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Existence of positive solutions for Lidstone boundary value problems on time scales

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Abstract

Let $\mathbb{T} \subseteq \mathbb{R}$ be a time scale. The purpose of this paper is to present sufficient conditions for the existence of multiple positive solutions of the following Lidstone boundary value problem on time scales:

$$(-1)^n y^{\Delta^{(2n)}}(t) = f(t, y(t)), \quad t \in [a,b]_{\mathbb{T}},$$

$$y^{\Delta^{(2i)}}(a) = y^{\Delta^{(2i)}}(\sigma^{2n-2i}(b)) = 0, \quad i = 0, 1, \ldots, n-1.$$

Existence of multiple positive solutions is established using fixed point methods. At the end some examples are also given to illustrate our results.

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Keywords: Krasnoselski˘ı's–Guo fixed point theorem; Time scale; Lidstone boundary value problem; Positive solution

1 Introduction

Let $\mathbb{T}$ be an arbitrary time scale (nonempty closed subset of $\mathbb{R}$). As usual, $\sigma : \mathbb{T} \to \mathbb{T}$ is the forward jump operator defined by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\}.$$

Also $y^\sigma(t) = y(\sigma(t))$, and $y^\Delta(t)$ denotes the time scale derivative of y. Higher order jump and derivative are defined inductively by $\sigma^j(t) = \sigma(\sigma^{j-1}(t))$ and $y^{\Delta^{(j)}}(t) = (y^{\Delta^{(j-1)}}(t))^\Delta$, $j \geq 1$. It is assumed that the reader is familiar with the time scale calculus. Some preliminary definitions and theorems on time scales can be found in [1–3].

Lidstone boundary value problems appear as a mathematical model of real world problems such as the study of bending of simply supported beams or suspended bridges [4–6]. The existence of positive solutions of the boundary value problems (BVPs) has created a great deal of interest due to wide applicability in both theory and applications [7, 8]. Some authors in the literature have obtained existence results about the solutions, positive solutions, or symmetric positive solutions of Lidstone type BVPs associated with ordinary differential equations, differential equations, and dynamic equations on time scales by using various methods (see [7–24] and the references therein).

In 2021 Graef and Yang investigated the following complementary Lidstone boundary value problem [25]:

$$\begin{aligned}
&(-1)^n y^{(2n+1)}(t) = g(t)f\big(y(t)\big), \quad t \in [0,1]\\
&y(0) = 0, y^{(2i-1)}(0) = y^{(2i-1)}(1) = 0, \quad i = 1,\dots,n.
\end{aligned}$$

They obtained sufficient conditions for the existence and nonexistence of positive solutions and some upper and lower bounds for positive solutions of the problem.

Cetin and Topal studied the following Lidstone boundary value problem on time scales [26]:

$$\begin{aligned}
&(-1)^n y^{\Delta^{(2n)}}(t) = f\big(t, y^{\sigma}(t)\big), \quad t \in [0,1],\\
&y^{\Delta^{(2i)}}(0) = y^{\Delta^{(2i)}}\big(\sigma(1)\big) = 0, \quad i = 1,\dots,n-1,
\end{aligned}$$

where $n \geq 1$ and $f : [0,\sigma(1)]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is continuous and $\sigma^j(1) = \sigma(1)$ for $j \geq 1$. They obtained sufficient conditions for the existence of solution by using Schauder's fixed point theorem in a cone and Krasnosel'skii's fixed point theorem. Also, the existence result for the problem was given by the monotone method.

In [17], the authors investigated the following complementary Lidstone boundary value problem on time scales [26]:

$$\begin{aligned}
&(-1)^n y^{\Delta^{(2n+1)}}(t) + q(t)f\big(t, y^{\sigma}(t)\big) = 0, \quad t \in [a,b]_{\mathbb{T}},\\
&y(a) = 0, y^{\Delta^{(2i-1)}}(a) = y^{\Delta^{(2i-1)}}\big(\sigma^{2n-2i+2}(b)\big) = 0, \quad i = 1,\dots,n,
\end{aligned}$$

where $n \geq 1$ and $f : [a,\sigma(b)]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ and $q : [a,\sigma(b)]_{\mathbb{T}} \to [0,\infty)$ are continuous. They gave the existence of one and two solutions by using fixed points methods.

Inspired by the aforementioned papers, the purpose of this paper is to study the existence of positive solutions to the Lidstone boundary value problem (LBVP) on time scales

$$\begin{aligned}
&(-1)^n y^{\Delta^{(2n)}}(t) = f\big(t, y(t)\big), \quad t \in [a,b]_{\mathbb{T}},\\
&y^{\Delta^{(2i)}}(a) = y^{\Delta^{(2i)}}\big(\sigma^{2n-2i}(b)\big) = 0, \quad i = 0,1,\dots,n-1,
\end{aligned} \tag{1.1}$$

where $n \geq 1$, $a, b \in \mathbb{T}$, and $f : [a,\sigma(b)]_{\mathbb{T}} \times \mathbb{R} \to \mathbb{R}$ is continuous.

In [17], while the authors studied complementary Lidstone boundary value problem, they reduced this problem to the Lidstone boundary value problem (1.1) and had the Green function of (1.1). For this reason we will use this Green function of LBVP obtained in [17] and its properties. Then we will give new results for LBVP (1.1). Also, although the authors considered the $2n$-order LBVP on time scales in [2, 27, 28], the boundary conditions in (1.1) are more general than the boundary conditions of the problem in [2, 27, 28]. In this paper, unlike [29], new sufficient conditions are obtained for the existence of solutions of LBVP (1.1) by using Schauder's fixed point theorem, Krasnosel'skii's fixed point theorem, the Leggett–Williams fixed point theorem, and the upper and lower solutions method.

Hereafter, we use the notation $[a,b]_{\mathbb{T}}$ to indicate the time scale interval $[a,b]\cap\mathbb{T}$. The intervals $[a,b)_{\mathbb{T}}$, $(a,b]_{\mathbb{T}}$, and $(a,b)_{\mathbb{T}}$ are similarly defined.

In Sect. 2, we develop some inequalities for certain Green's functions. In Sect. 3, using a variety of fixed point theorems, we establish the existence of a solution (not necessary positive), and we also discuss the existence of a nontrivial positive solution. Also, we give the existence results for two and three nontrivial positive solutions.

2 Preliminaries

To obtain a solution for LBVP (1.1), we require a mapping whose kernel $G_n^1(t,s)$ is the Green function of the homogeneous Lidstone boundary value problem

$$\begin{aligned} &(-1)^n y^{\Delta^{(2n)}}(t)=0, \quad t\in[a,b]_{\mathbb{T}}, \\ &y^{\Delta^{(2i)}}(a)=y^{\Delta^{(2i)}}\bigl(\sigma^{2n-2i}(b)\bigr)=0, \quad i=0,1,\ldots,n-1. \end{aligned} \tag{2.1}$$

The Green function for problem (2.1) is

$$G_n^1(t,s)=\int_a^{\sigma^{2n-1}(b)} G_n(t,r)G_{n-1}^1(r,s)\Delta r, \tag{2.2}$$

where

$$G_n(t,s)=\frac{-1}{\sigma^{2n}(b)-a}\begin{cases}(t-a)(\sigma^{2n}(b)-\sigma(s)), & t\le s,\\ (\sigma(s)-a)(\sigma^{2n}(b)-t), & \sigma(s)<t,\end{cases} \tag{2.3}$$

and

$$G_1^1(t,s)=G_1(t,s). \tag{2.4}$$

G_n is the Green function of the problem

$$y^{\Delta\Delta}(t)=0, \qquad y(a)=y\bigl(\sigma^{2n}(b)\bigr)=0.$$

Furthermore, it is easily seen that from (2.3) we have

$$G_n(t,s)\le 0, \quad (t,s)\in\bigl[a,\sigma^{2n}(b)\bigr]_{\mathbb{T}}\times\bigl[a,\sigma^{2n-2}(b)\bigr]_{\mathbb{T}}, \tag{2.5}$$

and from (2.5) and (2.2) we have

$$(-1)^n G_n^1(t,s)\ge 0, \quad (t,s)\in\bigl[a,\sigma^{2n}(b)\bigr]_{\mathbb{T}}\times[a,b]_{\mathbb{T}}. \tag{2.6}$$

Now let give some properties about the Green function $G_n^1(t,s)$, which can be found in reference [17].

