On the Origins of Cantor's Paradox: What Hilbert Left Unsaid at the 1900 ICM in Paris

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The first two of the twenty-three unsolved problems that David Hilbert famously proposed at the 1900 International Congress of Mathematicians (ICM) in 1900 dealt with issues associated with the real number continuum. The first problem concerned Cantor's continuum hypothesis, whereas the second dealt with Hilbert's attempt to establish the existence of the continuum by proving the consistency of his axioms for characterizing its properties. Few have noted, however, that Hilbert himself linked the larger goals of Cantor's theory of transfinite arithmetic with those of his own program for axiomatization. By carefully recounting Hilbert's interactions with Cantor from 1897 onward, this paper shows how Hilbert's understanding of "Cantor's paradox" influenced the views he expressed at the 1900 ICM.

Hilbert's First Two Paris Problems

Georg Cantor loved to talk about his latest mathematical ideas with anyone who would listen. One of those who did was David Hilbert, who probably met Cantor for the first time in Bremen in September 1890 at the annual meeting of the German Society of Natural Scientists and Physicians. Mathematicians rarely participated at these conferences, but several attended that year in answer to Cantor's well-publicized effort to enlist their support for an important undertaking. During the preceding year, he and his allies had prodded their fellow colleagues to join them in Bremen in order to discuss plans for founding a national mathematical society. Over thirty answered that call—including Hilbert, Hermann Minkowski, and Felix Klein—and this group succeeded in launching today's Deutsche Mathematiker-Vereinigung (DMV, or German Mathematical Society). None was more eager to do so than Hilbert.

Having grown up in the remotely located East Prussian city of Köngisberg, where he studied and taught at its "Albertina" (as the Albertus-Universität Königsberg was affectionately known), Hilbert was acutely aware that the DMV could play a critically important social function. Already before its founding, he freely offered this opinion to Klein by way of throwing his support behind such an organization:

> It seems to me that the mathematicians of today understand each other far too little and that they do not take an intense enough interest in one another. They also seem to know—so far as I can judge—too little of our classical authors [Klassiker]; many, moreover, spend much effort working on dead ends [21, p. 68].[1]

Hilbert also fully recognized the importance of reaping the kinds of rewards that only an organization like the DMV could bestow. One year later, Cantor hosted the DMV's inaugural meeting in Halle, on which occasion he unveiled one of his most famous ideas: the diagonal argument for proving that the set of real numbers is not countably infinite [5]. In fact, he showed that for any set M, the set of all its subsets, its power set $\mathscr{P}(M)$, has a greater cardinality than that of M. Ernst Zermelo, who introduced the axiom of power sets as part of the first axiomatization of set theory, would later dub this Cantor's theorem [56, p. 276].

A decade later, international congresses had become another standard venue at which mathematicians could celebrate their collective achievements and hawk their latest wares. At the first ICM, held in Zurich in 1897, Cantor took pleasure in hearing Adolf Hurwitz expound on the importance of set theory for recent research in analysis. Such praise had been late in coming, but on this occasion as well at the next two ICMs, in Paris (1900) and Heidelberg (1904), Cantor's name and his longstanding agenda were on prominent display. In Zurich he had the pleasure of meeting Émile Borel for the first time and informing him of Felix Bernstein's recent proof of the Cantor–Bernstein theorem. Borel was then preparing a textbook on function theory, to which he added an extensive note on Cantor's theory of cardinal numbers, only part of which he found to be solidly grounded [3, pp. 102–110]. This included Bernstein's proof, which Cantor communicated to him either during or after their encounter in Zurich.

Cantor had also originally planned to visit Paris for the second ICM, in 1900, but ultimately he decided not to attend, presumably due to illness. On that occasion, Hilbert underscored the importance of Cantor's novel ideas in his remarkable lecture entitled simply "Mathematical Problems."[2] He did so by describing what still remained to be proved, namely Cantor's conjectures about well ordering and the continuum hypothesis. Together these formed the

[1] All translations in the paper from original German sources are mine.

[2] In the congress proceedings, the text appeared in French translation under the title "Sur les Problèmes futurs des Mathématiques" [14, pp. 58–114].

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first of Hilbert's 23 Paris problems,[3] whereas the second dealt with Hilbert's own ideas for axiomatizing the real number continuum. For that problem, Hilbert claimed one could legitimize the mathematical existence of the real numbers—defined by the axioms for a complete, ordered Archimedean field—by proving that this system of axioms was consistent.

These three challenges—well ordering, consistency of arithmetic, and the continuum hypothesis—were among the most famous foundational problems ever posed, and their solutions evoke equally famous names: Ernst Zermelo, Kurt Gödel, and Paul Cohen.[4] Neither Cantor nor Hilbert foresaw the eventual outcomes, of course, it was far too early in the game, and their hunches were completely wrong.[5] But what ideas were in the air when Hilbert spoke in 1900 and why did he come to think of the first two Paris problems as strongly linked? To what extent were his ideas influenced by Cantor's own views and in what ways did he depart from them?

José Ferreirós has argued forcefully that Hilbert's early ideas relating to set theory and logic were closely aligned with those he found on reading Richard Dedekind's *Was sind und was sollen die Zahlen?* [9].[6] He cites Hilbert's reply to Cantor, who had alerted him that the class of all alephs was not a set [18, p. 51], a claim Hilbert initially resisted:

> The collection of all alephs can be conceived as a definite well-defined set, for certainly if any thing is given, it must always be possible to determine whether this thing is an aleph or not; and nothing more belongs to a well-defined set [38, p. 390].

As Ferreirós emphasizes:

> Dedekind would not have answered differently: it is enough that the *concept* of an aleph be well defined, this suffices (by the unstated comprehension principle) for the corresponding set to exist. Hilbert's answer is thus a perfect example of the dichotomic conception of sets [18, p. 51].

By the time Hilbert wrote his text for the 1900 Paris ICM, however, he had clearly accepted Cantor's view of the situation, as will be shown below in some detail.

Hilbert's friend Minkowski found the second Paris problem particularly provocative. On reading about it for the first time one month before the Paris congress took place, he sent Hilbert his reaction in the form of these prophetic words:

> It is, in any case, highly original to set out as a problem for the future one that mathematicians have long since believed to have already completely in their possession, such as the arithmetic axioms. What might the numerous laymen in the auditorium say to that? Will their respect for us grow? And you will also have a tough fight on your hands with the philosophers [39, p. 129].

Minkowski's premonition that some might feel baffled or provoked by this problem turned out to be right. In the discussion that followed, Giuseppe Peano made a pointed priority claim for his compatriot Alessandro Padoa, claiming that he had already solved the second problem [14, p. 21].[7] But no one, it seems, had any inkling as to what Hilbert meant when he asserted that "the system[8] of *all* cardinal numbers or of *all* Cantorian alephs" (which Cantor believed were identical) did not exist. Today this is called Cantor's paradox, but in 1900 only a very small number of mathematicians had any idea of such matters [17, pp. 290–296]. Hilbert learned about this thorny issue from Cantor himself, beginning with a conversation they had in Brunswick in September 1897.[9] The general circumstances surrounding this have been known since the publication of [47]. What has remained unclear, however, is what Hilbert thought about the matter after Cantor first informed him. That is something well worth knowing, but which he left unsaid at the ICM in Paris.

Later commentators who have written about Hilbert's awareness of logical problems in Cantorian set theory—but also in Dedekind's approach to the infinite in his grounding of arithmetic—have typically been persuaded by the optimistic rhetoric that pervaded Hilbert's Paris address. The consensus view has been that he and Ernst Zermelo were aware of the problem of set-theoretic antinomies already in the late 1890s, but that neither saw this as a problem of deep concern. Ferreirós thus writes, "it was only after Frege's reaction to the Zermelo–Russell paradox, published in 1903, that Hilbert came to doubt whether set theory really belongs to pure logic, and whether pure mathematics can be reduced to logic" [18, p. 49]. Publicly, at least, Hilbert first addressed the larger problems at stake in the paper he delivered at the Heidelberg ICM in [31].

