Mediterr. J. Math. (2023) 20:177
https://doi.org/10.1007/s00009-023-02380-8

Mediterranean Journal of Mathematics

# Spectral Properties of Differential–Difference Symmetrized Operators

Krzysztof Stempak

**Abstract.** We investigate some spectral properties of differential–difference operators, which are symmetrizations of differential operators of the form $(\mathfrak{d}^\dagger\mathfrak{d})^k$ and $(\mathfrak{d}\mathfrak{d}^\dagger)^k$, $k \geqslant 1$. Here, $\mathfrak{d} = p\frac{d}{\mathrm{d}x} + q$ and $\mathfrak{d}^\dagger$ stands for the formal adjoint of $\mathfrak{d}$ on $L^2((0,b), w\,\mathrm{d}x)$. In the simpliest case $k = 1$, this symmetrization brings in the operator $-\mathfrak{D}^2$, which can be seen as a 'Laplacian', and $\mathfrak{D}f := \mathfrak{D}_{\mathfrak{d}}f = \mathfrak{d}(f_{\rm even}) - \mathfrak{d}^\dagger(f_{\rm odd})$, a skew-symmetric operator in $L^2(I, w\,\mathrm{d}x)$, $I = (-b, 0) \cup (0, b)$, is the symmetrization of $\mathfrak{d}$. Investigated spectral properties include self-adjoint extensions, among them the Friedrichs extensions, of the symmetrized operators.

**Mathematics Subject Classification.** 47B25, 34B24.

**Keywords.** Differential–difference operator, symmetrized operator, weak derivative, weighted Sobolev space, self-adjoint extension, Friedrichs extension.

## 1. Introduction

In this paper, we propose a treatment of some class of differential–difference operators in dimension one from the spectral theory point of view. These operators emerge as symmetrizations of differential operators on $(0, b)$, $0 < b \leqslant \infty$, admitting, in the simplest case, a decomposition of type $\mathcal{L}_{\mathfrak{d}} = \mathfrak{d}^\dagger\mathfrak{d}$; see Sect. 2.

In some specific frameworks, the analysis of the so-called Jacobi–Dunkl operators of both compact and non-compact types (see Examples 2.1 and 2.2) was initiated by Ben Salem and Samaali [1]. Some aspects of harmonic analysis of a (first-order) differential–difference 'derivative', a building block of the Jacobi–Dunkl operator, in the compact case were investigated by Chouchene [3], and in the non-compact case by Chouchene et al. [4]; see again Examples 2.1 and 2.2.

Birkhäuser

Published online: 28 March 2023

Nowak and the author [8] established a general symmetrization procedure in the context of orthogonal expansions associated with a second-order differential operator $\mathcal{L}$, a 'Laplacian'. Roughly, the important point of the symmetrization procedure consisted in removing asymmetry of the decomposition $\mathcal{L} = \mathfrak{d}^\dagger \mathfrak{d} + a$, $a \in \mathbb{R}$, where $\mathfrak{d}$ is an associated 'derivative' and $\mathfrak{d}^\dagger$ is its formal adjoint. This general theory was constructed in the multi-dimensional setting and permitted to shed a new light on the theory of higher order Riesz transforms for orthogonal expansions. In some cases of orthogonal expansions, the theory was applied with emphasis on specific harmonic analysis issues. For instance, Langowski [7] studied the symmetrized Jacobi expansions with emphasis on potential and Sobolev spaces. See also [5]. Nowak, Szarek and the author [9] discussed the symmetrized Laguerre expansions with focus on some harmonic analysis operators.

Recently, the author [11] investigated spectral properties of ordinary differential operators admitting the beforementioned decomposition. The present paper continues this line of investigation but in the setting of symmetrized operators, i.e., differential–difference operators. Some aspects of such investigation were already undertaken by the author [12] in a specific case of the Jacobi–Dunkl operator of compact type. It is worth mentioning that the theory presented in this paper: (a) does not refer to orthogonal expansions, i.e., an associated orthonormal system is not postulated (as it was done in [8]); (b) includes not only second-order but also higher order differential–difference operators.

The Friedrichs extensions of differential operators, notably for the Sturm–Liouville operators, were widely investigated in the literature. See [13] and the references therein. In this paper, we describe the Friedrichs extension of the differential–difference operators $(-1)^k\mathfrak{D}^{2k}$ on $L^2(I, w)$; see Theorem 4.6. The description is given in terms of $\mathfrak{D}$-derivatives and $\mathfrak{D}$-Sobolev spaces, and for $k = 1$, this extension can be seen as the 'Dirichlet Laplacian'.

The paper is organized as follows. In Sect. 2, we recall the setting of Sturm–Liouville operators admitting a special decomposition and outline a symmetrization procedure leading to the setting of the corresponding differential–difference operators; the Liouville form of the latter operators is also discussed. This is then illustrated by two important examples of Jacobi differential operators and Jacobi–Dunkl differential–difference operators of both compact and non-compact types. In Sect. 3, we introduce and investigate weak $\mathfrak{D}$-derivatives and compare them with weak $\mathfrak{d}$- and $\mathfrak{d}^\dagger$-derivatives. In particular, we establish relations between the weak derivatives $\mathfrak{D}^{(2k)}_{\text{weak}}$ and $(\mathfrak{d}^\dagger\mathfrak{d})^k_{\text{weak}}$ or $(\mathfrak{d}\mathfrak{d}^\dagger)^k_{\text{weak}}$, $k \geqslant 1$, cf. Proposition 3.7. Section 4 is devoted to introducing and studying $\mathfrak{D}$-Sobolev spaces, subsequently applied to describing the minimal and maximal operators and the Friedrichs extensions. Again, we relate the Sobolev spaces in the $\mathfrak{D}$ and $\mathfrak{d}$ settings, cf. Proposition 4.4. All such relations, suggested by (2.3), find their cumulation in Proposition 4.7. Finally, Theorem 4.6 contains the main result on the Friedrichs extension of $(-1)^k\mathfrak{D}^{2k}$.

**Notation and Terminology**. Throughout the paper, we use standard notions and symbols. Thus, given a function $f$ on $(-b,0)\cup(0,b)$, $0<b\leqslant\infty$, we write $f_{\text{even}}$ and $f_{\text{odd}}$ for its even and odd parts, $f_{\text{even}}(x):=(f(x)+f(-x))/2$, $f_{\text{odd}}(x):=(f(x)-f(-x))/2$, respectively. We shall frequently use the fact that

$$\int_I f_{\text{even/odd}}\,(x)g(x)w(x)\mathrm{d}x=\int_I f(x)g_{\text{even/odd}}\,(x)w(x)\mathrm{d}x, \tag{1.1}$$

whenever the integral (on the left-hand side, say) exist. We also use fairly standard notation for (complex-valued) function spaces. For instance, $AC_{\text{loc}}$ $(\mathcal{O})$, where $\mathcal{O}$ stands for an open subset of $\mathbb{R}$, denotes the space of all functions $f$ on $\mathcal{O}$, such that $f\in AC[\alpha,\beta]$, for every bounded interval $[\alpha,\beta]\subset\mathcal{O}$. Weak derivatives of a function $f$ will be denoted by $f'_{\text{weak}}$, $f''_{\text{weak}}$, $f^{(n)}_{\text{weak}}$. The symbol $\langle\cdot,\cdot\rangle_{L^2}$ will mean the inner product in a relevant $L^2$ space. Given an open interval $J\subset\mathbb{R}$ and a set of Sturm–Liouville coefficients $\{v,r,s\}$, that is a triple of real-valued functions on $J$ satisfying some natural smoothness and positivity assumptions, the associated Sturm–Liouville differential expression is

$$\mathcal{L}_{\{v,r,s\}}=\frac{1}{v(x)}\Big(-\frac{d}{\mathrm{d}x}\big(r(x)\frac{d}{\mathrm{d}x}\big)+s(x)\Big).$$

One can associate with $\mathcal{L}_{\{v,r,s\}}$ a boundary value problem or an unbounded operator on $L^2(J,v(x)\,\mathrm{d}x)$.

We shall also apply the following convention: by affixing one of the superscripts '$-/+$' to an object originally considered on $(-b,0)\cup(0,b)$, we mean the restriction of this object to $(-b,0)$ or $(0,b)$, respectively. For instance, for a function $f$ on $(-b,0)\cup(0,b)$, $f^+$ stands for the restriction of $f$ to $(0,b)$.

## 2. Preliminaries

Let $0<b\leqslant\infty$ and $I=(-b,0)\cup(0,b):=I^-\cup I^+$ be given, and let $w$ be a weight function on $I^+$, by which we mean a real-valued positive $C^\infty$ function. For real-valued $p,q\in C^\infty(I^+)$, $p(x)>0$ for $x\in I^+$, consider the first-order linear differential expression

$$\mathfrak{d}=\mathfrak{d}_{\{w,p,q\}}=p(x)\frac{d}{\mathrm{d}x}+q(x),$$

treated as an operator on the Hilbert space $L^2(I^+,w\,\mathrm{d}x)$. We call $\mathfrak{d}$ *the delta-derivative associated with the triple* $\{w,p,q\}$. The formal adjoint to $\mathfrak{d}$ in $L^2(I^+,w)$, in the sense that

$$\langle\mathfrak{d}\varphi,\psi\rangle_{L^2(I^+,w)}=\langle\varphi,\mathfrak{d}^\dagger\psi\rangle_{L^2(I^+,w)},\qquad \varphi,\psi\in C_c^\infty(I^+), \tag{2.1}$$

is

$$\mathfrak{d}^\dagger=-p(x)\frac{d}{\mathrm{d}x}+q^\dagger(x),$$

where

$$q^\dagger(x)=q(x)-p(x)\frac{w'(x)}{w(x)}-p'(x).$$

Note that $\mathfrak{d}^\dagger$ is the delta-derivative associated with the *dual triple* $\{w, -p, q^\dagger\}$. Also, notice that $(q^\dagger)^\dagger = q$ and $(\mathfrak{d}^\dagger)^\dagger = \mathfrak{d}$ and, in general, skew-symmetry does not hold, $\mathfrak{d}^\dagger \neq -\mathfrak{d}$. This lack of skew-symmetry gave an impact for considering a symmetrization process, see [8].

