Then the process X^n defined by

$$X_t^n = X_{t\wedge\tau_n}, \qquad t \geq 0,$$

is a $(P, \mathbb{G})$-local martingale for $n \in \mathbb{N}$. Define Z^n by

$$Z_t^n = \sigma_1 \int_0^{t\wedge\tau_n} v_s^\alpha dB_s^1 + \sigma_2 \int_0^{t\wedge\tau_n} v_s^\alpha dB_s^2, \quad t \geq 0.$$

Then X^n is the stochastic exponential of Z^n, and since $[Z^n, Z^n]_t \leq n$ for all $t \geq 0$, X^n is a $(P, \mathbb{G})$-martingale for every $n \in \mathbb{N}$ by Novikov's condition and $(\tau_n)_{n\in\mathbb{N}}$ reduces X with respect to $(P, \mathbb{G})$. Since X^n stopped at $\hat{\tau}$ is also a $(P, \mathbb{G})$-martingale, we can define a new probability measure Q_n on $\mathcal{G}_T$ as

$$Q_n[A] = \frac{1}{X_0}\mathbb{E}[X_{T\wedge\tau_n\wedge\hat{\tau}} \mathbb{1}_A] \qquad \text{for all } A \in \mathcal{G}_T.$$

By dominated convergence,

$$\mathbb{E}[X_{T\wedge\hat{\tau}}] = \lim_{n\to\infty} \mathbb{E}\big[X_{T\wedge\tau_n\wedge\hat{\tau}} \mathbb{1}_{\{\tau_n\geq T\wedge\hat{\tau}\}}\big] = X_0 \lim_{n\to\infty} Q_n[\tau_n \geq T \wedge \hat{\tau}] \tag{6.8}$$

by the definition of Q_n. Moreover, Girsanov's theorem implies that the processes $B^{(n,1)}$, $B^{(n,2)}$ defined by

$$B_t^{(n,1)} = B_t^1 - \sigma_1 \int_0^t \mathbb{1}_{\{s\leq\tau_n\wedge\hat{\tau}\}} v_s^\alpha ds, \qquad t \geq 0,$$

$$B_t^{(n,2)} = B_t^2 - \sigma_2 \int_0^t \mathbb{1}_{\{s\leq\tau_n\wedge\hat{\tau}\}} v_s^\alpha ds, \qquad t \geq 0,$$

are Brownian motions under Q_n, $n \geq 0$. Therefore under Q_n, the process v has the dynamics

$$\begin{aligned} dv_t &= a_1 v_t dB_t^{(n,1)} + a_2 v_2 dB_t^{(n,2)} + a_3 v_t dB_t^3 + \rho(L - v_t)dt \\ &\quad + \mathbb{1}_{\{t\leq\tau_n\wedge\hat{\tau}\}}(a_1\sigma_1 + a_2\sigma_2)v_t^{\alpha+1}dt, \quad t \geq 0, \qquad v_0 = 1. \end{aligned}$$

Introduce the SDE

$$d\hat{v}_t = a_1\hat{v}_t dB_t^1 + a_2\hat{v}_t dB_t^2 + a_3\hat{v}_t dB_t^3 + \rho(L - \hat{v}_t)dt + (a_1\sigma_1 + a_2\sigma_2)\hat{v}_t^{\alpha+1}dt, \quad t \geq 0,$$

which admits a unique strong solution $\hat{v}$ by the same arguments as in the proof of Sin [31, Lemma 4.2], and consider the process $\hat{X}$ introduced in (6.7). Note that on $[\![0, \tau_n \wedge \hat{\tau}]\!]$, $(\hat{X}, v)$ has the same distribution under Q_n as $(B^1, \hat{v})$ under P. By the

Doob measurability theorem, there exists a measurable function $h : C([0,\infty)) \to \mathbb{R}_+$ such that $\hat{\tau} = h(\hat{X})$. Set $\tau = h(B^1)$. As $T \wedge \hat{\tau}$ is a $\sigma(\hat{X})$-stopping time, there exists, by the Doob measurability theorem again, a $\mathcal{B}(C([0,t]))$-measurable function Ψ_t such that $\mathbb{1}_{\{t \geq T\wedge\hat{\tau}\}} = \Psi_t(\hat{X}^t)$. Thus

$$\mathbb{1}_{\{\tau_n \geq T\wedge\hat{\tau}\}} = \Psi_{\tau_n}(\hat{X}^{\tau_n}), \quad n \in \mathbb{N}.$$

Analogously, by the construction of τ, we have

$$\mathbb{1}_{\{\hat{\tau}_n \geq T\wedge\tau\}} = \Psi_{\hat{\tau}_n}(B^{1,\hat{\tau}_n}), \quad n \in \mathbb{N},$$

where $(\hat{\tau}_n)_{n\in\mathbb{N}}$ are stopping times for the natural filtration of $\hat{v}$ defined by

$$\hat{\tau}_n = \inf\left\{t \geq 0 : |\sigma_1+\sigma_2|^2 \int_0^s \hat{v}_u^{2\alpha} du \geq n\right\}, \quad n \geq 1.$$

