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Stepanov-like doubly weighted pseudo almost automorphic mild solutions for fractional stochastic neutral functional differential equations

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Abstract

This paper first investigates the equivalence of the space and translation invariance of Stepanov-like doubly weighted pseudo almost automorphic stochastic processes for nonequivalent weight functions; secondly, based on semigroup theory, fractional calculations, and the Krasnoselskii fixed-point theorem, we obtain the existence and uniqueness of Stepanov-like doubly weighted pseudo almost automorphic mild solutions for a class of nonlinear fractional stochastic neutral functional differential equations under non-Lipschitz conditions. These results enrich the complex dynamics of Stepanov-like doubly weighted pseudo almost automorphic stochastic processes.

Keywords: Stepanov-like doubly weighted pseudo almost automorphic stochastic process; Translation invariance; Existence; Uniqueness

1 Introduction

Historically, a great many mathematical models for dynamic processes in the fields of engineering, biological, and physical sciences are elucidated by stochastic differential equations. Under the background of dynamical systems, functional differential equations have become an important focus of attention of investigation and research, in view of the ubiquity and persistence of time delays, see [1] and [2] for details. In [3], the author presented a comprehensive study of functional differential equations endowed with infinite delay. Since uncertainties and random factors are commonly encountered in differential equations, which are the key factors causing system instability, in recent years, the theory of stochastic functional differential equations has attracted the attention of more and more researchers, such as the related stability, ergodicity, etc. [4–7].

With the development of differential equations, the qualitative properties of fractional differential equations both with and without delays have long been an active topic in the interest of researchers. In particular, the Mittag–Leffler stability and asymptotic stability of solutions have been widely studied due to their importance in applications in the areas of engineering and applied sciences. For more detailed information on this subject, see

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[8–10]. As a combination of stochastic functional differential equations and fractional differential equations, the fractional stochastic neutral functional differential equation made its entrance into the hot topic [11–13]. Most importantly, noninteger-order stochastic differential equations possess the capability of describing the memory effects and real behavior that play a crucial role in mathematical models. Hence, it is significant and necessary to further explore this kind of equations.

In the course of studying the qualitative behavior of solutions to stochastic differential equations, the changes in the environment are not precisely periodic, therefore, the investigation of almost periodic solutions occupies an important position in the aspect of the qualitative theory of stochastic differential equations. As an extension of an almost periodic stochastic process, the almost automorphic stochastic process and other generalizations have developed rapidly and have been widely investigated in many publications due to its applications and significance in physics, mathematical biology, and mechanics [14–20], which constitute a significant part of mild solutions. In addition, while the authors in these papers studied the existence and uniqueness of solutions, the classical Banach fixed-point theorem is indispensable, but whether the properties of the solution will still hold if we replace the Banach fixed-point theorem with the more general Krasnoselskii fixed-point theorem under non-Lipschitz conditions [21] is an important question that needs to be studied.

It is worth mentioning that Chen and Lin introduced the concept of a weighted pseudo almost automorphic stochastic process and studied its translation invariance and composition theorem [22]. Further, Tang and Chang proposed the Stepanov-like weighted pseudo almost automorphic stochastic process that included the former as a special case, and investigated the existence and uniqueness of Stepanov-like weighted pseudo almost automorphic mild solutions in a real separable Hilbert space to a class of stochastic differential equations under global Lipschitz conditions [23]. Very recently, Yang and Zhu introduced the Stepanov-like doubly weighted pseudo almost automorphic stochastic process for nonequivalent weight functions, and explored its properties, such as the completeness, convolution invariance, etc.; further, the authors proved the existence and uniqueness for the stochastic differential equations driven by G-Brownian motion by using the Banach fixed-point theorem [24]. However, up to now, there are very few research results about Stepanov-like doubly weighted pseudo almost automorphic stochastic processes and still many properties have not been explored, let alone its applications to fractional stochastic functional differential equation, so it is necessary to further study this area.

Motivated by the above-mentioned works, the goal of our work is to investigate the theory for the p -mean Stepanov-like doubly weighted pseudo almost automorphic stochastic process and its applications to a class of nonlinear fractional stochastic neutral functional differential equation as follows:

$${}^c D_t^\kappa [x(t) - h(t, x_t)] = Ax(t) + f_1(t, x_t) + f_2(t, x_t) \frac{dw(t)}{dt}, \quad t \geq s, \tag{1}$$

where ${}^c D_t^\kappa$ is the Caputo fractional derivative of order $\kappa \in (\frac{1}{2}, 1)$; A is a sectorial linear operator and $-A$ is the infinitesimal generator of an analytic semigroup on Hilbert space [25]; f_1, f_2 , and h are suitable functions and $w(t)$ is a two-sided cylindrical Wiener process, which will be specified in Sect. 2.

The structure of this paper is as follows. Section 2 preliminarily introduces several definitions and related lemmas. Section 3 investigates the equivalence of the space and translation invariance of Stepanov-like doubly weighted pseudo almost automorphic stochastic processes for nonequivalent weight functions, which enrich the dynamics of Stepanov-like doubly weighted pseudo almost automorphic stochastic processes. In Sect. 4, based on semigroup theory and the famous Krasnoselskii fixed-point theorem, by using analytical skills of the Lebesgue dominated convergence theorem, Fubini theorem, Burkholder–Davis–Gundy inequality, etc., we obtain the existence and uniqueness of p -mean Stepanov-like doubly weighted pseudo almost automorphic mild solutions for a class of nonlinear fractional stochastic neutral functional differential equation under non-Lipschitz conditions. Moreover, an example is investigated to illustrate our conclusions.

2 Preliminaries

Let $(\Omega, \mathcal{F}, \mathbb{P})$ stand for a complete probability space with the filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions, and \mathbb{H} and \mathbb{K} are real separable Hilbert spaces. The family of all p -mean integrable \mathbb{H} -valued random variables is denoted by $L^p(\mathbb{P}, \mathbb{H})$ for $p \geq 2$, which is a Banach space equipped with the norm $\|\cdot\|_{L^p} = (\mathbb{E}\|\cdot\|^p)^{\frac{1}{p}} < \infty$ for the expectation \mathbb{E} . Denote by $\mathbb{L}_2(\mathbb{K}; \mathbb{H})$ the space of all Hilbert–Schmidt operators from \mathbb{K} to \mathbb{H} equipped with the Hilbert–Schmidt norm $\|\cdot\|_2$. We assume $\{w(t)\}_{t \in \mathbb{R}}$ is a \mathbb{K} -valued Q -Wiener process with the covariance operator $Q \in \mathbb{L}_2(\mathbb{K}; \mathbb{H})$. Let $\mathbb{K}_0 = Q^{\frac{1}{2}}\mathbb{K}$ and $\mathbb{L}_2^0(\mathbb{K}_0; \mathbb{H})$ endowed with the norm $\|\cdot\|_{\mathbb{L}_2^0}$. In addition, let $BC(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$ be the set of all stochastic bounded and continuous processes $X: \mathbb{R} \rightarrow L^p(\mathbb{P}, \mathbb{H})$, and $\varrho: (-\infty, 0] \rightarrow [0, +\infty)$ be a continuous function such that $\varrho^* := \int_{-\infty}^0 \varrho(s) ds < +\infty$. Define

$$\mathbb{B}_\varrho := \left\{ \sigma : (-\infty, 0] \rightarrow L^p(\mathbb{P}, \mathbb{H}) \mid \text{for any } s < 0, \|\sigma\|_{L^p} \text{ is bounded measurable} \right. \\ \left. \text{on } [s, 0] \text{ such that } \int_{-\infty}^0 \varrho(s) \int_t^{t+1} \sup_{s \leq \theta \leq 0} \mathbb{E}\|\sigma(\theta)\|^p du ds < +\infty \right\},$$

which is a Banach space endowed with the norm

$$\|\sigma\|_\varrho = \left(\int_{-\infty}^0 \varrho(s) \int_t^{t+1} \sup_{s \leq \theta \leq 0} \mathbb{E}\|\sigma(\theta)\|^p du ds \right)^{\frac{1}{p}} < +\infty,$$

it is not difficult to deduce that $\|x_t\|_\varrho = \varrho^* \|x\|_{SP}$, where $x_t(s) = x(t + s)$ for any $t \in \mathbb{R}$ and $s \in (-\infty, 0]$.

2.1 Stepanov-like doubly weighted pseudo almost automorphic stochastic process

The Bochner transform $x^b(t, s)$ for any $t \in \mathbb{R}$ and $s \in [0, 1]$ of a stochastic process x is denoted by $x^b(t, s) = x(t + s)$. Based on the Definitions 7–10 in [24], by replacing the Banach space $L_G^p(\Omega)$ with $L^p(\mathbb{P}, \mathbb{H})$, we present the next concepts.

Definition 2.1 A continuous stochastic process $Z: \mathbb{R} \rightarrow L^p(\mathbb{P}, \mathbb{H})$ is called p -mean almost automorphic if for every sequence of real numbers $\{\tau'_n\}_{n \in \mathbb{N}}$, there exists $\{\tau_n\}_{n \in \mathbb{N}} \subseteq \{\tau'_n\}_{n \in \mathbb{N}}$ and a stochastic process $Y: \mathbb{R} \rightarrow L^p(\mathbb{P}, \mathbb{H})$ such that $\lim_{n \rightarrow +\infty} (\mathbb{E}\|Z(t + \tau_n) - Y(t)\|^p)^{\frac{1}{p}} = 0$ and $\lim_{n \rightarrow +\infty} (\mathbb{E}\|Y(t - \tau_n) - Z(t)\|^p)^{\frac{1}{p}} = 0$ for each $t \in \mathbb{R}$.

Denote by $AA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$ the set of all such stochastic processes; this is a Banach space endowed with the norm $\|Z\|_\infty = \sup_{t \in \mathbb{R}} (\mathbb{E}\|Z(t)\|^p)^{\frac{1}{p}}$.