In this symmetrization of $\mathfrak{d} = \mathfrak{d}_{\{w,p,q\}}$ the functions from the triple $\{w, p, q\}$, initially considered on $I^+$, are extended to functions on $I$ by setting

$$w(-x) = w(x), \quad p(-x) = p(x), \quad q(-x) = -q(x), \qquad x \in I^+, \tag{2.2}$$

so that $w$ and $p$ are even and $q$ (and $q^\dagger$) is odd. Then, we let

$$\mathfrak{D}_{\mathfrak{d}} f := \mathfrak{d}(f_{\text{even}}) - \mathfrak{d}^\dagger(f_{\text{odd}}). \tag{2.3}$$

Here, we treat $\mathfrak{d}$ and $\mathfrak{d}^\dagger$ as operators acting on suitable functions on $I$, with $p$, $q$, and $w$ extended by (2.2). Notice that

- $\mathfrak{d} f$ (and $\mathfrak{d}^\dagger f$) is even/odd for $f$ odd/even,
- $\mathfrak{D}_{\mathfrak{d}} f$ is even/odd for $f$ odd/even,
- $(\mathfrak{D}_{\mathfrak{d}} f)^+ = \mathfrak{d}(f^+)$ for $f$ even and $(\mathfrak{D}_{\mathfrak{d}} f)^+ = -\mathfrak{d}^\dagger(f^+)$ for $f$ odd.

It is also worth observing that

$$\mathfrak{D}_{\mathfrak{d}} f = p\frac{\mathrm{d}f}{\mathrm{d}x} + q f_{\text{even}} - q^\dagger f_{\text{odd}} = p\frac{\mathrm{d}f}{\mathrm{d}x} + \frac{1}{2}(q - q^\dagger) f + \frac{1}{2}(q + q^\dagger)\check{f},$$

where $\check{f}$ denotes the reflection of $f$, $\check{f}(x) = f(-x)$, $x \in I$.

Checking that $\mathfrak{D} := \mathfrak{D}_{\mathfrak{d}}$ is skew-symmetric in $L^2(w) := L^2(I, w\,\mathrm{d}x)$ in the sense that

$$\langle \mathfrak{D}\varphi, \psi\rangle_{L^2(w)} = -\langle \varphi, \mathfrak{D}\psi\rangle_{L^2(w)}, \qquad \varphi, \psi \in C_c^\infty(I),$$

is not difficult using the above representations (as it was explained in [8]), but this can be also seen as a consequence of (2.1). Namely, for $\varphi, \psi \in C_c^\infty(I)$, using (2.3) and the identities (consequences of (1.1))

$$\langle f, g\rangle_{L^2(I,w)} = 2\langle f, g_{\text{even/odd}}\rangle_{L^2(I^+,w)},$$

valid for even/odd $f \in L^2(w)$, respectively, one gets

$$\begin{aligned}
\langle \mathfrak{D}\varphi, \psi\rangle_{L^2(I,w)} &= \langle \mathfrak{d}(\varphi_{\text{even}}), \psi\rangle_{L^2(I,w)} - \langle \mathfrak{d}^\dagger(\varphi_{\text{odd}}), \psi\rangle_{L^2(I,w)} \\
&= 2\big[\langle \mathfrak{d}(\varphi_{\text{even}}), \psi_{\text{odd}}\rangle_{L^2(I^+,w)} - \langle \mathfrak{d}^\dagger(\varphi_{\text{odd}}), \psi_{\text{even}}\rangle_{L^2(I^+,w)}\big] \\
&= 2\big[\langle \varphi_{\text{even}}, \mathfrak{d}^\dagger(\psi_{\text{odd}})\rangle_{L^2(I^+,w)} - \langle \varphi_{\text{odd}}, \mathfrak{d}(\psi_{\text{even}})\rangle_{L^2(I^+,w)}\big] \\
&= \langle \varphi, \mathfrak{d}^\dagger(\psi_{\text{odd}})\rangle_{L^2(I,w)} - \langle \varphi, \mathfrak{d}(\psi_{\text{even}})\rangle_{L^2(I,w)} \\
&= -\langle \varphi, \mathfrak{D}\psi\rangle_{L^2(I,w)}.
\end{aligned}$$

Since $\mathfrak{D}$ maps $C_c^\infty(I)$ into $C_c^\infty(I)$, we can consider, and we do this, the operators $\mathfrak{D}^n$, $n \geqslant 1$, as densely defined operators on $L^2(w)$ with domain $C_c^\infty(I)$. We let $L_{\mathfrak{D}}^{(k)} = (-1)^k \mathfrak{D}^{2k}$, $k \geqslant 1$, writing simply $L_{\mathfrak{D}}$ when $k = 1$. Notice that

- $L_{\mathfrak{D}}^{(k)}$ is symmetric and nonnegative on $L^2(w)$,
- for $f$ even, $\big(L_{\mathfrak{D}}^{(k)} f\big)^+ = (\mathfrak{d}^\dagger \mathfrak{d})^k (f^+)$,
- for $f$ odd, $\big(L_{\mathfrak{D}}^{(k)} f\big)^+ = (\mathfrak{d}\mathfrak{d}^\dagger)^k (f^+)$.

For further reference, we note (cf. [8]) that explicitly

$$L_{\mathfrak{D}} = -p^2\frac{d^2}{\mathrm{d}x^2} - \big(2pp' + p^2\frac{w'}{w}\big)\frac{d}{\mathrm{d}x} + qq^\dagger - pq'(\cdot)_{\mathrm{even}} + p(q^\dagger)'(\cdot)_{\mathrm{odd}}$$
$$= -p^2\frac{d^2}{\mathrm{d}x^2} - \big(2pp' + p^2\frac{w'}{w}\big)\frac{d}{\mathrm{d}x} + \big(qq^\dagger - \frac{1}{2}p(q' - (q^\dagger)')\big)$$
$$-\frac{1}{2}p\big(q' + (q^\dagger)'\big)\big(\,\cdot\,\big)\check{}. \tag{2.4}$$

We pause for a moment to point out a simple property of function spaces on $I = I^- \cup I^+$ that will be used throughout (usually without mention). For instance, for the spaces of smooth compactly supported functions, we identify $C_c^\infty(I)$ with the direct sum $C_c^\infty(I^-)\oplus C_c^\infty(I^+)$, in the sense that $\varphi \in C_c^\infty(I)$ is associated with $\varphi^\pm \in C_c^\infty(I^\pm)$. Analogous identification will concern $L^1_{\mathrm{loc}}(I)$, $L^2(I, w)$, $AC_{\mathrm{loc}}(I)$ and other function spaces.

Spectral analysis of operators of the form $\mathcal{L}_{\mathfrak{d}} = \mathfrak{d}^\dagger\mathfrak{d}$ (more generally $\mathcal{L}_{\mathfrak{d}} + a$, $a \in \mathbb{R}$) with domain $C_c^\infty(I^+)$, which are symmetric and nonnegative on $L^2(I^+, w)$, was recently performed by the author in [11]. Notice that each $\mathcal{L}_{\mathfrak{d}}$ is a Sturm–Liouville operator on $I^+$, corresponding to the Sturm–Liouville triple $\{w, r, s\}$ with $r := wp^2$ and $s := w(q^\dagger q - pq')$. For the Sturm–Liouville operators in divergent form $\mathcal{L}_{\{w,w,0\}}$, we have $\mathcal{L}_{\{w,w,0\}} = \mathcal{L}_{\mathfrak{d}}$ with

$$\mathfrak{d} = \frac{d}{\mathrm{d}x}, \qquad \mathfrak{d}^\dagger = -\frac{d}{\mathrm{d}x} - \frac{w'}{w} \quad \text{and} \quad L_{\mathfrak{d}} = -\frac{d^2}{\mathrm{d}x^2} - \frac{w'}{w}\frac{d}{\mathrm{d}x}. \tag{2.5}$$

Consequently, we have

$$\mathfrak{D}f = \frac{\mathrm{d}f}{\mathrm{d}x} + \frac{w'}{w}f_{\mathrm{odd}} \quad \text{and} \quad L_{\mathfrak{D}}f = -\frac{d^2 f}{\mathrm{d}x^2} - \frac{w'}{w}\frac{\mathrm{d}f}{\mathrm{d}x} - \Big(\frac{w'}{w}\Big)'f_{\mathrm{odd}}. \tag{2.6}$$

We now analyse two specific examples of differential operators which are in divergent form, and their symmetrizations, so that (2.5) and (2.6) are used.

