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Calculus of Variations

A spectral problem for the Laplacian in joined thin films

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Abstract

We consider a 3d multi-structure composed of two joined perpendicular thin films: a vertical one with small thickness h_n^a and a horizontal one with small thickness h_n^b. We study the asymptotic behavior, as h_n^a and h_n^b tend to zero, of an eigenvalue problem for the Laplacian defined on this multi-structure. We shall prove that the limit problem depends on the value $q = \lim_n \frac{h_n^b}{h_n^a}$. Precisely, we pinpoint three different limit regimes according to q belonging to $]0, +\infty[$, q equal to $+\infty$, or q equal to 0. We identify the limit problems and we also obtain H^1-strong convergence results.

Mathematics Subject Classification 35J05 · 35P05 · 35P20 · 74K20 · 74K30 · 74K35

1 Introduction

Let Ω_n, n in $\mathbb{N}$, be a $3d$ multi-structure composed of two joined perpendicular thin films (see Fig. 1): a vertical one Ω_n^a with small thickness h_n^a and a horizontal one Ω_n^b with small thickness h_n^b (from now on, the exponent 'a' stands for above, while 'b' for below).

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In Ω_n consider the following eigenvalue problem with mixed boundary conditions

$$\begin{cases} -\Delta U_n = \lambda U_n \text{ in } \Omega_n, \\ U_n = 0 \text{ on } \Gamma_n, \\ \dfrac{\partial U_n}{\partial \nu} = 0 \text{ on } \partial\Omega_n \setminus \Gamma_n, \end{cases} \tag{1.1}$$

where Γ_n denotes the part of the boundary of Ω_n having small thickness (see dotted area in Fig. 1) and ν denotes the exterior unit normal to Ω_n (see Sect. 2 for the rigorous definition of Ω_n and Γ_n, and for the weak formulation of problem (1.1)).

For any n in $\mathbb{N}$, problem (1.1) has a discrete positive spectrum $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ with corresponding eigenfunctions $\{U_{n,k}\}_{k\in\mathbb{N}}$ forming an orthonormal basis in $L^2(\Omega_n)$ (see Sect. 2), equipped with the inner product

$$(U, V) \in (L^2(\Omega_n))^2 \to \frac{1}{h_n^a} \int_{\Omega_n} UV dx.$$

This means that the following normalization

$$\|U_{n,k}\|^2_{L^2(\Omega_n)} = h_n^a, \quad \forall k \in \mathbb{N}, \tag{1.2}$$

is considered, but it does not restrict the generality of our results.

Problem (1.1) arises, for instance, from the Fourier analysis in the study of the heat problem or the propagation of sound waves (cf. [23], see also [14] in connection with elastic waves).

For reasons of simplicity and economy, especially from a numerical point of view, one tries to remodel the 3d problem with a problem defined on a multi-structure composed of 2d components. In this paper, it will be obtained by an asymptotic process based on the so-called "dimensional reduction", i.e., by the study of the asymptotic behavior of problem (1.1) as h_n^a and h_n^b tend to zero.

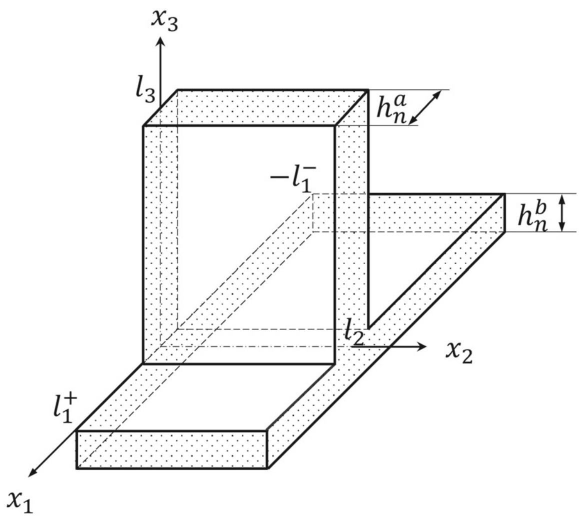

Fig. 1 The thin domain Ω_n

We shall prove that the limit problem depends on a nonegative parameter q defined by

$$q = \lim_n \frac{h_n^b}{h_n^a}.$$

Precisely, we pinpoint three different limit regimes according to q belonging to $]0, +\infty[$, q equal to $+\infty$, or q equal to 0.

• When q belongs to $]0, +\infty[$, i.e., when the thicknesses of the two thin films vanish with the same rate, we obtain the following limit eigenvalue problem,

$$\begin{cases} -\Delta_{x_2,x_3} u^a = \lambda u^a \text{ in } \omega^a, \\ -\Delta_{x_1,x_2} u_+^b = \lambda u_+^b \text{ in } \omega_+^b, \\ -\Delta_{x_1,x_2} u_-^b = \lambda u_-^b \text{ in } \omega_-^b, \\ u^a = 0 \text{ on } \gamma^a, \\ u_+^b = 0 \text{ on } \gamma_+^b, \\ u_-^b = 0 \text{ on } \gamma_-^b, \\ u^a = u_+^b = u_-^b \text{ on } \gamma, \\ \partial_{x_3} u^a = q(\partial_{x_1} u_-^b - \partial_{x_1} u_+^b) \text{ on } \gamma. \end{cases} \tag{1.3}$$

where ω^a is the cross-section of the vertical film, ω_+^b and ω_-^b are the two parts into which ω^b, the cross-section of the horizontal film, is divided by the intersection with $\partial\omega^a$ (see Fig. 2),

$$\gamma = \partial\omega^a \cap \partial\omega_+^b \cap \partial\omega_-^b, \quad \gamma^a = \partial\omega^a \setminus \gamma, \quad \gamma_+^b = \partial\omega_+^b \setminus \gamma, \quad \gamma_-^b = \partial\omega_-^b \setminus \gamma.$$

Problem (1.3) is a $2d - 2d - 2d$ eigenvalue problem with coupled conditions on γ (see the last two lines of (1.3)).

The weak formulation of (1.3) is given by (3.4) (see also (3.1), (3.2), and (3.3)). This problem has a discrete positive spectrum $\{\lambda_k\}_{k\in\mathbb{N}}$ with the corresponding eigenfunctions $\{(u_k^a, u_{k+}^b, u_{k-}^b)\}_{k\in\mathbb{N}}$ forming a basis in $L^2(\omega^a)\times L^2(\omega_+^b)\times L^2(\omega_-^b)$ subjected to the orthonormal condition

$$\int_{\omega^a} u_k^a u_h^a dx_2 dx_3 + q\left(\int_{\omega_+^b} u_{k+}^b u_{h+}^b dx_1 dx_2 + \int_{\omega_-^b} u_{k-}^b u_{h-}^b dx_1 dx_2\right) = \delta_{hk},$$

where $\delta_{h,k}$ denotes the Kronecker delta.

In Theorem 3.1 we prove the convergence of the eigenvalues of problem (1.1), as $n \to +\infty$, to the eigenvalues of problem (1.3) with conservation of the multiplicity. We prove also a strong H^1-convergence result for the corresponding eigenfunctions (see (3.5), (3.6), (3.7), and Corollary 3.2).

• When $q = +\infty$, i.e., when the thickness of the vertical thin film vanishes faster than the thickness of the horizontal thin film, the limit spectrum is the union of the spectra of the following two uncoupled $2d$ eigenvalue problems with homogeneous Dirichlet boundary

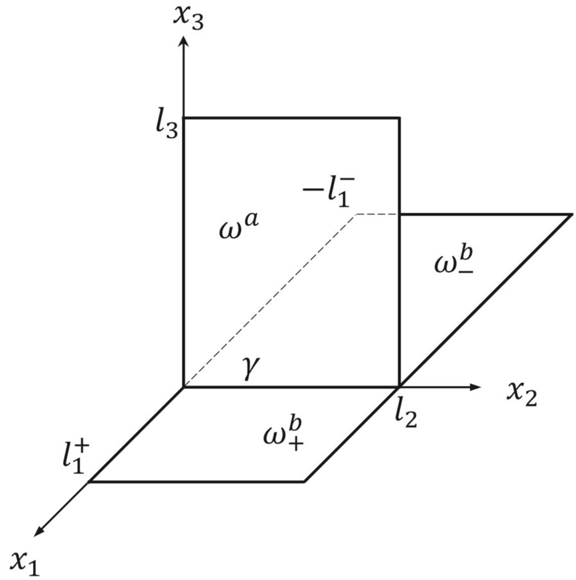

Fig. 2 The limit domain

condition

$$\begin{cases} -\Delta_{x_2,x_3} u^a = \lambda u^a \text{ in } \omega^a, \\ u^a = 0 \text{ on } \partial\omega^a, \end{cases} \qquad \begin{cases} -\Delta_{x_1,x_2} u^b = \lambda u^b \text{ in } \omega^b, \\ u^b = 0 \text{ on } \partial\omega^b. \end{cases}$$

Precisely, one has to collect together the eigenvalues of these two problems and order the obtained set in an increasing sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ with the convention of repeated eigenvalues. The corresponding eigenfunctions form an orthonormal basis in $L^2(\omega^a) \times L^2(\omega^b)$.

In Theorem 3.3 we prove the convergence of the eigenvalues of problem (1.1), as $n \to +\infty$, to the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ with conservation of the multiplicity. Moreover, by means of renormalization in Ω_n^b, we prove a strong H^1-convergence result for the corresponding eigenfunctions (see (3.10), (3.11), (3.12), and Corollary 3.5).

• When $q = 0$, i.e., when the thickness of the horizontal thin film vanishes faster than the thickness of the vertical thin film, we choose the sequence $\{U_{n,k}\}_{k\in\mathbb{N}}$ of eigenfunctions associated to the discrete positive spectrum $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ of problem (1.1) such that it forms an orthonormal basis in $L^2(\Omega_n)$ equipped with the inner product

$$(U, V) \in (L^2(\Omega_n))^2 \to \frac{1}{h_n^b} \int_{\Omega_n} UV dx,$$

i.e., the following normalization

$$\|U_{n,k}\|^2_{L^2(\Omega_n)} = h_n^b, \quad \forall k \in \mathbb{N}, \tag{1.4}$$

is considered.

In this case, the limit spectrum is the union of the spectra of the following three uncoupled $2d$ eigenvalue problems, the first one with mixed boundary condition, while the other two with homogeneous Dirichlet boundary condition

$$\begin{cases} -\Delta_{x_2,x_3} u^a = \lambda u^a \text{ in } \omega^a, \\ u^a = 0 \text{ on } \gamma^a, \\ \partial_{x_3} u^a = 0 \text{ on } \gamma, \end{cases} \qquad \begin{cases} -\Delta_{x_1,x_2} u^b_+ = \lambda u^b_+ \text{ in } \omega^b_+, \\ u^b_+ = 0 \text{ on } \partial\omega^b_+, \end{cases} \qquad \begin{cases} -\Delta_{x_1,x_2} u^b_- = \lambda u^b_- \text{ in } \omega^b_-, \\ u^b_- = 0 \text{ on } \partial\omega^b_-. \end{cases}$$

As above, one has to collect together the eigenvalues of these three problems and order the obtained set in an increasing sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ with the convention of repeated eigenvalues. The corresponding eigenfunctions form an orthonormal basis in $L^2(\omega^a)\times L^2(\omega^b_+)\times L^2(\omega^b_-)$.

In Theorem 3.6 we prove the convergence of the eigenvalues of problem (1.1), as $n \to +\infty$, to the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ with conservation of the multiplicity. Also in this case we prove a strong H^1-convergence result for the corresponding eigenfunctions, but by means of a renormalization in Ω^a_n (see (3.16), (3.17), (3.18), and Corollary 3.8).

Notice that, when q belongs to $]0, +\infty[$, choosing (1.2) or (1.4) as normalization leads to the same limit result. Instead, to obtain a meaningful result, normalization (1.2) must be used when q is $+\infty$ and normalization (1.4) when q is 0.

In Sect. 2, following [3], problem (1.1) is rescaled on a fixed domain. Section 4 is devoted to obtaining *a priori* estimates of the eigenvalues $\lambda_{n,k}$ of problem (1.1): below by a positive constant independent of n and k, and above by an explicit constant independent of n but dependent on k (see also Remark 4.2). The upper bound of $\lambda_{n,k}$ implies H^1-*a priori* estimates of the eigenfunctions. In Sect. 3, the main results are stated. Section 5 contains some results that are crucial for proving the main results, i.e., Theorems 3.1, 3.3, and 3.6. Precisely, in Proposition 5.1 we give a trace convergence result, written in a very general way, which will allow us to identify junction and boundary conditions in the limit problems. In Proposition 5.2, we prove a density result for approximating the elements of the space of setting of the limit problem, when q belongs to $]0, +\infty[$, with regular functions. Although this result was used in other works, to our knowledge, there are no previous proofs of it. Our proof is rather technical and it works also for domains which are not "symmetric". Proposition 5.3 is devoted to building a recovery sequence which will be used in the proof of all three main results. Sections 6, 7, and 8 are devoted to proving the main results in the case where q belongs to $]0, +\infty[$, q equal to $+\infty$, or q equal to 0, respectively. The three proofs follow the same pattern. In them, we highlight the novelties and refer to [9] and [22] for the classical parts.

In this paper we consider the Laplace operator in order to investigate the effect of the junction condition on the limit problem. It is of course possible to replace the Laplacian by an elliptic operator with a symmetric and positive definite thermal conductivity matrix. Taking into account our analysis and arguing as in [10] easily lead to the limit problem. Moreover, we just considered two perpendicular thin films. Of course, the whole analysis works with the appropriate modifications if the two films form an angle other than $\frac{\pi}{2}$. We leave the study of these cases to an interested reader.

The asymptotic behavior of a spectral problem for an homogeneous isotropic elastic body consisting of two folded and perpendicular plates with the same thickness h but with the requirement of large elastic coefficients, of order $O(h^{-2})$, was studied in [12] (see also [13]). This assumption technically avoids a rescaling of the eigenvalues and gives very different asymptotic behaviors from our problem. Also, we refer to [4], [11], [16], and [17] for different eigenvalue problems in plate theory.

The modelling of spectral problems for the Laplace operator in joined $1d-1d$ and $1d-2d$ multi-structures were obtained in [7], [9], [10], [15], and [20]. The modelling of the spectrum for the linear water-wave system in a joined $1d-2d$ multi-structure was obtained in [1].