Hilbert's lecture in Heidelberg picked up where he had left his second Paris problem four years earlier. His tone, however, was now very different, since this time he had to address recent controversies over the paradoxes of logic and set theory. He proposed to do so by simultaneously developing the laws of logic and arithmetic, an approach sometimes regarded as the initial step on Hilbert's way toward proof theory based on finitist principles. Before turning to his own proposal, though, Hilbert began by quickly disposing of earlier ideas offered by others. Mixing ridicule with praise, he attached a series of invidious labels to his rivals'

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[3]In the literature, they are usually referred to as the "Hilbert problems," but several were not original with him, of course.
[4]Sometimes, Gerhard Gentzen is credited with having given a positive solution to the second Paris problem in [22], but his methods went beyond Hilbert's strictures.
[5]For overviews of subsequent developments in set theory, see [13, 17, 42].
[6]Although the title is often translated *What Are Numbers and What Should They Be?* a more accurate rendering would be *What Are Numbers and What Is Their Meaning?*
[7]This remark was apparently just a misunderstanding, as in the published text of his talk, Padoa made no reference at all to Hilbert's axiom system or to the second Paris problem [14, pp. 249–256]; see further [25, pp. 98–101].
[8]Hilbert often employed the notion of a "system of things," just as used by Dedekind; see [18, p. 44].
[9]Hilbert later formulated a related paradox in a lecture course from 1905; see [45] and [46].

views, beginning with Kronecker, the *dogmatist*, who failed to examine the foundations of the number concept.[10] Helmholtz represented the *empiricist* approach, which was doomed to finitism, whereas Christoffel was one of numerous analysts, the *opportunists*, who tried in vain to save the notion of irrational number in the face of Kronecker's criticism.

Hilbert credited the *logicist* Frege with having started on the right track, since he "correctly recognized the essential properties of the notion of integer as well as the significance of inference by complete induction" [31, p. 175]; unfortunately, this entire effort had crumbled in the face of Russell's paradox. A similar fate befell the *transcendentalist* Dedekind,[11] who invoked the notion of a universe of objects, whereas Cantor adopted a *subjectivist* stance in trying to distinguish between "consistent" and "inconsistent" sets. Cantor was thus aware of the contradiction that arose in considering collections such as "the set of all sets," which led him to distinguish between "consistent" and "inconsistent" sets. The problem, however, was that "he provides no precise criterion for this distinction, [so] I must describe his conception of this matter as one that still leaves room for subjective opinion and hence furnishes no objective certainty" [31, p. 176].

Hilbert's brief synopsis of prior attempts to ground arithmetic on more primitive notions from both logic and set theory set the stage for his main theme, which was to suggest a way out of this morass. Not surprisingly, the signpost to which he next pointed was already familiar. "It is my opinion," he proclaimed, "that all the difficulties touched upon can be overcome and that we can provide a rigorous and completely satisfying foundation for the notion of number, and in fact by a method that I would call *axiomatic* ..." [31, p. 175]. Hilbert proceeded to describe some of the basic ideas behind his new method, beginning with the primitive notion of a thought-object (*Gedanken-ding*) and its designation by a sign. Recognizing that a proof of the consistency of arithmetic could not be attained by showing that the laws of arithmetic were reducible to logic, he proposed to develop the laws of logic and arithmetic simultaneously, thereby avoiding the paradoxes of set theory. This early program contained some of the germs of Hilbert's proof theory from the 1920s, in particular the idea of treating a mathematical proof as a formula in order to prove the consistency of arithmetic.

In Heidelberg, Hilbert sketched a proof for the consistency of a system of elementary axioms, and he went on to express his conviction that by such means, the resolution of his second Paris problem could not be long off. Speaking with his usual fervent enthusiasm, he ended his speech by taking some parting shots at Cantor's old nemesis, Leopold Kronecker:

> The existence of the totality of real numbers can be demonstrated in a way similar to that in which the existence of the smallest infinite can be proved; in fact, the axioms for real numbers as I have set them up [in "Über den Zahlbegriff" [29]] can be expressed by precisely such formulas as the axioms hitherto assumed. In particular, so far as the axiom I called the completeness axiom [*Vollständigkeitsaxiom*] is concerned, it expresses the fact that the totality of real numbers contains, in the sense of a one-to-one correspondence between elements, any other set whose elements also satisfy the axioms that precede; thus considered, the completeness axiom, too, becomes a stipulation expressible by formulas constructed like those above, and the axioms for the totality of real numbers do not differ qualitatively in any respect from, say, the axioms necessary for the definition of the integers. In the recognition of this fact lies, I believe, the real refutation of the conception of the foundations of arithmetic associated with L. Kronecker and characterized at the beginning of my lecture as dogmatic. In the same way we can show that the fundamental notions of Cantor's set theory, in particular Cantor's alephs, have a consistent existence [31, p. 185].

On Hilbert's Axiomatic Strategy

It may seem surprising that Hilbert wanted to identify himself so strongly with Cantor's new approach to the continuum based on a general theory of infinite sets. By the 1920s, when facing the new challenge posed by Brouwer's intuitionism, he was increasingly inclined to invoke Cantor's name in his rhetorical flourishes, famously declaring, "No one will cast us out from the paradise that Cantor has created for us" [32, p. 170]. Yet, as emphasized in [13], throughout his career, Hilbert's foundational interests reflected the concerns of "working mathematicians" rather than issues of a more philosophical nature, which he happily relegated to others.[12] As Burton Dreben and Akihiro Kanamori write:

> Hilbert did not make direct mathematical contributions toward the development of set theory. Although he liberally used nonconstructive arguments, his were still the concerns of mainstream mathematics, and he stressed concrete approaches and the eventual solvability of every mathematical problem. After its beginnings as the study of the transfinite numbers and definable collections of reals, set theory was becoming an open-ended axiomatic investigation of arbitrary collections and functions. For Hilbert, this was never to be a major concern, but he nonetheless exerted a strong influence on this development both through his broader mathematical approaches and

[10] The labels already appear in italics in [31].

[11] What Hilbert meant by this term seems hardly clear. Ferreirós argues that Dedekind's foundational program for arithmetic was essentially a logicist approach much like Frege's [18, p. 39]. Indeed, he sees Hilbert's work ca. 1900 as that of a logicist largely following in Dedekind's footsteps.

[12] The survey [13] also makes clear that Hilbert's general orientation toward foundational matters was strongly influenced by his early conflict with Paul Gordan, who famously objected to the nonconstructive nature of Hilbert's new results in invariant theory [13, pp. 78–80].

through his specific attempt to establish the continuum hypothesis [13, p. 77].

During the late 1890s, Hilbert developed an abstract axiomatic theory for the real number continuum, whereas Cantor never approached the continuum hypothesis from an axiomatic point of view. Yet in reading the text of Hilbert's Paris lecture carefully, one sees that he definitely wanted to establish this linkage in order to gain complete clarity with regard to the properties of the real number continuum. His strategy came down to proving the consistency of its axiom system, for if that could be done, the game would be won. The uniqueness of the reals would then follow from his final completeness axiom, which essentially asserted that the established model was categorical.[13]

For his second problem, Hilbert put the matter this way:

> ... the proof of the consistency of the axioms is at the same time the proof of the mathematical existence of the complete system of real numbers or of the continuum. Indeed, when the proof for the compatibility of the axioms shall be fully accomplished, the doubts which have been expressed occasionally as to the existence of the complete system of real numbers will become totally groundless [30, p. 301].