*Example* 2.1. Given parameters $\alpha, \beta \in \mathbb{R}$, let

$$w_{\alpha,\beta}(\theta) = \Big|\sin\frac{\theta}{2}\Big|^{2\alpha+1}\Big(\cos\frac{\theta}{2}\Big)^{2\beta+1}, \qquad \theta \in (-\pi, \pi).$$

Consider the *Jacobi operator*

$$\mathcal{J}_{\alpha,\beta} = -\frac{1}{w_{\alpha,\beta}}\Big(\frac{d}{\mathrm{d}x}\Big(w_{\alpha,\beta}\frac{d}{\mathrm{d}x}\Big)\Big)$$
$$= -\frac{d^2}{\mathrm{d}\theta^2} - \Big((\alpha+1/2)\cot\frac{\theta}{2} - (\beta+1/2)\tan\frac{\theta}{2}\Big)\frac{d}{\mathrm{d}\theta},$$

in the $L^2((0,\pi), w_{\alpha,\beta}\,\mathrm{d}\theta)$ setting. We have $\mathcal{J}_{\alpha,\beta} = \mathfrak{d}^\dagger_{\alpha,\beta}\mathfrak{d}_{\alpha,\beta}$ with

$$\mathfrak{d}_{\alpha,\beta} = \frac{d}{\mathrm{d}\theta}, \qquad \mathfrak{d}^\dagger_{\alpha,\beta} = -\frac{d}{\mathrm{d}\theta} - \Big((\alpha+1/2)\cot\frac{\theta}{2} - \Big(\beta+1/2)\tan\frac{\theta}{2}\Big).$$

The symmetrization of $\mathfrak{d}_{\alpha,\beta}$ brings in the skew-symmetric operator on $L^2((-\pi,\pi), w_{\alpha,\beta}\,\mathrm{d}\theta)$

$$\mathfrak{D}_{\alpha,\beta}f = \frac{\mathrm{d}f}{\mathrm{d}\theta} + \Big((\alpha+1/2)\cot\frac{\theta}{2} - \big(\beta+1/2\big)\tan\frac{\theta}{2}\Big)f_{\mathrm{odd}}, \tag{2.7}$$

and the symmetrized version of $\mathcal{J}_{\alpha,\beta}$ is $J_{\alpha,\beta} = -\mathfrak{D}^2_{\alpha,\beta}$,

$$J_{\alpha,\beta}f = -\frac{d^2 f}{\mathrm{d}\theta^2} - \frac{\alpha-\beta+(\alpha+\beta+1)\cos\theta}{\sin\theta}\frac{\mathrm{d}f}{\mathrm{d}\theta} + \frac{(\alpha+\beta+1)+(\alpha-\beta)\cos\theta}{\sin^2\theta} f_{\mathrm{odd}} .$$

This is a second-order differential–difference operator called the *Jacobi–Dunkl operator* of compact type. For $\alpha=\beta=-1/2$, $\mathcal{J}_{\alpha,\beta}$ and $J_{\alpha,\beta}$ reduce to $-\frac{d^2}{\mathrm{d}\theta^2}$ in the $L^2((0,\pi),\mathrm{d}\theta)$ and $L^2((-\pi,\pi),\mathrm{d}\theta)$ settings, respectively.

Some aspects of harmonic analysis of $\mathfrak{D}_{\alpha,\beta}$ in the compact case (and under restriction $\alpha \geqslant \beta \geqslant -1/2$, $\alpha \neq -1/2$) were investigated by Chouchene [3], who initiated the study of this operator (denoted in [3] and [1] by $\Lambda_{\alpha,\beta}$), and in the non-compact case (and under restriction $\alpha > -1/2$, $\beta \in \mathbb{R}$) by Chouchene et al. [4]. See also [2], it is convenient to observe here that

$$\frac{w'_{\alpha,\beta}}{w_{\alpha,\beta}} = \frac{\alpha-\beta+(\alpha+\beta+1)\cos\theta}{\sin\theta}, \qquad \Big(\frac{w'_{\alpha,\beta}}{w_{\alpha,\beta}}\Big)' = -\frac{(\alpha+\beta+1)+(\alpha-\beta)\cos\theta}{\sin^2\theta}.$$

*Example* 2.2. Given parameters $\alpha,\beta\in\mathbb{R}$, consider the *Jacobi (function) operator*

$$\hat{\mathcal{J}}_{\alpha,\beta} = -\frac{1}{\hat{w}_{\alpha,\beta}}\Big(\frac{d}{\mathrm{d}x}\big(\hat{w}_{\alpha,\beta}\frac{d}{\mathrm{d}x}\big)\Big) = -\frac{d^2}{\mathrm{d}x^2} - \big((2\alpha+1)\coth x + (2\beta+1)\tanh x\big)\frac{d}{\mathrm{d}x}$$

in the $L^2((0,\infty),\hat{w}_{\alpha,\beta}\,\mathrm{d}x)$ setting, where

$$\hat{w}_{\alpha,\beta}(x) = \big|\sinh x\big|^{2\alpha+1}(\cosh x)^{2\beta+1}, \qquad x\in(-\infty,\infty).$$

We have $\hat{\mathcal{J}}_{\alpha,\beta} = \mathfrak{d}^{\dagger}_{\alpha,\beta}\mathfrak{d}_{\alpha,\beta}$ with

$$\mathfrak{d}_{\alpha,\beta} = \frac{d}{\mathrm{d}x}, \qquad \mathfrak{d}^{\dagger}_{\alpha,\beta} = -\frac{d}{\mathrm{d}x} - \big((2\alpha+1)\coth x + (2\beta+1)\tanh x\big).$$

The symmetrization of $\mathfrak{d}_{\alpha,\beta}$ brings in the skew-symmetric operator on $L^2((-\infty,\infty),\hat{w}_{\alpha,\beta}\,\mathrm{d}x)$

$$\mathfrak{D}_{\alpha,\beta}f = \frac{\mathrm{d}f}{\mathrm{d}x} + \big((2\alpha+1)\coth x + (2\beta+1)\tanh x\big)f_{\mathrm{odd}},$$

and the symmetrized version of $\hat{\mathcal{J}}_{\alpha,\beta}$ is $\hat{J}_{\alpha,\beta} = -\mathfrak{D}^2_{\alpha,\beta}$. Explicitly (see [1, p. 368] or (2.4))

$$\hat{J}_{\alpha,\beta}f = -\frac{d^2 f}{\mathrm{d}x^2} - \big((2\alpha+1)\coth x + (2\beta+1)\tanh x\big)f + \Big(\frac{2\beta+1}{\cosh^2 x} - \frac{2\alpha+1}{\sinh^2 x}\Big)f_{\mathrm{odd}},$$

is a second-order differential–difference operator called the *Jacobi–Dunkl operator* of non-compact type. Again, for $\alpha=\beta=-1/2$, $\hat{\mathcal{J}}_{\alpha,\beta}$ and $\hat{J}_{\alpha,\beta}$ reduce to $-\frac{d^2}{\mathrm{d}x^2}$ on $L^2((0,\infty),\mathrm{d}x)$ and $L^2((-\infty,\infty),\mathrm{d}x)$, respectively.

The analysis of the Jacobi–Dunkl operators of both compact and non-compact types (under some restrictions on $\alpha$ and $\beta$) was initiated by Ben Salem and Samaali [1]. See also [12], where analysis of $J_{\alpha,\beta}$ was performed with emphasis on the so-called exotic cases.

### 2.1. Liouville Form

Although the term 'Liouville (normal) form' seems to be reserved for differential contexts, we adopt it here in the differential–difference framework.

We first recall the setting of Sturm–Liouville operators discussed in [11, Sect. 6.2]. Consider a weight $w \not\equiv 1$ on $I^+$. Together with an operator $\mathcal{L}_{\{w,r,s\}}$ acting on $L^2(I^+, w)$, its replique in the $L^2(I^+, \mathrm{d}x)$ setting defined through the unitary isomorphism $U^+ \colon L^2(I^+, w) \to L^2(I^+, \mathrm{d}x)$, $U^+ f = \sqrt{w} f$, is frequently considered. We have in mind the operator

$$\mathcal{L}^{\circ}_{\{w,r,s\}} := U^+ \circ \mathcal{L}_{\{w,r,s\}} \circ (U^+)^{-1},$$

called the Liouville form of $\mathcal{L}_{\{w,r,s\}}$. The operators $\mathcal{L}^{\circ}_{\{w,r,s\}}$ and $\mathcal{L}_{\{w,r,s\}}$, being unitarily intertwined by $U^+$, possess the same spectral properties.

A computation shows that for operators in divergent form, we have $\mathcal{L}^{\circ}_{\{w,w,0\}} = \mathcal{L}_{\{1,1,\hat{s}\}}$ with

$$\hat{s} = \frac{w''}{2w} - \left(\frac{w'}{2w}\right)^2,$$

so that $\mathcal{L}^{\circ}_{\{w,w,0\}}$ becomes a Schrödinger operator with potential $\hat{s}$. Clearly, for $\mathcal{L}_{\{w,w,0\}}$ of the form $\mathcal{L}_{\{w,w,0\}} = \mathcal{L}_{\mathfrak{d}}$, with $\mathfrak{d}$ and $\mathfrak{d}^{\dagger}$ given by (2.5), we have the decomposition $(\mathcal{L}_{\mathfrak{d}})^{\circ} = \mathfrak{d}^{\circ,\dagger}\mathfrak{d}^{\circ}$ with

$$\mathfrak{d}^{\circ} = U^+ \circ \mathfrak{d} \circ (U^+)^{-1}, \qquad \mathfrak{d}^{\circ,\dagger} = U^+ \circ \mathfrak{d}^{\dagger} \circ (U^+)^{-1}.$$

Explicitly

$$\mathfrak{d}^{\circ} = \frac{d}{\mathrm{d}x} - \frac{w'}{2w}, \qquad \mathfrak{d}^{\circ,\dagger} = -\frac{d}{\mathrm{d}x} - \frac{w'}{2w}.$$

Passing to the symmetrized case, i.e., to the case of $\mathfrak{D} = \mathfrak{D}_{\mathfrak{d}}$, consider the unitary isomorphism

$$U \colon L^2(I, w) \to L^2(I, \mathrm{d}x), \qquad Uf = \sqrt{w} f,$$

and let

$$\mathfrak{D}^{\circ} := U \circ \mathfrak{D} \circ U^{-1}, \qquad L^{\circ}_{\mathfrak{D}} := U \circ L_{\mathfrak{D}} \circ U^{-1}.$$

Then, $L^{\circ}_{\mathfrak{D}} = -(\mathfrak{D}^{\circ})^2$ is also a differential–difference operator. We call $L^{\circ}_{\mathfrak{D}}$ the Liouville form of $L_{\mathfrak{D}}$. Clearly, being unitarily intertwined, $L_{\mathfrak{D}}$ and $L^{\circ}_{\mathfrak{D}}$ possess analogous spectral properties and we can choose $L^{\circ}_{\mathfrak{D}}$ to analyse spectral properties of $L_{\mathfrak{D}}$. This remark allows to avoid double discussion of an operator and its twin, and, at the same moment, permits to analyse potentially easier form of an operator ($\mathcal{L}^{\circ}_{\mathfrak{D}}$ is a sum of a Schrödinger operator plus a reflection term). It should be remarked that $L^{\circ}_{\mathfrak{D}_{\mathfrak{d}}} = L_{\mathfrak{D}_{\mathfrak{d}^{\circ}}}$, and hence, $L^{\circ}_{\mathfrak{D}_{\mathfrak{d}}} = L_{\mathfrak{D}_{\mathfrak{d}^{\circ}}}$ (in words: the Liouville form of symmetrization is the symmetrization of Liouville form). Explicitly

$$\mathfrak{D}_{\mathfrak{d}^{\circ}} = \frac{d}{\mathrm{d}x} - \frac{w'}{2w}(\,\cdot\,)\check{}, \qquad L_{\mathfrak{D}_{\mathfrak{d}^{\circ}}} = -\frac{d^2}{\mathrm{d}x^2} + \left(\frac{w'}{2w}\right)^2 + \left(\frac{w'}{2w}\right)'(\,\cdot\,)\check{}.$$

We continue the analysis begun in Examples 2.1 and 2.2. Here, we exclude the case $\alpha = \beta = -1/2$, since then, the corresponding weight functions equal 1 identically. In the examples, $\mathfrak{d}^{\circ,\dagger}_{\alpha,\beta}$ stands for the formal adjoint of $\mathfrak{d}^{\circ}_{\alpha,\beta}$

on $L^2((0,\pi),\mathrm{d}\theta)$ or on $L^2((0,\infty),\mathrm{d}x)$, respectively, and $\mathfrak{D}^{\circ}_{\alpha,\beta}$ is skew-adjoint on $L^2((-\pi,\pi),\mathrm{d}\theta)$ or $L^2((-\infty,\infty),\mathrm{d}x)$, respectively.