Remark 2.1 The set of all Stepanov-like bounded stochastic processes is denoted by $BS^p(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$, which includes all stochastic processes $Z: \mathbb{R} \rightarrow L^p(\mathbb{P}, \mathbb{H})$ satisfying $Z^b \in L^\infty(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H})))$. This is a Banach space equipped with the norm

$$\begin{aligned} \|Z\|_{SP} &= \sup_{t \in \mathbb{R}} \left(\int_0^1 \mathbb{E}\|Z(t+s)\|^p ds \right)^{\frac{1}{p}} \\ &= \sup_{t \in \mathbb{R}} \left(\int_t^{t+1} \mathbb{E}\|Z(s)\|^p ds \right)^{\frac{1}{p}} := \sup_{t \in \mathbb{R}} \|Z^b(t, \cdot)\|_p. \end{aligned}$$

Definition 2.2 A stochastic process $Z \in BS^p(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$ is said to be Stepanov-like almost automorphic if $Z^b \in AA(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H})))$.

The collection of such functions is defined by $S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$; it is a Banach space under the norm $\|\cdot\|_{SP}$.

Let \mathcal{U} be the set of all locally integrable positive ρ on \mathbb{R} . For given $\rho \in \mathcal{U}$ and $r > 0$, assume $Q_r(\rho) = \int_{-r}^r \rho(t) dt$. Further, denote \mathcal{U}_∞ and \mathcal{U}_b by

$$\mathcal{U}_\infty = \left\{ \rho \in \mathcal{U} : \lim_{r \rightarrow +\infty} Q_r(\rho) = +\infty \right\}, \quad \mathcal{U}_b = \left\{ \rho \in \mathcal{U}_\infty : \rho \text{ is bounded, } \inf_{t \in \mathbb{R}} \rho(t) > 0 \right\}.$$

Obviously, $\mathcal{U}_b \subset \mathcal{U}_\infty \subset \mathcal{U}$.

Remark 2.2 A stochastic processes $Z \in BS^p(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$ is called Stepanov-like doubly weighted ergodic in $t \in \mathbb{R}$, if $h^b \in PAA_0(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H})), \rho, q)$, i.e.,

$$\lim_{r \rightarrow +\infty} \frac{1}{Q_r(\rho)} \int_{-r}^r \int_t^{t+1} \mathbb{E}\|h(s)\|^p ds q(t) dt = 0,$$

the set of all such functions will be labeled by $S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$.

Definition 2.3 Let $\rho, q \in \mathcal{U}_\infty$. A stochastic process $f \in BS^p(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$ is said to be Stepanov-like doubly weighted pseudo almost automorphic provided $f = g + h$, where $g \in S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$ and $h \in S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$.

The family of all such processes will be denoted by $S^pWPAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$, which is a Banach space with the norm $\|\cdot\|_{SP}$.

Similarly, $S^pWPAA(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}); L^p(\mathbb{P}, \mathbb{H}), \rho, q)$ can be defined, that is, for any $f \in S^pWPAA(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}); L^p(\mathbb{P}, \mathbb{H}), \rho, q)$, then $f = g + h$ with

$$\begin{aligned} g &\in S^pAA(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}); L^p(\mathbb{P}, \mathbb{H})) \\ &:= \{g(t, z) \in S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H})) : \text{for any } z \in L^p(\mathbb{P}, \mathbb{H})\} \end{aligned}$$

and

$$\begin{aligned} h &\in S^pPAA_0(\mathbb{R} \times L^p(\mathbb{P}, \mathbb{H}); L^p(\mathbb{P}, \mathbb{H}), \rho, q) \\ &:= \{h(t, z) \in S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q) : \text{for any } z \in L^p(\mathbb{P}, \mathbb{H})\}. \end{aligned}$$

Remark 2.3 If ρ is equivalent to q (i.e., $\rho \sim q$), it follows that $S^p WPAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q) = S^p WPAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho) = S^p WPAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), q)$. For the particular case of $p = 2$, one can refer to Definition 2.6 in [19] for more details.

This paper aims at studying the case of $p \geq 2$, ρ and q are nonequivalent for the Stepanov-like doubly weighted pseudo almost automorphic stochastic processes that admits more complex dynamics than the classical square-mean Stepanov-like weighted pseudo almost automorphic stochastic processes established in related papers.

Next, we introduce an indispensable Krasnoselskii fixed-point theorem used in Sect. 4.

Lemma 2.1 ([21]) *Let B be a bounded closed and convex subset of a Banach space X , J_1, J_2 be two maps of B into X such that $J_1x + J_2y \in B$ for $x, y \in B$. If J_1 is a contraction and J_2 is completely continuous, then there exists a $x \in B$ that satisfies $J_1x + J_2x = x$.*

2.2 Caputo derivative and fractional powers of sectorial operators

We recall the fractional integral of order κ for a function f defined as

$$I^\kappa f(t) := \frac{1}{\Gamma(\kappa)} \int_a^t (t-s)^{\kappa-1} f(s) ds, \quad \kappa > 0,$$

where Γ is the Gamma function, that is $\Gamma(\kappa) := \int_0^{+\infty} t^{\kappa-1} e^{-t} dt$.

For $0 < \kappa < 1$, the fractional Caputo's derivative of the function f with order κ is

$${}^c D_t^\kappa f(t) := \frac{1}{\Gamma(1-\kappa)} \int_a^t \frac{f'(s)}{(t-s)^\kappa} ds.$$

Next, we recall some knowledge of fractional powers of sectorial operators.

Definition 2.4 ([25]) Let X be a Banach space, a densely defined and closed linear operator $A: D(A) \subseteq X \rightarrow X$ is said to be sectorial if there exist constants $\zeta \in \mathbb{R}$, $\theta \in (0, \frac{\pi}{2})$, and $M > 0$ that satisfy

- (I) $\rho(A) \supseteq S_{\theta, \zeta} := \{\lambda \in \mathbb{C} : \lambda \neq \zeta, |\arg(\lambda - \zeta)| > \theta\}$;
- (II) $\|R(\lambda, A)\|_{L(X)} \leq \frac{M}{|\lambda - \zeta|}$ for each $\lambda \in S_{\theta, \zeta}$, where $\rho(A)$ and $R(\lambda, A)$ stand for the resolvent subset and the resolvent operator of A , respectively.

Let $\{S(t)\}_{t \in \mathbb{R}}$ be an analytic semigroup with infinitesimal generator $-A$ that satisfies $\|S(t)\|_{L(X)} \leq Me^{-\delta t}$ for $M \geq 1$ and $\delta \geq 0$, where A is a linear sectorial operator with $0 \in \rho(A)$, then the fractional powers of A is defined as

$$A^{-\kappa} := \frac{1}{\Gamma(\kappa)} \int_0^{+\infty} t^{\kappa-1} S(t) ds, \quad \kappa > 0,$$

clearly, $\{A^{-\kappa}\}$ is an operator semigroup and the next result holds.

Lemma 2.2 ([25]) *Let $0 < \alpha \leq \kappa$, then*

- (i) *The operator $A^{-\kappa}$ is one-to-one and denotes its inverse operator by A^κ . Moreover, the closed operator A^κ is also the fractional powers of linear operator A with range $X_\kappa := D(A^\kappa) = R(A^{-\kappa})$;*
- (ii) *X_κ is a Banach space equipped with the norm $\|x\|_\kappa := \|A^\kappa x\|_X$ for $x \in X_\kappa$, and the injection $X_\kappa \hookrightarrow X_\alpha$ is continuous;*

- (iii) There exists $M_\kappa > 0$ such that $\|A^\kappa S(t)\|_{L(X)} \leq M_\kappa t^{-\kappa} e^{-\eta t}$, where $t > 0$;
- (iv) For any $0 < \kappa \leq 1$, there exists $N_\kappa > 0$ such that $\|S(t)x - x\|_X \leq N_\kappa t^\kappa \|A^\kappa x\|_X$ with $t > 0$ and $x \in X_\kappa$.

3 Equivalence and translation invariance

For any set $D \subseteq \mathbb{R}$, denote its complementary set by D^c , then the following results hold.

Theorem 3.1 Let $\rho_i, q_i \in \mathcal{U}_\infty$ for $i = 1, 2$. Assume that there exist a measurable set $A_0 \subseteq \mathbb{R}$ and constants $m_i, M_j > 0$ ($j = 1, 2, 3$) that satisfy

$$m_1 \leq \frac{\rho_1(t)}{\rho_2(t)} \leq M_1, \quad m_2 \leq \frac{q_1(t)}{q_2(t)} \leq M_2, \quad t \notin A_0,$$

$$\max_{t \in A_0} \left\{ \frac{\rho_1(t)}{q_1(t)}, \frac{\rho_2(t)}{q_2(t)} \right\} \leq M_3, \quad \lim_{r \rightarrow +\infty} \frac{1}{Q_r(\rho_i)} \int_{[-r,r] \cap A_0} q_i(t) dt = 0,$$

then $S^p WPAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho_1, q_1) = S^p WPAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho_2, q_2)$.

Proof Based on the measurable set A_0 and its complementary set A_0^c in \mathbb{R} , we have

$$\begin{aligned} \frac{Q_r(\rho_1)}{Q_r(\rho_2)} &= \frac{1}{Q_r(\rho_2)} \int_{[-r,r] \cap A_0} \rho_1(t) dt + \frac{1}{Q_r(\rho_2)} \int_{[-r,r] \cap A_0^c} \rho_1(t) dt \\ &\leq \frac{Q_r(\rho_1)}{Q_r(\rho_2)} \sup_{t \in A_0} \frac{\rho_1(t)}{q_1(t)} \frac{1}{Q_r(\rho_1)} \int_{[-r,r] \cap A_0} q_1(t) dt \\ &\quad + \sup_{t \in A_0^c} \frac{\rho_1(t)}{\rho_2(t)} \frac{1}{Q_r(\rho_2)} \int_{[-r,r] \cap A_0^c} \rho_2(t) dt. \end{aligned}$$

Further, from $\sup_{t \in A_0} \frac{\rho_1(t)}{q_1(t)} \leq M_3$ and $\sup_{t \in A_0^c} \frac{\rho_1(t)}{\rho_2(t)} \leq M_1$, it follows that

$$\frac{Q_r(\rho_1)}{Q_r(\rho_2)} \left[1 - \frac{M_3}{Q_r(\rho_1)} \int_{[-r,r] \cap A_0} q_1(t) dt \right] \leq \frac{M_1}{Q_r(\rho_2)} \int_{[-r,r] \cap A_0^c} \rho_2(t) dt \leq M_1.$$