Insiders surely recognized that Hilbert's sweeping repudiation of the skeptics who cried *ignorabimus* was a convenient way to attack the views of the deceased Berlin algebraist Leopold Kronecker, an outspoken advocate of finitist constructive principles in mathematics. Cantor had long portrayed himself as a victim of Kronecker's machinations, though his claims in this respect were surely exaggerated. Yet be that as it may, within German mathematical circles, Kronecker was widely known as having been Cantor's nemesis, which made the subtext of Hilbert's address only that much clearer. With regard to the admissibility of infinite sets in mathematics, he intended to banish Kronecker's ghost from the mathematical world forever. At the center of this conflict stood the status of the continuum as a mathematical entity, the issue being whether it could be grasped "rigorously" as an infinite collection of objects. In Hilbert's view:

> The totality of real numbers ... is not the totality of all possible series in decimal fractions, or of all possible laws according to which the elements of a fundamental sequence may proceed. It is rather a system of things whose mutual relations are governed by the axioms set up and for which all propositions, and only those, are true which can be derived from the axioms by a finite number of logical processes. In my opinion, the concept of the continuum is strictly logically tenable in this sense only [30, p. 301].

Thus, once this was achieved, mathematicians could proceed to give a fully rigorous proof of Cantor's two conjectures, which Hilbert lumped together as the first Paris problem. Proving the continuum hypothesis would then establish that either every infinite subset of the real numbers would be denumerable or its cardinality would be identical to that of the continuum itself.[14] When we turn to Hilbert's very last comments in connection with the second problem, the linkage with Cantor's theory of transfinite numbers could not be clearer:

> The concept of the continuum or even that of the system of all functions exists, then, in exactly the same sense as the system of integers or rational numbers, for example, or as Cantor's higher classes of numbers and cardinal numbers. For I am convinced that the existence of the latter, just as that of the continuum, can be proved in the sense I have described; unlike the system of *all* cardinal numbers or of *all* Cantorian alephs, for which, as may be shown, a system of axioms, consistent in my sense, cannot be set up. These systems are, therefore, according to my terminology, mathematically nonexistent.

Hilbert's famous second problem was one of the ten he spoke about in Paris, so his audience presumably heard him make these remarks. Nevertheless, they and presumably the many more who read his text later likely missed the fact that Hilbert was calling for an axiomatization of Cantor's transfinite numbers parallel to the one he had already set forth for the real numbers. Many of Hilbert's contemporaries probably also overlooked the evident connection between his first and second Paris problems. Clearly, the second Paris problem should have taken precedence over the first, since it was first necessary to do away with "Kronecker's ghost," i.e., to prove that the totality of real numbers exists as a consistent set, before one could possibly exhibit it as a well-ordered set. Hilbert could have made the linkage between the two problems much clearer had he simply reversed their order. But that would have meant placing his own work before Cantor's, which would have undermined his larger goal of speaking in the name of mathematical researchers worldwide.[15] For those in the know—particularly certain members of the German mathematical community—Hilbert's address contained an obvious subtext, hinted at when he spoke of certain unnamed "doubters," who refused to accept the concept of the "set of all real numbers" (Hilbert used the word *Inbegriff* rather than *Menge*; this was an older Cantorian terminology for

[13]For an interpretation of Hilbert's completeness axiom that avoids models and reflects a parallelism with Dedekind's chain axiom, see [18, pp. 47–49].

[14]In Paris, Hilbert spoke only about this weaker formulation, long called the continuum problem. This was Cantor's original claim in 1878—that the continuum contains only two orders of infinity, which he later wrote as $\aleph_0$ and $2^{\aleph_0}$. In 1884, Cantor was actually able to prove this for closed subsets of the continuum. By that time, however, having made several futile attempts to solve the original problem, he adopted a different approach based on ordinal numbers (for a detailed account of this shift, see [41]).

[15]Thus, he introduced the first two problems by referring to the highly significant progress made during the last century in the foundations of analysis through "the arithmetic formulation of the concept of the continuum in the works of Cauchy, Bolzano, and Cantor" [30, p. 298]. The choice of names—representing France, Bohemia, and Germany—reflects the setting as well; had he been speaking to a German audience, Hilbert surely would not have overlooked Weierstrass or Dedekind.

what we now translate with the word "set"). That this was no mere passing remark can be seen from the fact that Hilbert elaborated at length on the broader significance of his second Paris problem.

In Cantorian language, the existence of the reals naturally led to the continuum hypothesis, namely that $2^{\aleph_0} = \aleph_1$. For both Cantor and Hilbert, this claim, along with the assertion that the continuum of real numbers can be reordered so as to form a well-ordered set, were central tenets of Cantor's set theory. Hilbert left little doubt that both of Cantor's claims were correct, adding that a direct proof of the latter was highly desirable, by which he meant an explicit well-ordering of the reals. Zermelo's proof in [55] by means of the axiom of choice was surely not what he had in mind, but Hilbert was nevertheless pleased that this at least salvaged Cantor's claim. Years later, commenting on attempts to achieve well ordering by means of a recursive procedure, Zermelo called such "well-known primitive attempts ... unsatisfying both intuitively and logically" [8, p. 352]. Hilbert's brief explanation of the continuum problem was also misleading in that it conflated Cantor's earlier formulation, sometimes called the weak continuum hypothesis, with the later version. He thus writes that if one could prove that every infinite subset of the continuum was either countable or had the same cardinality as the full continuum, then it would follow immediately "that the continuum has the next cardinal number beyond that of countable sets; the proof of this theorem would therefore form a new bridge between the continuum and countable sets" [30, pp. 298–299]. This overlooks that one must first establish that $2^{\aleph_0}$ is equivalent to an aleph, a result Cantor claimed he could prove starting from the assumption that every set can be well ordered.

If this may seem like nitpicking, let me reiterate that the present account merely aims to shed light on Hilbert's views at the time he delivered his famous Paris address. It goes without saying that the significance of his first two problems should be measured against the significant role they played in focusing attention on key foundational issues. My purpose here, on the other hand, is to look backward to the 1890s in an attempt to grasp how Hilbert came to link his own work on the axiomatization of geometry and arithmetic with Cantor's theory of transfinite arithmetic.

Cantor's Letters to Hilbert from 1897

Hilbert did not attend the Zurich congress, but one month later, in September 1897, he and Cantor met at the annual DMV conference, held in Brunswick. Richard Dedekind, a native of the city who taught at its Technische Hochschule, delivered the opening address on that occasion. Dedekind had already been informed about the Cantor paradox earlier that year, but it seems unlikely that he and Cantor discussed foundational matters during the course of this meeting. The official report [12] notes that Hilbert was one of the sixteen speakers, whereas Cantor's name was not among them; nor was set theory among the topics taken up at the conference. Nevertheless, we know that Cantor was present and took an active part on the sidelines of the meeting. In fact, he even gave a lecture about his latest results before a small group of attendees [19, p. 214].

In March 1897, Cantor had submitted the second part of his "Beiträge" [6] to *Mathematische Annalen*, just one month before Hilbert put the final touches on his *Zahlbericht*. Hilbert's famous preface contains a slight bow in Cantor's direction, though not in connection with set theory, but rather for his role in strengthening the foundations of analysis, about which Hilbert expressed his belief:

> ... that the modern development of pure mathematics takes place above all under the banner of number: Dedekind's and Weierstrass's[16] definitions of the fundamental concepts of arithmetic and Cantor's construction of the general concept of number lead to an arithmetization of function theory and serve to realize the principle that even in function theory one can only regard a result as ultimately proven when it has been reduced to relations between rational numbers [27, p. 66].

Hilbert surely heard Cantor's informal talk at the DMV conference in Brunswick, which no doubt intensified his interest in transfinite arithmetic. During this meeting, he informed Cantor that he had recently been appointed to the editorial board of *Mathematische Annalen*. Cantor evidently considered this a fortuitous turn of events, since he was trying to complete the third installment of his "Beiträge"[17] and could anticipate that Hilbert would be most interested in its contents. Toward the end of the conference, they may have spoken about these things, but in any event, Cantor was intent on answering a question Hilbert had posed to him.