Lemma 2.1 ([17]) *For* $(t,s)\in[a,\sigma^{2n}(b)]_{\mathbb{T}}\times[a,b]_{\mathbb{T}}$, *we have*

$$(-1)^n G_n^1(t,s)=\bigl|G_n^1(t,s)\bigr|\le\theta_n\bigl(\sigma(s)-a\bigr)\bigl(\sigma^2(b)-\sigma(s)\bigr), \tag{2.7}$$

where

$$\theta_n = \left[\prod_{i=1}^{n}\left(\sigma^{2i}(b) - a\right)\right]^{-1}\prod_{i=1}^{n-1} s_{2i},$$

$$s_j = \frac{1}{6}\left\{\left(\sigma^{j+2}(b) - a\right)^3 + \sum_{t\in A_j}\mu(t)^2\left[3\left(\sigma^{j+2}(b) + a\right) - 2\left(t + 2\sigma(t)\right)\right]\right.$$
$$\left. - \sum_{t\in B_j}\mu(t)^2\left[2\left(t + 2\sigma(t)\right) - 3\left(\sigma^{j+2}(b) + a\right)\right]\right\}, \quad j \geq 2, \tag{2.8}$$

and

$$A_j = \left[a, \frac{\sigma^{j+1}(b) + a}{2}\right]_{\mathbb{T}} - \left\{\max\left\{t : t \in \left[a, \frac{\sigma^{j+1}(b) + a}{2}\right]_{\mathbb{T}}\right\}\right\},$$
$$B_j = \left(\frac{\sigma^{j+1}(b) + a}{2}, \sigma^{j+1}(b)\right]_{\mathbb{T}} \cup \left\{\max\left\{t : t \in \left[a, \frac{\sigma^{j+1}(b) + a}{2}\right]_{\mathbb{T}}\right\}\right\}. \tag{2.9}$$

Remark 2.2 ([17]) If $\mathbb{T} = \mathbb{R}$, then from Lemma 2.1 we obtain for $(t,s) \in [a,b] \times [a,b]$

$$(-1)^n G_n^1(t,s) = \left|G_n^1(t,s)\right| \leq \left(\frac{(b-a)^2}{6}\right)^{n-1}\frac{(s-a)(b-s)}{b-a}. \tag{2.10}$$

Lemma 2.3 ([17]) *Let $\delta \in (0, \frac{1}{2})$ be a given constant. For $(t,s) \in [\alpha, \beta_n]_{\mathbb{T}} \times [a,b]_{\mathbb{T}}$, we have*

$$(-1)^n G_n(t,s) = \left|G_n(t,s)\right| \geq \psi_n(\delta)\left(\sigma(s) - a\right)\left(\sigma^2(b) - \sigma(s)\right), \tag{2.11}$$

where $\alpha = \min\{t \in [a, \sigma^{2n}(b)]_{\mathbb{T}} : a + \delta \leq t\}$, $\beta_j = \max\{t \in [a, \sigma^{2j}(b)]_{\mathbb{T}} : t \leq \sigma^{2j}(b) - \delta\}$,

$$\psi_n(\delta) = \delta^n\left[\prod_{i=1}^{n}\left(\sigma^{2i}(b) - a\right)\right]^{-1}\prod_{i=1}^{n-1} S_{i+1},$$

and

$$S_j = \frac{1}{6}\left\{(\beta_j - \alpha)(3\sigma^{2j}(b)(\beta_j + \alpha) + 3a\left(\beta_j + \alpha - 2\sigma^{2j}(b)\right) - 2\left(\beta_j^2 + \beta_j\alpha + \alpha^2\right)\right.$$
$$+ \sum_{t\in A_j - [a,\alpha)_{\mathbb{T}}}\mu(t)^2\left[3\left(\sigma^{2j}(b) + a\right) - 2\left(t + 2\sigma(t)\right)\right]$$
$$\left. - \sum_{t\in B_j - (\beta_j, \sigma^{2j}(b)]_{\mathbb{T}}}\mu(t)^2\left[2\left(t + 2\sigma(t)\right) - 3\left(\sigma^{2j}(b) + a\right)\right]\right\}, \quad j \geq 2.$$

Here, the sets A_j and B_j are defined as in (2.9).

Remark 2.4 ([17]) If $\mathbb{T} = \mathbb{R}$, then from Lemma 2.3 we obtain for $(t,s) \in [a+\delta, b-\delta] \times [a,b]$

$$(-1)^n G_n^1(t,s) = \left|G_n^1(t,s)\right| \geq \psi_n(\delta)\frac{(s-a)(b-s)}{b-a}, \tag{2.12}$$

where $\psi_n(\delta) = \frac{1}{6^{n-1}}(\frac{\delta}{b-a})^n((b-a)^2 - 6\delta^2 + \frac{4\delta^3}{b-a})^{n-1}$.

Remark 2.5 ([17]) From Lemmas 2.1 and 2.3, for $\delta = \frac{1}{4} \in (0, \frac{1}{2})$, we have

$$\begin{aligned} \min_{t\in[\alpha,\beta_n]_{\mathbb{T}}} \left|G_n^1(t,s)\right| &\geq \psi_n(1/4)G_1\bigl(\sigma(s),s\bigr) \geq \frac{\psi_n(\frac{1}{4})}{\theta_n} \max_{t\in[a,\sigma^{2n}(b)]_{\mathbb{T}}} \left|G_n^1(t,s)\right| \\ &\geq \gamma_n \max_{t\in[a,\sigma^{2n}(b)]_{\mathbb{T}}} \left|G_n^1(t,s)\right|, \end{aligned}$$

where

$$\gamma_n = \frac{\prod_{i=1}^{n-1} S_{i+1}}{4^n \prod_{i=1}^{n-1} s_{2i}}.$$

It is clear that $s_{2i} > S_{i+1}$, $1 \leq i \leq n-1$. Thus, we have $0 < \gamma_n < 1$.

In this section, we also state Schauder's and Krasnosel'skii's fixed point theorems in a cone [30, 31] to prove the existence of at least one and two positive solutions of the problem.

Theorem 2.6 *Let A be a closed convex subset of a Banach space $B = (B, \|\cdot\|)$, and assume there exists a continuous map T sending A to a countably compact subset $T(A)$ of A. Then T has a fixed point.*

Theorem 2.7 *Let $B = (B, \|\cdot\|)$ be a Banach space, $P \subset B$ be a cone in B. Suppose that Ω_1 and Ω_2 are open and bounded subsets of B with $0 \in \Omega_1$ and $\overline{\Omega_1} \subset \Omega_2$. Suppose further that $T : P \cap (\overline{\Omega_2} \setminus \Omega_1) \to P$ is a continuous and compact operator such that either*

(i) $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_1$, $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_2$, or

(ii) $\|Tu\| \geq \|u\|$ for $u \in P \cap \partial\Omega_1$, $\|Tu\| \leq \|u\|$ for $u \in P \cap \partial\Omega_2$

holds. Then T has a fixed point in $P \cap (\overline{\Omega_2} \setminus \Omega_1)$.

Finally, to prove the existence of at least three positive solutions of the problem, we now introduce the following fixed point theorem due to Leggett–Williams.

Theorem 2.8 *Let $\mathcal{P}$ be a cone in a real Banach space E. Set*

$$\begin{aligned} &\mathcal{P}_r := \bigl\{x \in \mathcal{P} : \|x\| < r\bigr\}, \\ &\mathcal{P}(\psi, a, b) := \bigl\{x \in \mathcal{P} : a \leq \psi(x), \|x\| \leq b\bigr\}. \end{aligned}$$

Suppose that $A : \overline{\mathcal{P}_r} \to \overline{\mathcal{P}_r}$ is a completely continuous operator and ψ is a nonnegative, continuous, concave functional on $\mathcal{P}$ with $\psi(u) \leq \|u\|$ for all $u \in \overline{\mathcal{P}_r}$. If there exist $0 < p < q < l \leq r$ such that the following conditions hold:

(i) $\{u \in \mathcal{P}(\psi, q, l) : \psi(u) > q\} \neq \emptyset$ and $\psi(Au) > q$ for all $u \in \mathcal{P}(\psi, q, l)$,

(ii) $\|Au\| < p$ for all $\|u\| \leq p$,

(iii) $\psi(Au) > q$ for $u \in \mathcal{P}(\psi, q, r)$ with $\|Au\| > l$.