For other problems in joined thin films, we refer to [2], [6], and [8].

Eventually, we refer to [5], [18], [19], and references therein, for problems on thin structures.

2 Position of the problem and rescalings

Let l_1^+, l_1^-, l_2, and l_3 be four positive real numbers such that

$$l_1^\pm > \frac{1}{2}.$$

Set (see Fig. 2)

$$\omega^a =]0, l_2[\times]0, l_3[, \quad \omega^b =]-l_1^-, l_1^+[\times]0, l_2[, \quad \omega^b_+ =]0, l_1^+[\times]0, l_2[, \quad \omega^b_- =]-l_1^-, 0[\times]0, l_2[,$$
$$\gamma^a = \partial\omega^a \setminus (]0, l_2[\times\{0\}).$$

Let $\left\{h_n^a\right\}_{n\in\mathbb{N}}$, $\left\{h_n^b\right\}_{n\in\mathbb{N}}$ be two sequences in $]0, 1[$ such that

$$\lim_n h_n^a = 0 = \lim_n h_n^b, \qquad \lim_n \frac{h_n^b}{h_n^a} = q \in [0, +\infty]. \tag{2.1}$$

For every n in $\mathbb{N}$ set (see Fig. 1)

$$\Omega_n^a = \left]-\frac{h_n^a}{2}, \frac{h_n^a}{2}\right[\times\omega^a, \quad \Omega_n^b = \omega^b \times \left]-h_n^b, 0\right[, \quad \Omega_n = \Omega_n^a \cup \Omega_n^b \cup \left(\left]-\frac{h_n^a}{2}, \frac{h_n^a}{2}\right[\times\left]0, l_2\right[\times\{0\}\right),$$
$$\Gamma_n^a = \left]-\frac{h_n^a}{2}, \frac{h_n^a}{2}\right[\times\gamma^a, \quad \Gamma_n^b = \partial\omega^b \times \left]-h_n^b, 0\right[, \quad \Gamma_n = \Gamma_n^a \cup \Gamma_n^b.$$

For every n in $\mathbb{N}$, consider the space $L^2(\Omega_n)$ equipped with the inner product

$$(U, V) \in (L^2(\Omega_n))^2 \to \frac{1}{h_n^a}\int_{\Omega_n} UV dx, \tag{2.2}$$

and the space

$$\mathcal{V}_n = \left\{V \in H^1(\Omega_n) \ : \ V = 0 \text{ on } \Gamma_n\right\} \tag{2.3}$$

equipped with the inner product

$$(U, V) \in \mathcal{V}_n \times \mathcal{V}_n \to \frac{1}{h_n^a}\int_{\Omega_n} DU DV dx. \tag{2.4}$$

The classical spectral theory (for instance, see [21]) ensures the existence of an increasing diverging sequence of positive numbers $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ forming the set of all the eigenvalues of Problem (1.1), i.e.,

$$\begin{cases} U_n \in \mathcal{V}_n, \\ \displaystyle\int_{\Omega_n} DU_n DV dx = \lambda \int_{\Omega_n} U_n V dx, \quad \forall v \in \mathcal{V}_n. \end{cases} \tag{2.5}$$

Moreover, there exists a $L^2(\Omega_n)$-Hilbert orthonormal basis $\{U_{n,k}\}_{k\in\mathbb{N}}$ such that, for every k in $\mathbb{N}$, $U_{n,k}$ belongs to $\mathcal{V}_n$ and it is an eigenvector of (2.5) with eigenvalue $\lambda_{n,k}$; hence, $\left\{\lambda_{n,k}^{-\frac{1}{2}} U_{n,k}\right\}_{k\in\mathbb{N}}$ is a $\mathcal{V}_n$-Hilbert orthonormal basis.

Set now

$$\Omega^a = \left]-\frac{1}{2}, \frac{1}{2}\right[\times \omega^a, \quad \Omega^b = \omega^b \times \left]-1, 0\right[, \quad \Gamma^a = \left]-\frac{1}{2}, \frac{1}{2}\right[\times \gamma^a, \quad \Gamma^b = \partial\omega^b \times \left]-1, 0\right[.$$

From now on,

$$H^1_{\Gamma^a}(\Omega^a) = \{v \in H^1(\Omega^a) : v = 0 \text{ on } \Gamma^a\}, \quad H^1_{\Gamma^b}(\Omega^b) = \{v \in H^1(\Omega^b) : v = 0 \text{ on } \Gamma^b\},$$
$$H^1_{\gamma^a}(\omega^a) = \{v \in H^1(\omega^a) : v = 0 \text{ on } \gamma^a\}.$$

As it is usual (see [3]), problem (2.5) will be reformulated on the fixed domain $\Omega^a \cup \Omega^b \cup \left(]-\frac{1}{2}, \frac{1}{2}[\times]0, l_2[\right)$ through the following maps

$$(x_1, x_2, x_3) \in \Omega^a \longrightarrow (h^a_n x_1, x_2, x_3) \in \Omega^a_n, \quad (x_1, x_2, x_3) \in \Omega^b \longrightarrow (x_1, x_2, h^b_n x_3) \in \Omega^b_n.$$

To this aim, for every n in $\mathbb{N}$, let H_n be the space $L^2(\Omega^a) \times L^2(\Omega^b)$ equipped with the inner product

$$\begin{gathered}(\cdot,\cdot)_n : (u, v) = ((u^a, u^b), (v^a, v^b)) \in \left(L^2(\Omega^a) \times L^2(\Omega^b)\right)^2 \longrightarrow \\ (u, v)_n = \int_{\Omega^a} u^a v^a dx + \frac{h^b_n}{h^a_n}\int_{\Omega^b} u^b v^b dx,\end{gathered} \tag{2.6}$$

and let V_n be the space defined by

$$\begin{aligned}V_n = \Big\{ & v = (v^a, v^b) \in H^1_{\Gamma^a}(\Omega^a) \times H^1_{\Gamma^b}(\Omega^b) : \\ & v^a(x_1, x_2, 0) = v^b(h^a_n x_1, x_2, 0) \text{ a.e. } (x_1, x_2) \in \left]-\frac{1}{2}, \frac{1}{2}\right[\times \left]0, l_2\right[\Big\}\end{aligned} \tag{2.7}$$

equipped with the inner product

$$\begin{aligned}& a_n : (u, v) = ((u^a, u^b), (v^a, v^b)) \in V_n^2 \longrightarrow a_n(u, v) = \\ & \int_{\Omega^a}\left(\frac{1}{(h^a_n)^2}\partial_{x_1}u^a\partial_{x_1}v^a + \partial_{x_2}u^a\partial_{x_2}v^a + \partial_{x_3}u^a\partial_{x_3}v^a\right)dx \\ & +\frac{h^b_n}{h^a_n}\int_{\Omega^b}\left(\partial_{x_1}u^b\partial_{x_1}v^b + \partial_{x_2}u^b\partial_{x_2}v^b + \frac{1}{(h^b_n)^2}\partial_{x_3}u^b\partial_{x_3}v^b\right)dx.\end{aligned} \tag{2.8}$$

Moreover, for every n and k in $\mathbb{N}$, set

$$u_{n,k} = \begin{cases} U_{n,k}(h^a_n x_1, x_2, x_3), \text{ a.e. in } \Omega^a, \\ U_{n,k}(x_1, x_2, h^b_n x_3), \text{ a.e. in } \Omega^b. \end{cases} \tag{2.9}$$

Then, for every n in $\mathbb{N}$, $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ is an increasing diverging sequence of positive numbers forming the set of all the eigenvalues of the following problem

$$\begin{cases} u_n \in V_n, \\ a_n(u_n, v) = \lambda(u_n, v)_n, \quad \forall v \in V_n, \end{cases} \tag{2.10}$$

$\{u_{n,k}\}_{k\in\mathbb{N}}$ is a H_n-Hilbert orthonormal basis such that, for every k in $\mathbb{N}$, $u_{n,k}$ belongs to V_n and it is an eigenvector of (2.10) with eigenvalue $\lambda_{n,k}$. Moreover, $\left\{\lambda_{n,k}^{-\frac{1}{2}} u_{n,k}\right\}_{k\in\mathbb{N}}$ is a V_n-Hilbert orthonormal basis. In particular, one has

$$\begin{cases} u_{n,k} \in V_n, \\ a_n(u_{n,k}, v) = \lambda_{n,k}(u_{n,k}, v)_n, \quad \forall v \in V_n, \end{cases} \qquad \forall n, k \in \mathbb{N}, \tag{2.11}$$

$$(u_{n,k}, u_{n,h})_n = \delta_{h,k}, \quad \forall n, k, h, \in \mathbb{N}, \tag{2.12}$$

$$a_n(\lambda_{n,k}^{-\frac{1}{2}} u_{n,k}, \lambda_{n,h}^{-\frac{1}{2}} u_{n,h}) = \delta_{h,k}, \quad \forall n, k, h, \in \mathbb{N}. \tag{2.13}$$

Furthermore, for every k in $\mathbb{N}$, $\lambda_{n,k}$ is characterized by the following min-max Principle

$$\lambda_{n,k} = \min_{\mathcal{E}_k\in\mathcal{F}_k} \max_{v\in\mathcal{E}_k,\ v\neq 0} \frac{a_n(v,v)}{(v,v)_n}, \tag{2.14}$$

where $\mathcal{F}_k$ is the set of the subspaces $\mathcal{E}_k$ of V_n with dimension k (for instance, see [21]).

Problem (2.10) is obtained from (2.5) by means of rescaling of variables, once multiplied by $\dfrac{1}{h_n^a}$.

3 The main results

This section is devoted to stating the main results of this paper.

The limit problem will depend on q defined by (2.1) which acts as a weight on ω^b in the scalar product. Precisely, three different limit regimes will appear according to q belonging to $]0, +\infty[$, q equal to $+\infty$, or q equal to 0.

3.1 The case q in $]0, +\infty[$

Fix q in $]0, +\infty[$.

Consider $L^2(\omega^a) \times L^2(\omega^b)$ equipped with the inner product

$$\begin{aligned} &[\cdot,\cdot]_q : (u, v) = ((u^a, u^b), (v^a, v^b)) \in \left(L^2(\omega^a) \times L^2(\omega^b)\right)^2 \\ &\longrightarrow \int_{\omega^a} u^a v^a dx_2 dx_3 + q \int_{\omega^b} u^b v^b dx_1 dx_2. \end{aligned} \tag{3.1}$$

Moreover, let

$$V = \left\{ v = (v^a, v^b) \in H^1_{\gamma^a}(\omega^a) \times H^1_0(\omega^b) \ :\ v^a(x_2, 0) = v^b(0, x_2) \text{ a.e. in }]0, l_2[\right\} \tag{3.2}$$

be equipped with the inner product

$$\begin{aligned} &\alpha_q : (u, v) = ((u^a, u^b), (v^a, v^b)) \in V \times V \longrightarrow \alpha_q(u, v) \\ &= \int_{\omega^a} \left(\partial_{x_2} u^a \partial_{x_2} v^a + \partial_{x_3} u^a \partial_{x_3} v^a\right) dx_2 dx_3 + q \int_{\omega^b} \left(\partial_{x_1} u^b \partial_{x_1} v^b + \partial_{x_2} u^b \partial_{x_2} v^b\right) dx_1 dx_2. \end{aligned} \tag{3.3}$$

Both are Hilbert spaces. Moreover, the norm induced on V by the inner product $\alpha_q(\cdot,\cdot)$ is equivalent to the usual $\big(H^1(\omega^a)\times H^1(\omega^b)\big)$-norm, and the norm induced on $L^2(\omega^a)\times L^2(\omega^b)$ by the inner product $[\cdot,\cdot]_q$ is equivalent to the usual $\big(L^2(\omega^a)\times L^2(\omega^b)\big)$-norm. Consequently, V is continuously and compactly embedded into $L^2(\omega^a)\times L^2(\omega^b)$. Furhtermore, V is dense in $L^2(\omega^a)\times L^2(\omega^b)$ since $C_0^\infty(\omega^a)\times\{v\in C_0^\infty(\omega^b): v=0 \text{ on } \{0\}\times]0,l_2[\}$ is included in V. Then, all classic results hold true for the eigenvalue problem (see [21])

$$\begin{cases} u\in V, \\ \\ \alpha_q(u,v)=\lambda[u,v]_q, \quad \forall v\in V, \end{cases} \tag{3.4}$$

and the following result will be proved.

Theorem 3.1 *For every n in $\mathbb{N}$, let H_n be the space $L^2(\Omega^a)\times L^2(\Omega^b)$ equipped with the inner product $(\cdot,\cdot)_n$ defined by (2.6) and V_n be the space defined by (2.7) equipped with the inner product $a_n(\cdot,\cdot)$ defined by (2.8).*

For every n in $\mathbb{N}$, let $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ be the increasing diverging sequence of all the eigenvalues of Problem (2.10) and let $\{u_{n,k}\}_{k\in\mathbb{N}}$ be a H_n-Hilbert orthonormal basis such that $\left\{\lambda_{n,k}^{-\frac{1}{2}}u_{n,k}\right\}_{k\in\mathbb{N}}$ is a V_n-Hilbert orthonormal basis and, for every $k\in\mathbb{N}$, $u_{n,k}=(u^a_{n,k},u^b_{n,k})$ is an eigenvector of Problem (2.10) with eigenvalue $\lambda_{n,k}$.

Assume that (2.1) holds true with q in $]0,+\infty[$.

Let $L^2(\omega^a)\times L^2(\omega^b)$ be equipped with the inner product $[\cdot,\cdot]_q$ defined by (3.1) and V be the space defined by (3.2) equipped with the inner product $\alpha_q(\cdot,\cdot)$ defined by (3.3).