On September 29, one day after the end of the conference, he sent Hilbert a letter in which he remarked:

> Unfortunately, the day before yesterday, due to the advancing noon hour, I had to interrupt our conversation on set theory at the Brunswick Polytechnicum, just when you had raised a concern as to whether all transfinite cardinal numbers are contained in the alephs, in other words, whether any definite **a** or **b** is a definite aleph.

[16]In the English edition, Leopold Kronecker's name mistakenly appears instead of Weierstrass's [35, ix]. We can be sure that Hilbert would never have committed this mistake—he had a deep respect for Weierstrass's work and regarded Kronecker's criticisms of it as scandalous.

[17]The first two were [6], translated in [7].

> That this is indeed the case can be rigorously proved. The totality of all alephs is namely of the kind that it cannot be comprehended as a definite well-defined finished [*fertige*] set. Were this the case, then a definite aleph would succeed this totality in size, and it would thus belong to this totality (as an element) as well as not belong, a contradiction.
>
> This having been said, I can rigorously prove: "If a definite well-defined finished set were to have a cardinality that was not an aleph, then it must contain subsets whose cardinality is the same as any given aleph, in others words, this set must then contain the totality of all alephs" [38, p. 388].

From this, Cantor drew the conclusion that the alephs are coextensive with all infinite cardinal numbers, so that the continuum of real numbers, whose cardinality is $2^{\aleph_0}$, must correspond to an aleph. His continuum hypothesis claimed further, in fact, that $2^{\aleph_0} = \aleph_1$. Cantor's assertions were grounded on the conviction that every set can be well ordered, which turned out to be equivalent to Zermelo's axiom of choice [40]. In his letter to Hilbert, Cantor elaborated on the above remarks by explaining:

> It follows from these results that the linear continuum, torn from its context [the natural ordering of the real numbers], is countable in a higher sense, that is, can be represented as a well-ordered set.
>
> Totalities which we cannot grasp as "sets" (like the example of the totality of all alephs, as was proved above) I have already many years ago named "absolute infinite" totalities and sharply distinguished from transfinite sets [38, p. 388].

As Walter Purkert has emphasized, Cantor attached great philosophical and religious significance to this distinction, since he identified such humanly incomprehensible totalities with the Absolute in the sense of Leibniz and Spinoza (see [48, pp. 57–61]).

Hilbert's reply to Cantor's letter no longer exists, but from the latter's counter-reply we can see that initially, the main point of Cantor's paradox had escaped him. Writing on October 2, Cantor began his explanation of its import by citing Hilbert's words: "The set [*Inbegriff*] of the alephs can be comprehended as a definite well-defined set, since for any given thing it must always be decidable whether or not that thing is an aleph; indeed, more than this is not required of a well-defined set" [38, p. 390]. Cantor then pointed out that this remark missed the point of his letter, which he underscored by putting his claim in the form of a theorem: the totality of all alephs cannot be comprehended as a definite well-defined and *also finished* set. He then emphatically added:

> I venture to designate this theorem, which is completely secure and is proved from the definition of the "totality of all alephs," as the most important and noble theorem of set theory. But one must understand the expression "finished" correctly. I say of a set that it can be thought of as finished, and name such sets, if they contain infinitely many elements, "transfinite" ... if it is possible without contradiction (as is the case for finite sets) to think of all their elements as existing together and hence the set itself as joined together as a thing by itself; or also, in other words, if it is possible to think of the set with the totality of its elements as presently existing [38, p. 390].

This brings to mind Cantor's famous definition at the very beginning of his "Beiträge": "A set is a collection of definite, distinguishable objects of perception or thought combined into a whole [*Zusammenfassung zu einem Ganzen*]" [6, p. 282]. Cantor varied his terminology a great deal over the years; nevertheless, it is striking that he formulated the above definition only in this, his final, work, which goes beyond the descriptive explanation in the first note to [4].

This was the era of so-called naive set theory, in which sets were conceived as arbitrary collections whose elements satisfy a given definition that was considered conceptually clear. One simply assumed that these elements belonged to some still larger collection, which was itself a set, thus suggesting the notion of a universal set of objects. The same viewpoint dominates in Arthur Schoenflies's official report on point set theory, which merely repeated what Cantor wrote about set comprehension back in 1882 [51, p. 5]. At that time, Cantor stated that a set of elements belonging to a given conceptual sphere is well defined when two conditions hold. First, its defining property must be such that the set is internally determined, meaning that the logical law of the excluded middle applies in answering whether a given element belongs to the set. Second, it must be possible to ascertain for any two elements whether they are identical. Cantor made it clear that these conditions were theoretical in nature and had nothing to do with the methodological issues involved in deciding whether a particular element might or might not happen to belong to a well-defined set. To illustrate this point, he noted that it was still not known whether π was a transcendental number.[18] This question, however, had no bearing on the definition of the set of all algebraic numbers, which was in any case well defined [8, pp. 150–151].

Hilbert thus invoked this idea in replying to Cantor's first letter when he asserted that the collection of all alephs was a well-defined set on the grounds that one could (in principle) always decide whether a given number was an aleph or was not. In the meantime, however, Cantor had come to realize that this naive conception based on the principle of comprehension was a grievous error, though his own earlier writings had served to entrench this very understanding. Lurking behind this, to be sure, were Dedekind's influential ideas in [9]. Thus, Ferreirós describes the shift in Hilbert's thinking as a shift away from Dedekind's views, once he came to recognize the import of Cantor's message:

[18]A few months later, Klein asked Cantor to referee Lindemann's paper containing such a proof [38, pp. 73–75]; see [50].

> Hilbert's views on "truth and existence" in mathematics emerged from a logicistic understanding of set theory in terms of the principle of comprehension. He was led to revising that contradictory principle in the light of Cantor's discovery of the antinomies of set theory; this is what triggered his noteworthy inversion of previous ideas about existence and consistency [18, p. 49].

Once he read Cantor's second letter, Hilbert must have realized the import of his message. In fact, it left him both puzzled and worried. Clearly, one could not admit these gigantic sets into a mathematical theory, but Cantor's approach lacked any fixed criteria for distinguishing between finished and unfinished collections. In his private notebook, Hilbert pondered what to do:

> Cantor's letters: "Fertig" [is] undefined. This evil becomes worse the more one thinks about it. The system of all irrational numbers should be thought of as "fertig" [b]ut not the system of all sets that result from the countable [sets] by the operations M^M and $\mathfrak{S}$ [26, p. 91].

These operations were used without restriction in Cantor's theory: M^M stands for the collection of all mappings (*Belegungen*) $M \to M$, whereas $\mathfrak{S}$ stands for the sum or union of a collection; both play a ubiquitous role in naive (nonaxiomatic) set theory.[19]

Hilbert's further reflections about how to finesse the problem Cantor had described are both illuminating and surprising:

> I see only the following way [out]: One examines the propositions and problems of set theory and first tries to formulate them without using the word set—as [this is] essentially only *façon de parler*. That there are as many points on the line as in the square is also easy. Yet even the problem of whether something lies between the countable and the linear continuum is very difficult. Also the following: one does not think of something existent behind a concept, but rather *façon de parler* should only signify that with a word I mean that I now want to use a certain fact that was previously established. Thus, an irrational number = thing for which I know that every rational number can be given a mark > [or] < in relation to this thing. This fact and only this fact shall apply when I speak of the concept. One must avoid speaking of the system of all irrationals, but one can still formulate general theorems about irrational numbers (e.g. there are infinitely many, *e* is irrational) in which the word thing = irrational number occurs. Then this shall merely mean a thing described by that property alone [26, p. 91].