Since $\lim_{r \rightarrow +\infty} \frac{1}{Q_r(\rho_1)} \int_{[-r,r] \cap A_0} q_1(t) dt = 0$, it follows that

$$\limsup_{r \rightarrow +\infty} \frac{Q_r(\rho_1)}{Q_r(\rho_2)} < +\infty. \tag{2}$$

For any $f^b \in PAA_0(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H})), \rho_1, q_1)$, based on $m_2 \leq \frac{q_1(t)}{q_2(t)}$, $t \notin A_0$, we have

$$\begin{aligned} &\frac{1}{Q_r(\rho_2)} \int_{-r}^r \int_t^{t+1} \mathbb{E} \|f(s)\|^p ds q_2(t) dt \\ &= \frac{1}{Q_r(\rho_2)} \int_{[-r,r] \cap A_0} \int_t^{t+1} \mathbb{E} \|f(s)\|^p ds q_2(t) dt \\ &\quad + \frac{1}{Q_r(\rho_2)} \int_{[-r,r] \cap A_0^c} \int_t^{t+1} \mathbb{E} \|f(s)\|^p ds q_2(t) dt \\ &\leq \frac{1}{m_2} \frac{Q_r(\rho_1)}{Q_r(\rho_2)} \frac{1}{Q_r(\rho_1)} \int_{-r}^r \int_t^{t+1} \mathbb{E} \|f(s)\|^p ds q_1(t) dt + \frac{\|f\|_{SP}}{Q_r(\rho_2)} \int_{[-r,r] \cap A_0} q_2(t) dt. \end{aligned}$$

Combined with (2) and $\lim_{r \rightarrow +\infty} \frac{1}{Q_r(\rho_2)} \int_{[-r,r] \cap A_0} q_2(t) dt = 0$, we conclude that $f^b \in PAA_0(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H})), \rho_2, q_2)$; moreover, $S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho_1, q_1) \subseteq S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho_2, q_2)$. Using a similar method as above, it follows that $S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho_2, q_2) \subseteq S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho_1, q_1)$. From Definition 2.3, this yields $S^pWPAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho_1, q_1) = S^pWPAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho_2, q_2)$. \square

Theorem 3.2 *Let $\rho_i, q_i \in \mathcal{U}_\infty$ for $i = 1, 2$. Assume that there exists a constant $0 < \alpha < 1$ and measurable set $A_0 \subseteq \mathbb{R}$ such that*

$$\begin{aligned} &\limsup_{r \rightarrow +\infty} \frac{Q_{\alpha r}(\rho_i)}{Q_r(\rho_i)} < 1, \\ &\lim_{r \rightarrow +\infty} \frac{\int_{A_r \cap A_0} q_i(t) dt}{Q_r(\rho_i)} = 0, \quad \text{where } A_r = \{t \in \mathbb{R} : \alpha r \leq |t - t_0| \leq r\}, \\ &\lim_{r \rightarrow +\infty} \max \left[\sup_{t \in \mathbb{R}} \frac{\rho_1(t)}{\rho_2(t)}, \sup_{t \in A_r \cap A_0^c} \frac{q_2(t)}{q_1(t)}, \sup_{t \in \mathbb{R}} \frac{\rho_2(t)}{\rho_1(t)}, \sup_{t \in A_r \cap A_0^c} \frac{q_1(t)}{q_2(t)} \right] < +\infty, \end{aligned}$$

then $S^pWPAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho_1, q_1) = S^pWPAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho_2, q_2)$.

Proof For any $h^b \in PAA_0(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H})), \rho_1, q_1)$, define

$$H(r) = \frac{1}{Q_r(\rho_2)} \int_{-r}^r \int_t^{t+1} \mathbb{E} \|h(s)\|^p ds q_2(t) dt,$$

then

$$H(r) = \frac{Q_{\alpha r}(\rho_2)}{Q_r(\rho_2)} H(\alpha r) + \frac{1}{Q_r(\rho_2)} \int_{A_r} \int_t^{t+1} \mathbb{E} \|h(s)\|^p ds q_2(t) dt. \tag{3}$$

Further, one obtains

$$\begin{aligned} &\frac{1}{Q_r(\rho_2)} \int_{A_r} \int_t^{t+1} \mathbb{E} \|h(s)\|^p ds q_2(t) dt \\ &\leq \sup_{t \in \mathbb{R}} \frac{\rho_1(t)}{\rho_2(t)} \sup_{t \in A_r \cap A_0^c} \frac{q_2(t)}{q_1(t)} \frac{1}{Q_r(\rho_1)} \int_{A_r \cap A_0^c} \int_t^{t+1} \mathbb{E} \|h(s)\|^p ds q_1(t) dt \\ &\quad + \frac{\|h\|_{S^p}}{Q_r(\rho_2)} \int_{A_r \cap A_0} q_2(t) dt. \end{aligned}$$

By using $\lim_{r \rightarrow +\infty} \sup_{t \in \mathbb{R}} \frac{\rho_1(t)}{\rho_2(t)} \sup_{t \in A_r \cap A_0^c} \frac{q_2(t)}{q_1(t)} < +\infty$, $\lim_{r \rightarrow +\infty} \frac{\int_{A_r \cap A_0} q_2(t) dt}{Q_r(\rho_2)} = 0$ and $h^b \in PAA_0(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H})), \rho_1, q_1)$, it follows that

$$\lim_{r \rightarrow +\infty} \frac{1}{Q_r(\rho_2)} \int_{A_r} \int_t^{t+1} \mathbb{E} \|h(s)\|^p ds q_2(t) dt = 0. \tag{4}$$

According to (3) and (4), one has

$$\limsup_{r \rightarrow +\infty} H(r) \leq \limsup_{r \rightarrow +\infty} \frac{Q_{\alpha r}(\rho_2)}{Q_r(\rho_2)} \limsup_{r \rightarrow +\infty} H(\alpha r).$$

Since $\limsup_{r \rightarrow +\infty} H(r) = \limsup_{r \rightarrow +\infty} H(\alpha r) < +\infty$ and $\limsup_{r \rightarrow +\infty} \frac{Q_{\alpha r}(\rho_2)}{Q_r(\rho_2)} < 1$, therefore

$$\limsup_{r \rightarrow +\infty} H(r) = \limsup_{r \rightarrow +\infty} H(\alpha r) = 0;$$

further, $\lim_{r \rightarrow +\infty} H(r) = 0$, which indicates that $h^b \in PAA_0(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H})), \rho_2, q_2)$ and

$$S^p PAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho_1, q_1) \subseteq S^p PAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho_2, q_2).$$

Similarly, it follows that $S^p PAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho_2, q_2) \subseteq S^p PAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho_1, q_1)$. Based on Definition 2.3, this completes the proof. \square

Corollary 3.1 *Let $\rho_i, q_i \in \mathcal{U}_\infty$ for $i = 1, 2$. Assume that there exist a constant $\alpha > 1$ and measurable set $A_0 \subseteq \mathbb{R}$ such that*

$$\begin{aligned} \limsup_{r \rightarrow +\infty} \frac{Q_r(\rho_i)}{Q_{\alpha r}(\rho_i)} &< 1, \\ \lim_{r \rightarrow +\infty} \frac{\int_{A_r \cap A_0} q_i(t) dt}{Q_{\alpha r}(\rho_i)} &= 0, \quad \text{where } A_r = \{t \in \mathbb{R} : r \leq |t| \leq \alpha r\}, \\ \lim_{r \rightarrow +\infty} \max \left[\sup_{t \in \mathbb{R}} \frac{\rho_1(t)}{\rho_2(t)}, \sup_{t \in A_r \cap A_0^c} \frac{q_2(t)}{q_1(t)}, \sup_{t \in \mathbb{R}} \frac{\rho_2(t)}{\rho_1(t)}, \sup_{t \in A_r \cap A_0^c} \frac{q_1(t)}{q_2(t)} \right] &< +\infty, \end{aligned}$$

then, $S^p WPAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho_1, q_1) = S^p WPAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho_2, q_2)$.

Next, we present the conclusion on the translation invariance.

Theorem 3.3 *Let $\rho, q \in \mathcal{U}_\infty$, and \mathcal{H} be a measurable set, if*

$$\limsup_{|t| \rightarrow +\infty} \frac{\rho(t + \tau)}{\rho(t)} < +\infty, \tag{5}$$

$$\sup_{t \in \mathbb{R}} \max \left\{ \frac{q(t)}{\rho(t)}, \frac{q(t - \tau)}{q(t)} \right\} < +\infty, \tag{6}$$

$$\lim_{r \rightarrow +\infty} \frac{1}{Q_r(t_0, \rho)} \int_{A_m^{r, \tau} \cap \mathcal{H}^c} q(t) dt = 0, \quad \text{where } A_m^{r, \tau} = \{t \in \mathbb{R} : m \leq |t| \leq r + |\tau|\}, \tag{7}$$

then $S^p WPAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$ is translation invariant.