Then, there exists an increasing diverging sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}}$, depending on q, such that

$$\lim_n \lambda_{n,k}=\lambda_k, \quad \forall k\in\mathbb{N},$$

and $\{\lambda_k\}_{k\in\mathbb{N}}$ is the set of all the eigenvalues of Problem (3.4). Moreover, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and a $(L^2(\omega^a)\times L^2(\omega^b),[\cdot,\cdot]_q)$-Hilbert orthonormal basis $\{u_k\}_{k\in\mathbb{N}}$ (depending possibly on the selected subsequence $\{n_i\}_{i\in\mathbb{N}}$ and q) such that, for every k in $\mathbb{N}$, u_k belongs to V and it is an eigenvector of Problem (3.4) with eigenvalue λ_k, and

$$u_{n_i,k}\to u_k \text{ strongly in } H^1(\Omega^a)\times H^1(\Omega^b), \quad \forall k\in\mathbb{N}, \tag{3.5}$$

as i diverges,

$$\frac{1}{h^a_n}\partial_{x_1}u^a_{n,k}\to 0 \text{ strongly in } L^2(\Omega^a), \quad \forall k\in\mathbb{N}, \tag{3.6}$$

$$\frac{1}{h^b_n}\partial_{x_3}u^b_{n,k}\to 0 \text{ strongly in } L^2(\Omega^b), \quad \forall k\in\mathbb{N}, \tag{3.7}$$

as n diverges. Furthermore, $\{\lambda_k^{-\frac{1}{2}}u_k\}_{k\in\mathbb{N}}$ is a (V,α_q)-Hilbert orthonormal basis.

As far as the original problem (1.1) is concerned, one has the following result which is an immediate corollary of Theorem 3.1, by change of variable.

Corollary 3.2 *For every n in $\mathbb{N}$, let $L^2(\Omega_n)$ be equipped with the inner product defined by (2.2) and let $\mathcal{V}_n$ be the space defined by (2.3) equipped with the inner product defined by (2.4).*

For every n in $\mathbb{N}$, let $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ be the increasing diverging sequence of all the eigenvalues of Problem (2.5) and let $\{U_{n,k}\}_{k\in\mathbb{N}}$ be a $L^2(\Omega_n)$-Hilbert orthonormal basis such that $\left\{\lambda_{n,k}^{-\frac{1}{2}}U_{n,k}\right\}_{k\in\mathbb{N}}$ is a $\mathcal{V}_n$-Hilbert orthonormal basis and, for every $k\in\mathbb{N}$, $U_{n,k}$ is an eigenvector of Problem (2.5) with eigenvalue $\lambda_{n,k}$.

Assume that (2.1) holds true with q in $]0,+\infty[$.

Let $L^2(\omega^a)\times L^2(\omega^b)$ be equipped with the inner product $[\cdot,\cdot]_q$ defined by (3.1) and V be the space defined by (3.2) equipped with the inner product $\alpha_q(\cdot,\cdot)$ defined by (3.3).

Then, there exists an increasing diverging sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}}$, depending on q, such that

$$\lim_n \lambda_{n,k} = \lambda_k, \quad \forall k\in\mathbb{N},$$

and $\{\lambda_k\}_{k\in\mathbb{N}}$ is the set of all the eigenvalues of Problem (3.4). Moreover, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and a $(L^2(\omega^a)\times L^2(\omega^b),[\cdot,\cdot]_q)$-Hilbert orthonormal basis $\{u_k\}_{k\in\mathbb{N}}$ (depending possibly on the selected subsequence $\{n_i\}_{i\in\mathbb{N}}$ and q) such that, for every k in $\mathbb{N}$, $u_k=(u_k^a,u_k^b)$ belongs to V and it is an eigenvector of Problem (3.4) with eigenvalue λ_k, and

$$\lim_i \fint_{\Omega^a_{n_i}} \left(|U_{n_i,k}-u_k^a|^2+|\partial_{x_1}U_{n_i,k}|^2+|\partial_{x_2}U_{n_i,k}-\partial_{x_2}u_k^a|^2+|\partial_{x_3}U_{n_i,k}-\partial_{x_3}u_k^a|^2\right)dx=0,$$

$$\lim_i \fint_{\Omega^b_{n_i}} \left(|U_{n_i,k}-u_k^b|^2+|\partial_{x_1}U_{n_i,k}-\partial_{x_1}u_k^b|^2+|\partial_{x_2}U_{n_i,k}-\partial_{x_2}u_k^b|^2+|\partial_{x_3}U_{n_i,k}|^2\right)dx=0,$$

where, from now on, $\fint_{\Omega^a_{n_i}}$ means $\frac{1}{|\Omega^a_{n_i}|}\int_{\Omega^a_{n_i}}$ and $\fint_{\Omega^b_{n_i}}$ means $\frac{1}{|\Omega^b_{n_i}|}\int_{\Omega^a_{n_i}}$.

Furthermore, $\{\lambda_k^{-\frac{1}{2}}u_k\}_{k\in\mathbb{N}}$ is a (V,α_q)-Hilbert orthonormal basis.

3.2 The case $q=+\infty$

Let $[\cdot,\cdot]_1$ be the inner product on $L^2(\omega^a)\times L^2(\omega^b)$ defined by (3.1) with $q=1$. Moreover, still denote by α_1 the inner product on $H_0^1(\omega^a)\times H_0^1(\omega^b)$ defined by (3.3) with $q=1$, i.e.,

$$\alpha_1:(u,v)=((u^a,u^b),(v^a,v^b))\in\left(H_0^1(\omega^a)\times H_0^1(\omega^b)\right)^2\longrightarrow\alpha_1(u,v)$$
$$=\int_{\omega^a}\left(\partial_{x_2}u^a\partial_{x_2}v^a+\partial_{x_3}u^a\partial_{x_3}v^a\right)dx_2dx_3+\int_{\omega^b}\left(\partial_{x_1}u^b\partial_{x_1}v^b+\partial_{x_2}u^b\partial_{x_2}v^b\right)dx_1dx_2. \tag{3.8}$$

Then, both are Hilbert spaces and all classic results hold true for the eigenvalue problem

$$\begin{cases} u\in H_0^1(\omega^a)\times H_0^1(\omega^b),\\ \alpha_1(u,v)=\lambda[u,v]_1, \quad \forall v\in H_0^1(\omega^a)\times H_0^1(\omega^b), \end{cases} \tag{3.9}$$

(see [21]) and the following result will be proved when q is equal to $+\infty$.

Theorem 3.3 *For every n in $\mathbb{N}$, let H_n be the space $L^2(\Omega^a)\times L^2(\Omega^b)$ equipped with the inner product $(\cdot,\cdot)_n$ defined by (2.6) and V_n be the space defined by (2.7) equipped with the inner product $a_n(\cdot,\cdot)$ defined by (2.8).*

For every n in $\mathbb{N}$, let $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ be the increasing diverging sequence of all the eigenvalues of Problem (2.10) and let $\{u_{n,k}\}_{k\in\mathbb{N}}$ be a H_n-Hilbert orthonormal basis such that $\left\{\lambda_{n,k}^{-\frac{1}{2}} u_{n,k}\right\}_{k\in\mathbb{N}}$ is a V_n-Hilbert orthonormal basis and, for every $k \in \mathbb{N}$, $u_{n,k} = (u^a_{n,k}, u^b_{n,k})$ is an eigenvector of Problem (2.10) with eigenvalue $\lambda_{n,k}$.

Assume that (2.1) holds true with $q = +\infty$.

Let $L^2(\omega^a) \times L^2(\omega^b)$ be equipped with the inner product $[\cdot,\cdot]_1$ defined by (3.1) and $H^1_0(\omega^a) \times H^1_0(\omega^b)$ be equipped with the inner product $\alpha_1(\cdot,\cdot)$ defined by (3.8).

Then, there exists an increasing diverging sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ such that

$$\lim_n \lambda_{n,k} = \lambda_k, \quad \forall k \in \mathbb{N},$$

and $\{\lambda_k\}_{k\in\mathbb{N}}$ is the set of all the eigenvalues of Problem (3.9). Moreover, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and a $(L^2(\omega^a) \times L^2(\omega^b), [\cdot,\cdot]_1)$-Hilbert orthonormal basis $\{u_k\}_{k\in\mathbb{N}}$ (depending possibly on the selected subsequence $\{n_i\}_{i\in\mathbb{N}}$) such that, for every k in $\mathbb{N}$, u_k belongs to $H^1_0(\omega^a) \times H^1_0(\omega^b)$ and it is an eigenvector of Problem (3.9) with eigenvalue λ_k, and

$$\left(u^a_{n_i,k}, \sqrt{\frac{h^b_{n_i}}{h^a_{n_i}}}\, u^b_{n_i,k}\right) \to u_k \text{ strongly in } H^1(\Omega^a) \times H^1(\Omega^b), \quad \forall k \in \mathbb{N}, \tag{3.10}$$

as i diverges,

$$\frac{1}{h^a_n}\partial_{x_1} u^a_{n,k} \to 0 \text{ strongly in } L^2(\Omega^a), \quad \forall k \in \mathbb{N}, \tag{3.11}$$

$$\frac{1}{h^b_n}\sqrt{\frac{h^b_n}{h^a_n}}\,\partial_{x_3} u^b_{n,k} \to 0 \text{ strongly in } L^2(\Omega^b), \quad \forall k \in \mathbb{N}, \tag{3.12}$$

as n diverges. Furthermore, $\{\lambda_k^{-\frac{1}{2}} u_k\}_{k\in\mathbb{N}}$ is a $(H^1_0(\omega^a) \times H^1_0(\omega^b), \alpha_1)$-Hilbert orthonormal basis.

Remark 3.4 Notice that (3.10) and (3.12) imply that

$$u^b_{n,k} \to 0 \text{ strongly in } H^1(\Omega^b), \quad \frac{1}{h^b_n}\partial_{x_3} u^b_{n,k} \to 0 \text{ strongly in } L^2(\Omega^b), \quad \forall k \in \mathbb{N}.$$

As far as the original problem (1.1) is concerned, one has the following result which is an immediate corollary of Theorem 3.3, by change of variable.

Corollary 3.5 *For every n in $\mathbb{N}$, let $L^2(\Omega_n)$ be equipped with the inner product defined by (2.2) and let $\mathcal{V}_n$ be the space defined by (2.3) equipped with the inner product defined by (2.4).*

For every n in $\mathbb{N}$, let $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ be the increasing diverging sequence of all the eigenvalues of Problem (2.5) and let $\{U_{n,k}\}_{k\in\mathbb{N}}$ be a $L^2(\Omega_n)$-Hilbert orthonormal basis such that $\left\{\lambda_{n,k}^{-\frac{1}{2}} U_{n,k}\right\}_{k\in\mathbb{N}}$ is a $\mathcal{V}_n$-Hilbert orthonormal basis and, for every $k \in \mathbb{N}$, $U_{n,k}$ is an eigenvector of Problem (2.5) with eigenvalue $\lambda_{n,k}$.

Assume that (2.1) holds true with $q = +\infty$.

Let $L^2(\omega^a) \times L^2(\omega^b)$ be equipped with the inner product $[\cdot,\cdot]_1$ defined by (3.1) and $H^1_0(\omega^a) \times H^1_0(\omega^b)$ be equipped with the inner product $\alpha_1(\cdot,\cdot)$ defined by (3.8).

Then, there exists an increasing diverging sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ *such that*

$$\lim_n \lambda_{n,k} = \lambda_k, \quad \forall k \in \mathbb{N},$$

and $\{\lambda_k\}_{k\in\mathbb{N}}$ *is the set of all the eigenvalues of Problem (3.9). Moreover, there exist an increasing sequence of positive integer numbers* $\{n_i\}_{i\in\mathbb{N}}$ *and a* $(L^2(\omega^a) \times L^2(\omega^b), [\cdot,\cdot]_1)$*-Hilbert orthonormal basis* $\{u_k\}_{k\in\mathbb{N}}$ *(depending possibly on the selected subsequence* $\{n_i\}_{i\in\mathbb{N}}$*) such that, for every k in* $\mathbb{N}$, $u_k = (u_k^a, u_k^b)$ *belongs to* $H_0^1(\omega^a)\times H_0^1(\omega^b)$ *and it is an eigenvector of Problem (3.9) with eigenvalue* λ_k*, and*

$$\lim_i \int_{\Omega^a_{n_i}} \left(|U_{n_i,k} - u_k^a|^2 + |\partial_{x_1} U_{n_i,k}|^2 + |\partial_{x_2} U_{n_i,k} - \partial_{x_2} u_k^a|^2 + |\partial_{x_3} U_{n_i,k} - \partial_{x_3} u_k^a|^2\right) dx = 0,$$

$$\lim_n \int_{\Omega^b_n} \left(\left|U_{n,k}\right|^2 + \left|DU_{n,k}\right|^2\right) dx = 0,$$

$$\lim_i \int_{\Omega^b_{n_i}} \left|\sqrt{\frac{h^b_{n_i}}{h^a_{n_i}}} U_{n_i,k} - u_k^b\right|^2 dx = 0,$$

$$\lim_i \int_{\Omega^b_{n_i}} \left(\left|\sqrt{\frac{h^b_{n_i}}{h^a_{n_i}}} \partial_{x_1} U_{n_i,k} - \partial_{x_1} u_k^b\right|^2 + \left|\sqrt{\frac{h^b_{n_i}}{h^a_{n_i}}} \partial_{x_2} U_{n_i,k} - \partial_{x_2} u_k^b\right|^2 + \left|\sqrt{\frac{h^b_{n_i}}{h^a_{n_i}}} \partial_{x_3} U_{n_i,k}\right|^2\right) dx = 0.$$

Furthermore, $\{\lambda_k^{-\frac{1}{2}} u_k\}_{k\in\mathbb{N}}$ *is a* $(H_0^1(\omega^a) \times H_0^1(\omega^b), \alpha_1)$*-Hilbert orthonormal basis.*

3.3 The case $q = 0$

Let $[\cdot,\cdot]_1$ be the inner product on $L^2(\omega^a) \times L^2(\omega^b)$ defined by (3.1) with $q = 1$.

Set

$$W_0 = \{v^b \in H_0^1(\omega^b) \, : \, v^b(0, x_2) = 0 \text{ a.e. in }]0, l_2[\}, \tag{3.13}$$

and still denote by α_1 the inner product on $H^1_{\gamma^a}(\omega^a) \times W_0$ defined by (3.3) with $q = 1$, i.e.,

$$\alpha_1 : (u, v) = ((u^a, u^b), (v^a, v^b)) \in \left(H^1_{\gamma^a}(\omega^a) \times W_0\right)^2 \longrightarrow \alpha_1(u, v)$$

$$= \int_{\omega^a} \left(\partial_{x_2} u^a \partial_{x_2} v^a + \partial_{x_3} u^a \partial_{x_3} v^a\right) dx_2 dx_3 + \int_{\omega^b} \left(\partial_{x_1} u^b \partial_{x_1} v^b + \partial_{x_2} u^b \partial_{x_2} v^b\right) dx_1 dx_2. \tag{3.14}$$

Then, both are Hilbert spaces and all classic results hold true for the following eigenvalue problem

$$\begin{cases} u \in H^1_{\gamma^a}(\omega^a) \times W_0, \\ \alpha_1(u, v) = \lambda[u, v]_1, \quad \forall v \in H^1_{\gamma^a}(\omega^a) \times W_0, \end{cases} \tag{3.15}$$

(see [21]) and the following result will be proved when $q = 0$.