Since on $[\![0, \tau_n \wedge \hat{\tau}]\!]$, $(\hat{X}, v)$ has the same law under Q_n as $(B^1, \hat{v})$ under P, we have that $\Psi_{\tau_n}(\hat{X}^{\tau_n})$ has the same law under Q_n as $\Psi_{\hat{\tau}_n}(B^{1,\hat{\tau}_n})$ under P. Thus we get from (6.8) that

$$\begin{aligned}
\mathbb{E}[X_{T\wedge\hat{\tau}}] &= X_0 \lim_{n\to\infty} Q_n[\tau_n \geq T\wedge\hat{\tau}] \\
&= X_0 \lim_{n\to\infty} E^{Q_n}[\Psi_{\tau_n}(\hat{X}^{\tau_n})] \\
&= X_0 \lim_{n\to\infty} E^{P}[\Psi_{\hat{\tau}_n}(B^{1,\hat{\tau}_n})] \\
&= X_0 \lim_{n\to\infty} P[\hat{\tau}_n \geq T\wedge\tau] \\
&= X_0 P[\hat{\tau}_n \geq T\wedge\tau \text{ for some } n] \\
&= X_0 P[\hat{v} \text{ does not explode to } +\infty \text{ up to time } T\wedge\tau],
\end{aligned}$$

and the proof is complete. □

We are now ready to give the following result.

Theorem 6.11 *Consider the stochastic volatility model defined by*

$$\begin{aligned}
dX_t &= \sigma_1 v_t^\alpha X_t dB_t^1 + \sigma_2 v_t^\alpha X_t dB_t^2, \quad t \geq 0, \qquad X_0 = x > 0, \qquad (6.9)\\
dv_t &= a_2 v_t dB_t^2 + \rho(L - v_t)dt, \quad t \geq 0, \qquad v_0 = 1,
\end{aligned}$$

i.e., the model introduced in (6.1) *and* (6.2) *with* $a_1 = a_3 = 0$, *and suppose that* $a_2\sigma_2 > 0$. *Consider the filtration* $\hat{\mathbb{F}} \subseteq \mathbb{G}$ *generated by the process* $\hat{X}$ *defined in* (6.7). *Then the* P*-optional projection of* X *on* $\hat{\mathbb{F}}$ *is not an* $\hat{\mathbb{F}}$*-local martingale.*

Proof The process X in (6.9) is a strict $(P, \mathbb{G})$-local martingale by Proposition 6.2. By Proposition 6.10, for every $\hat{\mathbb{F}}$-stopping time $\hat{\tau}$, there exists a $\sigma(B^1)$-stopping time τ such that

$$E[X_{T\wedge\hat{\tau}}] = X_0 P[\hat{v} \text{ does not explode to } +\infty \text{ up to time } T\wedge\tau], \qquad (6.10)$$

where $\hat{v}$ is now given by

$$d\hat{v}_t = a_2\hat{v}_t dB_t^2 + \rho(L - \hat{v}_t)dt + a_2\sigma_2\hat{v}_t^{\alpha+1}dt, \quad t \geq 0, \qquad \hat{v}_0 = 1.$$

Since $a_1\sigma_1 + a_2\sigma_2 = a_2\sigma_2 > 0$, Lemma 6.4 implies that

$$P[\hat{v} \text{ does not explode to } +\infty \text{ up to time } t] < 1 \qquad \text{for all } t > 0.$$

In particular,

$$P[\hat{v} \text{ does not explode to } +\infty \text{ up to time } T \wedge \eta] < 1$$

for every $\sigma(B^1)$-stopping time η with $P[\eta = \infty] < 1$, because $\hat{v}$ is independent of B^1. Together with (6.10), this implies that X cannot be localised by any sequence of $\hat{\mathbb{F}}$-stopping times. Consequently, the optional projection of X on $\hat{\mathbb{F}}$ cannot be an $\hat{\mathbb{F}}$-local martingale by Föllmer and Protter [11, Theorem 3.7]. □

Proposition 6.2 and Theorem 6.11 provide a further example of two probability measures P and Q, a P-local martingale X and a non-trivial filtration $\hat{\mathbb{F}} \subseteq \mathbb{G}$ such that the optional projection of X on $\hat{\mathbb{F}}$ under P is not a P-local martingale, but the optional projection of X on $\hat{\mathbb{F}}$ under Q is a Q-martingale. We conclude the section by considering projections on a delayed filtration.

Example 6.12 Let $X = (X_t)_{t\geq 0}$ be the process defined in (6.1) and (6.2), and let $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ be the natural filtration of the Brownian motions (B^1, B^2, B^3). Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be defined by $\mathcal{F}_t = \mathcal{G}_{t-h}$ for a given delay $h > 0$. Then Corollary 4.13 immediately implies that

$$\mathcal{M}_{\text{strict}}(X, \mathbb{G}) \cap \mathcal{M}^o_{\text{loc}}(X, \mathbb{F}) = \emptyset,$$

i.e., (P3) does not hold. On the other hand, from Proposition 6.2, we have that (P1) is satisfied, since projections of true martingales are true martingales.

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