Proof For any $f^b \in PAA_0(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H})), \rho, q)$, denote by $f_\tau(\cdot) = f(\cdot + \tau)$ for any $\tau \in \mathbb{R}$, it follows that

$$\begin{aligned} &\int_{-r}^r \int_t^{t+1} \mathbb{E} \|f_\tau(s)\|^p ds q(t) dt \\ &\leq \int_{-r-|\tau|}^{r+|\tau|} \int_t^{t+1} \mathbb{E} \|f(s)\|^p ds q(t - \tau) dt \\ &\leq \|f\|_{SP} \int_{-m}^m q(t - \tau) dt + \int_{A_m^{r, \tau} \cap \mathcal{H}} \int_t^{t+1} \mathbb{E} \|f(s)\|^p ds q(t - \tau) dt \\ &\quad + \|f\|_{SP} \int_{A_m^{r, \tau} \cap \mathcal{H}^c} q(t - \tau) dt, \end{aligned}$$

therefore

$$\begin{aligned} & \frac{1}{Q_r(\rho)} \int_{-r}^r \int_t^{t+1} \mathbb{E} \|f_\tau(s)\|^p dsq(t) dt \\ & \leq \frac{Q_{r+|\tau|}(\rho)}{Q_r(\rho)} \sup_{t \in \mathbb{R}} \frac{q(t)}{\rho(t)} \frac{\|f\|_{SP}}{Q_{r+|\tau|}(\rho)} \int_{-m-|\tau|}^{m+|\tau|} \rho(t) dt \\ & \quad + \sup_{t \in \mathbb{R}} \frac{q(t-\tau)}{q(t)} \frac{\|f\|_{SP}}{Q_r(\rho)} \int_{A_m^{r,\tau} \cap \mathcal{H}^c} q(t) dt \\ & \quad + \sup_{t \in \mathbb{R}} \frac{q(t-\tau)}{q(t)} \frac{Q_{r+|\tau|}(\rho)}{Q_r(\rho)} \frac{1}{Q_{r+|\tau|}(\rho)} \int_{-r-|\tau|}^{r+|\tau|} \int_t^{t+1} \mathbb{E} \|f(s)\|^p dsq(t) dt. \end{aligned}$$

It is not difficult to show that (5) implies

$$\limsup_{r \rightarrow +\infty} \frac{Q_{r+\tau}(\rho)}{Q_r(\rho)} < +\infty, \tag{8}$$

which, together with (6)–(8) and $f^b \in PAA_0(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H}))), \rho, q$, yields

$$\lim_{r \rightarrow +\infty} \frac{1}{Q_r(\rho)} \int_{-r}^r \int_t^{t+1} \mathbb{E} \|f_\tau(s)\|^p dsq(t) dt = 0,$$

that is, $S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$ is translation invariant. Moreover, the space of $S^pWPAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$ is translation invariant in view of the translation invariance of $S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$.

Denote by $\mathcal{U}_0 := \{\rho \in \mathcal{U}_\infty : \rho \sim \rho^\tau\}$ for any $\tau \in \mathbb{R}$ and $\rho^\tau(t) = \rho(t - \tau)$, then the next result holds. □

Theorem 3.4 *Let $\rho, q \in \mathcal{U}_0$, then $S^pWPAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$ is translation invariant.*

Proof For any $f^b \in PAA_0(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H}))), \rho, q$, denote by $f_\tau(\cdot) = f(\cdot + \tau)$ for any $\tau \in \mathbb{R}$, it follows that

$$\begin{aligned} & \frac{1}{Q_r(\rho)} \int_{-r}^r \int_t^{t+1} \mathbb{E} \|f_\tau(s)\|^p dsq(t) dt \\ & \leq \frac{Q_{r+|\tau|}(\rho^\tau)}{Q_r(\rho)} \frac{1}{Q_{r+|\tau|}(\rho^\tau)} \int_{-r-|\tau|}^{r+|\tau|} \int_t^{t+1} \mathbb{E} \|f(s)\|^p dsq^\tau(t) dt. \end{aligned}$$

From $\rho \in \mathcal{U}_0$, then $\rho \sim \rho^{2\tau}$, therefore

$$\begin{aligned} Q_{r+\tau}(\rho^\tau) &= \int_{-r-\tau}^{r+\tau} \rho(t - \tau) dt \\ &\leq \int_{-r}^{-r+2\tau} \rho^{2\tau}(t) dt + \int_{-r}^r \rho(t) dt \leq (m + 1)Q_r(\rho). \end{aligned}$$

According to $f^b \in PAA_0(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H}))), \rho, q$, it is not difficult to prove $f^b \in PAA_0(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H}))), \rho^\tau, q^\tau$, which implies $f_\tau^b \in PAA_0(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H}))), \rho, q$. Further, $S^pWPAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$ is translation invariant. □

Remark 3.1 For simplicity, denote

$$\mathcal{G}_\infty^* := \{ \rho \mid \rho \in \mathcal{U}_\infty \text{ such that } S^p PAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q) \text{ is translation invariant} \}.$$

4 Existence and uniqueness

Definition 4.1 A $\{\mathcal{F}_t\}_{t \in \mathbb{R}}$ progressively measurable process $\{x(t)\}_{t \in \mathbb{R}}$ is called a mild solution of Eq. (1) if $x(t)$ satisfies

$$\begin{aligned} x(t) &= \mathcal{S}_\kappa(t-s)[x(s) - h(s, x_s)] + \int_s^t (t-u)^{\kappa-1} A_\kappa(t-u)h(u, x_u) du \\ &\quad + h(t, x_t) + \int_s^t (t-u)^{\kappa-1} \mathcal{A}_\kappa(t-u)f_1(u, x_u) du \\ &\quad + \int_s^t (t-u)^{\kappa-1} \mathcal{A}_\kappa(t-u)f_2(u, x_u) dw(u), \end{aligned}$$

for all $t \geq s$ and for each $s \in \mathbb{R}$, where

$$\begin{aligned} \mathcal{S}_\kappa(t) &= \int_0^{+\infty} \zeta_\kappa(\theta)S(t^\kappa\theta) d\theta, & \mathcal{A}_\kappa(t) &= \kappa \int_0^{+\infty} \theta \zeta_\kappa(\theta)S(t^\kappa\theta) d\theta, \\ \zeta_\kappa(\theta) &= \frac{1}{\kappa} \theta^{-1-\frac{1}{\kappa}} \varpi\left(\theta^{-\frac{1}{\kappa}}\right) \geq 0, & \varpi_\kappa(\theta) &= \frac{1}{\pi} \sum_{n=1}^\infty \frac{(-1)^{n-1} \Gamma(\kappa n + 1)}{\theta^{\kappa n + 1} n!} \sin(n\kappa\pi) \end{aligned}$$

and ζ_κ is a probability density function defined on $(0, \infty)$ satisfying $\int_0^{+\infty} \theta \zeta_\kappa(\theta) d\theta = \frac{1}{\Gamma(1+\kappa)}$. Moreover, the next lemma holds.

Lemma 4.1 $\mathcal{S}_\kappa(t)$ and $\mathcal{A}_\kappa(t)$ are strongly continuous for $t \geq 0$ such that

- (1) $\|\mathcal{S}_\kappa(t)\| \leq M, \|\mathcal{A}_\kappa(t)\| \leq \frac{\kappa M}{\Gamma(1+\kappa)}$;
- (2) For any $t > 0$, $\mathcal{S}_\kappa(t)$ and $\mathcal{A}_\kappa(t)$ are compact operators if $S(t)$ is compact;
- (3) For any $0 < \alpha < 1$ and $0 < \beta \leq 1, x \in \mathbb{H}_\alpha := D((-A)^\alpha)$, one has

$$\begin{aligned} -A\mathcal{A}_\kappa(t)x &= (-A)^{1-\alpha} \mathcal{A}_\kappa(t)(-A)^\alpha x, \quad t \geq 0, \\ \|(-A)^\beta \mathcal{A}_\kappa(t)\|_{L(H)} &\leq \frac{\kappa M_\beta \Gamma(2-\beta)}{t^{\kappa\beta} \Gamma(1+\kappa(1-\beta))} e^{-\eta t}, \quad t > 0. \end{aligned}$$

Based on this lemma, the next conclusions hold.

Lemma 4.2 Let $\lambda_1 \in S^p AA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$, then

$$\mathcal{C}_1(t) := \int_{-\infty}^t (t-u)^{\kappa-1} \mathcal{A}_\kappa(t-u)\lambda_1(u) dw(u) \in S^p AA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$$

for $t \in \mathbb{R}$.

Proof Since $\lambda_1 \in S^p AA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$, for any real sequence $\{\tau_n\}_{n \in \mathbb{N}}$, there exists $\{\tau'_n\}_{n \in \mathbb{N}} \subseteq \{\tau_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ and stochastic processes $\tilde{\lambda}_1: \mathbb{R} \rightarrow L^p(\mathbb{P}, \mathbb{H})$ such that

$$\lim_{n \rightarrow +\infty} \left(\int_0^1 \mathbb{E} \|\lambda_1(t+s+\tau_n) - \tilde{\lambda}_1(t+s)\|^p ds \right)^{\frac{1}{p}} = 0. \tag{9}$$

Consider

$$\mathcal{E}_1^*(t) := \int_{-\infty}^t (t-u)^{\kappa-1} \mathcal{A}_\kappa(t-u) \tilde{\lambda}_1(u) dw(u)$$

and denote by $\tilde{w}(t) = w(t + \tau'_n) - w(\tau'_n)$ for $t \in \mathbb{R}$, then \tilde{w} is also a Brownian motion with the same distribution as w , therefore, from the Burkholder–Davis–Gundy inequality, we obtain

$$\begin{aligned} & \mathbb{E} \left\| \mathcal{E}_1(t + \tau'_n) - \mathcal{E}_1^*(t) \right\|^2 \\ & \leq \mathbb{E} \left(\sup_{s \leq t} \left\| \int_{-\infty}^s (s-u)^{\kappa-1} (-A)^{\alpha-1} (-A)^{1-\alpha} \mathcal{A}_\kappa(s-u) \right. \right. \\ & \quad \left. \left. \cdot [\lambda_1(u + \tau'_n) - \tilde{\lambda}_1(u)] d\tilde{w}(u) \right\|^2 \right) \\ & \leq 4M_0^2 \int_{-\infty}^t (t-u)^{2(\kappa\alpha-1)} e^{-2\eta(t-u)} \mathbb{E} \left\| \lambda_1(u + \tau'_n) - \tilde{\lambda}_1(u) \right\|_{\mathbb{L}_2}^2 du \end{aligned}$$

and for $p > 2$ that

$$\begin{aligned} & \mathbb{E} \left\| \mathcal{E}_1(t + \tau'_n) - \mathcal{E}_1^*(t) \right\|^p \\ & \leq \mathbb{E} \left(\sup_{s \leq t} \left\| \int_{-\infty}^s (s-u)^{\kappa-1} (-A)^{\alpha-1} (-A)^{1-\alpha} \mathcal{A}_\kappa(s-u) \right. \right. \\ & \quad \left. \left. \cdot [\lambda_1(u + \tau'_n) - \tilde{\lambda}_1(u)] d\tilde{w}(u) \right\|^p \right) \\ & \leq C_p M_0^p \mathbb{E} \left(\int_{-\infty}^t (t-u)^{2(\kappa\alpha-1)} e^{-2\eta(t-u)} \left\| \lambda_1(u + \tau'_n) - \tilde{\lambda}_1(u) \right\|_{\mathbb{L}_2}^2 du \right)^{\frac{p}{2}} \\ & \leq C_p M_1 \int_{-\infty}^t (t-u)^{2(\kappa\alpha-1)} e^{-2\eta(t-u)} \mathbb{E} \left\| \lambda_1(u + \tau'_n) - \tilde{\lambda}_1(u) \right\|^p du, \end{aligned}$$

where $C_p = [p^{p+1}/2(p-1)^{p-1}]^{p/2}$ and

$$M_0 = \frac{\kappa M_{1-\alpha} \Gamma(1 + \alpha) \|(-A)^{\alpha-1}\|_{\mathbb{L}}}{\Gamma(1 + \kappa\alpha)}, \quad M_1 = M_0^p \left[\frac{\Gamma(2\kappa\alpha - 1)}{(2\eta)^{2\kappa\alpha-1}} \right]^{\frac{p-2}{2}}.$$