Theorem 3.6 *With an abuse of notation, for every n in* $\mathbb{N}$*, let* $L^2(\Omega^a) \times L^2(\Omega^b)$ *be equipped with the inner product* $\frac{h^a_n}{h^b_n}(\cdot,\cdot)_n$*, where* $(\cdot,\cdot)_n$ *is defined by (2.6), still denoted by* H_n *and be the space defined by (2.7) equipped with the inner product* $\frac{h^a_n}{h^b_n} a_n(\cdot,\cdot)$*, where* $a_n(\cdot,\cdot)$ *is defined by (2.8), still denoted by* V_n*.*

For every n in $\mathbb{N}$, let $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ be the increasing diverging sequence of all the eigenvalues of Problem (2.10) and let $\{u_{n,k}\}_{k\in\mathbb{N}}$ be a H_n-Hilbert orthonormal basis such that $\left\{\lambda_{n,k}^{-\frac{1}{2}} u_{n,k}\right\}_{k\in\mathbb{N}}$ is a V_n-Hilbert orthonormal basis and, for every $k \in \mathbb{N}$, $u_{n,k} = (u^a_{n,k}, u^b_{n,k})$ is an eigenvector of Problem (2.10) with eigenvalue $\lambda_{n,k}$.

Assume that (2.1) holds true with $q = 0$.

Let $L^2(\omega^a) \times L^2(\omega^b)$ be the space equipped with the inner product $[\cdot,\cdot]_1$ defined by (3.1), W_0 be defined by (3.13), and $H^1_{\gamma^a}(\omega^a) \times W_0$ be the space equipped with the inner product $\alpha_1(\cdot,\cdot)$ defined by (3.14).

Then, there exists an increasing diverging sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ such that

$$\lim_n \lambda_{n,k} = \lambda_k, \quad \forall k \in \mathbb{N},$$

and $\{\lambda_k\}_{k\in\mathbb{N}}$ is the set of all the eigenvalues of Problem (3.15). Moreover, there exist an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and a $(L^2(\omega^a) \times L^2(\omega^b), [\cdot,\cdot]_1)$-Hilbert orthonormal basis $\{u_k\}_{k\in\mathbb{N}}$ (depending possibly on the selected subsequence $\{n_i\}_{i\in\mathbb{N}}$) such that, for every k in $\mathbb{N}$, u_k belongs to $H^1_{\gamma^a}(\omega^a) \times W_0$ and it is an eigenvector of Problem (3.15) with eigenvalue λ_k, and

$$\left(\sqrt{\frac{h^a_{n_i}}{h^b_{n_i}}}\, u^a_{n_i,k}, u^b_{n_i,k}\right) \to u_k \text{ strongly in } H^1(\Omega^a) \times H^1(\Omega^b), \quad \forall k \in \mathbb{N}, \tag{3.16}$$

as i diverges,

$$\frac{1}{h^a_n}\sqrt{\frac{h^a_n}{h^b_n}}\, \partial_{x_1} u^a_{n,k} \to 0 \text{ strongly in } L^2(\Omega^a), \quad \forall k \in \mathbb{N}, \tag{3.17}$$

$$\frac{1}{h^b_n} \partial_{x_3} u^b_{n,k} \to 0 \text{ strongly in } L^2(\Omega^b), \quad \forall k \in \mathbb{N}, \tag{3.18}$$

as n diverges. Furthermore, $\{\lambda_k^{-\frac{1}{2}} u_k\}_{k\in\mathbb{N}}$ is a $(H^1_{\gamma^a}(\omega^a) \times W_0, \alpha_1)$-Hilbert orthonormal basis.

Remark 3.7 Notice that (3.16) and (3.17) imply that

$$u^a_{n,k} \to 0 \text{ strongly in } H^1(\Omega^a), \quad \frac{1}{h^a_n} \partial_{x_1} u^a_{n,k} \to 0 \text{ strongly in } L^2(\Omega^a), \quad \forall k \in \mathbb{N}.$$

As far as the original problem (1.1) is concerned, one has the following result which is an immediate corollary of Theorem 3.6, by change of variable.

Corollary 3.8 *For every n in $\mathbb{N}$, let $L^2(\Omega_n)$ be equipped with the inner product defined*

$$(U, V) \in (L^2(\Omega_n))^2 \to \frac{1}{h^b_n} \int_{\Omega_n} UV dx$$

and let $\mathcal{V}_n$ be the space defined by (2.3) equipped with the inner product defined by

$$(U, V) \in \mathcal{V}_n \times \mathcal{V}_n \to \frac{1}{h^b_n} \int_{\Omega_n} DU DV dx.$$

For every n in $\mathbb{N}$, let $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ be the increasing diverging sequence of all the eigenvalues of Problem (2.5) and let $\{U_{n,k}\}_{k\in\mathbb{N}}$ be a $L^2(\Omega_n)$-Hilbert orthonormal basis such *that* $\left\{\lambda_{n,k}^{-\frac{1}{2}} U_{n,k}\right\}_{k\in\mathbb{N}}$ *is a* $\mathcal{V}_n$*-Hilbert orthonormal basis and, for every* $k \in \mathbb{N}$, $U_{n,k}$ *is an eigenvector of Problem* (2.5) *with eigenvalue* $\lambda_{n,k}$.

Assume that (2.1) holds true with $q = 0$.

Let $L^2(\omega^a) \times L^2(\omega^b)$ *be the space equipped with the inner product* $[\cdot,\cdot]_1$ *defined by* (3.1), W_0 *be defined by* (3.13)*, and* $H^1_{\gamma^a}(\omega^a) \times W_0$ *be the space equipped with the inner product* $\alpha_1(\cdot,\cdot)$ *defined by* (3.14).

Then, there exists an increasing diverging sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}}$ *such that*

$$\lim_n \lambda_{n,k} = \lambda_k, \quad \forall k \in \mathbb{N},$$

and $\{\lambda_k\}_{k\in\mathbb{N}}$ *is the set of all the eigenvalues of Problem (3.15). Moreover, there exist an increasing sequence of positive integer numbers* $\{n_i\}_{i\in\mathbb{N}}$ *and a* $(L^2(\omega^a) \times L^2(\omega^b), [\cdot,\cdot]_1)$*-Hilbert orthonormal basis* $\{u_k\}_{k\in\mathbb{N}}$ *(depending possibly on the selected subsequence* $\{n_i\}_{i\in\mathbb{N}}$*) such that, for every* $k \in \mathbb{N}$, $u_k = (u_k^a, u_k^b)$ *belongs to* $H^1_{\gamma^a}(\omega^a) \times W_0$ *and it is an eigenvector of Problem (3.15) with eigenvalue* λ_k*, and*

$$\lim_n \fint_{\Omega_n^a} \left(\left|U_{n,k}\right|^2 + \left|DU_{n,k}\right|^2\right) dx = 0,$$

$$\lim_i \fint_{\Omega_{n_i}^a} \left|\sqrt{\frac{h_{n_i}^a}{h_{n_i}^b}} U_{n_i,k} - u_k^a\right|^2 dx = 0,$$

$$\lim_i \fint_{\Omega_{n_i}^a} \left(\left|\sqrt{\frac{h_{n_i}^a}{h_{n_i}^b}} \partial_{x_1} U_{n_i,k}\right|^2 + \left|\sqrt{\frac{h_{n_i}^a}{h_{n_i}^b}} \partial_{x_2} U_{n_i,k} - \partial_{x_2} u_k^a\right|^2 + \left|\sqrt{\frac{h_{n_i}^a}{h_{n_i}^b}} \partial_{x_3} U_{n_i,k} - \partial_{x_3} u_k^a\right|^2\right) dx = 0,$$

$$\lim_i \fint_{\Omega_{n_i}^b} \left(|U_{n_i,k} - u_k^b|^2 + |\partial_{x_1} U_{n_i,k} - \partial_{x_1} u_k^b|^2 + |\partial_{x_2} U_{n_i,k} - \partial_{x_2} u_k^b|^2 + |\partial_{x_3} U_{n_i,k}|^2\right) dx = 0.$$

4 *A priori* estimates on the eigenvalues

This section is devoted to proving lower and upper bounds for the eigenvalues of Problem (2.10)

Proposition 4.1 *For every* n *in* $\mathbb{N}$*, let* H_n *be the space* $L^2(\Omega^a) \times L^2(\Omega^b)$ *equipped with the inner product* $(\cdot,\cdot)_n$ *defined by* (2.6)*,* V_n *be the space defined by* (2.7) *equipped with the inner product* $a_n(\cdot,\cdot)$ *defined by* (2.8)*, and* $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ *be the increasing diverging sequence of all the eigenvalues of Problem (2.10). Then,*

$$\lambda_{n,k} \geq \frac{1}{l_2^2}, \quad \forall k, n \in \mathbb{N}. \tag{4.1}$$

$$\forall k \in \mathbb{N}, \quad \exists c_k \in]0, +\infty[\ :\ \lambda_{n,k} \leq c_k, \quad \forall n \in \mathbb{N}, \tag{4.2}$$

where l_2 *is the positive real number involved in the definition of* ω^a *and* ω^b *(see Sect. 2).*

Proof As far as the proof of (4.1) is concerned, at first note that the boundary conditions on $u_{n,k}^a$ and $u_{n,k}^b$ provide that

$$\|u_{n,k}^a\|_{L^2(\Omega^a)} \leq l_2 \|\partial_{x_2} u_{n,k}^a\|_{L^2(\Omega^a)}, \quad \|u_{n,k}^b\|_{L^2(\Omega^b)} \leq l_2 \|\partial_{x_2} u_{n,k}^b\|_{L^2(\Omega^b)}, \quad \forall n, k \in \mathbb{N}, \tag{4.3}$$

where l_2 is the positive real number involved in the definition of ω^a and ω^b.

Combining now (2.13), (4.3), and (2.12) gives

$$\lambda_{n,k} = a_n(u_{n,k}, u_{n,k}) \geq \int_{\Omega^a} |\partial_{x_2} u^a_{n,k}|^2 dx + \frac{h^b_n}{h^a_n} \int_{\Omega^b} |\partial_{x_2} u^b_{n,k}|^2 dx$$

$$\geq \frac{1}{l_2^2} \left(\int_{\Omega^a} |u^a_{n,k}|^2 dx + \frac{h^b_n}{h^a_n} \int_{\Omega^b} |u^b_{n,k}|^2 dx \right) = \frac{1}{l_2^2} (u_{n,k}, u_{n,k})_n = \frac{1}{l_2^2} \quad \forall n, k \in \mathbb{N},$$

i.e., (4.1) holds true.

As far as the proof of (4.2) is concerned, let $\{\lambda_j\}_{j\in\mathbb{N}}$ be the increasing diverging sequence of all the eigenvalues of the following problem

$$\begin{cases} -\Delta y(x_2, x_3) = \lambda y(x_2, x_3) \text{ in } \omega^a, \\ y = 0 \text{ on } \partial\omega^a. \end{cases} \tag{4.4}$$

Then, for every $j \in \mathbb{N}$ there exists an eigenvector y_j in $H^1_0(\omega^a)$ of (4.4) with eigenvalue λ_j such that $\{y_j\}_{j\in\mathbb{N}}$ is a $L^2(\omega^a)$-Hilbert orthonormal basis and $\left\{\lambda_j^{-\frac{1}{2}} y_j\right\}_{j\in\mathbb{N}}$ is a $H^1_0(\omega^a)$-Hilbert orthonormal basis.

For every j in $\mathbb{N}$, set

$$\zeta_j(x_1, x_2, x_3) = \begin{cases} y_j(x_2, x_3), \text{ if } (x_1, x_2, x_3) \in \Omega^a, \\ 0, \text{ if } (x_1, x_2, x_3) \in \Omega^b. \end{cases}$$

Fix k in $\mathbb{N}$ and set

$$Z_k = \left\{ \sum_{j=1}^{k} \alpha_j \zeta_j : \alpha_1, \cdots, \alpha_k \in \mathbb{R} \right\}.$$

Then, for every n in $\mathbb{N}$, Z_k is a subspace of V_n with dimension k. Consequently, the min-max Principle (2.14) provides that

$$\lambda_{n,k} \leq \max_{\zeta\in Z_k-\{0\}} \frac{a_n(\zeta, \zeta)}{(\zeta, \zeta)_n} = \max_{(\alpha_1,\cdots,\alpha_k)\in\mathbb{R}^k-\{0\}} \frac{\sum_{j=1}^k \alpha_j^2 \lambda_j}{\sum_{j=1}^k \alpha_j^2} \leq \lambda_k, \quad \forall n \in \mathbb{N},$$

i.e. (4.2) holds true with $c_k = \lambda_k$. □

Remark 4.2 It is possible to give an estimate of the constant c_k in Proposition 4.1. Indeed, it is well known that the set of all the eigenvalues of problem (4.4) is given by

$$\left\{ \left(\frac{i^2}{l_2^2} + \frac{m^2}{l_3^2} \right) \pi^2 \right\}_{i,m\in\mathbb{N}}.$$

Then,

$$\forall k \in \mathbb{N}, \quad c_k \leq \left(\frac{k^2}{l_2^2} + \frac{k^2}{l_3^2} \right) \pi^2.$$

Recall that $A_{im} \sin\left(\frac{i\pi}{l_2}x_2\right) \sin\left(\frac{m\pi}{l_3}x_3\right)$, with A_{im} in $\mathbb{R}$, is an eigenfunction of (4.4) with eigenvalue $\left(\frac{i^2}{l_2^2}+\frac{m^2}{l_3^2}\right)\pi^2$.

Remark 4.3 Proposition 4.1 is independent of the asymptotic behavior of $\{h_n^a\}_{n\in\mathbb{N}}$ and $\{h_n^b\}_{n\in\mathbb{N}}$.