Then A has at least three positive solutions u_1, u_2, and u_3 in $\overline{\mathcal{P}_r}$ satisfying

$$\|u_1\| < p, \qquad \psi(u_2) > q, \qquad p < \|u_3\| \quad \text{with } \psi(u_3) < q.$$

3 Existence of positive solutions

Let the Banach space $B = \mathcal{C}[a,\sigma^{2n}(b)]_{\mathbb{T}}$ be equipped with the norm $\|y\| = \max_{t\in[a,\sigma^{2n}(b)]_{\mathbb{T}}}|y(t)|$ for $y \in B$. We now define a mapping $T : \mathcal{C}[a,\sigma^{2n}(b)]_{\mathbb{T}} \to \mathcal{C}[a,\sigma^{2n}(b)]_{\mathbb{T}}$ by

$$Ty(t) = \int_a^{\sigma(b)} (-1)^n G_n^1(t,s) f\bigl(s,y(s)\bigr)\Delta s, \tag{3.1}$$

where $G_n^1(t,s)$ is the Green function given in (2.2).

Let

$$K = \bigl\{y \in B : y(t) \geq 0, t \in \bigl[a,\sigma^{2n}(b)\bigr]_{\mathbb{T}}\bigr\}.$$

We now give a condition, which will be used in some results in this paper:

(C_1) f is continuous on $[a,\sigma(b)]_{\mathbb{T}} \times \mathbb{R}$ with $f(t,y) \geq 0$ for $(t,y) \in [a,\sigma(b)]_{\mathbb{T}} \times K$.

Our first result is an existence criterion for a solution (it need not be positive).

Theorem 3.1 *Let (C_1) hold and let f be continuous. If $M > 0$ satisfies $\theta_n Q s_0 \leq M$, where $Q > 0$ satisfies*

$$Q \geq \max_{\|y\|\leq M}\bigl|f\bigl(t,y(t)\bigr)\bigr| \quad \textit{for } t \in \bigl[a,\sigma(b)\bigr]_{\mathbb{T}}$$

and the numbers θ_n and s_0 are defined in Lemma 2.1 and $(2.8)|_{j=0}$ respectively, then LBVP (1.1) has a solution $y(t)$.

Proof Let $K_1 = \{y \in B : \|y\| \leq M\}$. We will apply Schauder's fixed point theorem. The solutions of LBVP (1.1) are the fixed points of the operator T. A standard argument guarantees that $T : K_1 \to B$ is continuous. Next we show $T(K_1) \subset K_1$. For $y \in K_1$, we obtain

$$\begin{aligned}
\bigl|Ty(t)\bigr| &= \left|\int_a^{\sigma(b)} (-1)^n G_n^1(t,s) f(s,y(s)\Delta s\right| \\
&\leq \int_a^{\sigma(b)} \bigl|G_n^1(t,s)\bigr|\bigl|f\bigl(s,y(s)\bigr)\bigr|\Delta s \\
&\leq \theta_n Q \int_a^{\sigma(b)} \bigl(\sigma(s)-a\bigr)\bigl(\sigma^2(b)-\sigma(s)\bigr)\Delta s \\
&\leq \theta_n Q s_0 \\
&\leq M
\end{aligned}$$

for all $t \in [a,\sigma^{2n}(b)]_{\mathbb{T}}$. This implies that $\|Ty\| \leq M$. A standard argument, via the Arzela–Ascoli theorem, guarantees that $T : K_1 \to K_1$ is a compact operator. Hence T has a fixed point $y \in K_1$ by Schauder's fixed point theorem. □

Corollary 3.2 *If f is continuous and bounded on $[a,\sigma(b)]_{\mathbb{T}} \times \mathbb{R}$, then LBVP (1.1) has a solution.*

Now, we will give the existence of positive solutions by the monotone method, and we define the set

$$D := \{y : y^{\Delta^{(2n)}} \text{ is continuous on } [a,b]_{\mathbb{T}}\}.$$

For any $u, v \in D$, we define the sector $[u,v]$ by

$$[u,v] := \{w \in D : u \le v \le w\}.$$

Definition 3.3 A real-valued function $u(t) \in D$ on $[a,b]_{\mathbb{T}}$ is a lower solution for LBVP (1.1) if

$$\begin{aligned} &(-1)^n u^{\Delta^{(2n)}}(t) \le f(t,u(t)), \quad t \in [a,b]_{\mathbb{T}}, \\ &(-1)^i u^{\Delta^{(2i)}}(a) \le 0, \qquad (-1)^i u^{\Delta^{2i}}(\sigma^{2n-2i}(b)) \le 0, \quad 0 \le i \le n-1. \end{aligned}$$

Similarly, a real-valued function $v(t) \in D$ on $[a,b]_{\mathbb{T}}$ is an upper solution for LBVP (1.1) if

$$\begin{aligned} &(-1)^n v^{\Delta^{(2n)}}(t) \ge f(t,v(t)), \quad t \in [a,b]_{\mathbb{T}}, \\ &(-1)^i v^{\Delta^{(2i)}}(a) \ge 0, \qquad (-1)^i v^{\Delta^{(2i)}}(\sigma^{2n-2i}(b)) \ge 0, \quad 0 \le i \le n-1. \end{aligned}$$

Lemma 3.4 *Assume that $w(t) \in C^2[a,b]$ and $w(t)$ satisfies $-w^{\Delta\Delta}(t) \le 0$ on $[a,b]_{\mathbb{T}}$, $w(a) \le 0$, $w(\sigma^2(b)) \le 0$. Then $w(t) \le 0$ on $[a,\sigma^2(b)]_{\mathbb{T}}$.*

Proof Since $-w^{\Delta\Delta}(t) \le 0$, then $w^{\Delta\Delta}(t) \ge 0$ on $[a,b]_{\mathbb{T}}$. By the mean value theorem on time scales, there exists $\tau_1 \in [a,t)_{\mathbb{T}}$ such that

$$w(t) - w(a) \le w^{\Delta}(\tau_1)(t-a).$$

For all $t \in [a,\sigma^2(b)]_{\mathbb{T}}$, we take $t = \lambda_1 a + \lambda_2 \sigma^2(b)$ with $\lambda_1 + \lambda_2 = 1$ and $\lambda_1, \lambda_2 \ge 0$. So we have

$$\begin{aligned} w(t) - w(a) &\le w^{\Delta}(\tau_1)(\lambda_1 a + \lambda_2 \sigma^2(b) - a) \\ &= w^{\Delta}(\tau_1)(a(\lambda_1 - 1) + \lambda_2 \sigma^2(b)) \\ &= w^{\Delta}(\tau_1)(-\lambda_2 a + \lambda_2 \sigma^2(b)) \\ &= w^{\Delta}(\tau_1)\lambda_2(\sigma^2(b) - a). \end{aligned}$$

Similarly, there exists $\tau_2 \in [t,\sigma^2(b))_{\mathbb{T}}$ such that

$$\begin{aligned} w(\sigma^2(b)) - w(t) &\ge w^{\Delta}(\tau_2)(\sigma^2(b) - t) \\ &= w^{\Delta}(\tau_2)(\sigma^2(b) - \lambda_1 a - \lambda_2 \sigma^2(b)) \\ &= w^{\Delta}(\tau_2)((1-\lambda_2)\sigma^2(b) - \lambda_1 a) \\ &= w^{\Delta}(\tau_2)\lambda_1(\sigma^2(b) - a). \end{aligned}$$

Combining these inequalities, we get

$$\lambda_2 w\big(\sigma^2(b)\big) + \lambda_1 w(a) - (\lambda_1 + \lambda_2)w(t) \geq \big[w^{\Delta}(\tau_2) - w^{\Delta}(\tau_1)\big]\lambda_1\lambda_2\big(\sigma^2(b) - a\big).$$

Again, using the mean value theorem on $[\tau_1, \tau_2]_{\mathbb{T}}$, we have

$$\lambda_2 w\big(\sigma^2(b)\big) + \lambda_1 w(a) - (\lambda_1 + \lambda_2)w(t) \geq w^{\Delta\Delta}(\tau)(\tau_2 - \tau_1)\lambda_1\lambda_2\big(\sigma^2(b) - a\big),$$

where $\tau \in [\tau_1, \tau_2)_{\mathbb{T}}$.