Denote by $C_p^* = C_p$ for $p > 2$ and $C_p^* = 4$ for $p = 2$, which yields

$$\begin{aligned} & \mathbb{E} \left\| \mathcal{E}_1(t + \tau'_n) - \mathcal{E}_1^*(t) \right\|^p \\ & \leq C_p^* M_1 \int_0^{+\infty} m^{2(\kappa\alpha-1)} e^{-2\eta m} \mathbb{E} \left\| \lambda_1(t - m + \tau'_n) - \tilde{\lambda}_1(t - m) \right\|^p dm. \end{aligned}$$

From the famous Fubini theorem, we obtain

$$\begin{aligned} & \int_0^1 \mathbb{E} \left\| \mathcal{E}_1(t + s + \tau'_n) - \mathcal{E}_1^*(t + s) \right\|^p ds \\ & \leq C_p^* M_1 \int_0^{+\infty} m^{2(\kappa\alpha-1)} e^{-2\eta m} \int_0^1 \mathbb{E} \left\| \lambda_1(t + s - m + \tau'_n) \right. \\ & \quad \left. - \tilde{\lambda}_1(t + s - m) \right\|^p ds dm. \end{aligned}$$

Based on the translation invariance of $S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$, (9) and the Lebesgue dominated convergence theorem, it follows that

$$\left(\int_0^1 \mathbb{E} \|\mathcal{C}_1(t+s+\tau'_n) - \mathcal{C}_1^*(t+s)\|^p ds \right)^{\frac{1}{p}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Similarly, it follows that $\lim_{n \rightarrow \infty} \left(\int_0^1 \mathbb{E} \|\mathcal{C}_1^*(t+s-\tau'_n) - \mathcal{C}_1(t+s)\|^p ds \right)^{\frac{1}{p}} = 0$. □

Lemma 4.3 *Let $\lambda_2 \in S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}_\alpha))$, then*

$$\mathcal{C}_2(t) := \int_{-\infty}^t (t-u)^{\kappa-1} A \mathcal{A}_\kappa(t-u) \lambda_2(u) du \in S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H})) \quad \text{for } t \in \mathbb{R}.$$

Proof Since $\lambda_2 \in S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$, for any real sequence $\{\tau_n\}_{n \in \mathbb{N}}$, there exists $\{\tau'_n\}_{n \in \mathbb{N}} \subseteq \{\tau_n\}_{n \in \mathbb{N}} \subseteq \mathbb{R}$ and stochastic processes $\tilde{\lambda}_2: \mathbb{R} \rightarrow L^p(\mathbb{P}, \mathbb{H})$ such that

$$\lim_{n \rightarrow +\infty} \left(\int_0^1 \mathbb{E} \|(-A)^\alpha \lambda_2(t+s+\tau'_n) - (-A)^\alpha \tilde{\lambda}_2(t+s)\|^p ds \right)^{\frac{1}{p}} = 0. \tag{10}$$

Assume

$$\mathcal{C}_2^*(t) := \int_{-\infty}^t (t-u)^{\kappa-1} A \mathcal{A}_\kappa(t-u) \tilde{\lambda}_2(u) du,$$

further, it follows that

$$\begin{aligned} & \mathbb{E} \|\mathcal{C}_2(t+\tau'_n) - \mathcal{C}_2^*(t)\|^p \\ & \leq \mathbb{E} \left\| \int_{-\infty}^s (s-u)^{\kappa-1} (-A)^{1-\alpha} \mathcal{A}_\kappa(s-u) [(-A)^\alpha \lambda_2(u+\tau'_n) - (-A)^\alpha \tilde{\lambda}_2(u)] du \right\|^p \\ & \leq \frac{M_0^p}{\|(-A)^{\alpha-1}\|^p} \mathbb{E} \left(\int_{-\infty}^t \frac{(t-u)^{\kappa\alpha-1}}{e^{\eta(t-u)}} \|(-A)^\alpha \lambda_2(u+\tau'_n) - (-A)^\alpha \tilde{\lambda}_2(u)\| du \right)^p \\ & \leq \frac{M_0^p}{\|(-A)^{\alpha-1}\|^p} \left[\frac{\Gamma(\kappa\alpha)}{\eta^{\kappa\alpha}} \right]^{p-1} \int_{-\infty}^t (t-u)^{\kappa\alpha-1} e^{-\eta(t-u)} \\ & \quad \cdot \mathbb{E} \|(-A)^\alpha \lambda_2(u+\tau'_n) - (-A)^\alpha \tilde{\lambda}_2(u)\|^p du \\ & \leq M_2 \int_0^{+\infty} m^{\kappa\alpha-1} e^{-\eta m} \mathbb{E} \|(-A)^\alpha \lambda_2(t-m+\tau'_n) - (-A)^\alpha \tilde{\lambda}_2(t-m)\|^p dm, \end{aligned}$$

where $M_2 = \left[\frac{M_0}{\|(-A)^{\alpha-1}\|} \right]^p \left[\frac{\Gamma(\kappa\alpha)}{\eta^{\kappa\alpha}} \right]^{p-1}$. From the famous Fubini theorem, we obtain

$$\begin{aligned} & \int_0^1 \mathbb{E} \|\mathcal{C}_2(t+s+\tau'_n) - \mathcal{C}_2^*(t+s)\|^p ds \\ & \leq M_2 \int_0^{+\infty} m^{\kappa\alpha-1} e^{-\eta m} \int_0^1 \mathbb{E} \|(-A)^\alpha \lambda_2(t+s-m+\tau'_n) \\ & \quad - (-A)^\alpha \tilde{\lambda}_2(t+s-m)\|^p ds dm. \end{aligned}$$

Based on the translation invariance of $S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}_\alpha))$, (10), and the Lebesgue dominated convergence theorem, it follows that

$$\left(\int_0^1 \mathbb{E} \|\mathcal{C}_2(t+s+\tau'_n) - \mathcal{C}_2^*(t+s)\|^p ds \right)^{\frac{1}{p}} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Similarly, it follows that $\lim_{n \rightarrow \infty} \left(\int_0^1 \mathbb{E} \|\mathcal{C}_2^*(t+s-\tau'_n) - \mathcal{C}_2(t+s)\|^p ds \right)^{\frac{1}{p}} = 0$. □

Corollary 4.1 *Let $\lambda_3 \in S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}_\alpha))$, then*

$$\mathcal{C}_2(t) = \int_{-\infty}^t (t-u)^{\kappa-1} \mathcal{A}_\kappa(t-u) \lambda_3(u) du \in S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$$

for $t \in \mathbb{R}$.

Based on Lemmas 4.2 and 4.3, to establish the existence and uniqueness of a Stepanov-like doubly weighted pseudo almost automorphic mild solution of Eq. (1), the following hypotheses are necessary.

(H1) Assume $\rho, q \in \mathcal{G}_\infty^*$, and $f_i = \phi_i + \psi_i \in S^pWPAA(\mathbb{R} \times \mathbb{B}_\varrho; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$, where $\phi_i \in S^pAA(\mathbb{R} \times \mathbb{B}_\varrho; L^p(\mathbb{P}, \mathbb{H}))$ and $\psi_i^b \in PAA_0(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H})), \rho, q)$, there exists a positive constant L that satisfies

$$\left(\int_t^{t+1} \mathbb{E} \|\phi_i(s, x_s) - \phi_i(s, y_s)\|^p ds \right)^{\frac{1}{p}} \leq L \|x_t - y_t\|_\varrho, \quad t \in \mathbb{R}, i = 1, 2 \tag{11}$$

for any $x, y \in \mathbb{B}_\varrho$. Furthermore, there exist $\gamma \in S^pPAA_0(\mathbb{R}; \mathbb{R}^+)$ and a nondecreasing function $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ that satisfies for all $x \in L^p(\mathbb{P}, \mathbb{B}_\varrho)$ with $\|x\|_\infty \leq \delta$, which yields

$$\int_t^{t+1} \mathbb{E} \|\psi_i(s, x_s)\|^p ds \leq \varphi^p(\delta) \int_t^{t+1} \mathbb{E} \|\gamma(s)\|^p ds \quad \text{and} \quad \liminf_{\delta \rightarrow +\infty} \frac{\varphi(\delta)}{\delta} = b. \tag{12}$$

(H2) Let $\rho, q \in \mathcal{G}_\infty^*$ and $h = h_1 + h_2 \in S^pWPAA(\mathbb{R} \times \mathbb{B}_\varrho; L^p(\mathbb{P}, \mathbb{H}_\alpha), \rho, q)$, where $h_1 \in S^pAA(\mathbb{R} \times \mathbb{B}_\varrho; L^p(\mathbb{P}, \mathbb{H}_\alpha))$ and $h_2^b \in PAA_0(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H})), \rho, q)$, there exists a positive constant L that satisfies

$$\left(\int_t^{t+1} \mathbb{E} \|(-A)^\alpha h_1(s, x_s) - (-A)^\alpha h_1(s, y_s)\|^p ds \right)^{\frac{1}{p}} \leq L \|x_t - y_t\|_\varrho, \quad t \in \mathbb{R} \tag{13}$$

for any $x, y \in \mathbb{B}_\varrho$. Furthermore, there exist $\gamma \in S^pPAA_0(\mathbb{R}; \mathbb{R}^+)$ and a nondecreasing function $\varphi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ that satisfies for all $x \in L^p(\mathbb{P}, \mathbb{B}_\varrho)$ with $\|x\|_\infty \leq \delta$, which satisfies

$$\int_t^{t+1} \mathbb{E} \|(-A)^\alpha h_2(s, x_s)\|^p ds \leq \varphi^p(\delta) \int_t^{t+1} \mathbb{E} \|\gamma(s)\|^p ds \quad \text{and} \tag{14}$$

$$\liminf_{\delta \rightarrow +\infty} \frac{\varphi(\delta)}{\delta} = b.$$

Denote $a = \|\gamma\|_{S^p}$, thus the next result holds.