Choosing $k = h$ in (2.13) and taking into account Proposition 4.1 provide the following result.

Corollary 4.4 *For every n in $\mathbb{N}$, let H_n be the space $L^2(\Omega^a) \times L^2(\Omega^b)$ equipped with the inner product $(\cdot,\cdot)_n$ defined by (2.6) and V_n be the space defined by (2.7) equipped with the inner product $a_n(\cdot,\cdot)$ defined by (2.8).*

For every n in $\mathbb{N}$, let $\{\lambda_{n,k}\}_{k\in\mathbb{N}}$ be the increasing diverging sequence of all the eigenvalues of Problem (2.10) and let $\{u_{n,k}\}_{k\in\mathbb{N}}$ be a H_n-Hilbert orthonormal basis such that $\left\{\lambda_{n,k}^{-\frac{1}{2}}u_{n,k}\right\}_{k\in\mathbb{N}}$ is a V_n-Hilbert orthonormal basis and, for every $k\in\mathbb{N}$, $u_{n,k}$ is an eigenvector of Problem (2.10) with eigenvalue $\lambda_{n,k}$. Then,

$$\forall k\in\mathbb{N},\quad \exists c_k\in]0,+\infty[\,:\; a_n(u_{n,k},u_{n,k}) = \lambda_{n,k} \le c_k,\quad \forall n,k\in\mathbb{N}. \tag{4.5}$$

5 Some preliminary results

This section contains some results that are crucial for proving Theorems 3.1, 3.3, and 3.6. Precisely, Proposition 5.1 will give a trace convergence result, written in a very general way, which will allow us to identify junction and boundary conditions in the limit problems. Proposition 5.2 will give a density result for approximating the elements of V defined in (3.2) by regular functions. Although this result was used in other works, to our knowledge, there are no previous proofs of it. Our proof is rather technical and it works also for domains which are not "symmetric". Proposition 5.3 is devoted to building a recovery sequence which will be used in the proof of all three main results.

Proposition 5.1 *Let $\{h_i\}_{i\in\mathbb{N}}$ be a sequence in $]0,+\infty[$ such that*

$$\lim_i h_i = 0. \tag{5.1}$$

Let $\{w_i\}_{i\in\mathbb{N}}$ be sequence in $H^1(\Omega^b)$ such that

$$\lim_i \left(\frac{1}{h_i}\int_{\Omega^b} |\partial_{x_3} w_i(x)|^2 dx\right) = 0, \tag{5.2}$$

and

$$\exists w\in H^1(\Omega^b):\quad w_i \rightharpoonup w \text{ weakly in } H^1(\Omega^b),\ \text{as } i\to+\infty, \tag{5.3}$$

Then,

$$\lim_i \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[} w_i(h_i x_1, x_2, 0)\varphi(x_2)dx_1dx_2 = \int_0^{l_2} w(0,x_2)\varphi(x_2)dx_2,\quad \forall\varphi\in C_0^\infty(]0,l_2[). \tag{5.4}$$

Notice that assumption (5.2) ensures that the function w given by (5.3) is independent of x_3, i.e.,

$$w(x_1, x_2, x_3) = w(x_1, x_2), \text{ for a.e. } (x_1, x_2, x_3) \in \Omega^b, \quad \text{for a.e. } (x_1, x_2) \in \omega^b. \tag{5.5}$$

Then, it makes sense to write $w(0, x_2)$ in (5.4).

Proof At first, one proves the existence of $\overline{x}_3$ in $]-1, 0[$ and of an increasing sequence of positive integer numbers $\{i_j\}_{j\in\mathbb{N}}$ such that

$$w_{i_j}(\cdot, \cdot, \overline{x}_3) \rightharpoonup w \text{ weakly in } H^1\left(]-\tfrac{1}{2}, \tfrac{1}{2}[\times]0, l_2[\right), \tag{5.6}$$

as j diverges.

Indeed, set

$$\rho_i(x_3) = \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[} \left(|w_i(x_1, x_2, x_3)|^2 + \left|\partial_{x_1} w_i(x_1, x_2, x_3)\right|^2 + \left|\partial_{x_2} w_i(x_1, x_2, x_3)\right|^2\right) dx_1 dx_2$$

for x_3 a.e. in $]-1, 0[, \quad \forall i \in \mathbb{N}$.

Then, Fatou's Lemma combined with assumption (5.3) provides that

$$\int_{-1}^{0} \liminf_i \rho_i(x_3) dx_3 \le \liminf_i \int_{-1}^{0} \rho_i(x_3) dx_3 < +\infty.$$

Consequently, there exist two constants c in $]0, +\infty[$ and $\overline{x}_3$ in $]-1, 0[$, and an increasing sequence of positive integer numbers $\{i_j\}_{j\in\mathbb{N}}$ such that

$$\rho_{i_j}(\overline{x}_3) < c, \quad \forall j \in \mathbb{N},$$

which provides (5.6), thanks to (5.3) and (5.5).

Now, for proving (5.4), fix φ in $C_0^\infty(]0, l_2[)$ and split the first integral in (5.4), written with index i_j, as

$$\begin{aligned}
&\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[} w_{i_j}(h_{i_j} x_1, x_2, 0)\varphi(x_2) dx_1 dx_2 \\
&= \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[} \left(w_{i_j}(h_{i_j} x_1, x_2, 0) - w_{i_j}(h_{i_j} x_1, x_2, \overline{x}_3)\right) \varphi(x_2) dx_1 dx_2 \\
&+ \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[} \left(w_{i_j}(h_{i_j} x_1, x_2, \overline{x}_3) - w_{i_j}(0, x_2, \overline{x}_3)\right) \varphi(x_2) dx_1 dx_2 \\
&+ \int_0^{l_2} w_{i_j}(0, x_2, \overline{x}_3)\varphi(x_2) dx_2. \quad \forall j \in \mathbb{N}.
\end{aligned} \tag{5.7}$$

One will pass to the limit, as j diverges, in each term of this decomposition.

As far as the first integral on the right-hand side of (5.7) is concerned, assumption (5.2) implies that

$$\begin{aligned}
&\left|\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[}\left(w_{i_j}(h_{i_j}x_1,x_2,0)-w_{i_j}(h_{i_j}x_1,x_2,\overline{x}_3)\right)\varphi(x_2)dx_1dx_2\right|\\
&=\left|\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[}\left(\int_{\overline{x}_3}^{0}\partial_{x_3}w_{i_j}(h_{i_j}x_1,x_2,x_3)dx_3\right)\varphi(x_2)dx_1dx_2\right|\\
&\leq\|\varphi\|_{L^\infty(]0,l_2[)}|\Omega^b|^{\frac{1}{2}}\left(\int_{\Omega^b}|\partial_{x_3}w_{i_j}(h_{i_j}x_1,x_2,x_3)|^2dx\right)^{\frac{1}{2}}\\
&\leq\|\varphi\|_{L^\infty(]0,l_2[)}|\Omega^b|^{\frac{1}{2}}\left(\frac{1}{h_{i_j}}\int_{\Omega^b}|\partial_{x_3}w_{i_j}(x_1,x_2,x_3)|^2dx\right)^{\frac{1}{2}}\to 0,\ \text{as } j\to+\infty.
\end{aligned}\tag{5.8}$$

As far as the second integral on the right-hand side of (5.7) is concerned, assumption (5.1) and (5.6) imply

$$\begin{aligned}
&\left|\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[}\left(w_{i_j}(h_{i_j}x_1,x_2,\overline{x}_3)-w_{i_j}(0,x_2,\overline{x}_3)\right)\varphi(x_2)dx_1dx_2\right|\\
&=\left|\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[}\left(\int_{0}^{h_{i_j}x_1}\partial_t w_{i_j}(t,x_2,\overline{x}_3)dt\right)\varphi(x_2)dx_1dx_2\right|\\
&\leq\frac{1}{2}\|\varphi\|_{L^\infty(]0,l_2[)}\int_0^{l_2}\left(\int_0^{\frac{h_{i_j}}{2}}\left|\partial_t w_{i_j}(t,x_2,\overline{x}_3)\right|dt\right)dx_2\\
&+\frac{1}{2}\|\varphi\|_{L^\infty(]0,l_2[)}\int_0^{l_2}\left(\int_{-\frac{h_{i_j}}{2}}^{0}\left|\partial_t w_{i_j}(t,x_2,\overline{x}_3)\right|dt\right)dx_2\\
&\leq\|\varphi\|_{L^\infty(]0,l_2[)}\sqrt{l_2\frac{h_{i_j}}{2}}\left(\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[}|\partial_{x_1}w_{i_j}(x_1,x_2,\overline{x}_3)|^2dx_1dx_2\right)^{\frac{1}{2}}\to 0,\\
&\text{as } j\to+\infty.
\end{aligned}\tag{5.9}$$

As far as the last integral on the right-hand side of (5.7) is concerned, (5.6) implies

$$\lim_j\int_0^{l_2}w_{i_j}(0,x_2,\overline{x}_3)\varphi(x_2)dx_2=\int_0^{l_2}w(0,x_2)\varphi(x_2)dx_2.\tag{5.10}$$

Eventually, passing to the limit in (5.7), as j diverges, and taking into account (5.8), (5.9), and (5.10) give (5.4) for the subsequence $\{n_{i_j}\}_{j\in\mathbb{N}}$. Notice that (5.4) holds true for the whole subsequence $\{n_i\}_{i\in\mathbb{N}}$ too, since the limit φw does not depend on $\{n_{i_j}\}_{j\in\mathbb{N}}$. □

The following proposition is devoted to approximating the elements of the space V defined by (3.2) by more regular functions belonging to the space V_{reg} defined by

$$\begin{cases} V_{reg} = \Big\{ \left(v^a, v^b\right) \in C_0^\infty \left(]0, l_2[\times[0, l_3[\right) \times C_0\left(\omega^b\right) : \\ v^b_{|_{[-l_1^-,0]\times[0,l_2]}} \in C^\infty\left([-l_1^-, 0] \times [0, l_2]\right), \quad v^b_{|_{[0,l_1^+]\times[0,l_2]}} \in C^\infty\left([0, l_1^+] \times [0, l_2]\right), \\ v^a(x_2, 0) = v^b\left(0, x_2\right) \text{ in }]0, l_2[\Big\}. \end{cases} \tag{5.11}$$

Proposition 5.2 *Let V and V_{reg} be defined by (3.2) and (5.11), respectively. Then, V_{reg} is dense in V.*

Proof Fix $\left(v^a, v^b\right)$ in V. The goal is to find a sequence $\left\{\left(v_n^a, v_n^b\right)\right\}_{n\in\mathbb{N}}$ in V_{reg} such that

$$\left(v_n^a, v_n^b\right) \to \left(v^a, v^b\right) \text{ strongly in } H^1\left(\omega^a\right) \times H^1\left(\omega^b\right). \tag{5.12}$$

The proof of (5.12) will be split into two steps.
Step 1. The first step is devoted to proving (5.12) when

$$l_1^+ = l_1^-.$$

Split v^b in the even part and in the odd part with respect to x_1, i.e.,

$$v^b(x_1, x_2) = v^e(x_1, x_2) + v^o(x_1, x_2), \text{ a.e. in } \omega^b, \tag{5.13}$$

where

$$v^e(x_1, x_2) = \frac{v^b(x_1, x_2) + v^b(-x_1, x_2)}{2}, \quad v^o(x_1, x_2) = \frac{v^b(x_1, x_2) - v^b(-x_1, x_2)}{2}, \text{ a.e. in } \omega^b.$$

As far as the approximation of v^o is concerned, since it belongs to $H_0^1(\omega^b)$ and

$$v^o(0, x_2) = 0, \text{ a.e. in }]0, l_2[,$$

one has that $v^o_{|\omega^b_-}$ belongs to $H_0^1(\omega^b_-)$ and $v^o_{|\omega^b_+}$ belongs to $H_0^1(\omega^b_+)$ (see Sect. 2 for the definition of ω^b_+ and ω^b_-). Consequently, there exist two sequences $\left\{v_n^{o-}\right\}_{n\in\mathbb{N}}$ in $C_0^\infty\left(\omega^b_-\right)$ and $\left\{v_n^{o+}\right\}_{n\in\mathbb{N}}$ in $C_0^\infty\left(\omega^b_+\right)$ such that

$$v_n^{o-} \to v^o_{|\omega^b_-} \text{ strongly in } H^1\left(\omega^b_-\right), \quad v_n^{o+} \to v^o_{|\omega^b_+} \text{ strongly in } H^1\left(\omega^b_+\right).$$

Then, setting for every n in $\mathbb{N}$

$$v_n^o : (x_1, x_2) \in \omega^b \to \begin{cases} v_n^{o+}(x_1, x_2), & \text{if } (x_1, x_2) \in \omega^b_+, \\ 0, & \text{if } (x_1, x_2) \in \{0\}\times]0, l_2[, \\ v_n^{o-}(x_1, x_2), & \text{if } (x_1, x_2) \in \omega^b_-, \end{cases}$$

one has that

$$v_n^o \in C_0^\infty\left(\omega^b\right), \quad \forall n \in \mathbb{N}, \tag{5.14}$$

$$v_n^o(0, x_2) = 0, \text{ if } x_2 \in]0, l_2[, \quad \forall n \in \mathbb{N}, \tag{5.15}$$

and

$$v_n^o \to v^o \text{ strongly in } H^1\left(\omega^b\right). \tag{5.16}$$

As far as the approximation of v^a and v^e is concerned, set

$$\omega_R^a =]-l_3, 0]\times]0, l_2[,$$
$$v_R^a : (x_1, x_2) \in \omega_R^a \to v_R^a(x_1, x_2) = v^a(x_2, -x_1), \tag{5.17}$$
$$\widehat{v} : (x_1, x_2) \in \omega_+^b \cup \omega_R^a \to \begin{cases} v^e(x_1, x_2), \text{ if } (x_1, x_2) \in \omega_+^b, \\ \\ v_R^a(x_1, x_2), \text{ if } (x_1, x_2) \in \omega_R^a. \end{cases} \tag{5.18}$$