Since $w^{\Delta\Delta}(t) \geq 0$ on $[a,b]_{\mathbb{T}}$, we get $\lambda_2 w(\sigma^2(b)) + \lambda_1 w(a) - (\lambda_1 + \lambda_2)w(t) \geq 0$, so $w(t) \leq 0$ on $[a, \sigma^2(b)]$. □

Lemma 3.5 *Assume that $w(t) \in C^{2n}[a,b]$ and $w(t)$ satisfies $(-1)^n w^{\Delta^{(2n)}}(t) \leq 0$ on $[a,b]_{\mathbb{T}}$, $(-1)^i w^{\Delta^{(2i)}}(a) \leq 0$, $(-1)^i w^{\Delta^{(2i)}}(\sigma^{2n-2i}(b)) \leq 0$, for $0 \leq i \leq n-1$. Then $w(t)$ is nonpositive on $[a, \sigma^{2n}(b)]_{\mathbb{T}}$.*

Proof Let us define $z_{n-1}(t) := (-1)^{n-1} w^{\Delta^{2(n-1)}}(t)$. Then $-z_{n-1}^{\Delta\Delta}(t) \leq 0$ on $[a,b]_{\mathbb{T}}$, and by the boundary condition we get $z_{n-1}(a) \leq 0$, $z_{n-1}(\sigma^2(b)) \leq 0$. It follows from Lemma 3.4 that $z_{n-1}(t) \leq 0$. Similarly, let us define $z_{n-2}(t) := (-1)^{n-2} w^{\Delta^{2(n-2)}}(t)$. Then $-z_{n-2}^{\Delta\Delta}(t) \leq 0$ on $[a,b]_{\mathbb{T}}$, and from the boundary condition we get $z_{n-2}(a) \leq 0$, $z_{n-2}(\sigma^4(b)) \leq 0$. Thus we have $z_{n-2}(t) \leq 0$ on $[a, \sigma^4(b)]$ by Lemma 3.4.

The conclusion of the lemma follows by an induction argument. □

In this part of the section, we will prove that when the lower and upper solutions are given well order, i.e., $u \leq v$, LBVP (1.1) admits a solution lying between the lower and upper solutions.

Theorem 3.6 *Let f be continuous on $[a, \sigma(b)]_{\mathbb{T}} \times \mathbb{R}$. Assume that there exist a lower solution u and an upper solution v for LBVP (1.1) such that $u \leq v$ on $[a, \sigma^{2n}(b)]_{\mathbb{T}}$. Then LBVP (1.1) has a solution $y \in [u, v]$ on $[a, \sigma^{2n}(b)]_{\mathbb{T}}$.*

Proof Consider the LBVP

$$\begin{aligned} &(-1)^n y^{\Delta^{(2n)}}(t) = F\big(t, y(t)\big), \quad t \in [a,b]_{\mathbb{T}}, \\ &y^{\Delta^{(2i)}}(a) = y^{\Delta^{(2i)}}\big(\sigma^{2n-2i}(b)\big) = 0, \quad i = 0, 1, \ldots, n-1, \end{aligned} \tag{3.2}$$

where

$$F(t,\xi) = \begin{cases} f(t, v(t)) - \frac{\xi - v(t)}{1+|\xi|}, & \xi \geq v(t), \\ f(t,\xi), & u(t) \leq \xi \leq v(t), \\ f(t, u(t)) + \frac{u(t) - \xi}{1+|\xi|}, & \xi \leq u, \end{cases}$$

for $t \in [a, \sigma(b)]_{\mathbb{T}}$.

Clearly, the function F is bounded for $t \in [a, \sigma(b)]_{\mathbb{T}}$ and $\xi \in \mathbb{R}$, and is continuous in ξ. Thus, by Corollary 3.2, there exists a solution $y(t)$ of LBVP (3.2).

We claim $y(t) \leq v(t)$ for $t \in [a, \sigma^{2n}(b)]_{\mathbb{T}}$. If not, we know that $y(t) - v(t) > 0$ for $t \in [a, \sigma^{2n}(b)]_{\mathbb{T}}$ and

$$\begin{aligned}(-1)^n y^{\triangle^{(2n)}}(t) &= F\big(t, y(t)\big) \\ &= f\big(t, v(t)\big) - \frac{y - v(t)}{1 + |y|} \\ &< f\big(t, v(t)\big) \\ &\leq (-1)^n v^{\triangle^{(2n)}}(t).\end{aligned}$$

Hence, we have

$$(-1)^n (y - v)^{\triangle^{(2n)}}(t) \leq 0,$$

and from the boundary conditions we get

$$(-1)^i (y - v)^{\triangle^{(2i)}}(a) \leq 0 \quad \text{and} \quad (-1)^i (y - v)^{\triangle^{(2i)}}\big(\sigma^{2n-2i}(b)\big) \leq 0, \quad 0 \leq i \leq n - 1.$$

Using Lemma 3.5, we have that

$$y - v \leq 0 \quad \text{on } [a, \sigma^{2n}(b)]_{\mathbb{T}},$$

which is a contradiction. It follows that $y(t) \leq v(t)$ on $[a, \sigma^{2n}(b)]_{\mathbb{T}}$.

Similarly, we get easily $u \leq y$ on $[a, \sigma^{2n}(b)]_{\mathbb{T}}$.

Thus $y(t)$ is a solution of LBVP (1.1) and lies between u and v. □

Next let

$$P = \Big\{ y \in B : \min_{t \in [a, \sigma^{2n}(b)]_{\mathbb{T}}} y(t) \geq 0 \text{ and } \min_{t \in [\alpha, \beta_n]_{\mathbb{T}}} y(t) \geq \gamma_n \|y\| \Big\} \subset K. \tag{3.3}$$

It is easy to check that P is a cone of nonnegative functions in $\mathcal{C}[a, \sigma^{2n}(b)]_{\mathbb{T}}$. Now assume (C_1). Next we will apply Theorem 2.7. First we show $T : P \to P$ (see (3.1) for the definition of T). Now (C_1) and $y \in P$ implies that $Ty(t) \geq 0$ on $[a, \sigma^{2n}(b)]_{\mathbb{T}}$ and

$$\begin{aligned}\min_{t \in [\alpha, \beta_n]_{\mathbb{T}}} Ty(t) &= \int_a^{\sigma(b)} \min_{t \in [\alpha, \beta_n]_{\mathbb{T}}} (-1)^n G_n^1(t, s) f\big(s, y(s)\big) \Delta s \\ &\geq \int_a^{\sigma(b)} \gamma_n \max_{t \in [a, \sigma^{2n}(b)]_{\mathbb{T}}} \big|G_n(t, s)\big| f\big(s, y(s)\big) \Delta s.\end{aligned}$$

It follows that

$$\min_{t \in [\alpha, \beta_n]_{\mathbb{T}}} Ty(t) \geq \gamma_n \|Ty\|.$$

Thus $Ty \in P$ so $T(P) \subset P$. A standard argument, via the Arzela–Ascoli theorem, guarantees that $T : P \to P$ is continuous and completely continuous.

Theorem 3.7 *Let* (C_1) *hold. Also assume*

(C_2) $\lim_{y\to 0^+}\frac{f(t,y)}{y}=0$, $\lim_{y\to+\infty}\frac{f(t,y)}{y}=+\infty$ *for* $t\in[a,\sigma(b)]_{\mathbb{T}}$.

Then LBVP (1.1) has at least one positive solution.

Proof We will apply Theorem 2.7 with the cone P defined in (3.3). Since $\lim_{y\to 0^+}\frac{f(t,y)}{y}=0$, there exists $r_1>0$ such that

$$f(t,y)\le \eta y,\quad 0\le y\le r_1, a\le t\le\sigma(b), \tag{3.4}$$

where $\eta=\frac{1}{\theta_n s_0}$ and the number s_0 is defined in $(2.8)|_{j=0}$. Let $\Omega_1=\{y\in B:\|y\|<r_1\}$.