These remarks, although undated, were surely written long before August 1900, when Hilbert spoke at length about Cantor's conjectures at the Paris ICM. He could have been reacting to the two letters cited above, although a later date in 1899 is also possible, since Cantor wrote him several times then about the same topic [38, pp. 399–431]. In any case, the reaction cited above gives a clear impression of Hilbert's state of mind during the time that the antinomies in set theory first emerged. They also contain hints of what was to come. Rather than imposing restrictions on the operations of set theory, Hilbert considered the more radical approach of treating mathematical concepts on two different levels, only one of which would have ontological significance.

Wakter Purkert recognized the importance of Cantor's two letters to Hilbert when he first published them in [47]. They reveal that Cantor fully grasped the problem posed by antinomies in the theory of infinite sets well before other mathematicians had attained this insight. Certainly he became aware of this problem independently of Cesare Burali-Forti, who in 1897 published a paper related to the paradox that today bears his name.[20] Retrospectively, this led to the realization that the set of all ordinal numbers, were it to exist, would lead to an immediate contradiction, since one could then define a still greater ordinal that would not belong to the "set" containing all of them. In principle, this was the very same difficulty Cantor had encountered and communicated to both Hilbert and Dedekind that same year. It remains unclear exactly when Cantor realized that one needed somehow to restrict set-theoretic operations in order to avoid forming inconsistent sets, but in his letters to Hilbert from this time, he kept returning to the distinction between finished and unfinished, or consistent and inconsistent, sets (the terminology he introduced a few years later). Yet Cantor never made these mathematical issues transparent in his publications, preferring instead to inform sympathetic colleagues, like Hilbert, through personal communications. Nor did he tackle the continuum hypothesis—his conjecture that $2^{\aleph_0} = \aleph_1$—though he surely planned to do so in the third installment of his "Beiträge," which he was working on at the time he spoke with Hilbert in Brunswick.

Minkowski, who always read and followed Hilbert's work with intense pleasure, had in the meantime accepted a professorship in Zurich, where he taught alongside Hurwitz. One of his students from this time—remembered later as a "real lazybones"—was a young fellow by the name of Albert Einstein.[21] In the summer of 1898, Hilbert and his wife visited the Minkowskis in Zurich. The two friends planned to see each other again at the forthcoming DMV conference in Düsseldorf, where Minkowski gave a lecture on his recent work in number theory. Hilbert's name was not among those listed on the program of speakers, but he

[19]*Belegungen* are not literally mappings but rather coverings. Cantor introduced this notion in 1892, and it plays a key role in his "Beiträge"; see Jourdain's comments in [7, pp. 81–82]. The symbol 2^M, which denotes the set of mappings $M \to \{0, 1\}$, is commonly used for the power set of M.

[20]As shown in [43], however, the so-called Burali-Forti paradox does not appear in that paper.

[21]Minkowski later told his assistant Max Born that "relativity came as a tremendous surprise, for in his student days Einstein was a real lazybones. He never bothered about mathematics at all" (translated from [52, p. 45]).

undoubtedly took part as a listener. This we know from a letter Cantor wrote to Hilbert on October 6, 1898, roughly a fortnight after the Düsseldorf meeting took place.

Cantor had decided not to attend that event, preferring instead to spend time vacationing with his family. When he arrived home in Halle, he was delighted to find a letter from Hilbert awaiting him. It was dated September 16, 1898, thus three days before the DMV conference began, and was evidently written with the hope and expectation that they would be able to pick up their discussion of fundamental problems in set theory at the DMV conference. In his reply, Cantor expressed his regrets that he had been unable to meet Hilbert in Düsseldorf, but also wanted him to know how happy he was "for the interest you devote to set theory." He then added:

> How often during the past year my thoughts have involuntarily turned to you with the question whether the active participation [Theilnahme] in these researches that you showed me in Brunswick would continue.
>
> Nothing could be more welcome or dearer to me than to discuss the elements of set theory with you, as this promises not only to bring profit for the matter itself but also instruction and motivation for me [38, p. 393].

Curiously, Hilbert had still not fully absorbed the message Cantor had tried to convey to him one year earlier, and in the remainder of this letter, Cantor merely recapitulated what he had already written him then. After explaining the main idea once again, he closed by saying, "In the example you present, however, the set of all alephs is assumed to be a finished set, which thus solves and explains the contradiction you were led to by applying theorems that are only proven valid for finished sets" [38, p. 393].

After October 1898, there seems to be no evidence of further contacts between Cantor and Hilbert until the spring of 1899. In early May of that year, Hilbert sent Cantor a postcard, to which the latter replied with some brief but significant remarks. Concerning terminology, Cantor informed him that he no longer spoke of "finished" (*fertige*) sets, preferring instead to use the adjective "consistent." He then wrote that he and Hilbert shared the conviction that the arithmetic continuum is a consistent set, though the question remained "whether this truth is provable or whether it is an axiom. I now incline more to the latter alternative, although I would gladly be convinced by you of the former" [38, p. 399]. These words reflect a sense of pessimism that one rarely finds in earlier correspondence. Two years earlier, Cantor had shown Hilbert an argument for why every cardinal number must be an aleph,[22] and he had long believed that the cardinality of the continuum was $\aleph_1$. Now he seemed unsure whether an axiom might be needed in order to assert that the continuum of real numbers was even a consistent set. Thus, the possibilities for axiomatizing set theory had begun to creep into his speculations, and Cantor must have realized that his theory stood in need of stronger foundations. Hilbert clearly recognized this, too.

Climactic Events of 1899

By this time, Hilbert had firmly established his reputation as the era's leading authority in both invariant theory and the theory of number fields, two formerly distinct disciplines that had now been brought under the same algebraic roof through his work. But then came a most unexpected turn of events. Many years later, in [2, p. 402], Otto Blumenthal recalled the buzzing chatter among the students when they read Hilbert's announcement for a course on "Grundlagen der Euklidischen Geometrie" [36, pp. 185–406], which he was offering for the winter semester of 1898–1899. Blumenthal and the older students, those who had been accompanying Hilbert on weekly walks, had never heard him talk about geometry, only number fields. Little did they realize that Hilbert had been contemplating the foundations of geometry ever since his years as a Privatdozent in Königsberg, as evidenced by recent historical studies (see, in particular, [36, 53], and the commentaries by Klaus Volkert in [28]). Hilbert's lecture course that semester surprised them even more, for in it he sought to lay out the fundamental structures underlying Euclidean geometry as no one had ever done before. The following spring, following a request from Klein, he revised this material and presented his "Grundlagen der Geometrie" for a Festschrift commemorating the unveiling of the Gauss–Weber monument in Göttingen in June 1899.

One often reads that Hilbert's principal goal in his Festschrift article was to prove the relative consistency of Euclidean geometry by showing that its axiom system depended only on the consistency of the axioms for the real numbers. This faulty reading stems from viewing his Festschrift contribution through the prism of Hilbert's second Paris problem, which Kurt Gödel later showed could not be solved within the framework of proof theory as designed by Hilbert and Paul Bernays. This was the upshot of Gödel's second incompleteness theorem, published in "On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems I" [23]. This classic paper also contains Gödel's first incompleteness theorem, which roughly states that any consistent formal system F strong enough to contain elementary arithmetic (e.g., that of Whitehead and Russell's *Principia Mathematica*) is incomplete, meaning there are statements in F that can be neither proved nor disproved in the language of F. These famous results lie far beyond the chronological bounds of the present essay, but they were rooted, of course, in the problems dealt with here. The distinction that needs to be made clear, though, concerns the challenge of proving the consistency of the axioms for the real numbers, on the one hand, and proving the relative consistency of the axioms for Euclidean geometries, on the other. The proof of the latter appears at the beginning of Chapter 2 in [28, pp. 19–21/95–97].

[22] He presented his argument in more detail in his long letter to Dedekind from August 3 [38, pp. 407–411].