*Example* 2.3. The Liouville form of $\mathcal{J}_{\alpha,\beta}$ is

$$\mathcal{J}^{\circ}_{\alpha,\beta} = -\frac{d^2}{\mathrm{d}\theta^2} + \frac{\alpha^2-1/4}{4\sin^2\frac{\theta}{2}} - \frac{\beta^2-1/4}{4\cos^2\frac{\theta}{2}},$$

and it has the decomposition $\mathcal{J}^{\circ}_{\alpha,\beta} = \mathfrak{d}^{\circ,\dagger}_{\alpha,\beta}\mathfrak{d}^{\circ}_{\alpha,\beta}$, where (see, e.g., [7, p. 3])

$$\mathfrak{d}^{\circ}_{\alpha,\beta} = \frac{d}{\mathrm{d}\theta} - \Big(\frac{\alpha+1/2}{2}\cot\frac{\theta}{2} - \frac{\beta+1/2}{2}\tan\frac{\theta}{2}\Big),$$
$$\mathfrak{d}^{\circ,\dagger}_{\alpha,\beta} = -\frac{d}{\mathrm{d}\theta} - \Big(\frac{\alpha+1/2}{2}\cot\frac{\theta}{2} - \frac{\beta+1/2}{2}\tan\frac{\theta}{2}\Big).$$

The Liouville form of $J_{\alpha,\beta}$, and at the same moment the symmetrization of $\mathcal{J}^{\circ}_{\alpha,\beta}$, is

$$J^{\circ}_{\alpha,\beta}f = -\frac{d^2 f}{\mathrm{d}\theta^2} + \Big(\frac{\alpha+1/2}{2}\cot\frac{\theta}{2} - \frac{\beta+1/2}{2}\tan\frac{\theta}{2}\Big)^2 f$$
$$-\Big(\frac{\alpha+1/2}{4\sin^2\frac{\theta}{2}} + \frac{\beta+1/2}{4\cos^2\frac{\theta}{2}}\Big)\big(f_{\mathrm{even}} - f_{\mathrm{odd}}\big),$$

with the corresponding decomposition $J^{\circ}_{\alpha,\beta} = -(\mathfrak{D}^{\circ}_{\alpha,\beta})^2$, where (with notation $\mathbb{D}_{\alpha,\beta}$ in [7, p. 3])

$$\mathfrak{D}^{\circ}_{\alpha,\beta}f = \frac{\mathrm{d}f}{\mathrm{d}\theta} - \Big(\frac{\alpha+1/2}{2}\cot\frac{\theta}{2} - \frac{\beta+1/2}{2}\tan\frac{\theta}{2}\Big)\big(f_{\mathrm{even}} - f_{\mathrm{odd}}\big).$$

*Example* 2.4. The Liouville form of $\hat{\mathcal{J}}_{\alpha,\beta}$ is (see [11, Sect. 7.4])

$$\mathcal{J}^{\circ}_{\alpha,\beta} = -\frac{d^2}{\mathrm{d}x^2} + \big(\alpha^2 - \frac{1}{4}\big)\coth^2 x + \big(\beta^2 - \frac{1}{4}\big)\tanh^2 x + c_{\alpha,\beta},$$

$c_{\alpha,\beta} = (\alpha+1/2)(\beta+1/2) + \alpha + \beta + 1$, and it has the decomposition $\hat{\mathcal{J}}^{\circ}_{\alpha,\beta} = \mathfrak{d}^{\circ,\dagger}_{\alpha,\beta}\mathfrak{d}^{\circ}_{\alpha,\beta}$, where

$$\mathfrak{d}^{\circ}_{\alpha,\beta} = \frac{d}{\mathrm{d}\theta} - \Big(\big(\alpha+\frac{1}{2}\big)\coth x + \big(\beta+\frac{1}{2}\big)\tanh x\Big),$$
$$\mathfrak{d}^{\circ,\dagger}_{\alpha,\beta} = -\frac{d}{\mathrm{d}\theta} - \Big((\alpha+\frac{1}{2})\coth x + (\beta+\frac{1}{2})\tanh x\Big).$$

The Liouville form of $\hat{J}_{\alpha,\beta}$ is

$$\hat{J}^{\circ}_{\alpha,\beta}f = -\frac{d^2 f}{\mathrm{d}x^2} + \Big(\big(\alpha+\frac{1}{2}\big)\coth x + \big(\beta+\frac{1}{2}\big)\tanh x\Big)^2 f$$
$$-\Big(\frac{\alpha+1/2}{\sinh^2\frac{\theta}{2}} - \frac{\beta+1/2}{\cosh^2\frac{\theta}{2}}\Big)\big(f_{\mathrm{even}} - f_{\mathrm{odd}}\big),$$

and it has the decomposition $\hat{J}^{\circ}_{\alpha,\beta} = -(\mathfrak{D}^{\circ}_{\alpha,\beta})^2$, where

$$\mathfrak{D}^{\circ}_{\alpha,\beta}f = \frac{\mathrm{d}f}{\mathrm{d}x} - \Big(\big(\alpha+\frac{1}{2}\big)\coth x + \big(\beta+\frac{1}{2}\big)\tanh x\Big)\big(f_{\mathrm{even}} - f_{\mathrm{odd}}\big).$$

## 3. $\mathfrak{D}$-derivatives

Until the end of this and the next section, $I$, $w$, $\mathfrak{d}$, and thus $\mathfrak{D} := \mathfrak{D}_{\mathfrak{d}}$, are fixed. In several places, we shall tacitly use the fact that for a continuous and positive function $w$ on $I$, one has $L^1_{\text{loc}}(I, w) = L^1_{\text{loc}}(I, \mathrm{d}x)$; for this space, we shall write $L^1_{\text{loc}}(w)$ for short.

We begin with notion of the *weak* $\mathfrak{D}^{(k)}$*-derivative*, $k \geqslant 1$.

**Definition 3.1.** Let $f \in L^1_{\text{loc}}(w)$ and $k \geqslant 1$. We say that weak $\mathfrak{D}^{(k)}$*-derivative of* $f$ exists provided that there is $g_k \in L^1_{\text{loc}}(w)$, such that

$$\int_I \mathfrak{D}^k \varphi(x) \overline{f(x)} w(x)\,\mathrm{d}x = (-1)^k \int_I \varphi(x) \overline{g_k(x)} w(x)\,\mathrm{d}x, \qquad \varphi \in C_c^\infty(I).$$

Then, we set $\mathfrak{D}^{(k)}_{\text{weak}} f := g_k$ and call it the weak $\mathfrak{D}^{(k)}$-derivative of $f$.

Following the general definition for open subsets of $\mathbb{R}$ and specified to $I$, we say that $f \in L^1_{\text{loc}}(I, \mathrm{d}x)$ has a weak derivative of order $k$, $k \geqslant 1$, provided that there exists $h_k \in L^1_{\text{loc}}(I, \mathrm{d}x)$, such that

$$\int_I \varphi^{(k)}(x) \overline{f(x)}\,\mathrm{d}x = (-1)^k \int_I \varphi(x) \overline{h_k(x)}\,\mathrm{d}x, \qquad \varphi \in C_c^\infty(I).$$

Then, we call $h_k$ the weak derivative of $f$ on $I$ and write $f^{(k)}_{\text{weak}} := h_k$; for $k = 1, 2$, we shall simply write $f'_{\text{weak}}$, $f''_{\text{weak}}$. It is obvious that existence of $f^{(k)}_{\text{weak}}$ on $I$ implies existence of $(f^\pm)^{(k)}_{\text{weak}}$ on $I^\pm$ and *vice versa*. In such cases, $(f^\pm)^{(k)}_{\text{weak}}$ are restrictions of $f^{(k)}_{\text{weak}}$ to $I^\pm$, respectively, and *vice versa*, $f^{(k)}_{\text{weak}}$, a function on $I$, is glued from $(f^\pm)^{(k)}_{\text{weak}}$ as functions on $I^\pm$.

Recall that existence of $k$th weak derivative of an $f \in L^1_{\text{loc}}(I, \mathrm{d}x)$ implies existence of weak derivatives of $f$ of all lower orders. We shall also need the fact that for $h \in C^\infty(I)$, existence of $f^{(k)}_{\text{weak}}$ implies existence of $(hf)^{(k)}_{\text{weak}}$, and with additional assumption $h > 0$, the opposite holds: if $(hf)^{(k)}_{\text{weak}}$ exists, then also $f^{(k)}_{\text{weak}}$ exists. See, for instance, [11] for details.