Using Lemma 2.1 and (3.4), we find for $t\in[a,\sigma^{2n}(b)]_{\mathbb{T}}$ that

$$\begin{aligned}
Ty(t) &= \int_a^{\sigma(b)}(-1)^n G_n^1(t,s)f\bigl(s,y(s)\bigr)\Delta s\\
&\le \eta\int_a^{\sigma(b)}\theta_n G_1\bigl(\sigma(s),s\bigr)y(s)\Delta s\\
&\le \eta\int_a^{\sigma(b)}\theta_n G_1\bigl(\sigma(s),s\bigr)r_1\Delta s\\
&\le \eta\theta_n r_1\int_a^{\sigma(b)}\bigl(\sigma(s)-a\bigr)\bigl(\sigma^2(b)-\sigma(s)\bigr)\Delta s\\
&\le \theta_n\eta r_1 s_0=r_1=\|y\|,
\end{aligned}$$

and so

$$\|Ty\|\le\|y\|\quad \text{for all } y\in P\cap\partial\Omega_1.$$

Since $\lim_{y\to+\infty}\frac{f(t,y)}{y}=+\infty$, there exists $\overline{R}>0$ such that

$$f(t,y)\ge\mu y,\quad y\ge\overline{R}, a\le t\le\sigma(b), \tag{3.5}$$

where $\mu=(\gamma_n\psi_n(1/4)\int_\alpha^{\beta_n}G_1(\sigma(s),s)\Delta s)^{-1}$.

Let $R_1=\max\{2r_1,\frac{\overline{R}}{\gamma_n}\}$ and $\Omega_2=\{y\in B:\|y\|<R_1\}$. For $y\in P\cap\partial\Omega_2$, we have

$$\min_{[\alpha,\beta_n]_{\mathbb{T}}} y(t)\ge\gamma_n\|y\|=\gamma_n R_1=\overline{R}.$$

Using Lemma 2.3 and (3.5), we find for $t_0\in[\alpha,\beta_n]_{\mathbb{T}}$ that

$$\begin{aligned}
Ty(t_0) &= \int_a^{\sigma(b)}(-1)^n G_n^1(t_0,s)f\bigl(s,y(s)\bigr)\Delta s\\
&\ge \int_\alpha^{\beta_n}\psi_n(1/4)G_1\bigl(\sigma(s),s\bigr)f\bigl(s,y(s)\bigr)\Delta s\\
&\ge \psi_n(1/4)\int_\alpha^{\beta_n}G_1\bigl(\sigma(s),s\bigr)\mu y(s)\Delta s\\
&\ge \psi_n(1/4)\int_\alpha^{\beta_n}G_1\bigl(\sigma(s),s\bigr)\mu\gamma_n R_1\Delta s
\end{aligned}$$

$$\geq \psi_n(1/4)\mu\gamma_n R_1 \int_\alpha^{\beta_n} G_1\bigl(\sigma(s),s\bigr)\Delta s$$
$$\geq R_1 = \|y\|,$$

and so

$$\|Ty\| \geq \|y\| \quad \text{for all } y \in P \cap \partial\Omega_2.$$

Consequently, Theorem 2.7 guarantees that T has a fixed point $y \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. □

Theorem 3.8 *Let* (C_1) *hold. Also assume*

(C_3) $\lim_{y\to 0^+} \frac{f(t,y)}{y} = +\infty$, $\lim_{y\to +\infty} \frac{f(t,y)}{y} = 0$ *for* $t \in [a,\sigma(b)]_{\mathbb{T}}$.

Then LBVP (1.1) *has at least one positive solution.*

Proof Since $\lim_{y\to 0^+} \frac{f(t,y)}{y} = +\infty$, there exists $r_2 > 0$ such that

$$f(t,y) \geq \overline{\mu} y, \quad 0 < y \leq r_2, a \leq t \leq \sigma(b),$$

where $\overline{\mu} \geq \mu$; here μ is given in the proof of Theorem 3.7.

Let $\Omega_1 = \{y \in B : \|y\| < r_2\}$. For $y \in P \cap \partial\Omega_1$, we have for $t_0 \in [\alpha,\beta_n]_{\mathbb{T}}$ that

$$\begin{aligned} Ty(t_0) &= \int_a^{\sigma(b)} (-1)^n G_n^1(t_0,s) f\bigl(s,y(s)\bigr)\Delta s \\ &\geq \int_\alpha^{\beta_n} \psi_n(1/4) G_1\bigl(\sigma(s),s\bigr) f\bigl(s,y(s)\bigr)\Delta s \\ &\geq \psi_n(1/4)\int_\alpha^{\beta_n} G_1\bigl(\sigma(s),s\bigr)\overline{\mu} y(s)\Delta s \\ &\geq \psi_n(1/4)\int_\alpha^{\beta_n} G_1\bigl(\sigma(s),s\bigr)\overline{\mu}\gamma_n R_1\Delta s \\ &\geq \psi_n(1/4)\overline{\mu}\gamma_n r_2 \int_\alpha^{\beta_n} G_1\bigl(\sigma(s),s\bigr)\Delta s \\ &\geq r_2 = \|y\|, \end{aligned}$$

and so $\|Ty\| \geq |y\|$ for all $y \in P \cap \partial\Omega_1$.

Since $\lim_{y\to\infty} \frac{f(t,y)}{y} = 0$, there exists $\overline{r_2}$ such that

$$f(t,y) \leq \overline{\eta} y, \quad y \geq \overline{r_2}, a \leq t \leq \sigma(b), \tag{3.6}$$

where $\overline{\eta} \leq \eta$.

We consider two cases.

Case 1. Suppose that f is bounded. Then there exists some $N > 0$ such that

$$f(t,y) \leq N, \quad t \in \bigl[a,\sigma(b)\bigr]_{\mathbb{T}}, y \in [0,\infty). \tag{3.7}$$

Let $r_3 = \max\{r_2 + 1, N\theta_n s_0\}$ and $\Omega_2 = \{y \in B : \|y\| < r_3\}$. For $y \in P \cap \partial\Omega_2$, using Lemma 2.1 and (3.7), we get

$$\begin{aligned} Ty(t) &= \int_a^{\sigma(b)} (-1)^n G_n^1(t,s) f\bigl(s, y(s)\bigr)\Delta s \\ &\leq N\theta_n \int_a^{\sigma(b)} G_1\bigl(\sigma(s), s\bigr)\Delta s \\ &\leq N\theta_n s_0 \leq r_3 = \|y\|. \end{aligned}$$

Hence, $\|Ty\| \leq \|y\|$ for all $y \in P \cap \partial\Omega_2$.

Case 2. Suppose that f is unbounded. In this case let

$$g(r) := \max\bigl\{f(t,y) : t \in \bigl[a, \sigma(b)\bigr]_{\mathbb{T}}, 0 \leq y \leq r\bigr\}$$

such that $\lim_{r\to\infty} g(r) = \infty$. We choose $r_3 > \max\{2r_2, \frac{\overline{r_2}}{\gamma_n}\}$ such that $g(r_3) \geq g(r)$ and let $\Omega_2 = \{y \in B : \|y\| < r_3\}$. For $y \in P \cap \partial\Omega_2$, using Lemma 2.1 and (3.6), we have

$$\begin{aligned} Ty(t) &= \int_a^{\sigma(b)} (-1)^n G_n^1(t,s) f\bigl(s, y[s]\bigr)\Delta s \\ &\leq \overline{\eta}\theta_n \int_a^{\sigma(b)} G_1\bigl(\sigma(s), s\bigr) f\bigl(s, y[s]\bigr)\Delta s \\ &\leq \overline{\eta}\theta_n \|y\| \int_a^{\sigma(b)} G_1\bigl(\sigma(s), s\bigr)\Delta s \\ &\leq \overline{\eta}\theta_n s_0 \|y\| = \|y\|, \end{aligned}$$

and so $\|Ty\| \leq \|y\|$ for all $y \in P \cap \partial\Omega_2$. It follows from Theorem 2.7 that T has a fixed point $y \in P \cap (\overline{\Omega}_2 \setminus \Omega_1)$. □

Theorem 3.9 *Let (C_1) hold. Also assume*

(C_4) $\lim_{y\to 0^+} \frac{f(t,y)}{y} = +\infty$, $\lim_{y\to+\infty} \frac{f(t,y)}{y} = +\infty$ *for* $t \in [a, \sigma(b)]_{\mathbb{T}}$,

(C_5) *There exists a constant ρ_1 such that $f(t,y) \leq \Gamma\rho_1$ for $y \in [0, \rho_1]_{\mathbb{T}}$,*

where $\Gamma \leq \eta$.