Theorem 4.1 Assume (H_1) and (H_2) hold and

$$\varrho^*(ab + L)[M_3(1 + \|(-A)^{\alpha-1}\|) + M_4 + \|(-A)^{-\alpha}\|] < 1, \tag{15}$$

where $M_4 = \frac{\sqrt[p]{C_p^* M_{1-\alpha} \Gamma(\alpha) \|(-A)^{\alpha-1}\|}}{\Gamma(\kappa\alpha)} [\frac{\Gamma(2\kappa\alpha-1)}{(2\eta)^{2\kappa\alpha-1}}]^{\frac{1}{2}}$, $M_3 = \frac{M_{1-\alpha} \Gamma(\alpha)}{\eta^{\kappa\alpha}}$, then Eq. (1) admits a unique Stepanov-like doubly weighted pseudo almost automorphic mild solution.

Proof According to (15), there exists a constant $d > 0$ that satisfies

$$\varrho^*[a\varphi(d + \|\omega\|_\varrho) + Ld][M_3(1 + \|(-A)^{\alpha-1}\|) + M_4 + \|(-A)^{-\alpha}\|] \leq d. \tag{16}$$

To finish the proof, we will complete it in several steps.

Step 1. For above $d > 0$, let

$$\mathfrak{S}_d := \{ \xi \in S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q) : \|\xi_t\|_\varrho \leq d \},$$

obviously, \mathfrak{S}_d is a bounded closed and convex subset of $S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$. For any $\omega \in S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$, $\xi \in \mathfrak{S}_d$, assume the operator $(\mathcal{B}\xi)(t) = \sum_{i=1}^6 (\mathcal{B}_i\xi)(t)$, where

$$(\mathcal{B}_1\xi)(t) = h_2(t, \omega_t + \xi_t) + \int_{-\infty}^t (t-u)^{\kappa-1} A \mathcal{A}_\kappa(t-u) h_2(u, \omega_u + \xi_u) du,$$

$$(\mathcal{B}_2\xi)(t) = [h_1(t, \omega_t + \xi_t) - h_1(t, \omega_t)] + \int_{-\infty}^t (t-u)^{\kappa-1} A \mathcal{A}_\kappa(t-u) [h_1(u, \omega_u + \xi_u) - h_1(u, \omega_u)] du,$$

$$(\mathcal{B}_3\xi)(t) = \int_{-\infty}^t (t-u)^{\kappa-1} \mathcal{A}_\kappa(t-u) \psi_1(u, \omega_u + \xi_u) du,$$

$$(\mathcal{B}_4\xi)(t) = \int_{-\infty}^t (t-u)^{\kappa-1} \mathcal{A}_\kappa(t-u) [\phi_1(u, \omega_u + \xi_u) - \phi_1(u, \omega_u)] du,$$

$$(\mathcal{B}_5\xi)(t) = \int_{-\infty}^t (t-u)^{\kappa-1} \mathcal{A}_\kappa(t-u) \psi_2(u, \omega_u + \xi_u) dw(u),$$

$$(\mathcal{B}_6\xi)(t) = \int_{-\infty}^t (t-u)^{\kappa-1} \mathcal{A}_\kappa(t-u) [\phi_2(u, \omega_u + \xi_u) - \phi_2(u, \omega_u)] dw(u).$$

By applying (11)–(14), we deduce that

$$\left(\int_t^{t+1} \mathbb{E} \|(-A)^\alpha h_1(u, \omega_u + \xi_u) - (-A)^\alpha h_1(u, \omega_u)\|^p du \right)^{\frac{1}{p}} \leq L \|\xi_t\|_\varrho, \tag{17}$$

$$\int_t^{t+1} \mathbb{E} \|(-A)^\alpha h_2(u, \omega_u + \xi_u)\|^p du \leq \varphi^p(d + \|\omega\|_\varrho) \int_t^{t+1} \mathbb{E} \|\gamma(s)\|^p ds, \tag{18}$$

and

$$\left(\int_t^{t+1} \mathbb{E} \|\phi_i(u, \omega_u + \xi_u) - \phi_i(u, \omega_u)\|^p du \right)^{\frac{1}{p}} \leq L \|\xi_t\|_\varrho, \tag{19}$$

$$\int_t^{t+1} \mathbb{E} \|\psi_i(u, \omega_u + \xi_u)\|^p du \leq \varphi^p(d + \|\omega\|_\varrho) \int_t^{t+1} \mathbb{E} \|\gamma(s)\|^p ds, \tag{20}$$

which indicates based on $\gamma \in S^pPAA_0(\mathbb{R}; \mathbb{R}^+)$ and $\xi \in S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$ that $(-A)^\alpha h_1(u, \omega_u + \xi_u) - (-A)^\alpha h_1(u, \omega_u) \in S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$, $\psi_i(u, \omega_u + \xi_u) \in S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$, $(-A)^\alpha h_2(u, \omega_u + \xi_u) \in S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$, and $\phi_i(u, \omega_u + \xi_u) - \phi_i(u, \omega_u) \in S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$.

Let $\Psi(t) = \phi_2(t, \omega_t + \xi_t) - \phi_2(t, \omega_t)$, then $\Psi^b(\cdot) \in PAA_0(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H})), \rho, q)$, further

$$\begin{aligned} & \lim_{r \rightarrow +\infty} \frac{1}{Q(r, \rho)} \int_{-r}^r \int_t^{t+1} \mathbb{E} \|(\mathcal{B}_6 \xi)(s)\|^p ds q(t) dt \\ & \leq \lim_{r \rightarrow +\infty} \frac{1}{Q(r, \rho)} \int_{-r}^r \int_t^{t+1} \mathbb{E} \left(\sup_{m \leq s} \left\| \int_{-\infty}^m (m-u)^{\kappa-1} (-A)^{\alpha-1} (-A)^{1-\alpha} \right. \right. \\ & \quad \left. \left. \cdot \mathcal{A}_\kappa(m-u) \Psi(u) dw(u) \right\|^p \right) dm q(t) dt \\ & \leq C_p^* M_1 \lim_{r \rightarrow +\infty} \frac{1}{Q(r, \rho)} \int_{-r}^r \int_t^{t+1} \int_{-\infty}^s \frac{(s-u)^{2(\kappa\alpha-1)}}{e^{2\eta(s-u)}} \mathbb{E} \|\Psi(u)\|^p du ds q(t) dt \\ & \leq C_p^* M_1 \int_0^{+\infty} \frac{m^{2(\kappa\alpha-1)}}{e^{2\eta m}} \lim_{r \rightarrow +\infty} \frac{1}{Q(r, \rho)} \int_{-r}^r \int_0^1 \mathbb{E} \|\Psi(t+\tau-m)\|^p d\tau q(t) dt dm, \end{aligned}$$

where C_p^*, M_1 defined as in Lemma 4.3. Combining the Lebesgue dominated convergence theorem with $\Psi^b(\cdot) \in PAA_0(\mathbb{R}; L^p(0, 1; L^p(\mathbb{P}, \mathbb{H})), \rho, q)$, we deduce

$$\lim_{r \rightarrow +\infty} \frac{1}{Q(r, \rho)} \int_{-r}^r \int_t^{t+1} \mathbb{E} \|(\mathcal{B}_6 \xi)(s)\|^p ds q(t) dt = 0,$$

that is, $(\mathcal{B}_6 \xi)(\cdot) \in S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$. Taking a similar argument, we obtain that $(\mathcal{B}_i \xi)(\cdot) \in S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$ for $i = 1, \dots, 5$.

Let $(\Lambda_1 \xi)(t) = (\mathcal{B}_1 \xi)(t) + (\mathcal{B}_3 \xi)(t) + (\mathcal{B}_5 \xi)(t)$, $(\Lambda_2 \xi)(t) = (\mathcal{B}_2 \xi)(t) + (\mathcal{B}_4 \xi)(t) + (\mathcal{B}_6 \xi)(t)$ for $t \in \mathbb{R}$, obviously, this gives $(\Lambda_i \xi)(\cdot) \in S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$ for $i = 1, 2$, based on the Burkholder–Davis–Gundy inequality, we obtain

$$\begin{aligned} & \|(\mathcal{B}_2 \xi)^b(t, m)\|_p \\ & \leq \|(-A)^{-\alpha} \left[\int_t^{t+1} \mathbb{E} \|(-A)^\alpha h_1(m, \omega_m + \xi_m) - (-A)^\alpha h_1(m, \omega_m)\|^p dm \right]^{\frac{1}{p}} \\ & \quad + \left[M_2 \int_0^{+\infty} r^{\kappa\alpha-1} e^{-\eta r} \int_t^{t+1} \mathbb{E} \|(-A)^\alpha h_1(m-r, \omega_{m-r} + \xi_{m-r}) \right. \\ & \quad \left. - (-A)^\alpha h_1(m-r, \omega_{m-r})\|^p dm dr \right]^{\frac{1}{p}} \\ & \leq Ld \|(-A)^{-\alpha}\| + Ld \left[M_2 \int_0^{+\infty} r^{\kappa\alpha-1} e^{-\eta r} dr \right]^{\frac{1}{p}} \\ & \leq Ld (\|(-A)^{-\alpha}\| + M_3), \\ & \|(\mathcal{B}_4 \xi)^b(t, m)\|_p + \|(\mathcal{B}_6 \xi)^b(t, m)\|_p \\ & \leq \|(-A)^{\alpha-1} \left[M_2 \int_0^{+\infty} r^{\kappa\alpha-1} e^{-\eta r} \int_t^{t+1} \mathbb{E} \|\psi_1^*(m-r, \omega_{m-r}, \xi_{m-r})\|^p dm dr \right]^{\frac{1}{p}} \end{aligned}$$