Before discussing connections between $\mathfrak{D}^{(k)}$-derivatives and weak derivatives, we need the following convenient representation of $\mathfrak{D}^k$, the formal $k$-fold composition of $\mathfrak{D}$.

**Proposition 3.2.** *Let* $k \geqslant 1$*. Then,* $\mathfrak{D}^k$ *can be represented as*

$$\mathfrak{D}^k = \sum_{j=0}^{k} p^{[k]}_{k-j} \Big(\frac{d}{\mathrm{d}x}\Big)^{k-j} + \sum_{j=1}^{k} \Big[ q^{[k,1]}_{k-j} \Big( \Big(\frac{d}{\mathrm{d}x}\Big)^{k-j} \Big)_{\text{even}} + q^{[k,2]}_{k-j} \Big( \Big(\frac{d}{\mathrm{d}x}\Big)^{k-j} \Big)_{\text{odd}} \Big], \tag{3.1}$$

*where* $p^{[k]}_{k-j}$ *and* $q^{[k,s]}_{k-j}$*,* $s = 1, 2$*, are real-valued* $C^\infty$ *functions on* $I$*, even when* $j$ *is even and odd when* $j$ *is odd. Moreover*

$$p^{[1]}_1 = p, \quad p^{[1]}_0 = 0, \quad q^{[1,1]}_0 = q, \quad q^{[1,2]}_0 = -q^\dagger,$$

*and for $k \geqslant 2$, we have the relations*

$$p_k^{[k]} = p^k, \quad p_m^{[k]} = p\Big((p_m^{[k-1]})' + p_{m-1}^{[k-1]}\Big), \quad m = k-1, \dots, 1, 0,$$

$$q_{k-1}^{[k,1]} = pq_{k-2}^{[k-1,2]} + qp^{k-1},$$

$$q_m^{[k,1]} = p\big(q_m^{[k-1,1]}\big)' + pq_{m-1}^{[k-1,2]} + qp_m^{[k-1]} + qq_m^{[k-1,1]}, \quad m = k-2, \dots, 1, 0,$$

*and*

$$q_{k-1}^{[k,2]} = pq_{k-2}^{[k-1,1]} - q^\dagger p^{k-1},$$

$$q_m^{[k,2]} = p\big(q_m^{[k-1,2]}\big)' + pq_{m-1}^{[k-1,1]} - q^\dagger p_m^{[k-1]} - q^\dagger q_m^{[k-1,2]}, \quad m = k-2, \dots, 1, 0.$$

*Here, by convention, $p_{-1}^{[k-1]} = q_{-1}^{[k-1,1]} = q_{-1}^{[k-1,2]} = 0$.*

*Proof.* For $k = 1$, one has $\mathfrak{D} = p\frac{d}{\mathrm{d}x} + q(\cdot)_{\mathrm{even}} - q^\dagger(\cdot)_{\mathrm{odd}}$, with $p$ even and $q, q^\dagger$ odd, $C^\infty$ and real-valued, as required. In the induction step, one obtains the relevant recurrence relations as direct consequences of $\mathfrak{D}^k = \mathfrak{D} \circ \mathfrak{D}^{k-1}$, $k \geqslant 2$. Checking the required properties of emerging function coefficients is easily performed using the induction hypothesis and the identities

$$\frac{d}{\mathrm{d}x}\big(f_{\mathrm{even/odd}}\big) = \left(\frac{d}{\mathrm{d}x}f\right)_{\mathrm{odd/even}},$$

that include the fact that the derivative of an even/odd function is an odd/even function. □

We shall also need the following simple lemma used in the proof of Proposition 3.4.

**Lemma 3.3.** *Let $k \geqslant 2$, $f \in L^1_{\mathrm{loc}}(w)$ and $\mathfrak{D}^{(k)}_{\mathrm{weak}} f$ exists. If for some $1 \leqslant m \leqslant k-1$ also $\mathfrak{D}^{(m)}_{\mathrm{weak}} f$ exists, then $\mathfrak{D}^{(k-m)}_{\mathrm{weak}}(\mathfrak{D}^{(m)}_{\mathrm{weak}} f)$ exists and equals $\mathfrak{D}^{(k)}_{\mathrm{weak}} f$.*

*Proof.* The proof is straightforward. By assumption

$$\forall \varphi \in C_c^\infty(I) \quad \int_I \mathfrak{D}^m \varphi(x)\overline{f(x)}w(x)\,\mathrm{d}x = (-1)^m \int_I \varphi(x)\overline{\mathfrak{D}^{(m)}_{\mathrm{weak}} f(x)}w(x)\,\mathrm{d}x \tag{3.2}$$

and

$$\forall \psi \in C_c^\infty(I) \quad \int_I \mathfrak{D}^k \psi(x)\overline{f(x)}w(x)\,\mathrm{d}x = (-1)^k \int_I \psi(x)\overline{\mathfrak{D}^{(k)}_{\mathrm{weak}} f(x)}w(x)\,\mathrm{d}x.$$

Taking any $\psi \in C_c^\infty(I)$ and inserting $\varphi = \mathfrak{D}^{k-m}\psi$ into (3.2) give

$$\int_I \mathfrak{D}^k \psi(x)\overline{f(x)}w(x)\,\mathrm{d}x = (-1)^m \int_I \mathfrak{D}^{k-m}\psi(x)\overline{\mathfrak{D}^{(m)}_{\mathrm{weak}} f(x)}w(x)\,\mathrm{d}x.$$

Comparing the last two identities gives

$$\forall \psi \in C_c^\infty(I) \quad \int_I \mathfrak{D}^{k-m}\psi(x)\overline{\mathfrak{D}^{(m)}_{\mathrm{weak}} f(x)}w(x)\,\mathrm{d}x$$
$$= (-1)^{k-m} \int_I \psi(x)\overline{\mathfrak{D}^{(k)}_{\mathrm{weak}} f(x)}w(x)\,\mathrm{d}x,$$

as required. □

It would be desirable to know, in analogy with [11, Proposition 1], that existence of $\mathfrak{D}^{(k)}_{\text{weak}}f$ for some $k \geqslant 2$ implies existence of $\mathfrak{D}^{(r)}_{\text{weak}}f$ for all $1 \leqslant r \leqslant k-1$. Unfortunately, currently, this remains to be an open question for the author.

**Proposition 3.4.** *Let $f \in L^1_{\text{loc}}(w)$ and $k \geqslant 1$. If $\mathfrak{D}^{(r)}_{\text{weak}}f$ exist for $r = 1, \ldots, k$, then $f^{(r)}_{\text{weak}}$ exist for $r = 1, \ldots, k$, and*

$$\forall r \in \{1, \ldots, k\} \quad \mathfrak{D}^{(r)}_{\text{weak}}f = \sum_{j=0}^{r} p^{[r]}_{r-j} f^{(r-j)}_{\text{weak}} + \sum_{j=1}^{r} \left[ q^{[r,1]}_{r-j} \left( f^{(r-j)}_{\text{weak}} \right)_{\text{even}} + q^{[r,2]}_{r-j} \left( f^{(r-j)}_{\text{weak}} \right)_{\text{odd}} \right], \tag{3.3}$$

*where $p^{[r]}_{r-j}$ and $q^{[r,s]}_{r-j}$, $s = 1, 2$, are as in (3.1). Conversely, if $f^{(k)}_{\text{weak}}$ exists, then $\mathfrak{D}^{(r)}_{\text{weak}}f$ exists for $r = 1, \ldots, k$, and one has (3.3).*

*Proof.* For the first claim, we proceed by induction and begin with $k = 1$. Let $\mathfrak{D}^{(1)}_{\text{weak}}f = g_1 \in L^1_{\text{loc}}(w)$. This means that for all $\varphi \in C^\infty_c(I)$, we have

$$\int_I \mathfrak{D}\varphi \overline{f}\, w\mathrm{d}x = -\int_I \varphi \overline{g_1}\, w\mathrm{d}x.$$

After routine manipulations (recall that $q$ and $q^\dagger$ are odd), this becomes

$$\int_I \varphi' \overline{fpw}\, \mathrm{d}x = -\int_I \varphi \overline{(g_1 + qf_{\text{odd}} - q^\dagger f_{\text{even}})w}\, \mathrm{d}x,$$

which means that the weak derivative $(fpw)'_{\text{weak}}$ exists (and equals $(g_1 + qf_{\text{odd}} - q^\dagger f_{\text{even}})w$). However, $pw > 0$ on $I$ is a $C^\infty$ function, and hence, also $f'_{\text{weak}}$ exists. It remains to justify (3.3) for $k = 1$. Since we know that $(fpw)'_{\text{weak}}$ and $f'_{\text{weak}}$ exist, we have $(fpw)'_{\text{weak}} = f'_{\text{weak}}pw + f(pw)'$, and hence, we can write

$$\begin{aligned} \int_I \mathfrak{D}\varphi \overline{f}\, w\mathrm{d}x &= -\int_I \varphi \overline{(f'_{\text{weak}}p + fp' + fpw'/w - qf_{\text{odd}} + q^\dagger f_{\text{even}})}\, w\mathrm{d}x \\ &= -\int_I \varphi \overline{(pf'_{\text{weak}} + qf_{\text{even}} - q^\dagger f_{\text{odd}})}\, w\mathrm{d}x. \end{aligned}$$

To proceed with the induction step, we introduce the notation

$$\langle F, G \rangle_w := \int_I F(x)\overline{G(x)}w(x)\, \mathrm{d}x,$$

whenever the integral on the right-hand side exists; we skip the subscript $w$ when $w \equiv 1$.