Then LBVP (1.1) has at least two positive solutions y_1 and y_2 such that

$$0 < \|y_1\| \leq \rho_1 < \|y_2\|.$$

Proof Since $\lim_{y\to 0^+} \frac{f(t,y)}{y} = +\infty$, there exists $\rho_* \in (0, \rho_1)$ such that

$$f(t,y) \geq \mu_1 y \quad \text{for } 0 \leq y \leq \rho_*, a \leq t \leq \sigma(b), \tag{3.8}$$

where $\mu_1 \geq \mu$; here μ is given in the proof of Theorem 3.7. Set $\Omega_1 = \{y \in B : \|y\| < \rho_*\}$. For $y \in P \cap \partial\Omega_1$, using Lemma 2.3 and (3.8), we find for $t_0 \in [\alpha, \beta_n]_{\mathbb{T}}$ that

$$Ty(t_0) = \int_a^{\sigma(b)} (-1)^n G_n^1(t_0, s) f\bigl(s, y(s)\bigr)\Delta s$$

$$\begin{aligned}
&\geq \int_{\alpha}^{\beta_n} \psi_n(1/4) G_1\bigl(\sigma(s),s\bigr) f\bigl(s,y(s)\bigr)\Delta s \\
&\geq \psi_n(1/4)\int_{\alpha}^{\beta_n} G_1\bigl(\sigma(s),s\bigr)\mu_1 y(s)\Delta s \\
&\geq \psi_n(1/4)\gamma_n\rho_*\mu_1\int_{\alpha}^{\beta_n} G_1\bigl(\sigma(s),s\bigr)\Delta s \\
&\geq \rho_* = \|y\|,
\end{aligned}$$

and so

$$\|Ty\| \geq \|y\| \quad \text{for all } y \in P\cap \partial\Omega_1. \tag{3.9}$$

Since $\lim_{y\to+\infty}\frac{f(t,y)}{y} = +\infty$, there exists $\rho^* > \rho_1$ such that

$$f(t,y) \geq \mu_2 y \quad \text{for } y \geq \rho^*, \tag{3.10}$$

where $\mu_2 \geq \mu$; here μ is given in the proof of Theorem 3.7.

Choose $\overline{\rho^*} > \max\{\frac{\rho^*}{\gamma_n}, \rho_1\}$ and set $\Omega_2 = \{y \in B : \|y\| < \overline{\rho^*}\}$. For any $y \in P\cap\partial\Omega_2$, we get

$$y(t) \geq \min_{t\in[\alpha,\beta_n]_{\mathbb{T}}} y(t) \geq \gamma_n\|y\| = (\nu-\alpha)\gamma_n\overline{\rho^*} > \rho^*. \tag{3.11}$$

Using Lemma 2.1, (3.10) and (3.11), for $t_0 \in [\alpha,\beta_n]_{\mathbb{T}}$ we have

$$\begin{aligned}
Ty(t_0) &= \int_a^{\sigma(b)} (-1)^n G_n^1(t_0,s) f\bigl(s,y(s)\bigr)\Delta s \\
&\geq \int_{\alpha}^{\beta_n} \psi_n(1/4) G_1\bigl(\sigma(s),s\bigr) f\bigl(s,y(s)\bigr)\Delta s \\
&\geq \psi_n(1/4)\int_{\alpha}^{\beta_n} G_1\bigl(\sigma(s),s\bigr))\mu_2 y(s)\Delta s \\
&\geq \psi_n(1/4)\gamma_n\overline{\rho^*}\mu_2\int_{\alpha}^{\beta_n} G_1\bigl(\sigma(s),s\bigr)\Delta s \\
&\geq \overline{\rho^*} = \|y\|,
\end{aligned}$$

which yields

$$\|Ty\| \geq \|y\| \quad \text{for all } y \in P\cap \partial\Omega_2. \tag{3.12}$$

Let $\Omega_3 = \{y \in B : \|y\| < \rho_1\}$. For $y \in P\cap\partial\Omega_3$ from (C_5) we obtain

$$\begin{aligned}
Ty(t) &= \int_a^{\sigma(b)} (-1)^n G_n^1(t,s) f\bigl(s,y(s)\bigr)\Delta s \\
&\leq \int_a^{\sigma(b)} \theta_n G_1\bigl(\sigma(s),s\bigr) f\bigl(s,y(s)\bigr)\Delta s \\
&\leq \theta_n\int_a^{\sigma(b)} G_1\bigl(\sigma(s),s\bigr)\Gamma\rho_1\Delta s
\end{aligned}$$

$$\leq \theta_n \Gamma \rho_1 \int_a^{\sigma(b)} G_1(\sigma(s), s)\Delta s$$
$$\leq \theta_n s_0 \Gamma \rho_1 = \rho_1 = \|y\|,$$

which yields

$$\|Ty\| \leq \|y\| \quad \text{for all } y \in P \cap \partial\Omega_3. \tag{3.13}$$

Hence, since $\rho_* \leq \rho_1 < \rho^*$ and from (3.9), (3.12), and (3.13) it follows from Theorem 2.7 that T has a fixed point y_1 in $P \cap (\overline{\Omega_3} \backslash \Omega_1)$ and a fixed point y_2 in $P \cap (\overline{\Omega_2} \backslash \Omega_3)$. Note that both are positive solutions of LBVP (1.1) satisfying

$$0 < \|y_1\| \leq \rho_1 < \|y_2\|. \qquad \square$$

Theorem 3.10 *Let* (C_1) *hold. Also assume*

(C_6) $\lim_{y\to 0^+} \frac{f(t,y)}{y} = 0$, $\lim_{y\to +\infty} \frac{f(t,y)}{y} = 0$ *for* $t \in [a, \sigma(b)]_{\mathbb{T}}$;

(C_7) *There exists a constant* ρ_2 *such that* $f(t,y) \geq \Theta\rho_2$ *for* $y \in [\gamma_n\rho_2, \rho_2]_{\mathbb{T}}$,

where $\Theta \geq \mu\gamma_n$.

Then LBVP (1.1) *has at least two positive solutions* y_1 *and* y_2 *such that*

$$0 < \|y_1\| \leq \rho_2 < \|y_2\|.$$

Let us define the functional

$$\omega(y) = \min_{t\in[\alpha,\beta_n]_{\mathbb{T}}} |y(t)|$$

and the numbers $N_1 \leq \eta = \frac{1}{\theta_n s_0}$, $N_2 = \frac{1}{\gamma_n s_0}$.

Theorem 3.11 *Let* (C_1) *hold and there exist constants* A, B, C, D *with* $0 < A < B < C = D$ *such that the following conditions hold*:

(C_8) $f(t,y) \leq N_1 A$ *for all* $(t,y) \in [a,\sigma(b)] \times [0,A]$,

(C_9) $f(t,y) \geq N_2 B$ *for all* $(t,y) \in [\alpha,\beta_n] \times [B,C]$,

(C_{10}) $f(t,y) \leq N_1 C$ *for all* $(t,y) \in [a,\sigma(b)] \times [0,C]$.

Then LBVP (1.1) *has at least three positive solutions* y_1, y_2, y_3 *such that*

$$\max_{t\in[a,\sigma^{2n}(b)]} |y_1(t)| < A, \qquad B < \min_{t\in[\alpha,\beta_n]} |y_2(t)| < \max_{t\in[a,\sigma^{2n}(b)]} |y_2(t)| \leq C,$$
$$\min_{t\in[\alpha,\beta_n]} |y_3(t)| < B, \qquad A < \max_{t\in[a,\sigma^{2n}(b)]} |y_3(t)| \leq C.$$

Proof Let $\mathcal{P}_C$, then $\|y\| \leq C$. So we get

$$\|Ty\| = \max_{t\in[a,\sigma^{2n}(b)]} |Ty(t)|$$
$$= \max_{t\in[a,\sigma^{2n}(b)]} \left| \int_a^{\sigma(b)} (-1)^n G_n^1(t,s) f(s,y(s))\Delta s \right|$$
$$\leq \int_a^{\sigma(b)} \theta_n G_1(\sigma(s),s) f(s,y(s))\Delta s$$

$$
\begin{aligned}
&\le \int_a^{\sigma(b)} \theta_n G_1\big(\sigma(s),s\big) N_1 C \Delta s \\
&\le \theta_n N_1 C \int_a^{\sigma(b)} \big(\sigma(s)-a\big)\big(\sigma^2(b)-\sigma(s)\big)\Delta s \\
&\le \theta_n N_1 C s_0 = C = \|y\|.
\end{aligned}
$$

In the same way, we can show that if (C_8) holds, then $T\overline{\mathcal{P}_A} \subset P_A$. Hence condition (ii) of Theorem 2.8 is satisfied.