For Hilbert, establishing relative consistency was, of course, an important aim, but he accomplished that goal without any appeal to the field of real numbers. In fact, for the system of axioms he presented in the original edition of *Grundlagen der Geometrie*, this was only one among several possible models for a Euclidean geometry (geometries in which the axiom of parallels holds). Initially, Hilbert identified continuity with a single axiom—the axiom of Archimedes—so what he called Cartesian geometry, i.e., analytic geometry over the reals, was only one particular case. Since this model would have required coming to grips with the properties of the real number continuum, Hilbert sensibly opted to give a proof of consistency using a smaller, indeed countable, model that satisfies all of his original axioms. He introduced this as a plane analytic geometry over the number field Ω, an extension of $\mathbb{Q}$ generated by the four arithmetic operations and square roots of the form $\sqrt{1+\omega^2}$. This meant that Ω was the smallest possible Pythagorean number field, and its existence proved the consistency of the axioms for "complex number systems," although Hilbert took the consistency of Ω itself simply for granted. His concluding Chapter 7 was devoted to constructions corresponding to this model, which he carried out with two virtual instruments: a compass and a device for transporting line segments. The conventional constructions with straightedge and compass found in Euclid's *Elements* correspond to an extension of $\mathbb{Q}$ allowing for arbitrary square roots $\sqrt{\omega}$, as Descartes had essentially already observed in 1637. For a brief overview of Hilbert's models, see Volkert's commentary [28, pp. 247–251].

The unveiling of the Gauss–Weber monument took place on June 17, 1899, and two of those who came to Göttingen to attend that ceremony were Cantor and Minkowski. Cantor was curious to learn more about the status of the "arithmetic axioms" (his quotation marks) in Hilbert's Festschrift, if possible even before meeting Hilbert in Göttingen [38, p. 399]. What he may have learned about these no one can say, but Hilbert definitely spoke about this very topic with Minkowski, who alluded to it in a thank-you letter, written one week after the festivities in Göttingen:

> Dear friend,
>
> Now that I've returned to the reality of Zurich, the wonderful days in Göttingen seem today like a dream to me, and yet one can as little doubt their existence as that of your $18 = 17 + 1$ axioms of arithmetic. I felt especially comfortable in your warm home, and I've been reporting here repeatedly with pleasure about the exciting time I spent there with you. ...
>
> Anyone who experienced these days in Göttingen will hardly get over their astonishment over the liveliness in the Göttingen mathematical circle, and at the moment this is entirely due to you. Spending time in such air gives a person higher ambitions and an impulse to more intensive creativity [39, pp. 116–117].

Minkowski was here alluding to the full set of axioms that Hilbert would present three months later at the DMV conference in Munich.

Over the course of Hilbert's career, this conference stands out as an early and significant personal triumph. He was one of some eighty mathematicians who gathered in Munich for the annual DMV conference, despite floods that had left the city's transportation system largely paralyzed. The impressive turnout and upbeat atmosphere set the mood as the DMV's presiding officer, Max Noether, greeted the assembled throng. In opening the meeting, Noether mentioned some noteworthy accomplishments of the past years, not least of which was the publication of Hilbert's *Zahlbericht*. In this brief period, the DMV had indeed come into its own, emerging as the core structure around which a fast-growing German mathematical community was taking form. Hilbert, still not yet forty, was about to move into its spotlight.

It was at this Munich meeting that he first presented the second Paris problem, his plan for a rigorous axiomatic approach to the real number continuum, one that entailed proving the consistency of the eighteen axioms he used to characterize the set of all real numbers. Hilbert also delivered a second talk on the Dirichlet principle, a topic that would prove even more important as a leitmotiv for many of his leading pupils. Their methods proved to be spectacularly successful, and they helped usher in a major new subfield in analysis [54, pp. 380–382].

As noted above, Hilbert's argument for the logical consistency of Euclidean geometry rested on assuming the consistency of the number field Ω. Furthermore, in his Festschrift he had only analyzed the consistency and *independence* of his axiom system. In his Munich lecture [29], Hilbert shifted the focus to consistency and *completeness*, while emphasizing that these were the two general goals of the axiomatic method. Thus, it was only at this stage that the full continuum became the object of investigation. In the case of Cartesian geometry, he had merely noted its relative consistency; it thus remained to provide a proof of absolute consistency, which now hinged on proving this directly for the axiom system of the real numbers. In emphasizing the importance of this problem, Hilbert linked it directly with the very issues he had been discussing with Cantor for the last two years. He furthermore claimed that to prove this consistency required only a suitable modification of known methods of argument,[23] "a proof that establishes the existence of the real numbers or, in the terminology of G. Cantor, shows that the system of real numbers is a consistent (finished) set" [29, p. 184]. Cantor almost surely attended this lecture, and since he and Hilbert spoke at least briefly during this conference [38, p. 425], they presumably discussed this aspect of consistency as well. One year later, in presenting his famous second Paris problem,

[23]For a discussion of what Hilbert probably meant by this, see [18, pp. 60–61].

Hilbert would repeat the very same conviction using almost the identical words.

Cantor's Forgotten Pupil

During his many years in Halle, Cantor never offered a course on set theory, nor did he supervise any of the students who took doctoral degrees there. Interest in Cantor's theory nevertheless gradually spread, and in Germany, three talented young postdocs taught special courses dealing with this subject. The first to do so, in the winter semester of 1900/01, was Ernst Zermelo, then a Privatdozent working closely with Hilbert in Göttingen. One year later, Felix Hausdorff followed suit in Leipzig, and in 1902, Edmund Landau taught this new theory to a large group of auditors in Berlin [42, p. 44]. A little later, in 1905, Hilbert offered a lecture course on "Logical Principles of Mathematical Thought," his first serious foray into mathematical logic (for details, see [45, pp. 50–58]). Zermelo's course attracted seven students, one of whom, Felix Bernstein, was working on a dissertation in set theory in which he hoped to take a significant step toward solving Cantor's continuum problem.

Bernstein grew up in Halle, where his father, Julius Bernstein, was a professor of physiology whose friendship with Cantor extended to their respective families. Their wives were both trained singers and talented musicians, so *Hausmusik* was surely a common bond. As a highly precocious gymnasium pupil, Felix had already begun to study Cantor's theory, aided no doubt by his mathematics teacher, Friedrich Meyer [1]. Although this was only one of his many intellectual interests, he never abandoned it throughout his turbulent life. During his university studies in Munich, Halle, Berlin, and Göttingen, he took courses in subjects ranging from mathematics and physics to philosophy and art history.

Many of his teachers were distinguished professors, though ultimately Cantor exerted the greatest influence on the young Felix Bernstein, whose name is remembered from the Cantor–Bernstein theorem. This classic result provides a criterion for A and B to have the same cardinality, namely, if A is equipollent with a proper subset of B, and B is equipollent with a proper subset of A.[24] Cantor had long sought to prove a closely related theorem that could serve as a stepping stone toward confirming the trichotomy law for cardinal numbers in the realm of well-ordered sets. In 1882, he discussed these matters with Dedekind, who found a way to prove the theorem, though he revealed this to Cantor only much later (see [17, pp. 239–240]). In 1897, however, Dedekind learned directly from Bernstein that the latter had found a proof himself. The circumstances surrounding this meeting were, indeed, quite remarkable.

In early June of that year, at age nineteen, Bernstein went to visit Dedekind in Bad Harzburg, an event he recalled more than three decades later in a letter to Emmy Noether [10, 3: p. 449]. Bad Harzburg, a tourist town in the Harz Mountains, lies fifty kilometers south of Brunswick, where Dedekind lived and taught. Its nearby location and fresh air drew many wealthier city dwellers, and in 1853, Dedekind's father bought a house there on the Herzog-Julius-Strasse. Richard and his older sister Julie often stayed there, though in later years they frequented the hotel on the Burgberg above the town. Cantor spent a week in Bad Harzburg visiting Dedekind in September 1882, probably the most eventful of their various encounters.