$$\begin{aligned}
 & + \left[C_p^* M_1 \int_0^{+\infty} r^{2(\kappa\alpha-1)} e^{-2\eta r} \int_t^{t+1} \mathbb{E} \|\psi_2^*(m-r, \omega_{m-r}, \xi_{m-r})\|^p dm dr \right]^{\frac{1}{p}} \\
 & \leq Ld \|(-A)^{\alpha-1}\| \left[M_2 \int_0^{+\infty} r^{\kappa\alpha-1} e^{-\eta r} dr \right]^{\frac{1}{p}} \\
 & \quad + Ld \left[C_p^* M_1 \int_0^{+\infty} r^{2(\kappa\alpha-1)} e^{-2\eta r} dr \right]^{\frac{1}{p}} \leq Ld [M_3 \|(-A)^{\alpha-1}\| + M_4],
 \end{aligned}$$

where $\psi_i^*(m-r, \omega_{m-r}, \xi_{m-r}) = \psi_i(m-r, \omega_{m-r} + \xi_{m-r}) - \psi_i(m-r, \omega_{m-r})$ for $i = 1, 2$, $M_4 = \frac{p \sqrt[p]{C_p^* M_1 - \alpha} \Gamma(\alpha) \|(-A)^{\alpha-1}\|}{\Gamma(\kappa\alpha)} \left[\frac{\Gamma(2\kappa\alpha-1)}{(2\eta)^{2\kappa\alpha-1}} \right]^{\frac{1}{2}}$, $M_3 = \frac{M_{1-\alpha} \Gamma(\alpha)}{\eta^{\kappa\alpha}}$. Therefore,

$$\|(\Lambda_2 \xi)(m)\|_{S^p} \leq Ld [M_3 (1 + \|(-A)^{\alpha-1}\|) + M_4 + \|(-A)^{-\alpha}\|].$$

Analogously, this gives

$$\|(\Lambda_1 \xi)(m)\|_{S^p} \leq a\varphi(d + \|\omega\|_e) [M_3 (1 + \|(-A)^{\alpha-1}\|) + M_4 + \|(-A)^{-\alpha}\|].$$

From (16), it follows that $\|\Lambda_i \xi\|_e \leq d$, furthermore, Λ_i maps \mathfrak{S}_d into \mathfrak{S}_d for $i = 1, 2$.

Step 2. Λ_2 is a contraction mapping and Λ_1 is completely continuous on \mathfrak{S}_d .

For any $\hat{\xi}, \tilde{\xi} \in \mathfrak{S}_d$, we obtain

$$\begin{aligned}
 & \|(\Lambda_2 \tilde{\xi})(t) - (\Lambda_2 \hat{\xi})(t)\|_p \\
 & \leq \|(\mathcal{B}_2 \tilde{\xi})(t) - (\mathcal{B}_2 \hat{\xi})(t)\|_p + \|(\mathcal{B}_4 \tilde{\xi})(t) - (\mathcal{B}_4 \hat{\xi})(t)\|_p + \|(\mathcal{B}_6 \tilde{\xi})(t) - (\mathcal{B}_6 \hat{\xi})(t)\|_p \\
 & \leq \|(-A)^{-\alpha}\| \left[\int_t^{t+1} \mathbb{E} \|(-A)^\alpha h_1(m, \omega_m + \tilde{\xi}_m) - (-A)^\alpha h_1(m, \omega_m + \hat{\xi}_m)\|^p dm \right]^{\frac{1}{p}} \\
 & \quad + \left[M_2 \int_0^{+\infty} r^{\kappa\alpha-1} e^{-\eta r} \int_t^{t+1} \mathbb{E} \|(-A)^\alpha h_1(m-r, \omega_{m-r} + \tilde{\xi}_{m-r}) \right. \\
 & \quad \left. - (-A)^\alpha h_1(m-r, \omega_{m-r} + \hat{\xi}_{m-r})\|^p dm dr \right]^{\frac{1}{p}} \\
 & \quad + \|(-A)^{\alpha-1}\| \left[M_2 \int_0^{+\infty} r^{\kappa\alpha-1} e^{-\eta r} \int_t^{t+1} \mathbb{E} \|\psi_1(m-r, \omega_{m-r} + \tilde{\xi}_{m-r}) \right. \\
 & \quad \left. - \psi_1(m-r, \omega_{m-r} + \hat{\xi}_{m-r})\|^p dm dr \right]^{\frac{1}{p}} \\
 & \quad + \left[C_p^* M_1 \int_0^{+\infty} r^{2(\kappa\alpha-1)} e^{-2\eta r} \int_t^{t+1} \mathbb{E} \|\psi_2(m-r, \omega_{m-r} + \tilde{\xi}_{m-r}) \right. \\
 & \quad \left. - \psi_2(m-r, \omega_{m-r} + \hat{\xi}_{m-r})\|^p dm dr \right]^{\frac{1}{p}} \\
 & \leq LQ^* \|(-A)^{-\alpha}\| \|\tilde{\xi} - \hat{\xi}\|_{S^p} \\
 & \quad + L(1 + \|(-A)^{\alpha-1}\|) \left[M_2 \int_0^{+\infty} r^{\kappa\alpha-1} e^{-\eta r} \|\tilde{\xi}_{m-r} - \hat{\xi}_{m-r}\|_e^p dr \right]^{\frac{1}{p}}
 \end{aligned}$$

$$\begin{aligned}
 &+ L \left[C_p^* M_1 \int_0^{+\infty} r^{2(\kappa\alpha-1)} e^{-2\eta r} \|\tilde{\xi}_{m-r} - \bar{\xi}_{m-r}\|_{\mathbb{Q}}^p dr \right]^{\frac{1}{p}} \\
 &\leq L_{\mathbb{Q}}^* [M_3(1 + \|(-A)^{\alpha-1}\|) + M_4 + \|(-A)^{-\alpha}\|] \|\tilde{\xi} - \bar{\xi}\|_{S^p},
 \end{aligned}$$

and based on (15), we have

$$\begin{aligned}
 &\|(\Lambda_2 \tilde{\xi})(t) - (\Lambda_2 \bar{\xi})(t)\|_{S^p} \\
 &\leq L_{\mathbb{Q}}^* [M_3(1 + \|(-A)^{\alpha-1}\|) + M_4 + \|(-A)^{-\alpha}\|] \|\tilde{\xi} - \bar{\xi}\|_{S^p} < \|\tilde{\xi} - \bar{\xi}\|_{S^p}.
 \end{aligned}$$

Since for any $\xi_0 \in \mathfrak{S}_d$, this yields $\|(\Lambda_1 \xi)\|_{\mathbb{Q}} \leq d$, therefore, Λ_1 is uniformly bounded. Based on the Arzela–Ascoli theorem, it is not difficult to derive that Λ_1 is compact, further, Λ_1 is completely continuous on \mathfrak{S}_d .

Step 3. Let \mathcal{H} on $S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$ satisfy

$$\begin{aligned}
 (\mathcal{H}\omega)(t) &= \int_{-\infty}^t (t-u)^{\kappa-1} A \mathcal{A}_{\kappa}(t-u) h_1(u, \omega_u) du \\
 &+ \int_{-\infty}^t (t-u)^{\kappa-1} \mathcal{A}_{\kappa}(t-u) \phi_1(u, \omega_u) du \\
 &+ \int_{-\infty}^t (t-u)^{\kappa-1} \mathcal{A}_{\kappa}(t-u) \phi_2(u, \omega_u) dW(u) + h_1(t, \omega_t).
 \end{aligned}$$

From $\omega \in S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$ and $h_1 \in S^pAA(\mathbb{R} \times \mathbb{B}_{\mathbb{Q}}; L^p(\mathbb{P}, \mathbb{H}))$, we can extract a real sequence $\{\tau_n\}_{n \in \mathbb{N}}$ such that stochastic processes $\tilde{\omega}: \mathbb{R} \rightarrow L^p(\mathbb{P}, \mathbb{H})$ and $\tilde{h}_1: \mathbb{R} \times \mathbb{B}_{\mathbb{Q}} \rightarrow L^p(\mathbb{P}, \mathbb{H})$ such that

$$\lim_{n \rightarrow +\infty} \left(\int_0^1 \mathbb{E} \|\omega(t+s+\tau_n) - \tilde{\omega}(t+s)\|^p ds \right)^{\frac{1}{p}} = 0 \tag{21}$$

and

$$\lim_{n \rightarrow +\infty} \left(\int_0^1 \mathbb{E} \|h_1(t+s+\tau_n, x) - \tilde{h}_1(t+s, x)\|^p ds \right)^{\frac{1}{p}} = 0, \tag{22}$$

where $x \in \mathbb{B}_{\mathbb{Q}}$, therefore

$$\begin{aligned}
 &\|(-A)^{\alpha} h_1^b(u + \tau_n, \omega_{u+\tau_n}) - (-A)^{\alpha} \tilde{h}_1^b(u, \tilde{\omega}_u)\|_p \\
 &\leq \|(-A)^{\alpha} h_1^b(u + \tau_n, \omega_{u+\tau_n}) - (-A)^{\alpha} h_1^b(u + \tau_n, \tilde{\omega}_u)\|_p \\
 &\quad + \|(-A)^{\alpha} h_1^b(u + \tau_n, \tilde{\omega}_u) - (-A)^{\alpha} \tilde{h}_1^b(u, \tilde{\omega}_u)\|_p \\
 &\leq L_{\mathbb{Q}}^* \|\omega_{t+\tau_n} - \tilde{\omega}_t\|_{S^p} + \|(-A)^{\alpha} h_1(u + \tau_n, \tilde{\omega}_u) - (-A)^{\alpha} \tilde{h}_1(u, \tilde{\omega}_u)\|_p = 0,
 \end{aligned}$$

which indicates from (21) and (22) that

$$\lim_{n \rightarrow +\infty} \|(-A)^{\alpha} h_1(u + \tau_n, \omega_{u+\tau_n}) - (-A)^{\alpha} \tilde{h}_1(u, \tilde{\omega}_u)\|_p = 0.$$

Further, we have $h_1(t, \omega_t) \in S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}_{\alpha}))$ for $t \in \mathbb{R}$. Analogously, this gives $\phi_i(t, \omega_t) \in S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$ for $i = 1, 2$. Denote by $\lambda_1(\cdot) = \phi_2(\cdot, \omega)$, $\lambda_2(\cdot) = h_1(\cdot, \omega)$, $\lambda_3(\cdot) = \phi_1(\cdot, \omega)$,

based on Lemmas 4.2 and 4.3 and Corollary 4.1, we deduce that \mathcal{H} maps $S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$ into itself.