To show condition (i) of Theorem 2.8, we choose $y_0(t) = \frac{B+C}{2}$ for all $t \in [a, \sigma^{2n}(b)]_{\mathbb{T}}$. It is easy to see that $y_0 \in \mathcal{P}$ and $\|y_0\| = \frac{B+C}{2} > B$. That is, $y_0 \in \{y \in \mathcal{P}(\omega, B, D) : \omega(y) > B\} \neq \emptyset$.

Moreover, if $y \in \mathcal{P}(\omega, B, D)$, we have $B \le y(t) \le C$ for $t \in [\alpha, \beta_n]_{\mathbb{T}}$. By (C_9) and Remark 2.5, we have

$$
\begin{aligned}
\omega(Ty) &= \min_{t\in[\alpha,\beta_n]_{\mathbb{T}}} \big|Ty(t)\big| \\
&\ge \int_a^{\sigma(b)} \min_{t\in[\alpha,\beta_n]_{\mathbb{T}}} \big|(-1)^n G_n^1(t,s) f\big(s,y(s)\big)\big| \Delta s \\
&\ge \gamma_n \int_\alpha^{\beta_n} \max_{t\in[a,\sigma^{2n}(b)]_{\mathbb{T}}} \big|G_n^1(t,s)\big|\big|f\big(s,y(s)\big)\big| \Delta s \\
&\ge \gamma_n s_0 N_2 B = B.
\end{aligned}
$$

Hence condition (i) of Theorem 2.8 is satisfied.

Since $D = C$, condition (i) implies condition (iii) of Theorem 2.8.

To sum up, all the hypotheses of Theorem 2.8 are satisfied. The proof is complete. □

Example 3.12 Let $\mathbb{T} = \mathbb{Z}$. We consider the following complementary Lidstone boundary value problem on $\mathbb{T}$:

$$
\begin{aligned}
&-y^{\Delta^{(6)}}(t) = f\big(t,y(t)\big), \quad t \in [0,5]_{\mathbb{T}}, \\
&y(0) = y(11) = y^{\Delta^{(2)}}(0) = y^{\Delta^{(2)}}(9) = y^{\Delta^{(4)}}(0) = y^{\Delta^{(4)}}(7) = 0.
\end{aligned}
\tag{3.14}
$$

Note that (3.14) is a particular case of (1.1) with $2n = 6$. Since $\mathbb{T} = \mathbb{Z}$, $\sigma(t) = t + 1$, $\sigma^j(t) = t + j$ and $x^{\Delta}(t) = \Delta x(t)$, $x^{\Delta^{(j)}}(t) = \Delta^j x(t)$. We notice that our Lidstone boundary value problem is the following difference Lidstone boundary value problem:

$$
\begin{aligned}
&\Delta^6 y(t) + f\big(t,y(t)\big) = 0, \quad t = 0,1,\dots,5 \\
&y(0) = y(11) = \Delta^2 y(0) = \Delta^2 y(9) = \Delta^4 y(0) = \Delta^4 y(7) = 0.
\end{aligned}
$$

The Green function $G_3^1(t,s)$ is

$$
G_3^1(t,s) = \int_0^{\sigma^5(5)} G_3(t,r) G_2^1(r,s)\Delta r = \sum_{r=0}^{9} G_3(t,r) G_2^1(r,s),
$$

where

$$G_3(t,s) = \frac{-1}{\sigma^6(5)} \begin{cases} t(\sigma^6(5) - \sigma(s)), & t \le s, \\ \sigma(s)(\sigma^6(5) - t), & \sigma(s) \le t \end{cases} = \frac{-1}{11} \begin{cases} t(10 - s), & t \le s, \\ (s+1)(11 - t), & s+1 \le t, \end{cases}$$

$$G_2^1(t,s) = \int_0^{\sigma^3(5)} G_2(t,r) G_1^1(r,s) \Delta r = \sum_{r=0}^{7} G_2(t,r) G_1^1(r,s),$$

$$G_2(t,s) = \frac{-1}{\sigma^4(5)} \begin{cases} t(\sigma^4(5) - \sigma(s)), & t \le s, \\ \sigma(s)(\sigma^4(5) - t), & \sigma(s) \le t \end{cases} = \frac{-1}{9} \begin{cases} t(8 - s), & t \le s, \\ (s+1)(9 - t), & s+1 \le t, \end{cases}$$

and

$$G_1^1(t,s) = G_1(t,s) = \frac{-1}{\sigma^2(5)} \begin{cases} t(\sigma^2(5) - \sigma(s)), & t \le s, \\ \sigma(s)(\sigma^2(5) - t), & \sigma(s) \le t \end{cases} = \frac{-1}{7} \begin{cases} t(6 - s), & t \le s, \\ (s+1)(7 - t), & s+1 \le t. \end{cases}$$

In Lemma 2.1, we find $\theta_3 = \frac{s_2 s_4}{\sigma^2(5)\sigma^4(5)\sigma^6(5)} = \frac{s_2 s_4}{7\times 9\times 11}$, where

$$\begin{aligned} s_2 &= \frac{1}{6}\Big\{ \big(\sigma^4(5)\big)^3 + \sum_{t\in A_2} \mu(t)^2 \big[3\big(\sigma^4(5)\big) - 2\big(t + 2\sigma(t)\big)\big] \\ &\quad - \sum_{t\in B_2} \mu(t)^2 \big[2\big(t + 2\sigma(t)\big) - 3\big(\sigma^4(5)\big)\big] \Big\} \\ &= \frac{1}{6}\Big\{ 9^3 + \sum_{t=0}^{3} 23 - 6t - \sum_{t=4}^{8} 6t - 23 \Big\} = 120 \end{aligned}$$

and

$$\begin{aligned} s_4 &= \frac{1}{6}\Big\{ \big(\sigma^6(5)\big)^3 + \sum_{t\in A_4} \mu(t)^2 \big[3\big(\sigma^6(5)\big) - 2\big(t + 2\sigma(t)\big)\big] \\ &\quad - \sum_{t\in B_4} \mu(t)^2 \big[2\big(t + 2\sigma(t)\big) - 3\big(\sigma^6(5)\big)\big] \Big\} \\ &= \frac{1}{6}\Big\{ 11^3 + \sum_{t=0}^{4} 29 - 6t - \sum_{t=5}^{10} 6t - 29 \Big\} = 252 \end{aligned}$$

with $A_2 = \{0,1,2,3\}$, $B_2 = \{4,5,\dots,8\}$, $A_4 = \{0,1,2,3,4\}$, and $B_4 = \{5,\dots,9,10\}$.

So $\theta_3 = \frac{120\times 252}{7\times 9\times 11} \cong 43.63$.

Also, choosing $\alpha = 1, \beta_3 = 10, \beta_2 = 8, \xi = 6, \nu = 4$, we find $\psi_3(\delta) = \delta^3 \frac{S_2 S_3}{\sigma^2(5)\sigma^4(5)\sigma^6(5)} = \frac{S_2 S_3}{7\times 9\times 11}$, where

$$\begin{aligned} S_2 &= \frac{1}{6}\Big\{ (\beta_2 - \alpha) 3\sigma^4(5)(\beta_2 + \alpha) - 2\big(\beta_2^2 + \beta_2\alpha + \alpha^2\big) \\ &\quad + \sum_{t\in A_2 - [0,1)} \mu(t)^2 \big[3\big(\sigma^4(5)\big) - 2\big(t + 2\sigma(t)\big)\big] \\ &\quad - \sum_{t\in B_2 - (\beta_2, \sigma^4(5)]} \mu(t)^2 \big[2\big(t + 2\sigma(t)\big) - 3\big(\sigma^4(5)\big)\big] \Big\} \end{aligned}$$

$$= \frac{1}{6}\left\{7 \times 27 \times 9 - 2(64 + 8 + 1) + \sum_{t=1}^{3} 23 - 6t - \sum_{t=4}^{8} 6t - 23\right\} \cong 211.6$$

and

$$S_3 = \frac{1}{6}\{(\beta_3 - \alpha)3\sigma^6(5)(\beta_3 + \alpha) - 2\left(\beta_3^2 + \beta_3\alpha + \alpha^2\right) + \sum_{t\in A_3-[0,1)} \left[3\sigma^6(5) - 2\left(t + 2\sigma(t)\right)\right]$$

$$- \sum_{t\in B_3-(\beta_3,\sigma^6(5)]} \left[2\left(t + 2\sigma(t)\right) - 3\sigma^6(5)\right]$$

$$= \frac{1}{6}\left\{9.33.11 - 2(100 + 10 + 1) + \sum_{t=1}^{3} 27 - 6t - \sum_{t=4}^{9} 6t - 29\right\} = 449$$

with $A_3 = \{0, 1, 2, 3\}$, $B_3 = \{4, 5, \ldots, 8, 9\}$, $A_4 = \{0, 1, 2, 3, 4\}$, and $B_4 = \{5, \ldots, 9, 10\}$.