Since

$$v_R^a(0, x_2) = v^a(x_2, 0) = v^b(0, x_2) = v^e(0, x_2), \text{ a.e. in }]0, l_2[,$$

it is easy to see that $\widehat{v}$ belongs to $H_0^1(\omega_+^b \cup \omega_R^a)$. Consequently, there exists a sequence $\{\widehat{v}_n\}_{n\in\mathbb{N}}$ in $C_0^\infty(\omega_+^b \cup \omega_R^a)$, such that

$$\widehat{v}_n \to \widehat{v} \text{ strongly in } H^1\left(\omega_+^b \cup \omega_R^a\right),$$

which implies, thanks to definition (5.18), that

$$\widehat{v}_{n_{|\omega_+^b}} \to v^e_{|\omega_+^b} \text{ strongly in } H^1\left(\omega_+^b\right), \tag{5.19}$$

and

$$\widehat{v}_{n_{|\omega_R^a}} \to v_R^a \text{ strongly in } H^1\left(\text{Interior}(\omega_R^a)\right). \tag{5.20}$$

Set now, for every n in $\mathbb{N}$,

$$v_n^a : (x_2, x_3) \in]0, l_2[\times[0, l_3[\to v_n^a(x_2, x_3) = \widehat{v}_{n_{|\omega_R^a}}(-x_3, x_2). \tag{5.21}$$

Then, the sequence $\left\{v_n^a\right\}_{n\in\mathbb{N}}$ is included in C_0^∞ (]0, $l_2[\times[0, l_3[$) and, thanks to (5.21), (5.20), and (5.17), it converges strongly in $H^1(\omega^a)$ to the function given by

$$v_R^a(-x_3, x_2) = v^a(x_2, x_3), \text{ a.e. in } \omega^a,$$

i.e.

$$v_n^a \to v^a \text{ strongly in } H^1\left(\omega^a\right).$$

Moreover, setting for every n in $\mathbb{N}$,

$$v_n^e : (x_1, x_2) \in \omega^b \to v_n^e(x_2, x_3) = \begin{cases} \widehat{v}_n(x_1, x_2), & \text{if } (x_1, x_2) \in \omega_+^b, \\ \\ \widehat{v}_n(0, x_2), & \text{if } x_2 \in]0, l_2[, \\ \\ \widehat{v}_n(-x_1, x_2), & \text{if } (x_1, x_2) \in \omega_-^b. \end{cases} \tag{5.22}$$

one has

$$v_n^e \in C_0\left(\omega^b\right), \quad v^e_{n_{|\overline{\omega_+^b}}} \in C^\infty\left(\overline{\omega_+^b}\right) \quad v^e_{n_{|\overline{\omega_-^b}}} \in C^\infty\left(\overline{\omega_-^b}\right), \quad \forall n \in \mathbb{N}, \tag{5.23}$$
$$v_n^e(0, x_2) = \widehat{v}_n(0, x_2) = v_n^a(x_2, 0), \text{ if } x_2 \in]0, l_2[, \quad \forall n \in \mathbb{N}, \tag{5.24}$$

and by virtue of (5.19)

$$v_n^e \to v^e \text{ strongly in } H^1\left(\omega^b\right). \tag{5.25}$$

Now, setting for every n in $\mathbb{N}$,

$$v_n^b : (x_1, x_2) \in \omega^b \to v_n^e(x_1, x_2) + v_n^o(x_1, x_2),$$

(5.13), (5.14), (5.15), (5.16), (5.23), (5.24), and (5.25) imply that

$$v_n^b \in C_0\left(\omega^b\right), \quad v^b_{n|_{\overline{\omega^b_+}}} \in C^\infty\left(\overline{\omega^b_+}\right) \quad v^b_{n|_{\overline{\omega^b_-}}} \in C^\infty\left(\overline{\omega^b_-}\right), \quad \forall n \in \mathbb{N},$$
$$v_n^b(0, x_2) = v_n^a(x_2, 0), \text{ in }]0, l_2[, \quad \forall n \in \mathbb{N},$$
$$v_n^b \to v^b \text{ strongly in } H^1\left(\omega^b\right).$$

Eventually, the sequence $\left\{\left(v_n^a, v_n^b\right)\right\}_{n\in\mathbb{N}}$, so built, is in V_{reg} and satisfies (5.12).
Step 2. The second step is devoted to proving (5.12) when

$$l_1^+ \neq l_1^-.$$

For instance, assume

$$l_1^- > l_1^+.$$

Let $\widetilde{v}_b$ be the function defined on $]-l_1^-, l_1^-[\times]0, l_2[$ by

$$\widetilde{v}^b(x_1, x_2) = \begin{cases} v^b(x_1, x_2), & \text{if } x_1 < 0, \\ v^b\left(\dfrac{l_1^+}{l_1^-} x_1, x_2\right), & \text{if } x_1 > 0. \end{cases}$$

By virtue of the previous step, there exists a sequence $\left\{\left(v_n^a, \widetilde{v}_n^b\right)\right\}_{n\in\mathbb{N}} \subset C_0^\infty(]0, l_2[\times [0, l_3[\times C_0\left(]-l_1^-, l_1^-[\times]0, l_2[\right)$ such that

$$\widetilde{v}^b_{n|_{[-l_1^-,0]\times[0,l_2]}} \in C^\infty\left([-l_1^-, 0] \times [0, l_2]\right), \quad \widetilde{v}^b_{n|_{[0,l_1^-]\times[0,l_2]}} \in C^\infty\left([0, l_1^-] \times [0, l_2]\right),$$
$$v^a(x_2, 0) = \widetilde{v}_n^b(0, x_2) \text{ in }]0, l_2[,$$

for every $n \in \mathbb{N}$, and

$$\left(v_n^a, \widetilde{v}_n^b\right) \to \left(v^a, \widetilde{v}^b\right) \text{ strongly in } H^1\left(\omega^a\right) \times H^1\left(]-l_1^-, l_1^-[\times]0, l_2[\right).$$

Now, for every $n \in \mathbb{N}$, let v_n^b be the function defined on $\omega^b =]-l_1^-, l_1^+[\times]0, l_2[$ by

$$v_n^b = \begin{cases} \widetilde{v}_n^b(x_1, x_2), & \text{if } x_1 < 0, \\ \widetilde{v}_n^b\left(\dfrac{l_1^-}{l_1^+} x_1, x_2\right), & \text{if } x_1 > 0. \end{cases}$$

Then, the sequence $\left\{\left(v_n^a, v_n^b\right)\right\}_{n\in\mathbb{N}}$ belongs to V_{reg} and satisfies (5.12).
The proof of (5.12) is similar if $l_1^- < l_1^+$. □

This section concludes with the building of a recovery sequence for functions in V_{reg} with functions in V_n defined by (2.7).

Proposition 5.3 *Let V_{reg} be defined by (5.11). Let $v = (v^a, v^b)$ be in V_{reg}. Then, there exists a sequence $\{g_n\}_{n\in\mathbb{N}} \subset H^1_{\Gamma^a}(\Omega^a)$ such that*

$$\begin{cases} g_n \to v^a \text{ strongly in } L^2(\Omega^a), \text{ as } n \to +\infty, \\ \left(\frac{1}{h_n^a}\partial_{x_1} g_n, \partial_{x_2} g_n, \partial_{x_3} g_n\right) \to \left(0, \partial_{x_2} v^a, \partial_{x_3} v^a\right) \text{ strongly in } \left(L^2(\Omega^a)\right)^3, \\ \qquad\qquad\qquad\qquad \text{as } n \to +\infty, \\ g_n(x_1, x_2, 0) = v^b(h_n^a x_1, x_2), \text{ for } (x_1, x_2) \in]-\frac{1}{2}, \frac{1}{2}[\times]0, l_2[, \quad \forall n \in \mathbb{N}. \end{cases} \tag{5.26}$$

Proof For every $n \in \mathbb{N}$ set

$$g_n(x) = \begin{cases} v^a(x_2, x_3), & \text{if } x = (x_1, x_2, x_3) \in]-\frac{1}{2}, \frac{1}{2}[\times]0, l_2[\times]h_n^a, l_3[, \\ v^a(x_2, h_n^a)\dfrac{x_3}{h_n^a} + v^b(h_n^a x_1, x_2)\dfrac{h_n^a - x_3}{h_n^a}, & \text{if } x = (x_1, x_2, x_3) \in]-\frac{1}{2}, \frac{1}{2}[\times]0, l_2[\times[0, h_n^a]. \end{cases}$$

Obviously, $\{g_n\}_{n\in\mathbb{N}}$ is included in $H^1_{\Gamma^a}(\Omega^a)$ and the last line of (5.26) is satisfied. Moreover, by the definition of V_{reg}, it is easy to see that

$$\begin{aligned} &\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[\times]0,h_n^a[} |g_n|^2\, dx \le 2(\|v^a\|^2_{L^\infty(\omega^a)} + \|v^b\|^2_{L^\infty(\omega^b)}) l_2 h_n^a \to 0, \\ &\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[\times]0,h_n^a[} \left|\frac{1}{h_n^a}\partial_{x_1} g_n\right|^2 dx \le \|v^b\|^2_{W^{1,\infty}(\omega^b)} l_2 h_n^a \to 0, \\ &\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[\times]0,h_n^a[} \left|\partial_{x_2} g_n\right|^2 dx \le 2(\|v^a\|^2_{W^{1,\infty}(\omega^a)} + \|v^b\|^2_{W^{1,\infty}(\omega^b)}) l_2 h_n^a \to 0, \end{aligned}$$

and

$$\begin{aligned} &\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[\times]0,h_n^a[} \left|\partial_{x_3} g_n\right|^2 dx = \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[} \frac{1}{h_n^a}\left|v^a(x_2, h_n^a) - v^b(h_n^a x_1, x_2)\right|^2 dx_1 dx_2 \\ &= \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[} \frac{1}{h_n^a}\left|v^a(x_2, h_n^a) - v^a(x_2, 0) + v^b(0, x_2) - v^b(h_n^a x_1, x_2)\right|^2 dx_1 dx_2 \\ &\le 2\left(\|v^a\|^2_{W^{1,\infty}(\omega^a)} + \|v^b\|^2_{W^{1,\infty}(\omega^b)}\right) l_2 h_n^a \to 0, \end{aligned}$$

as n diverges, which imply the convergences in (5.26). □

Eventually, introduce the space

$$\begin{aligned} \widetilde{V} &= \left\{v = (v^a, v^b) \in H^1_{\Gamma^a}(\Omega^a) \times H^1_{\Gamma^b}(\Omega^b) \;:\; v^a \text{ indep. of } x_1, \quad v^b \text{ indep. of } x_3\right\} \\ &\simeq H^1_{\gamma^a}(\omega^a) \times H^1_0(\omega^b). \end{aligned} \tag{5.27}$$

which will be used in the following sections.

6 Proof of Theorem 3.1

The proof will be split into several steps.

Step 1. The first step is devoted to proving the existence of an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$, an increasing sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}}$, a sequence $\left\{u_k = (u_k^a, u_k^b)\right\}_{k\in\mathbb{N}}$ in $\widetilde{V}$, where $\widetilde{V}$ is the space defined by (5.27), and a sequence $\left\{(\xi_k^a, \xi_k^b)\right\}_{k\in\mathbb{N}}$ in $L^2(\Omega^a)\times L^2(\Omega^b)$ (depending possibly on the selected subsequence $\{n_i\}_{i\in\mathbb{N}}$) such that, for every k in $\mathbb{N}$,

$$\lim_i \lambda_{n_i,k} = \lambda_k, \tag{6.1}$$

$$(u_{n_i,k}^a, u_{n_i,k}^b) \rightharpoonup (u_k^a, u_k^b) \text{ weakly in } H^1(\Omega^a)\times H^1(\Omega^b) \text{ and strongly in } L^2(\Omega^a)\times L^2(\Omega^b), \tag{6.2}$$

$$\left(\frac{1}{h_{n_i}^a}\partial_{x_1} u_{n_i,k}^a, \frac{1}{h_{n_i}^b}\partial_{x_3} u_{n_i,k}^b\right) \rightharpoonup (\xi_k^a, \xi_k^b) \text{ weakly in } L^2(\Omega^a)\times L^2(\Omega^b), \tag{6.3}$$

as i diverges, and

$$[u_k, u_h]_q = \delta_{h,k}, \quad \forall k, h \in \mathbb{N}. \tag{6.4}$$

Estimates in (4.1) and in (4.5), assumption (2.1) with q in $]0, +\infty[$, and a diagonal argument ensure that (6.1), (6.2), and (6.3) hold true for a suitable increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$ and for suitable sequences $\{\lambda_k\}_{k\in\mathbb{N}}$ in $[\frac{1}{l_2^2}, +\infty[$, $\left\{u_k = (u_k^a, u_k^b)\right\}_{k\in\mathbb{N}}$ in $\widetilde{V}$ and $\left\{(\xi_k^a, \xi_k^b)\right\}_{k\in\mathbb{N}}$ in $L^2(\Omega^a)\times L^2(\Omega^b)$.

Eventually, (6.4) follows by passing to the limit in

$$(u_{n_i,k}, u_{n_i,h})_n = \delta_{h,k}, \quad \forall i, k, h, \in \mathbb{N},$$

as i diverges, thanks to assumption (2.1) with q in $]0, +\infty[$ and the strong L^2-convergence in (6.2).

For asserting that $u_k = (u_k^a, u_k^b)$ belongs to V, it remains to prove the following result.

Step 2.

$$u_k^a(x_2, 0) = u_k^b(0, x_2) \text{ a.e. in }]0, l_2[, \quad \forall k \in \mathbb{N}. \tag{6.5}$$

Fix k in $\mathbb{N}$.