Let $k \geqslant 2$ and assume inductively that the claim holds for $k-1$. Next, take $f \in L^1_{\text{loc}}(w)$ and assume that for $1 \leqslant r \leqslant k$, $\mathfrak{D}^{(k)}_{\text{weak}}f$ exist; in particular

$$\langle \mathfrak{D}^k\varphi, f \rangle = (-1)^k \langle \varphi, g_k \rangle, \qquad \varphi \in C^\infty(I),$$

for some $g_k \in L^1_{\rm loc}(w)$. Using the representation (3.1), we also have

$$\begin{aligned}\langle \mathfrak{D}^k\varphi, f\rangle_w = \langle \varphi^{(k)}, p_k^{[k]} f w\rangle \\ &+ \sum_{j=1}^{k} \Big[\big\langle \varphi^{(k-j)}, p_{k-j}^{[k]} f w\big\rangle + \langle (\varphi^{(k-j)})_{\rm even}, q_{k-j}^{[k,1]} f w\rangle \\ &+ \big\langle (\varphi^{(k-j)})_{\rm odd}, q_{k-j}^{[k,2]} f w\big\rangle\Big].\end{aligned}$$

From this, one gets

$$\begin{aligned}\langle \varphi^{(k)}, p^k f w\rangle &= (-1)^k \langle \varphi, g_k\rangle \\ &- \sum_{j=1}^{k} \big\langle \varphi^{(k-j)}, p_{k-j}^{[k]} f w + (q_{k-j}^{[k,1]} f w)_{\rm odd} + (q_{k-j}^{[k,2]} f w)_{\rm even}\big\rangle.\end{aligned}$$

By the induction hypothesis, the weak derivatives of $f$ of order less than $k$ exist, and hence, the same is true for each of the terms $p_{k-j}^{[k]} f w$, $(q_{k-j}^{[k,1]} f w)_{\rm odd}$ and $(q_{k-j}^{[k,2]} f w)_{\rm even}$, $j = 1, \ldots, k$, so

$$\begin{aligned}\langle \varphi^{(k)}, p^k f w\rangle = \big\langle \varphi, (-1)^k g_k - \sum_{j=1}^{k} (-1)^{k-j} \big[p_{k-j}^{[k]} f w + (q_{k-j}^{[k,1]} f w)_{\rm odd} \\ + (q_{k-j}^{[k,2]} f w)_{\rm even}\big]^{(k-j)}_{\rm weak}\big\rangle.\end{aligned}$$

This means that the weak derivative of order $k$ of $p^k f w$ exists, and hence, also $f^{(k)}_{\rm weak}$ exists.

It remains to justify that the identity in (3.3) holds for $r = k$. We have

$$\forall \varphi \in C_c^\infty(I) \quad \int_I \mathfrak{D}^k \varphi(x) \overline{f(x)} w(x)\, {\rm d}x = (-1)^k \int_I \varphi(x) \overline{\mathfrak{D}^{(k)}_{\rm weak} f(x)} w(x)\, {\rm d}x.$$

Our aim is now to show that

$$\forall \varphi \in C_c^\infty(I) \quad \int_I \mathfrak{D}^k \varphi(x) \overline{f(x)} w(x)\, {\rm d}x = (-1)^k \int_I \varphi(x) \overline{\mathcal{R}_k f(x)} w(x)\, {\rm d}x, \tag{3.4}$$

where, for any $r \in \{1, \ldots, k\}$, $\mathcal{R}_r f$ denotes the right-hand side of the identity in (3.3). Then, combining the two above identities will give (3.3) for $r = k$.

To verify (3.4), we write the sequence of equalities

$$\begin{aligned}\int_I \mathfrak{D}^k \varphi\, \overline{f}\, w\, {\rm d}x &= (-1)^k \int_I \varphi\, \overline{\mathfrak{D}^{(1)}_{\rm weak}(\mathfrak{D}^{(k-1)}_{\rm weak} f)}\, w\, {\rm d}x \\ &= (-1)^k \int_I \varphi\, \overline{\mathfrak{D}^{(1)}_{\rm weak}(\mathcal{R}_{k-1} f)}\, w\, {\rm d}x \\ &= (-1)^k \int_I \varphi\, \overline{\mathcal{R}_k f}\, w\, {\rm d}x.\end{aligned}$$

The first equality is due to the induction hypothesis and Lemma 3.3, and the second is again due to the induction hypothesis. Finally, the third equality is obtained using the result from the first step of the induction procedure and combining it with relations for the coefficients involved in $\mathcal{R}_k f$ and $\mathcal{R}_{k-1} f$.

This finishes the induction step and hence the proof of the first claim. The proof of the converse claim, also inductive, relies on appropriate reversing of just used arguments, and thus, it is omitted. The proof of the proposition is therefore completed. □

Finally, we notice that the weak derivative $f'_{\rm weak}$ of $f$ on $I$ exists if and only if $f \in AC_{\rm loc}(I)$, and then $f'_{\rm weak}(x) = f'(x)$ for almost every $x \in I$. It is clear that $AC_{\rm loc}(I) = AC_{\rm loc}(I^-) \oplus AC_{\rm loc}(I^+)$ in the sense that $f \in AC_{\rm loc}(I)$ if and only if $f^\pm \in AC_{\rm loc}(I^\pm)$.

We can reformulate Proposition 3.4 to the following.

**Corollary 3.5.** *Let $f \in L^1_{\rm loc}(w)$. If $\mathfrak{D}^{(1)}_{\rm weak} f$ exists, then $f \in AC_{\rm loc}(I)$ and*

$$\mathfrak{D}^{(1)}_{\rm weak} f = pf' + qf_{\rm even} - q^\dagger f_{\rm odd}, \tag{3.5}$$

*a.e. on $I$. Conversely, if $f \in AC_{\rm loc}(I)$, then $\mathfrak{D}^{(1)}_{\rm weak} f$ exists, and one has (3.5) a.e. on $I$.*

*More generally, if $\mathfrak{D}^{(k)}_{\rm weak} f$ exists, $k \geqslant 2$, then $f \in C^{k-1}(I)$ and $f^{(k-1)} \in AC_{\rm loc}(I)$, and*

$$\mathfrak{D}^{(k)}_{\rm weak} f = \sum_{j=0}^{k} p^{[k]}_{k-j} f^{(k-j)} + \sum_{j=1}^{k} \left[ q^{[k,1]}_{k-j} \left( f^{(k-j)} \right)_{\rm even} + q^{[k,2]}_{k-j} \left( f^{(k-j)} \right)_{\rm odd} \right], \tag{3.6}$$

*a.e. on $I$. Conversely, if $f \in C^{k-1}(I)$ and $f^{(k-1)} \in AC_{\rm loc}(I)$, then $\mathfrak{D}^{(k)}_{\rm weak} f$ exists and one has (3.6) a.e. on $I$.*

The end of this section is devoted to explaining relations between $\mathfrak{D}^{(1)}_{\rm weak}$-derivatives and $\mathfrak{d}_{\rm weak}$- and $\mathfrak{d}^\dagger_{\rm weak}$-derivatives, and moreover, between $\mathfrak{D}^{(2k)}_{\rm weak}$-derivatives and $(\mathfrak{d}^\dagger\mathfrak{d})^k_{\rm weak}$- and $(\mathfrak{d}^\dagger\mathfrak{d})^k_{\rm weak}$-derivatives. We begin with notion of *weak delta-derivatives*. Recall, cf. [11], that given $f \in L^1_{\rm loc}(I^+, w)$, we say that weak $\mathfrak{d}$-*derivative of* $f$ exists provided that there is $g \in L^1_{\rm loc}(I^+, w)$, such that

$$\int_{I^+} \mathfrak{d}^\dagger \varphi(x) \overline{f(x)} w(x)\, {\rm d}x = \int_{I^+} \varphi(x) \overline{g(x)} w(x)\, {\rm d}x, \qquad \varphi \in C^\infty_c(I^+).$$

Then, we set $\mathfrak{d}_{\rm weak} f := g$ and call $g$ the weak $\mathfrak{d}$-derivative of $f$. If in the above equality, $\mathfrak{d}^\dagger$ is replaced by $\mathfrak{d}$, then we call $g$ the weak $\mathfrak{d}^\dagger$-derivative of $f$ and set $\mathfrak{d}^\dagger_{\rm weak} f := g$. Analogously, $g_k \in L^1_{\rm loc}(I^+, w)$ is called the weak $(\mathfrak{d}^\dagger\mathfrak{d})^k_{\rm weak}$-derivative of $f$ provided that

$$\int_{I^+} (\mathfrak{d}^\dagger\mathfrak{d})^k \varphi(x) \overline{f(x)} w(x)\, {\rm d}x = \int_{I^+} \varphi(x) \overline{g_k(x)} w(x)\, {\rm d}x, \qquad \varphi \in C^\infty_c(I^+).$$

We then set $(\mathfrak{d}^\dagger\mathfrak{d})^k_{\rm weak} f := g_k$. When the order of $\mathfrak{d}^\dagger$ and $\mathfrak{d}$ is reversed, we say about the $(\mathfrak{d}\mathfrak{d}^\dagger)^k_{\rm weak}$-derivative and write $(\mathfrak{d}\mathfrak{d}^\dagger)^k_{\rm weak} f := g_k$.

We shall need a simple lemma.