Next, we prove \mathcal{H} is a contraction mapping on $S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$. For any $\tilde{\omega}, \bar{\omega} \in \mathbb{B}_\rho$, similar to the proof of Step 2, it follows that

$$\begin{aligned} & \|(\mathcal{H}\tilde{\omega})(t) - (\mathcal{H}\bar{\omega})(t)\|_p \\ & \leq L(1 + \|(-A)^{\alpha-1}\|) \left[M_2 \int_0^{+\infty} r^{\kappa\alpha-1} e^{-\eta r} \|\tilde{\omega}_{m-r} - \bar{\omega}_{m-r}\|_\rho^p dr \right]^{\frac{1}{p}} \\ & \quad + L \left[C_p^* M_1 \int_0^{+\infty} r^{2(\kappa\alpha-1)} e^{-2\eta r} \|\tilde{\omega}_{m-r} - \bar{\omega}_{m-r}\|_\rho^p dr \right]^{\frac{1}{p}} \\ & \quad + L \|(-A)^{-\alpha}\| \|\tilde{\omega} - \bar{\omega}\|_\rho \\ & \leq L \rho^* [M_3(1 + \|(-A)^{\alpha-1}\|) + M_4 + \|(-A)^{-\alpha}\|] \|\tilde{\omega} - \bar{\omega}\|_{S^p}, \end{aligned}$$

and based on (15), we have

$$\begin{aligned} & \|(\mathcal{H}\tilde{\omega}) - (\mathcal{H}\bar{\omega})\|_{S^p} \\ & \leq L \rho^* [M_3(1 + \|(-A)^{\alpha-1}\|) + M_4 + \|(-A)^{-\alpha}\|] \|\tilde{\omega} - \bar{\omega}\|_{S^p} < \|\tilde{\omega} - \bar{\omega}\|_{S^p}. \end{aligned}$$

From what has been discussed above, based on the results of step 1, step 2, and the Krasnoselskii fixed-point theorem, there exists a fixed point $\xi^* \in S^pPAA_0(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$. Combining step 3 with the Banach fixed-point theorem, it follows that \mathcal{H} admits a unique fixed point ω^* in $S^pAA(\mathbb{R}; L^p(\mathbb{P}, \mathbb{H}))$. Consider the coupled system

$$\left\{ \begin{aligned} \omega^*(t) &= \int_{-\infty}^t (t-u)^{\kappa-1} A \mathcal{A}_\kappa(t-u) h_1(u, \omega_u^*) du \\ & \quad + \int_{-\infty}^t (t-u)^{\kappa-1} \mathcal{A}_\kappa(t-u) \phi_1(u, \omega_u^*) du \\ & \quad + \int_{-\infty}^t (t-u)^{\kappa-1} \mathcal{A}_\kappa(t-u) \phi_2(u, \omega_u^*) dw(u) + h_1(t, \omega_t^*), \\ \xi^*(t) &= h_2(t, \omega_t^* + \xi_t^*) + \int_{-\infty}^t (t-u)^{\kappa-1} A \mathcal{A}_\kappa(t-u) h_2(u, \omega_u^* + \xi_u^*) du \\ & \quad + [h_1(t, \omega_t^* + \xi_t^*) - h_1(t, \omega_t^*)] \\ & \quad + \int_{-\infty}^t (t-u)^{\kappa-1} A \mathcal{A}_\kappa(t-u) [h_1(u, \omega_u^* + \xi_u^*) - h_1(u, \omega_u^*)] du \\ & \quad + \int_{-\infty}^t (t-u)^{\kappa-1} \mathcal{A}_\kappa(t-u) \psi_1(u, \omega_u^* + \xi_u^*) du \\ & \quad + \int_{-\infty}^t (t-u)^{\kappa-1} \mathcal{A}_\kappa(t-u) [\phi_1(u, \omega_u^* + \xi_u^*) - \phi_1(u, \omega_u^*)] du \\ & \quad + \int_{-\infty}^t (t-u)^{\kappa-1} \mathcal{A}_\kappa(t-u) \psi_2(u, \omega_u^* + \xi_u^*) dw(u) \\ & \quad + \int_{-\infty}^t (t-u)^{\kappa-1} \mathcal{A}_\kappa(t-u) [\phi_2(u, \omega_u^* + \xi_u^*) - \phi_2(u, \omega_u^*)] dw(u); \end{aligned} \right.$$

further, $x^*(t) = \omega^*(t) + \xi^*(t) \in S^pWPAA(\mathbb{R} \times \mathbb{B}_\rho; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$, which is a Stepanov-like doubly weighted pseudo almost automorphic mild solution of (1).

Substituting the assumptions (H_1) and (H_2) for the following (H_1^*) and (H_2^*) , respectively, that is

(H₁^{*}) Assume $\rho, q \in \mathcal{G}_\infty^*$, and $f_i \in S^p WPAA(\mathbb{R} \times \mathbb{B}_\rho; L^p(\mathbb{P}, \mathbb{H}), \rho, q)$, there exists a positive constant L that satisfies

$$\left(\int_t^{t+1} \mathbb{E} \|f_i(s, x_s) - f_i(s, y_s)\|^p ds \right)^{\frac{1}{p}} \leq L \|x_t - y_t\|_\rho, \quad t \in \mathbb{R}, i = 1, 2$$

for any $x, y \in \mathbb{B}_\rho$.

(H₂^{*}) Let $\rho, q \in \mathcal{G}_\infty^*$, and $h \in S^p WPAA(\mathbb{R} \times \mathbb{B}_\rho; L^p(\mathbb{P}, \mathbb{H}_\alpha), \rho, q)$, there exists a positive constant L that satisfies

$$\left(\int_t^{t+1} \mathbb{E} \|(-A)^\alpha h(s, x_s) - (-A)^\alpha h(s, y_s)\|^p ds \right)^{\frac{1}{p}} \leq L \|x_t - y_t\|_\rho, \quad t \in \mathbb{R},$$

for any $x, y \in \mathbb{B}_\rho$.

Similar to the discussion in Theorem 4.1, by utilizing the Banach fixed-point theorem, it is not difficult to show the next conclusion holds. □

Corollary 4.2 *Let (H₁^{*}) and (H₂^{*}) hold. Then, Eq. (1) admits a unique Stepanov-like doubly weighted pseudo almost automorphic mild solution provided that*

$$\varrho^* L [M_3(1 + \|(-A)^{\alpha-1}\|) + M_4 + \|(-A)^{-\alpha}\|] < 1, \tag{23}$$

where $M_4 = \frac{p \sqrt[p]{C_p^* M_{1-\alpha}} \Gamma(\alpha) \|(-A)^{\alpha-1}\|}{\Gamma(\kappa\alpha)} \left[\frac{\Gamma(2\kappa\alpha-1)}{(2\eta)^{2\kappa\alpha-1}} \right]^{\frac{1}{2}}$, $M_3 = \frac{M_{1-\alpha} \Gamma(\alpha)}{\eta^{\kappa\alpha}}$.

Remark 4.1 By comparing Theorem 4.1 and Corollary 4.2, it is obvious that the condition (15) is more accurate than (23), which indicates the discussion and computation in Theorem 4.1 based on the Krasnoselskii fixed-point theorem is more complex and challenging; therefore, Theorem 4.1 is significant compared to the relevant existence and uniqueness of the Stepanov-like doubly weighted pseudo almost automorphic mild solution by using the Banach fixed-point theorem.

Example 4.1 Consider the following special one-dimensional stochastic neutral differential equation of the form

$$\begin{cases} \partial_t^\kappa [x(t, \zeta) - h(t, x(t - \tau(t), \zeta))] = \partial_\zeta^2 x(t, \zeta) + f_1(t, x(t - \tau(t), \zeta)) \\ \quad + f_2(t, x(t - \tau(t), \zeta)) \frac{dw(t)}{dt}, \\ x(t, 0) = x(t, 1) = 0, \quad t \in \mathbb{R}, \end{cases} \tag{24}$$

where

$$\begin{aligned} h(t, x(t - \tau(t), \zeta)) &= \frac{1}{100} \sin\left(\frac{1}{1 + \cos t + \cos \sqrt{2}t}\right) \cos x(t - \tau, \zeta) \\ &\quad + e^{-|t|} x(t, \zeta) \sin^2 x(t, \zeta), \\ f_i(t, x(t - \tau(t), \zeta)) &= \frac{1}{100} \sin\left(\frac{1}{2 + \cos t + \cos \sqrt{3}t}\right) \cos x(t - \tau, \zeta) \\ &\quad + e^{-|t|} x(t, \zeta) \sin^2 x(t, \zeta). \end{aligned}$$

Let $\mathbb{H} = L^2[0, 1]$ and $A: D(A) \subseteq \mathbb{H} \rightarrow \mathbb{H}$ with $(Ax)(\zeta) = x''(\zeta)$, A is an infinitesimal generator of an analytic semigroup $\{S(t)\}_{t \geq 0}$ such that $\|S(t)\| \leq e^{-t}$ and

$$(-A)^{\frac{1}{2}}x = \sum_{n=0}^{\infty} n \langle x, x_n \rangle x_n, x \in D((-A)^{\frac{1}{2}}) = \left\{ x \in \mathbb{H} : \sum_{n=0}^{\infty} n \langle x, x_n \rangle x_n \in \mathbb{H} \right\}.$$

Assume

$$\rho(t) = q(t) = \begin{cases} 1, & t > 0, \\ e^{-t^2}, & t \leq 0, \end{cases}$$

then Eq. (24) can be formulated in abstract form as Eq. (1) and the conditions (H_1) and (H_2) hold, where $L = \frac{1}{100}$, $\varphi(\delta) = \delta$, $b = 1$, $\gamma = e^{-|t|}$, $\kappa = \alpha = \frac{1}{2}$, $\|(-A)^{-\frac{1}{2}}\| = 1$, $\Gamma(\frac{1}{2}) = \sqrt{\pi}$, $M = \eta = 1$, it follows that (24) admits a unique square-mean Stepanov-like doubly weighted pseudo almost automorphic mild solution.

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