The transmission condition in (2.7) gives

$$\begin{aligned}&\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[} u_{k,n_i}^a(x_1, x_2, 0)\varphi(x_2)dx_1dx_2\\ &= \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[} u_{k,n_i}^b(h_{n_i}^a x_1, x_2, 0)\varphi(x_2)dx_1dx_2, \quad \forall i \in \mathbb{N}, \quad \forall \varphi \in C_0^\infty(]0, l_2[).\end{aligned} \tag{6.6}$$

As far as the first integral in (6.6) is concerned, the weak H^1-convergence in (6.2) and the fact that u_k^a is independent of x_1 imply

$$\begin{aligned}&\lim_i \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[} u_{k,n_i}^a(x_1, x_2, 0)\varphi(x_2)dx_1dx_2\\ &= \int_0^{l_2} u_k^a(x_2, 0)\varphi(x_2)dx_2, \quad \forall \varphi \in C_0^\infty(]0, l_2[).\end{aligned} \tag{6.7}$$

As far as the last integral in (6.6) is concerned, note that estimate in (4.5) provides that

$$\frac{1}{h_n^a}\int_{\Omega^b}|\partial_{x_3}u_{k,n}^b(x)|^2dx \le c_k h_n^b \to 0,\ \text{as } n\to+\infty. \tag{6.8}$$

Then, combining (6.8) with the weak H^1-convergence in (6.2) and using Proposition 5.1 yield

$$\begin{aligned}&\lim_i \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[} u_{k,n_i}^b(h_{n_i}^a x_1, x_2, 0)\varphi(x_2)dx_1dx_2\\ &= \int_0^{l_2} u_k^b(0,x_2)\varphi(x_2)dx_2, \quad \forall\varphi\in C_0^\infty(]0,l_2[)\,.\end{aligned} \tag{6.9}$$

Eventually, the junction condition in (6.5) follows from (6.6), (6.7), and (6.9).
Step 3. This step is devoted to proving that

$$\alpha_q(u_k,v) = \lambda_k[u_k,v]_q, \quad \forall v\in V, \qquad \forall k\in\mathbb{N}, \tag{6.10}$$

$$\alpha_q(\lambda_k^{-\frac{1}{2}}u_k, \lambda_h^{-\frac{1}{2}}u_h) = \delta_{h,k}, \quad \forall k,h\in\mathbb{N}, \tag{6.11}$$

$$\lim_k \lambda_k = +\infty. \tag{6.12}$$

Fix k in $\mathbb{N}$. To prove (6.10), by following the classic idea of $\Gamma-$convergence, a recovery sequence will be constructed for regular function $v=(v^a,v^b)$ in V_{reg}, where V_{reg} is defined by (5.11). Precisely, for $v=(v^a,v^b)$ in V_{reg}, let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence in $H^1_{\Gamma^a}(\Omega^a)$ satisfying (5.26) in Proposition 5.3. Choosing (g_{n_i}, v^b) as test function in (2.11) written with index n_i yields

$$\begin{aligned}&\int_{\Omega^a}\left(\frac{1}{h_{n_i}^a}\partial_{x_1}u_{n_i,k}^a\frac{1}{h_{n_i}^a}\partial_{x_1}g_{n_i} + \partial_{x_2}u_{n_i,k}^a\partial_{x_2}g_{n_i} + \partial_{x_3}u_{n_i,k}^a\partial_{x_3}g_{n_i}\right)dx\\ &+\frac{h_{n_i}^b}{h_{n_i}^a}\int_{\Omega^b}\left(\partial_{x_1}u_{n_i,k}^b\partial_{x_1}v^b + \partial_{x_2}u_{n_i,k}^b\partial_{x_2}v^b\right)dx\\ &= \lambda_{n_i,k}\left(\int_{\Omega^a}u_{n_i,k}^a g_{n_i}dx + \frac{h_{n_i}^b}{h_{n_i}^a}\int_{\Omega^b}u_{n_i,k}^b v^b dx\right), \quad \forall i\in\mathbb{N}.\end{aligned} \tag{6.13}$$

Passing to the limit, as i diverges, in (6.13) and using (2.1) with q in $]0,+\infty[$, (6.1), (6.2), (6.3), and (5.26) provide that

$$\begin{aligned}&\int_{\omega^a}\left(\partial_{x_2}u_k^a\partial_{x_2}v^a + \partial_{x_3}u_k^a\partial_{x_3}v^a\right)dx_2dx_3 + q\int_{\omega^b}\left(\partial_{x_1}u_k^b\partial_{x_1}v^b + \partial_{x_2}u_k^b\partial_{x_2}v^b\right)dx_1dx_2\\ &= \lambda_k\int_{\omega^a}u_k^a v^a dx_2dx_3 + q\int_{\omega^b}u_k^b v^b dx_1dx_2, \quad \forall(v^a,v^b)\in V_{reg},\end{aligned}$$

which implies (6.10), thanks to the density of V_{reg} in V proved in Proposition 5.2.

Relations in (6.11) follow from (6.10), (6.4), and from the fact that λ_k are all positive.

As far as (6.12) is concerned, either (6.12) holds true, or $\{\lambda_k\}_{k\in N}$ is a finite set. In the second case, by virtue of (6.4), Problem (6.10) would admit an eigenvalue of infinite multiplicity. But this is not possible, due to the Fredholm's alternative Theorem.

Step 4. This step is devoted to proving (3.5), (3.6), and (3.7).

Fix k in $\mathbb{N}$.

Combining (2.13), (6.1), and (6.11) gives the convergence of the energies

$$\lim_{n_i} a_{n_i}(u_{n_i,k}, u_{n_i,k}) = \lim_{n_i} \lambda_{n_i,k} = \lambda_k = \alpha_q(u_k, u_k),$$

which implies (3.5), (3.6), and (3.7), thanks to (2.1) with q in $]0, +\infty[$, (6.2), and (6.3).

Step 5. Conclusion.

It is proved that $\{\lambda_k\}_{k\in\mathbb{N}}$ is included in $[\frac{1}{l_2^2}, +\infty[$ and it is an increasing and diverging sequence of eigenvalues of Problem (6.10), $\{u_k\}_{k\in\mathbb{N}}$ is an orthonormal sequence in $(L^2(\omega^a)\times L^2(\omega^b), [\cdot,\cdot]_q)$, $\{\lambda_k^{-\frac{1}{2}} u_k\}_{k\in\mathbb{N}}$ is an orthonormal sequence in (V, α_q), for every $k \in \mathbb{N}$ u_k is an eigenvector for Problem (6.10), with eigenvalue λ_k, and convergences (3.5), (3.6), and (3.7) hold true.

Moreover, arguing as in [9] (see step 2 in the proof of Theorem 2.5) or as in [22] (see Theorem 9.2), one can prove that there does not exist $(\overline{u}, \overline{\lambda}) \in V \times \mathbb{R}$ satisfying the following problem

$$\begin{cases} \overline{u} \in V, \\ \alpha_q(\overline{u}, v) = \overline{\lambda}[\overline{u}, v]_q, \quad \forall v \in V, \\ [\overline{u}, u_k]_q = 0, \quad \forall k \in \mathbb{N}, \\ [\overline{u}, \overline{u}]_q = 1. \end{cases}$$

As in [9] (see step 3 in the proof of Theorem 2.5), this implies that the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ forms the whole set of the eigenvalues of Problem (3.4), that $\{\lambda_k^{-\frac{1}{2}} u_k\}_{k\in\mathbb{N}}$ is a (V, α_q)-Hilbert orthonormal basis, and that $\{u_k\}_{k\in\mathbb{N}}$ is a $(L^2(\omega^a)\times L^2(\omega^b), [\cdot,\cdot]_q)$-Hilbert orthonormal basis.

In conclusion, since the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ can be characterized by the min-max Principle, for every $k \in \mathbb{N}$ convergence (6.1) holds true for the whole sequence $\{\lambda_{n,k}\}_{n\in\mathbb{N}}$.

7 Proof of Theorem 3.3

The proof will be split into several steps.

Step 1. The first step is devoted to proving the existence of an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$, an increasing sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}}$, a sequence $\left\{u_k = (u_k^a, u_k^b)\right\}_{k\in\mathbb{N}}$ in $\widetilde{V}$, where $\widetilde{V}$ is the space defined by (5.27), and a sequence $\left\{(\xi_k^a, \xi_k^b)\right\}_{k\in\mathbb{N}}$ in $L^2(\Omega^a)\times L^2(\Omega^b)$ (depending possibly on the selected subsequence $\{n_i\}_{i\in\mathbb{N}}$)

such that, for every k in $\mathbb{N}$,

$$\lim_i \lambda_{n_i,k} = \lambda_k, \tag{7.1}$$

$$\left(u^a_{n_i,k}, \sqrt{\frac{h^b_{n_i}}{h^a_{n_i}}}\, u^b_{n_i,k}\right) \rightharpoonup (u^a_k, u^b_k) \text{ weakly in } H^1(\Omega^a) \times H^1(\Omega^b) \text{ and} \\ \text{strongly in } L^2(\Omega^a) \times L^2(\Omega^b), \tag{7.2}$$

$$\left(\frac{1}{h^a_{n_i}} \partial_{x_1} u^a_{n_i,k}, \frac{1}{h^b_{n_i}} \sqrt{\frac{h^b_{n_i}}{h^a_{n_i}}}\, \partial_{x_3} u^b_{n_i,k}\right) \rightharpoonup (\xi^a_k, \xi^b_k) \text{ weakly in } L^2(\Omega^a) \times L^2(\Omega^b), \tag{7.3}$$

as i diverges, and

$$[u_k, u_h]_1 = \delta_{h,k}, \quad \forall k, h \in \mathbb{N}. \tag{7.4}$$

Estimates in (4.1) and in (4.5), and a diagonal argument ensure that (7.1), (7.2), and (7.3) hold true for a suitable increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$, for suitable sequences $\{\lambda_k\}_{k\in\mathbb{N}}$ in $[\frac{1}{l_2^2}, +\infty[$ and for suitable sequences $\left\{u_k = (u^a_k, u^b_k)\right\}_{k\in\mathbb{N}}$ in $\widetilde{V}$ and $\left\{(\xi^a_k, \xi^b_k)\right\}_{k\in\mathbb{N}}$ in $L^2(\Omega^a) \times L^2(\Omega^b)$.

Eventually, (7.4) follows by passing to the limit in

$$(u_{n_i,k}, u_{n_i,h})_n = \delta_{h,k}, \quad \forall i, k, h, \in \mathbb{N},$$

as i diverges, thanks to the strong L^2-convergence in (7.2).

For asserting that u^a_k belongs to $H^1_0(\omega^a)$, it remains to prove the following result.

Step 2.

$$u^a_k(x_2, 0) = 0 \text{ a.e. in }]0, l_2[, \quad \forall k \in \mathbb{N}. \tag{7.5}$$

Fix k in $\mathbb{N}$.

The transmission condition in (2.7) ensures that

$$\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[} u^a_{k,n_i}(x_1, x_2, 0)\varphi(x_2)dx_1dx_2 \\ = \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[} u^b_{k,n_i}(h^a_{n_i}x_1, x_2, 0)\varphi(x_2)dx_1dx_2, \quad \forall i \in \mathbb{N}, \quad \forall \varphi \in C^\infty_0(]0, l_2[). \tag{7.6}$$

As far as the first integral in (7.6) is concerned, the weak H^1-convergence in (7.2) and the fact that u^a_k is independent of x_1 imply

$$\lim_i \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[} u^a_{k,n_i}(x_1, x_2, 0)\varphi(x_2)dx_1dx_2 \\ = \int_0^{l_2} u^a_k(x_2, 0)\varphi(x_2)dx_2, \quad \forall \varphi \in C^\infty_0(]0, l_2[). \tag{7.7}$$

As far as the last integral in (7.6) is concerned, note that estimate in (4.5) provides that

$$\frac{1}{h^a_n}\int_{\Omega^b} |\partial_{x_3} u^b_{k,n}(x)|^2 dx \le c_k h^b_n \to 0, \text{ as } n \to +\infty, \tag{7.8}$$

moreover, weak H^1-convergences in (7.2) and assumption (2.1) with $q = +\infty$ provide

$$u^b_{n_i,k} \to 0 \text{ strongly in } H^1(\Omega^b), \tag{7.9}$$

as i diverges. Then, combining (7.8) with (7.9) and using Proposition 5.1 yield

$$\lim_i \int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[} u^b_{k,n_i}(h^a_{n_i}x_1, x_2, 0)\varphi(x_2)dx_1dx_2 = 0, \quad \forall \varphi \in C_0^\infty(]0, l_2[)\,. \tag{7.10}$$

Eventually, boundary condition (7.5) follows from (7.6), (7.7), and (7.10).

Step 3. This step is devoted to proving that

$$\alpha_1(u_k, v) = \lambda_k[u_k, v]_1, \quad \forall v = (v^a, v^b) \in H^1_0(\omega^a) \times H^1_0(\omega^b). \tag{7.11}$$

Fix k in $\mathbb{N}$.

To obtain (7.11), it is enough to prove that

$$\int_{\omega^a} \left(\partial_{x_2}u^a_k\partial_{x_2}v^a + \partial_{x_3}u^a_k\partial_{x_3}v^a\right)dx_2dx_3 = \lambda_k \int_{\omega^a} u^a_k v^a dx_2dx_3, \quad \forall v^a \in H^1_0(\omega^a), \tag{7.12}$$

$$\int_{\omega^b} \left(\partial_{x_1}u^b_k\partial_{x_1}v^b + \partial_{x_2}u^b_k\partial_{x_2}v^b\right)dx_1dx_2 = \lambda_k \int_{\omega^b} u^b_k v^b dx_1dx_2, \quad \forall v^b \in H^1_0(\omega^b), \tag{7.13}$$

and to add (7.12) and (7.13).

Equation (7.12) follows immediately by passing to the limit, as i diverges, in (2.11) written with index n_i and with a test function $v = (v^a, 0)$, v^a in $H^1_0(\omega^a)$, and using (7.1) and (7.2).