**Lemma 3.6.** *Let $f \in L^1_{\rm loc}(w)$ and $k \geqslant 1$. Assume that $\mathfrak{D}^{(k)}_{\rm weak} f$ exists.*

1. *If $f$ is even, then for $k$ even/odd, $\mathfrak{D}^{(k)}_{\rm weak} f$ is even/odd;*
2. *if $f$ is odd, then for $k$ odd/even, $\mathfrak{D}^{(k)}_{\rm weak} f$ is even/odd.*

*Proof.* It is obvious that the above rules apply for $\varphi \in C_c^\infty(I)$ with $\mathfrak{D}^{(k)}_{\rm weak}$ replaced by $\mathfrak{D}^k$. By assumption

$$\forall \varphi \in C_c^\infty(I) \qquad \int_I \mathfrak{D}^k\varphi\, \overline{f}\, w\, {\rm d}x = (-1)^k \int_I \varphi\, \overline{\mathfrak{D}^{(k)}_{\rm weak} f}\, w\, {\rm d}x.$$

Consider, for instance, $f$ even. For $k$ even/odd, take in the above line odd/even functions $\varphi$. This gives

$$\forall \varphi \in C_c^\infty(I) \qquad 0 = (-1)^k \int_{I^+} \varphi\, \overline{(\mathfrak{D}^{(k)}_{\rm weak} f)_{\rm odd/even}}\, w\, {\rm d}x,$$

and hence, $(\mathfrak{D}^{(k)}_{\rm weak} f)_{\rm odd/even} = 0$ $x$-a.e. for $k$ even/odd as required. For $f$ odd, we argue analogously. □

**Proposition 3.7.** *Let $f \in L^1_{\rm loc}(w)$. Then:*

1. *for $f$ odd: $\mathfrak{D}^{(1)}_{\rm weak} f$ exists if and only if $\mathfrak{d}^\dagger_{\rm weak}(f^+)$ exists; moreover, $\mathfrak{d}^\dagger_{\rm weak}(f^+) = -(\mathfrak{D}^{(1)}_{\rm weak} f)^+$;*
2. *for $f$ even: $\mathfrak{D}^{(1)}_{\rm weak} f$ exists if and only if $\mathfrak{d}_{\rm weak}(f^+)$ exists; moreover, $\mathfrak{d}_{\rm weak}(f^+) = (\mathfrak{D}^{(1)}_{\rm weak} f)^+$;*
3. *for $f$ odd: $\mathfrak{D}^{(2k)}_{\rm weak} f$ exists if and only if $(\mathfrak{d}^\dagger\mathfrak{d})^k_{\rm weak}(f^+)$ exists; moreover, $(\mathfrak{d}^\dagger\mathfrak{d})^k_{\rm weak}(f^+) = (-1)^k(\mathfrak{D}^{(2k)}_{\rm weak} f)^+$;*
4. *for $f$ even: $\mathfrak{D}^{(2k)}_{\rm weak} f$ exists if and only if $(\mathfrak{d}\mathfrak{d}^\dagger)^k_{\rm weak}(f^+)$ exists; moreover, $(\mathfrak{d}\mathfrak{d}^\dagger)^k_{\rm weak}(f^+) = (-1)^k(\mathfrak{D}^{(2k)}_{\rm weak} f)^+$.*

*Proof.* Consider, for instance, $f$ odd and prove (1). For $\Rightarrow$, we have

$$\forall \varphi \in C_c^\infty(I) \qquad \int_I \mathfrak{D}\varphi\, \overline{f}\, w\, {\rm d}x = -\int_I \varphi\, \overline{\mathfrak{D}^{(1)}_{\rm weak} f}\, w\, {\rm d}x.$$

Taking above even functions $\varphi$ gives (by Lemma 3.6, $\mathfrak{D}^{(1)}_{\rm weak} f$ is even)

$$2\int_{I^+} \mathfrak{d}\varphi\, \overline{f}\, w\, {\rm d}x = -2\int_{I^+} \varphi\, \overline{\mathfrak{D}^{(1)}_{\rm weak} f}\, w\, {\rm d}x.$$

This implies the required conclusion with accompanying identity. The opposite implication in 1. goes by reversing the above arguments.

Property 2. is proved analogously. Also, 3. and 4. require similar arguments. For instance, for $f$ odd, proving $\Rightarrow$ in 3., we begin with

$$\forall \varphi \in C_c^\infty(I) \qquad \int_I \mathfrak{D}^{2k}\varphi\, \overline{f}\, w\, {\rm d}x = \int_I \varphi\, \overline{\mathfrak{D}^{(2k)}_{\rm weak} f}\, w\, {\rm d}x.$$

Taking above odd functions $\varphi$ and noting that $\mathfrak{D}^{2k}\varphi = (-1)^k(\mathfrak{d}^\dagger\mathfrak{d})^k\varphi$ gives, by Lemma 3.6,

$$2(-1)^k \int_{I^+} (\mathfrak{d}^\dagger\mathfrak{d})^k\varphi\, \overline{f}\, w\, {\rm d}x = 2\int_{I^+} \varphi\, \overline{\mathfrak{D}^{(2k)}_{\rm weak} f}\, w\, {\rm d}x,$$

and hence, the required conclusion with accompanying identity follows. The opposite implication in 3. again goes by reversing the above argument. □

## 4. $\mathfrak{D}$-Sobolev Spaces and the Friedrichs Extensions

We now come to introducing and discussing Sobolev spaces associated with powers of the differential–difference operator $\mathfrak{D}$. These spaces can be regarded in the broader context of theory of Sobolev spaces. They are particularly well suited to describe some objects like minimal and maximal operators, the Friedrichs extensions, etc., in the $\mathfrak{D}$-framework.

**Definition 4.1.** The $\mathfrak{D}^{(k)}$-Sobolev space $H^k_{\mathfrak{D}}(I,w)$, $k \geqslant 1$, is the space

$$H^k_{\mathfrak{D}}(I,w) = \{f \in L^2(w)\colon \mathfrak{D}^{(k)}_{\mathrm{weak}} f \text{ exists and is in } L^2(w)\}$$

equipped with the inner product

$$\langle f,g\rangle_{H^k_{\mathfrak{D}}(I,w)} := \langle f,g\rangle_{L^2(w)} + \langle \mathfrak{D}^{(k)}_{\mathrm{weak}} f, \mathfrak{D}^{(k)}_{\mathrm{weak}} g\rangle_{L^2(w)}.$$

The closure of $C^\infty_c(I)$ in $H^k_{\mathfrak{D}}(I,w)$ with respect to the norm generated by $\langle\cdot,\cdot\rangle_{H^k_{\mathfrak{D}}(I,w)}$, that is

$$\|f\|_{H^k_{\mathfrak{D}}(I,w)} := \left(\|f\|^2_{L^2(w)} + \|\mathfrak{D}^{(k)}_{\mathrm{weak}} f\|^2_{L^2(w)}\right)^{1/2},$$

is denoted $H^k_{\mathfrak{D},0}(I,w)$.

Since $C^\infty_c(I) \subset H^k_{\mathfrak{D},0}(I,w) \subset H^k_{\mathfrak{D}}(I,w)$, $H^k_{\mathfrak{D},0}(I,w)$ and $H^k_{\mathfrak{D}}(I,w)$ are dense in $L^2(w)$.

Sobolev spaces in the context of the Jacobi operator $\mathcal{J}_{\alpha,\beta}$ and the Jacobi–Dunkl operator $J_{\alpha,\beta}$, $\alpha,\beta > -1$, were defined and investigated by Langowski [6,7]. These spaces were denoted $W^{p,m}_{\alpha,\beta}$ (for $1 \leqslant p \leqslant \infty$ and order $m \geqslant 1$) and their relation to potential spaces was studied. In Definition 4.1 specified to $\mathfrak{D}_{\alpha,\beta}$, there is no restriction on $\alpha$ and $\beta$.

The following proposition has a relatively standard proof, and hence, we omit it.

**Proposition 4.2.** *The $\mathfrak{D}^{(k)}$-Sobolev space $H^k_{\mathfrak{D}}(I,w)$ is a Hilbert space.*

Consequently, also $H^k_{\mathfrak{D},0}(I,w)$ is a Hilbert space.

A comment is in order on relations between our Definition 4.1 and the definition of Sobolev spaces that appeared in [7] in the context of the Jacobi operator $J^\circ_{\alpha,\beta}$; see Example 2.3. To be precise in [7, p. 3], $J^\circ_{\alpha,\beta}$ is shifted by a constant term, so that $\mathbb{J}_{\alpha,\beta} := J^\circ_{\alpha,\beta} + A^2_{\alpha,\beta}$ is considered with some restrictions on $\alpha$ and $\beta$ (and $\mathbb{D}_{\alpha,\beta} = \mathfrak{D}^\circ_{\alpha,\beta}$, see Example 2.3), but this change is immaterial from the spectral theory point of view. The main motivation in establishing a suitable definition of Sobolev spaces in [7] was to obtain, as a prize, an isomorphism between these Sobolev spaces and the potential spaces with properly chosen parameters. This was indeed achieved; see [7, Definition 3.2 and Theorem 3.3]. Our definition, in the general setting, is seemingly the most natural and allows to achieve our main goal, a characterization of Friedrichs extensions.

It is natural to ask when we can claim that

$$H^1_{\mathfrak{D},0}(I,w) = H^1_{\mathfrak{D}}(I,w).$$

Of course, the answer depends on $w$, compactness of $\overline{I}$, and $\mathfrak{D}$, and may be difficult in concrete settings. The 'classical' setting is described in the following.

*Example* 4.3. Let $w \equiv 1$, $p \equiv 1$, $q \equiv 0$, so that $\mathfrak{D} = \frac{d}{dx}$ on $I = (-b, 0) \cup (0, b)$, $0 < b \leqslant \infty$. Then, $H^1(I)$ coincides with the direct sum $H^1(I^-) \oplus H^1(I^+)$. Similarly, $H^1_0(I) = H^1_0(I^-) \oplus H^1_0(I^+)$. Hence, the two spaces in question, $H^1(I)$ and $H^1_0(I)$, differ. Additionally, we note that $H^1_0(I)$ coincides with the space of restrictions to $I$ of functions from the space $\{f \in H^1_0(-b, b) \colon f(0) = 0\}$.

Following the line of thoughts in Sect. 3, we now relate the introduced Sobolev spaces with those defined and studied in [11] and connected to $\mathfrak{d}$, $\mathfrak{d}^\dagger$, and their compositions. To recall, the delta-Sobolev space $H_\mathfrak{d}$ was defined in [11] (where slightly different notation was used) as

$$H_\mathfrak{d}(I^+, w) = \{f \in L^2(I^+, w) \colon \mathfrak{d}_{\text{weak}} f \text{ exists and is in } L^2(I^+, w)\}.$$

Analogous definition applies to $\mathfrak{d}^\dagger$. For higher order derivatives, we shall consider

$$H_{(\mathfrak{d}^\dagger\mathfrak{d})^k}(I^+, w) = \{f \in L^2(I^+, w) \colon (\mathfrak{d}^\dagger\mathfrak{d})^k_{\text{weak}} f \text{ exists and is in } L^2(I^+, w)\},$$

$k \geqslant 1$, and its analogue, $H_{(\mathfrak{d}\mathfrak{d}^\dagger)^k}(I^+, w)$, with the roles of $\mathfrak{d}$ and $\mathfrak{d}^\dagger$ reversed (in [11] an arbitrary composition of $\mathfrak{d}$ and $\mathfrak{d}^\dagger$ was admitted). Inner products and norms in these spaces are given in a way analogous to that in Definition 4.1 and the corresponding closures of $C^\infty_c(I^+)$ are denoted $H_{\mathfrak{d},0}(I^+, w)$, $H_{(\mathfrak{d}^\dagger\mathfrak{d})^k,0}(I^+, w)$, and so on. Also, the closed subspaces of $H^1_\mathfrak{D}(I, w)$, $H^{2k}_\mathfrak{D}(I, w)$ and their counterparts, $H^1_{\mathfrak{D},0}(I, w)$, $H^{2k}_{\mathfrak{D},0}(I, w)$, consisting of even/odd functions will be denoted by adding the affix *even/odd*.