According to Bernstein's recollections, Cantor had written to Dedekind in early 1897 to inform him about the problem of paradoxes in set theory. This revelation had a direct bearing on Dedekind's attempt to ground the number concept on purely logical principles, because in Section V of [9], he considered "the totality of all things that can be objects of thought," which itself belonged to this system. Dedekind argued that this entity satisfied his definition of an infinite set, namely that it can be mapped one-to-one onto a proper subset of itself (see Müller-Stach's commentary in [9, pp. 143–146]). Cantor had no sympathy at all for Dedekind's effort to found arithmetic on the basis of logic and set theory, but he badly wanted the latter's approval or at least a sign of recognition that his arguments were sound. Having received no reply, Cantor asked the young Bernstein to visit the old man. Dedekind was then 65 and suffering from poor health; he had already resigned his professorship at the Brunswick Institute of Technology three years earlier. Bernstein recalled only that Dedekind had little to say, other than that he had pondered Cantor's letter and was almost in despair over its implications; still, he had reached no definite conclusions and apparently he never would, at least not in print form.

Although the sources relating to this incident are very thin, we can certainly assume that Bernstein conveyed Dedekind's reaction to Cantor shortly after this meeting took place. José Ferreirós commented about this as follows:

> Bernstein says that Cantor had immediately realized that the contradiction affected the "system of all things" that underlay Dedekind's theorem of infinity. In his correspondence of 1899, Cantor formulates the paradoxes using Dedekind's terminology of systems, and he says explicitly that they affect the collection of everything thinkable [8, p. 443]. Dedekind had stated that every system is a thing, but this now became untenable, and his theorem that there is an infinite set vanished, bringing into question his whole logicistic project.[25] Otherwise said, Cantor had shown that Dedekind's logicistic notion of set is not sufficient as a basis for set theory, for it allows both "finished" and "unfinished" collections. According to Bernstein, in 1897 Dedekind had not arrived at a definite opinion, but in his reflections he had almost come to doubt whether human thought is completely rational [17, p. 292].

[24] In the literature, this is sometimes called the Schröder–Bernstein theorem, though the proof published by Ernst Schröder in 1898 turned out to be incorrect.

[25] As noted above, Hilbert would later make this explicit in his lecture at the 1904 Heidelberg ICM.

In all likelihood, this episode was entirely forgotten until many years after the deaths of Cantor and Dedekind, when information about it surfaced in the course of research on their respective works and correspondence. Emmy Noether, whose admiration for Dedekind only grew as she edited his collected works [10], was working on Volume 3 at the same time that Zermelo was preparing the edition of Cantor's collected works [8], and she initially tried to convince him to publish all of their letters. Thus, on May 12, 1930, she wrote Zermelo, "I would be very pleased if the Cantor–Dedekind correspondence would be published in your Cantor edition" [15, p. 161]. She reasoned that Cantor's letters, unlike Dedekind's, were full of mathematical claims, but Zermelo was not persuaded, and so he published only some excerpts from their correspondence. Jean Cavaillès later came to Göttingen, and he and Noether soon reached agreement on a joint project to publish the entire Cantor–Dedekind correspondence up to 1882; this edition, however, only appeared two years after her death, in [44].

References to Bernstein's visit can be found in two letters from August 1899, apparently the last exchanges between the two mathematicians. On August 29, Dedekind sent Cantor a proof of the Cantor–Bernstein theorem (see [8, p. 449]). This was prefaced by a remark recalling Bernstein's visit and how the young man had been taken aback when Dedekind informed him that this theorem was easy to prove using his method of chains in [9]. Noether published this part of the letter along with a manuscript from 1887 that Cavaillès found among Dedekind's papers [10, 3: pp. 447–448]. Dedekind had indeed proved the theorem already then, but he had neglected to include this in his booklet, which appeared the following year. Ferreirós has suggested that this was something like a "cat and mouse" game, which Dedekind played to defend himself against Cantor's entreaties [17, pp. 239–240]. The day after Dedekind sent him his proof, Cantor wrote to thank him for it. He also mentioned that Bernstein had presented his proof in the Halle seminar around Easter of 1897. Cantor then sketched his ideas for proving the trichotomy law and described his goal of showing that every cardinal number is an aleph. The weaknesses in his argument were analyzed by Zermelo in his commentary attached to this letter [8, pp. 449–451].

The contents of these two letters have thus long been known, whereas little has been written about the surrounding circumstances. Emmy Noether was curious to know more, and so she asked Felix Bernstein, a fellow colleague in Göttingen, about his visit to Bad Harzburg long ago. Like everyone who had known these two famous mathematicians, Bernstein was surely impressed by the striking differences in their physical presence and manner of expression—Cantor's flamboyance versus Dedekind's reticence—and he added an interesting note to this by recalling the sharp contrast in the metaphors they used in explaining how they imagined an infinite set. During this visit, Dedekind remarked that an infinite set was like a closed sack containing certain definite things, though one could not see or know them, except to say they were inside. Bernstein then recalled how Cantor had responded when asked the same question: he rose to his feet, lifted his arms and eyes upward and declared, "A set is to me like a giant abyss" [10, 3: p. 449].

Cantor Versus Dedekind

Over the course of the summer of 1899, Cantor reestablished contact with Dedekind, during which time he sent him a barrage of letters in hopes of inducing him to take up the matter of inconsistent sets.[26] Dedekind eventually answered and invited Cantor to visit him, but his response made clear that he had absolutely no desire to discuss the fundamental issues Cantor had raised. Part of the reason surely stemmed from personal misgivings, as documented in [16]; another, however, was purely mathematical. Just as Hilbert would later remark in his 1904 Heidelberg lecture, Dedekind pointed to that fact that Cantor failed to offer a criterion for distinguishing between consistent and inconsistent sets:

> Highly honored friend!
>
> Your visit will always be welcome to me and my sister, but I am by no means prepared for a discussion of your communication, which at this stage would be completely fruitless! You will certainly appreciate this when I tell you openly that, although I have read through your letter from August 3rd many times, I am still not clear as to your division of sets [Inbegriffe] into consistent and inconsistent; I do not know what you mean by "association of all elements of a multiplicity" [Zusammensein aller Elemente einer Vielheit] and by its opposite. I don't doubt that a light would come on if I were to study your letter more carefully, as I have great trust in your deep and sharp-minded researches. But because of the unending flood of proofs I have had to correct, I have until now not had the time or the tranquility needed to reflect on these matters. Now only revisions remain, and I promise to use this greater period of quiet for this reflection.
>
> … I have not busied myself with these interesting things at all for many years, and since my step-by-step thinking [Treppen-Verstand] was always very slow, it will not be easy for me to work my way into your researches [38, p. 413].

Cantor seems not to have taken this disappointment badly, as can be seen from a letter he wrote to Hilbert on November 15. None of Hilbert's letters to Cantor have survived, but from the latter's reply, it seems clear that Hilbert had written about Dedekind's *What Are Numbers and What Is Their Meaning?* [9], confirming his agreement with Cantor's

[26] The significance of these letters was appreciated much later by Zermelo, who published lengthy excerpts from them followed by his commentary in [8, pp. 443–451].

opinion. "From your worthy writing," Cantor replied, "I see to my joy that you recognize the significance, which the open publication of the foundation of my set-theoretic researches must have *precisely for him* [Dedekind]" [38, p. 414]. Here Cantor added the curious remark that Hilbert would find this completely clear foundational declaration in the closing notes to [4], even though intentionally somewhat hidden. Cantor was greatly delighted to read that Hilbert now acknowledged the flaw in Dedekind's theory. He went on to explain how he had long been hoping that Dedekind would reach the same conclusion and admit his mistake, namely that his theory was "based on the naive assumption that any well-defined property determines a consistent set." Referring to his recent meeting with Dedekind, Cantor even took the trouble to copy out for Hilbert the long letter he had sent Dedekind on August 4 [38, pp. 407–411]. Evidently he was still hopeful that the latter would soon reach the same conclusion as had Hilbert. Cantor's letter is the closest he ever came to completing the promised third installment of his "Beiträge."