As far as the proof of (7.13) is concerned, for v^b in $C_0^\infty(\omega^b)$, it is easy to construct v^a in $C_0^\infty(]0, l_2[\times[0, l_3[)$ such that

$$v^a(x_2, 0) = v^b(0, x_2) \text{ in }]0, l_2[.$$

Then, $v = (v^a, v^b)$ belongs to V_{reg}, where V_{reg} is defined by (5.11). Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence in $H^1_{\Gamma^a}(\Omega^a)$ satisfying (5.26) in Proposition 5.3. Choosing $\left(\sqrt{\frac{h^a_{n_i}}{h^b_{n_i}}}\, g_{n_i}, \sqrt{\frac{h^a_{n_i}}{h^b_{n_i}}}\, v^b\right)$ as test function in (2.11) written with index n_i yields

$$\begin{aligned}
&\sqrt{\frac{h^a_{n_i}}{h^b_{n_i}}} \int_{\Omega^a} \left(\frac{1}{h^a_{n_i}}\partial_{x_1}u^a_{n_i,k}\frac{1}{h^a_{n_i}}\partial_{x_1}g_{n_i} + \partial_{x_2}u^a_{n_i,k}\partial_{x_2}g_{n_i} + \partial_{x_3}u^a_{n_i,k}\partial_{x_3}g_{n_i}\right)dx \\
&+ \int_{\Omega^b}\left(\partial_{x_1}\left(\sqrt{\frac{h^b_{n_i}}{h^a_{n_i}}}\,u^b_{n_i,k}\right)\partial_{x_1}v^b + \partial_{x_2}\left(\sqrt{\frac{h^b_{n_i}}{h^a_{n_i}}}\,u^b_{n_i,k}\right)\partial_{x_2}v^b\right)dx = \\
&= \lambda_{n_i,k}\left(\sqrt{\frac{h^a_{n_i}}{h^b_{n_i}}}\int_{\Omega^a} u^a_{n_i,k}g_{n_i}dx + \int_{\Omega^b}\sqrt{\frac{h^b_{n_i}}{h^a_{n_i}}}\,u^b_{n_i,k}v^b dx\right), \quad \forall i \in \mathbb{N}.
\end{aligned} \tag{7.14}$$

Passing to the limit, as i diverges, in (7.14) and using (2.1) with $q = +\infty$, (5.26), (7.1), (7.2), and (7.3) provide (7.13) with v^b in $C_0^\infty(\omega^b)$. Then, (7.13) holds true for any v^b in $H^1_0(\omega^b)$, by a density argument.

Step 4. Conclusion.

By arguing as in the proof of Theorem 3.1, one proves that

$$\alpha_1(\lambda_k^{-\frac{1}{2}}u_k, \lambda_h^{-\frac{1}{2}}u_h) = \delta_{h,k}, \quad \forall k, h \in \mathbb{N},$$
$$\lim_k \lambda_k = +\infty,$$

and that (3.10), (3.11), and (3.12) hold true.

Moreover in a classical way (for instance, see [9] or [22]) one can prove that the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ forms the whole set of the eigenvalues of Problem (3.9), that $\{\lambda_k^{-\frac{1}{2}}u_k\}_{k\in\mathbb{N}}$ is a $(H_0^1(\omega^a)\times H_0^1(\omega^b),\alpha_1)$-Hilbert orthonormal basis, and that $\{u_k\}_{k\in\mathbb{N}}$ is a $(L^2(\omega^a)\times L^2(\omega^b),[\cdot,\cdot]_1)$-Hilbert orthonormal basis.

In conclusion, since the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ can be characterized by the min-max Principle, for every $k\in\mathbb{N}$ convergence (7.1) holds true for the whole sequence $\{\lambda_{n,k}\}_{n\in\mathbb{N}}$.

8 Proof of Theorem 3.6

The proof will be split into several steps.

Step 1. The first step is devoted to proving the existence of an increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$, an increasing sequence of positive numbers $\{\lambda_k\}_{k\in\mathbb{N}}$, a sequence $\left\{u_k=(u_k^a,u_k^b)\right\}_{k\in\mathbb{N}}$ in $\widetilde{V}$, where $\widetilde{V}$ is the space defined by (5.27), and a sequence $\left\{(\xi_k^a,\xi_k^b)\right\}_{k\in\mathbb{N}}$ in $L^2(\Omega^a)\times L^2(\Omega^b)$ (depending possibly on the selected subsequence $\{n_i\}_{i\in\mathbb{N}}$) such that, for every k in $\mathbb{N}$,

$$\lim_i \lambda_{n_i,k}=\lambda_k, \tag{8.1}$$

$$\left(\sqrt{\frac{h_{n_i}^a}{h_{n_i}^b}}\,u_{n_i,k}^a,\,u_{n_i,k}^b\right)\rightharpoonup(u_k^a,u_k^b)\ \text{ weakly in } H^1(\Omega^a)\times H^1(\Omega^b)\text{ and} \tag{8.2}$$

$$\text{strongly in } L^2(\Omega^a)\times L^2(\Omega^b),$$

$$\left(\frac{1}{h_{n_i}^a}\sqrt{\frac{h_{n_i}^a}{h_{n_i}^b}}\,\partial_{x_1}u_{n_i,k}^a,\,\frac{1}{h_{n_i}^b}\partial_{x_3}u_{n_i,k}^b\right)\rightharpoonup(\xi_k^a,\xi_k^b)\ \text{weakly in } L^2(\Omega^a)\times L^2(\Omega^b), \tag{8.3}$$

as i diverges, and

$$[u_k,u_h]_1=\delta_{h,k},\quad \forall k,h\in\mathbb{N}. \tag{8.4}$$

Thanks to Proposition 4.1,

$$\forall k\in\mathbb{N},\quad \exists c_k\in]0,+\infty[\ :\ \frac{h_n^a}{h_n^b}a_n(u_{n,k},u_{n,k})=\lambda_{n,k}\le c_k,\quad \forall n,k\in\mathbb{N}. \tag{8.5}$$

Then, a diagonal argument ensures that (8.1), (8.2), and (8.3) hold true for a suitable increasing sequence of positive integer numbers $\{n_i\}_{i\in\mathbb{N}}$, for suitable sequences $\{\lambda_k\}_{k\in\mathbb{N}}$ in $[\frac{1}{l_2^2},+\infty[$ and for suitable sequences $\{u_k\}_{k\in\mathbb{N}}$ in $\widetilde{V}$ and $\left\{(\xi_k^a,\xi_k^b)\right\}_{k\in\mathbb{N}}$ in $L^2(\Omega^a)\times L^2(\Omega^b)$.

Eventually, (8.4) follows by passing to the limit in

$$\frac{h_{n_i}^a}{h_{n_i}^b}(u_{n_i,k},u_{n_i,h})_n=\delta_{h,k},\quad \forall i,k,h,\in\mathbb{N},$$

as i diverges, and using the strong L^2-convergence in (8.2).

For asserting that u_k^b belongs to W_0, it remains to prove the following result.

Step 2.

$$u_k^b(0,x_2)=0 \text{ a.e. in }]0,l_2[,\quad \forall k\in\mathbb{N}. \tag{8.6}$$

Fix k in $\mathbb{N}$.
The transmission conditions in (2.7) gives

$$\sqrt{\frac{h^b_{n_i}}{h^a_{n_i}}}\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[}\sqrt{\frac{h^a_{n_i}}{h^b_{n_i}}}\,u^a_{k,n_i}(x_1,x_2,0)\varphi(x_2)dx_1dx_2$$
$$=\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[}u^b_{k,n_i}(h^a_{n_i}x_1,x_2,0)\varphi(x_2)dx_1dx_2,\quad \forall i\in\mathbb{N},\quad \forall\varphi\in C^\infty_0(]0,l_2[)\,. \tag{8.7}$$

As far as the first integral in (8.7) is concerned, the weak H^1-convergence in (8.2), the fact that u^a_k is independent of x_1 and that assumption (2.1) holds true with $q=0$ imply

$$\lim_i\left(\sqrt{\frac{h^b_{n_i}}{h^a_{n_i}}}\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[}\sqrt{\frac{h^a_{n_i}}{h^b_{n_i}}}\,u^a_{k,n_i}(x_1,x_2,0)\varphi(x_2)dx_1dx_2\right)$$
$$=0\cdot\int_0^{l_2}u^a_k(x_2,0)\varphi(x_2)dx_2=0,\quad\forall\varphi\in C^\infty_0(]0,l_2[)\,. \tag{8.8}$$

As fas as the last integral in (8.7) is concerned, note that estimate (8.5) and assumption (2.1) with $q=0$ provide that

$$\frac{1}{h^a_n}\int_{\Omega^b}|\partial_{x_3}u^b_{k,n}(x)|^2dx=\frac{h^b_n}{h^a_n}\frac{1}{h^b_n}\int_{\Omega^b}|\partial_{x_3}u^b_{k,n}(x)|^2dx\le\frac{h^b_n}{h^a_n}c_kh^b_n\to 0,\ \text{as } n\to+\infty. \tag{8.9}$$

Then, combining the weak H^1-convergence in (8.2) with (8.9), and using Proposition 5.1 yield

$$\lim_i\int_{]-\frac{1}{2},\frac{1}{2}[\times]0,l_2[}u^b_{k,n_i}(h^a_{n_i}x_1,x_2,0)\varphi(x_2)dx_1dx_2=\int_0^{l_2}u^b_k(0,x_2)\varphi(x_2)dx_2,\quad\forall\varphi\in C^\infty_0(]0,l_2[)\,. \tag{8.10}$$

Eventually, boundary condition (8.6) follows from (8.7), (8.8), and (8.10).
Step 3. This step is devoted to proving that

$$\alpha_1(u_k,v)=\lambda_k[u_k,v]_1,\quad \forall v=(v^a,v^b)\in H^1_{\gamma^a}(\omega^a)\times W_0. \tag{8.11}$$

Fix k in $\mathbb{N}$.
To obtain (8.11), it is enough to prove that

$$\int_{\omega^a}\left(\partial_{x_2}u^a_k\partial_{x_2}v^a+\partial_{x_3}u^a_k\partial_{x_3}v^a\right)dx_2dx_3=\lambda_k\int_{\omega^a}u^a_kv^adx_2dx_3,\quad\forall v^a\in H^1_{\gamma^a}(\omega^a), \tag{8.12}$$
$$\int_{\omega^b}\left(\partial_{x_1}u^b_k\partial_{x_1}v^b+\partial_{x_2}u^b_k\partial_{x_2}v^b\right)dx_1dx_2=\lambda_k\int_{\omega^b}u^b_kv^bdx_1dx_2,\quad\forall v^b\in W_0(\omega^b), \tag{8.13}$$

and to add (8.12) and (8.13).

As far as the proof of (8.12) is concerned, for v^a in $C^\infty_0(]0,l_2[\times[0,l_3[)$, it is easy to construct v^b in $C^\infty_0(\omega^b)$ such that

$$v^a(x_2,0)=v^b(0,x_2)\ \text{ in }]0,l_2[.$$

Then, $v = (v^a, v^b)$ belongs to V_{reg}, where V_{reg} is defined by (5.11). Let $\{g_n\}_{n\in\mathbb{N}}$ be a sequence in $H^1_{\Gamma^a}(\Omega^a)$ satisfying (5.26) in Proposition 5.3. Choosing $\left(\sqrt{\frac{h^a_{n_i}}{h^b_{n_i}}}\, g_{n_i}, \sqrt{\frac{h^a_{n_i}}{h^b_{n_i}}}\, v^b\right)$ as test function in (2.11) written with index n_i yields

$$\int_{\Omega^a}\left(\frac{1}{h^a_{n_i}}\sqrt{\frac{h^a_{n_i}}{h^b_{n_i}}}\partial_{x_1}u^a_{n_i,k}\frac{1}{h^a_{n_i}}\partial_{x_1}g_{n_i} + \partial_{x_2}\left(\sqrt{\frac{h^a_{n_i}}{h^b_{n_i}}}u^a_{n_i,k}\right)\partial_{x_2}g_{n_i} + \partial_{x_3}\left(\sqrt{\frac{h^a_{n_i}}{h^b_{n_i}}}u^a_{n_i,k}\right)\partial_{x_3}g_{n_i}\right)dx$$

$$+\sqrt{\frac{h^b_{n_i}}{h^a_{n_i}}}\int_{\Omega^b}\left(\partial_{x_1}u^b_{n_i,k}\partial_{x_1}v^b + \partial_{x_2}u^b_{n_i,k}\partial_{x_2}v^b\right)dx =$$

$$= \lambda_{n_i,k}\left(\int_{\Omega^a}\sqrt{\frac{h^a_{n_i}}{h^b_{n_i}}}u^a_{n_i,k}g_{n_i}dx + \sqrt{\frac{h^b_{n_i}}{h^a_{n_i}}}\int_{\Omega^b}u^b_{n_i,k}v^b dx\right), \quad \forall i \in \mathbb{N}. \tag{8.14}$$

Passing to the limit, as i diverges, in (8.14) and using (2.1) with $q = 0$, (5.26), (8.1), (8.2), and (8.3) provide (8.12) with v^a in $C^\infty_0(]0, l_2[\times[0, l_3[)$. Then, (8.12) holds true for any v^a in $H^1_{\gamma_a}(\omega^a)$, by a density argument.

As far as the proof of (8.13) is concerned, set

$$\widetilde{W}_0 = \{v \in C^\infty_0(\omega^b) : \quad v^b_{|\,\omega^b_-} \in C^\infty_0(\omega^b_-), \quad v^b_{|\,\omega^b_+} \in C^\infty_0(\omega^b_+)\}$$

(see Sect. 2 for the definition of ω^b_+ and ω^b_-). Obviously, $\widetilde{W}_0$ is dense in W_0.

Passing to the limit, as i diverges, in (2.11) written with index n_i and with a test function $\frac{h^a_{n_i}}{h^b_{n_i}}(0, v^b)$, v^b in $\widetilde{W}_0$ (note that $(0, v^b)$ belong to V_{n_i}, for i large enough), and using (8.1) and (8.2) provide (8.13) with v^b in $\widetilde{W}_0$. Then, (8.13) holds true for any v^b in W_0, by a density argument.

Step 4. Conclusion.

By arguing as in the proof of Theorem 3.1, one proves that

$$\alpha_1(\lambda_k^{-\frac{1}{2}}u_k, \lambda_h^{-\frac{1}{2}}u_h) = \delta_{h,k}, \quad \forall k, h \in \mathbb{N},$$
$$\lim_k \lambda_k = +\infty,$$

and that (3.16), (3.17), and (3.18) hold true.

Moreover in a classical way (for instance, see [9] or [22]) one can prove that the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ forms the whole set of the eigenvalues of Problem (3.15), that $\{\lambda_k^{-\frac{1}{2}}u_k\}_{k\in\mathbb{N}}$ is a $(H^1_{\gamma^a}(\omega^a) \times W_0, \alpha_1)$-Hilbert orthonormal basis, and that $\{u_k\}_{k\in\mathbb{N}}$ is a $(L^2(\omega^a) \times L^2(\omega^b), [\cdot,\cdot]_1)$-Hilbert orthonormal basis.

In conclusion, since the sequence $\{\lambda_k\}_{k\in\mathbb{N}}$ can be characterized by the min-max Principle, for every $k \in \mathbb{N}$ convergence (8.1) holds true for the whole sequence $\{\lambda_{n,k}\}_{n\in\mathbb{N}}$.

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