So $\psi_3(\delta) \cong \delta^3 137$ and $\gamma_3 = \frac{S_2 S_3}{4^3 s_2 s_4} \cong 0.04$.

Besides these, also find

$$s_0 = \frac{1}{6}\left\{7^3 + \sum_{t=0}^{2} 17 - 6t - \sum_{t=3}^{6} 6t - 17\right\} \cong 33.3.$$

We note that

$$0 < \int_4^6 G_1\left(\sigma(s), s\right)\Delta s = \int_4^6 (s + 1)(8 - s)\Delta s = \sum_{s=4}^{6}\left(-s^2 + 6s + 7\right) = 34.$$

(i) Consider the Lidstone dynamic equation (3.14) with the function $f(t, y) = \frac{y}{10^5(1+y^2)}$. It is easy to see that f satisfies condition (C_1). If we choose $M = 10^6$, we can easily see that the condition $\theta_3 Q s_0 \leq M$ is satisfied for $Q = 11$. Therefore, according to Theorem 3.1, Lidstone BVP (3.14) has a solution $y(t)$.

(ii) Consider the Lidstone dynamic equation (3.14) with the function $f(t, y) = 1 - \sin^2 y$. It is easy to see that f satisfies condition (C_1). Also the continuous function f is bounded on $[0, 6] \times \mathbb{R}$. Therefore, according to Corollary 3.2, Lidstone BVP (3.14) has a solution $y(t)$. Also, $u(t) = 0$ is a lower solution and $v(t) = \frac{\pi}{2}$ is an upper solution for LBVP (3.14). Thus, according to Theorem 3.6, Lidstone BVP (3.14) has a solution $y \in [0, \frac{\pi}{2}]$ on $t \in [1, 11]$.

(iii) Consider the Lidstone dynamic equation (3.14) with the function $f(t, y) = y^2(t + y)$. It is easy to see that f satisfies condition (C_1). Since

$$\lim_{y\to 0^+} \frac{f(t, y)}{y} = \lim_{y\to 0^+} \frac{y^2(t + y)}{y} = 0,$$

$$\lim_{y\to +\infty} \frac{f(t, y)}{y} = \lim_{y\to +\infty} \frac{y^2(t + y)}{y} = +\infty \quad \text{for } t \in [0, 8]_{\mathbb{T}},$$

condition (C_2) is fulfilled. Therefore, according to Theorem 3.7, Lidstone BVP (3.14) has at least one positive solution.

(iv) Consider the Lidstone dynamic equation (3.14) with the function $f(t, y) = \sqrt{y(t)} + t^2$. It is easy to see that f satisfies condition (C_1). Also we obtain

$$\lim_{y\to 0^+} \frac{f(t, y)}{y} = \lim_{y\to 0^+} \frac{\sqrt{y} + t^2}{y} = +\infty,$$

$$\lim_{y\to+\infty}\frac{f(t,y)}{y}=\lim_{y\to+\infty}\frac{\sqrt{y}+t^2}{y}=0\quad\text{for } t\in[0,8]_{\mathbb{T}},$$

so condition (C_3) is fulfilled. From Theorem 3.8, Lidstone BVP (3.14) has at least one positive solution.

(v) Consider the Lidstone dynamic equation (3.14) with the function

$$f(t,y)=\begin{cases}\frac{y^3}{4^3 10^5(1+y)}, & y\geq 4;\\ \frac{\sqrt{y}}{10^6}, & 0\leq y<4.\end{cases}$$

The function f is continuous on $[0,5]_{\mathbb{T}}\times\mathbb{R}$ and nondecreasing in the second argument with $f(t,y)\geq 0$ for $(t,x)\in[0,5]_{\mathbb{T}}\times K$. We can easily see that condition (C_1) is fulfilled. Also we have

$$\lim_{y\to 0^+}\frac{f(t,y)}{y}=\lim_{y\to 0^+}\frac{\sqrt{y}}{10^6 y}=+\infty,$$
$$\lim_{y\to+\infty}\frac{f(t,y)}{y}=\lim_{y\to+\infty}\frac{y^2}{4^3 10^5(1+y)}=+\infty\quad\text{for } t\in[0,8]_{\mathbb{T}}.$$

Thus (C_4) is satisfied. Furthermore, we find $\Gamma\leq\frac{1}{\theta_3 s_0}=\frac{99}{210\times 120}$. If we choose $\rho_1=\frac{1}{10^4}$ and $\Gamma=\frac{390}{34\times 120\times 252}$, we have

$$f(t,y)=\frac{\sqrt{y}}{10^6}\leq\Gamma\rho_1,\quad\text{for } 0\leq y\leq 10^{-4}, t\in[0,5]_{\mathbb{T}},$$

so condition (C_5) is satisfied. Thus all the conditions of Theorem 3.9 are satisfied, so the LBVP has at least two positive solutions.

(vi) Consider the Lidstone dynamic equation (3.14) with the function

$$f(t,y)=\begin{cases}\sqrt{y-1}+\frac{101}{2}, & y\geq 1;\\ \frac{101y^2}{1+y}, & 0\leq y<1.\end{cases}$$

The function f is continuous on $[0,5]_{\mathbb{T}}\times\mathbb{R}$ and nondecreasing in the second argument with $f(t,x)\geq 0$ for $(t,x)\in[0,5]_{\mathbb{T}}\times K$. We can easily see that condition (C_1) is fulfilled. Also we have

$$\lim_{y\to 0^+}\frac{f(t,y)}{y}=\lim_{y\to 0^+}\frac{101y}{1+y}=0,$$
$$\lim_{y\to+\infty}\frac{f(t,y)}{y}=\lim_{y\to+\infty}\frac{\sqrt{y-1}+\frac{101}{2}}{y}=0\quad\text{for } t\in[0,8]_{\mathbb{T}}.$$

Thus (C_6) is satisfied. Now, if we calculate the number Θ in Theorem 3.10, we obtain $\Theta\cong 0.04$. If we choose $\rho_2=\frac{1}{3}$, and noting $\gamma_3\cong 0.04$, we have

$$f(t,y)=\frac{101y^2}{1+y}\geq\frac{4}{100}\times\frac{1}{3},\quad\text{for } \gamma_3\rho_2\leq y\leq\rho_2, t\in[0,5]_{\mathbb{T}},$$

so condition (C_7) is satisfied. Thus all the conditions of Theorem 3.10 are satisfied, so the LBVP has at least two positive solutions.

4 Conclusion

In this paper, we obtain sufficient conditions that guarantee the existence of solutions for LBVP (1.1) on time scales. Firstly, by using Schauder's fixed point theorem, the existence of a solution is proved, and by using this theorem and lower and upper solutions method, the other existence result is also given. Later, by using Krasnosel'skii's fixed point theorem the existence of one and two positive solutions is proved. Finally, by using the Leggett–Williams fixed point theorem, the existence of three positive solutions is proved. Although the studies [2, 15, 27, 28] worked on limited time scales, which satisfies that $[0,1]_{\mathbb{T}}$ and $\sigma(1)$ is right dense, $\sigma^j(1) = \sigma(1)$ for $j \geq 1$, this study works on $[a, \sigma^{2n}(b)]_{\mathbb{T}}$ where $\mathbb{T}$ is any time scale. Therefore this work generalizes papers about the existence of solutions for LBVP. This study demonstrates the combining and generalizing properties of time scale theory.

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