A tragic cloud hovers over this whole phase of Cantor's life, as his behavior became ever more erratic and frenzied. In a letter to Hilbert from January 27, 1900, he attempted to reassure his younger supporter that his intense pursuit of historical studies—especially his passionate interest in proving that Francis Bacon was the true author of Shakespeare's works—in no way distracted him from pursuing his mathematical work. By this time, Hilbert must surely have come to doubt that Cantor would ever deliver the third installment of his "Beiträge," and this letter could hardly have restored his confidence. Cantor had in the meantime come up with a rather odd-sounding title for it, namely, "Aphoristic Grounding of a Theory of Finite and Transfinite Ordinal and Cardinal Numbers" [38, p. 425]. He then explained that:

> the reason why I have hesitated for so long (I can say for years) in laying forth [my latest work] is this, that its *essential and principal* conclusion places me in opposition to two great authorities, to Gauss and Dedekind, and to both in completely different ways .... This is especially unpleasant with regard to the latter because I know that my theory stands opposed to his favorite ideas [*Lieblingsideen*], which he developed most carefully in *Was sind und was sollen die Zahlen?* I placed this before him for his consideration in a mailing from August 1899, explaining *everything essential*, but he only answered evasively and what he then said showed that he had not grasped the essence of the matter. He wanted to think it over further, but now I have heard nothing from him for the last five months [38, pp. 425–426].

This was a time of great duress for Cantor, who began to suffer his first serious attack of mental illness [24, pp. 365–368]. Well before this, though, he had been diagnosed as suffering from manic depression, although from this time forth his bouts of listlessness would become longer and more severe. On November 10, 1899, thus five days before he first informed Hilbert of his recent contact with Dedekind, Cantor had become so discouraged that he petitioned the Prussian government for a position outside academia, either in the diplomatic service or perhaps at a library. This request was passed to the Ministry of Education, which then sent an inquiry to the Kurator, its local representative at Halle University. The latter spoke with Cantor's personal physician, Dr. Mekus, who informed him that this episode had begun when Cantor learned that his daughter's fiancé had broken off their engagement. Mekus strongly advised not to act on Cantor's petition, but rather to grant him a leave of absence until he recovered. That process could best be furthered, according to the physician, by encouraging him to continue his researches in set theory, which deflected his mind from troubling thoughts. By the following summer semester, however, Cantor was still not well enough to resume teaching; not until November 1900 did he finally return to the lecture hall [49, pp. 193–195].

Richard Dedekind, who avoided mathematical stages both large and small, apparently never voiced any public opinion about the status of antinomies in set theory. The closest he came to doing so was in the preface to the third edition of *Was sind und was sollen die Zahlen?* [9], issued in 1911. There he expressed his conviction that his approach to the logical foundations of arithmetic was sound, even though he had not been able to address a weak point that he left unmentioned. Emmy Noether alluded to this weakness in her commentary by noting that Zermelo's axiom of infinity as well as the axiom of choice was required to secure the arguments in Dedekind's text [10, 3: p. 391]. Many years before, Dedekind had written the following remarks in the preface to the first edition:

> Numbers are free creations of the human mind; they serve as a means of apprehending more easily and more sharply the difference of things. It is only through the purely logical process of building up the science of numbers and by thus acquiring the continuous number-domain that we are prepared accurately to investigate our notions of space and time by bringing them into relation with this number-domain created in our mind. If we scrutinize closely what is done in counting an aggregate or number of things, we are led to consider the ability of the mind to relate things to things, to let a thing correspond to a thing, an ability without which no thinking is possible. Upon this unique and therefore absolutely indispensable foundation, as I have already affirmed in an announcement of this work, must, in my judgment, the whole science of numbers be established [11, pp. 53–54].

This passage may bring to mind a famous saying of Cantor: "the essence of mathematics lies in its freedom" [4, p. 182]. But in fact, this latter expression reflects an entirely different understanding. Cantor, the inspired visionary who was often called the inventor of set theory, saw himself in a very different light. He thought of himself as a discoverer and an explorer, a kind of mathematical Moses, to whom God had shown the promised land. Dedekind's mathematics, much like Hilbert's, reflected the unfettered freedom of rational thought; Cantor's stemmed from a search for the divine.

These differences were also reflected in their personalities [16]. Dedekind seems to have modeled himself after his mentor, Peter Gustav Lejeune Dirichlet, whom he greatly admired [37]. This applies to the exacting standards found in Dirichlet's published works, but also to the personal dignity he exuded when in the company of others. Dedekind was nearly fifteen years older than Cantor, who always felt he had to fight an uphill battle for recognition from the establishment in Berlin, whereas Dedekind was content to work quietly offstage at Brunswick Polytechnic (elevated to a *Technische Hochschule* in 1877). After the death of his friend Bernhard Riemann in 1866, he became the last living representative of an earlier Göttingen tradition dating back to Gauss. Indeed, much of his subsequent work was inspired by or connected with the names of Gauss and Riemann, but especially Dirichlet, though he soon went beyond them, creating his own personal legacy so deeply appreciated by Emmy Noether. This solidity was reflected in Dedekind's personal life, in his urbane manners, and in his letters to family and friends, such as those he wrote to his collaborator Heinrich Weber, which abound in warmth and genuine humility [10, 3: pp. 483–490]. He expressed his views with utmost clarity, sometimes sharply, but almost never in sarcastic language that might give offense to another party.

On Cantor's Final Years

Cantor exhibited no such inhibitions; his letters were often full of prodding and pleading, unflattering gossip, or on occasion violent outbursts. He wrote openly about his unhappiness in Halle and how he felt victimized by Kronecker and the Berlin clique, who longed for nothing more than to suppress his divinely inspired ideas. Knowing that he would eventually succumb to manic depression, one cannot read his letters without sensing his mental fragility and the signs pointing to the tragic last years of his life. Whereas Dedekind stood proudly and independently as a representative of a great legacy, maintaining a noble distance from the fray, Cantor desperately longed for the support of friends and allies. After taking the initiative to launch the DMV, he took little interest in the mundane affairs of the organization. Although he usually attended its annual conferences and occasionally stepped forward to deliver a lecture, Cantor was no doubt happiest when he could hold forth about his latest ideas before a small group of avid listeners, as he did in Brunswick in 1897 when Hilbert first fell under his spell.

Hilbert's public remarks about inconsistent sets from his lectures in Munich and Paris apparently never drew any significant notice, whereas others had in the meantime noted similar paradoxes already. It has sometimes been suggested that none of these mathematicians reacted with alarm because the massive constructs that led to these inconsistencies had no apparent bearing for applications of set theory to conventional mathematical theories. Yet the private correspondence and recollections of conversations described above fail to square with this interpretation. Already in 1897, Cantor had brought forth the problem of antinomies forcefully, and the evidence strongly suggests that both Dedekind and Hilbert grasped very well the implications this had for the foundations of set theory. None of these insiders, however, had any idea how to resolve these difficulties.

As Abraham Fraenkel wrote, the last decades of Cantor's life were filled with growing recognition for his work [20]. Hilbert's Paris lecture was an important milestone along that path, and one can easily see how Hilbert's own rising fame was reflected in Cantor's legacy, which he continued to extol long afterward. Never at a loss for superlatives, Hilbert found these words of praise to commemorate him:

> It gives me great joy to contribute to the remembrance of Georg Cantor, who stands as one of the first in the succession of masters of our science. In originality and boldness of thought, there is no mathematician in history—from Euclid to Einstein—who surpassed him. He created something completely new: set theory. Its conceptual methods and applications have by now become the common property of all mathematicians, although I believe that it is only in recent decades that the deepest thoughts of his theory have had their greatest effects [47, p. 326].

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