The following result is a direct consequence of Proposition 3.7; the proof of the proposition is straightforward, and hence, we skip it. Below, if $X$ is a linear space of functions on $I$, then $X^+$ stands for the space of restrictions of all functions from $X$ to $I^+$.

**Proposition 4.4.** *We have*

$$H_\mathfrak{d}(I^+, w) = H^1_{\mathfrak{D},\text{even}}(I, w)^+ \quad \text{and} \quad H_{\mathfrak{d}^\dagger}(I^+, w) = H^1_{\mathfrak{D},\text{odd}}(I, w)^+.$$

*Moreover, for* $k \geqslant 1$*, we have*

$$H_{(\mathfrak{d}^\dagger\mathfrak{d})^k}(I^+, w) = H^{2k}_{\mathfrak{D},\text{odd}}(I, w)^+ \quad \text{and} \quad H_{(\mathfrak{d}\mathfrak{d}^\dagger)^k}(I^+, w) = H^{2k}_{\mathfrak{D},\text{even}}(I, w)^+.$$

*Analogous identities hold when* $H_\mathfrak{d}$ $H_{\mathfrak{d}^\dagger}$, $H_{(\mathfrak{d}^\dagger\mathfrak{d})^k}$, $H_{(\mathfrak{d}\mathfrak{d}^\dagger)^k}$, $H^1_{\mathfrak{D},\text{even/odd}}$ *and* $H^{2k}_{\mathfrak{D},\text{even/odd}}$*, are replaced by* $H_{\mathfrak{d},0}$, $H_{(\mathfrak{d}^\dagger\mathfrak{d})^k,0}$, $H^{2k}_{\mathfrak{D},0,\text{even/odd}}$*, etc., respectively.*

### 4.1. Minimal and Maximal Operators

We now define the minimal and maximal operators related to the symmetric operator $T^{(k)}_\mathfrak{D} := (-i\mathfrak{D})^k$ with domain $C^\infty_c(I)$, $k \geqslant 1$, (so that $T^{(2k)}_\mathfrak{D}$ is just $L^{(k)}_\mathfrak{D}$ in our former notation). We follow the well-known path of constructing these two operators. The minimal and maximal operators are important, because self-adjoint extensions of $T^{(k)}_\mathfrak{D}$, if exist, lie in between. Notice that for $k$ even, $T_{\mathfrak{D}}^{(k)}$ is nonnegative, and hence, self-adjoint extensions of $T_{\mathfrak{D}}^{(k)}$ do exist.

Define $T_{\mathfrak{D},\min}^{(k)}$ as the closure $\overline{T_{\mathfrak{D}}^{(k)}}$; $T_{\mathfrak{D}}^{(k)}$ is closable, since $\mathrm{Dom}\big((T_{\mathfrak{D}}^{(k)})^*\big)$ is dense in $L^2(w)$. Define $T_{\mathfrak{D},\max}^{(k)}$ as the operator with domain $\mathrm{Dom}(T_{\mathfrak{D},\max}^{(k)}) = H_{\mathfrak{D}}^k(I,w)$, given by the rule

$$T_{\mathfrak{D},\max}^{(k)} f = (-\mathrm{i})^k \mathfrak{D}_{\mathrm{weak}}^{(k)} f.$$

We have $T_{\mathfrak{D},\min}^{(k)} \subset T_{\mathfrak{D},\max}^{(k)}$, since $T_{\mathfrak{D},\max}^{(k)}$ is closed. Indeed, assume that $f_n \in H_{\mathfrak{D}}^k(I,w)$, and for some $f, g \in L^2(w)$, we have $f_n \to f$ and $(-\mathrm{i})^k \mathfrak{D}_{\mathrm{weak}}^{(k)} f_n \to g$. This means that $\{f_n\}$ and $\{(-\mathrm{i})^k \mathfrak{D}_{\mathrm{weak}}^{(k)} f_n\}$ are Cauchy sequences in $L^2(w)$, and hence, $\{f_n\}$ is a Cauchy sequence in $H_{\mathfrak{D}}^k(I,w)$. By Proposition 4.2, there is $F \in H_{\mathfrak{D}}^k(I,w)$, such that $f_n$ converges to $F$ in $H_{\mathfrak{D}}^k(I,w)$. Consequently, $f_n \to F$ in $L^2(w)$, and hence, $f = F \in H_{\mathfrak{D}}^k(I,w)$. Since also $(-\mathrm{i})^k \mathfrak{D}_{\mathrm{weak}}^{(k)} f_n \to g$ in $L^2(w)$, a simple argument then shows that $g = (-\mathrm{i})^k \mathfrak{D}_{\mathrm{weak}}^{(k)} f$.

**Proposition 4.5.** *We have* $(T_{\mathfrak{D},\min}^{(k)})^* = T_{\mathfrak{D},\max}^{(k)}$ *and* $(T_{\mathfrak{D},\max}^{(k)})^* = T_{\mathfrak{D},\min}^{(k)}$. *Moreover, every self-adjoint extension* $T$ *of* $T_{\mathfrak{D}}^{(k)}$ *satisfies* $T_{\mathfrak{D},\min}^{(k)} \subset T \subset T_{\mathfrak{D},\max}^{(k)}$.

*Proof.* We begin with the first equality and prove the inclusion $(T_{\mathfrak{D},\min}^{(k)})^* \supset T_{\mathfrak{D},\max}^{(k)}$. Let $f \in \mathrm{Dom}(T_{\mathfrak{D},\max}^{(k)})$. This means that $\mathfrak{D}_{\mathrm{weak}}^{(k)} f$ exists and belongs to $L^2(w)$. In other words, it holds

$$\forall \varphi \in C_c^\infty(I) \quad \langle (-\mathrm{i}\mathfrak{D})^k \varphi, f \rangle_{L^2(w)} = \langle \varphi, T_{\mathfrak{D},\max}^{(k)} f \rangle_{L^2(w)},$$

which means that $f \in \mathrm{Dom}\big((T_{\mathfrak{D},\max}^{(k)})^*\big) = \mathrm{Dom}\Big(\big(\overline{T_{\mathfrak{D},\max}^{(k)}}\big)^*\Big)$ and $T_{\mathfrak{D},\max}^{(k)} f = (T_{\mathfrak{D}}^{(k)})^* f = \Big(\overline{T_{\mathfrak{D}}^{(k)}}\Big)^* f$.

To prove the opposite $(T_{\mathfrak{D},\min}^{(k)})^* \subset T_{\mathfrak{D},\max}^{(k)}$, let $f \in \mathrm{Dom}\big((T_{\mathfrak{D},\min}^{(k)})^*\big)$ and set $g = (T_{\mathfrak{D},\min}^{(k)})^* f$. This means, in particular, that

$$\forall \varphi \in C_c^\infty(I) \quad \langle (-\mathrm{i}\mathfrak{D})^k \varphi, f \rangle_{L^2(w)} = \langle \varphi, g \rangle_{L^2(w)}.$$

Consequently, $\mathfrak{D}_{\mathrm{weak}}^{(k)} f$ exists and equals $(-\mathrm{i})^k g$. Hence, $f \in \mathrm{Dom}(T_{\mathfrak{D},\max}^{(k)})$ and $g = T_{\mathfrak{D},\max}^{(k)} f$.

Since $T_{\mathfrak{D},\min}^{(k)}$ is closed, we have $(T_{\mathfrak{D},\min}^{(k)})^{**} = T_{\mathfrak{D},\min}^{(k)}$, and thus, the second claimed equality is a consequence of the first one. The last claim of the proposition is obvious. ☐

## 4.2. Friedrichs Extensions

To continue, we need to recall basic facts on the Friedrichs extension. It is well known that any densely defined symmetric and nonnegative operator $S$ has a self-adjoint extension which is also nonnegative. The construction of this operator (which works in a more general setting of lower semibounded operators), denoted $S_F$, was given by Friedrichs in 1933 and is nowadays called the Friedrichs extension of $S$.

The construction of $S_F$ is based on the theory of (sesquilinear) forms. An important result in this theory says that for a given $(H, \langle\cdot,\cdot\rangle_H)$, there is a one-to-one correspondence between the set of all densely defined Hermitian nonnegative closed forms and the set of all self-adjoint nonnegative operators on $H$. If $\mathfrak{s}$ is such a form, then in this correspondence, $A_\mathfrak{s}$ denotes the relevant operator; if $A$ is such an operator, then $\mathfrak{s}_A$ denotes the relevant form. Moreover, for every $A$, we have $A_{(\mathfrak{s}_A)} = A$, and for every $\mathfrak{s}$, we have $\mathfrak{s}_{(A_\mathfrak{s})} = \mathfrak{s}$.

More precisely, the associated operator $A_\mathfrak{s}$ is defined by first determining its domain

$$\mathrm{Dom}(A_\mathfrak{s}) = \{h \in \mathrm{Dom}(\mathfrak{s})\colon \exists u_h \in H \;\; \forall h' \in \mathrm{Dom}(\mathfrak{s}) \;\; \mathfrak{s}[h,h'] = \langle u_h, h'\rangle_H\},$$

and then by setting its action on $h \in \mathrm{Dom}(A_\mathfrak{s})$ by $A_\mathfrak{s} h = u_h$. See [10, Chapter 10 and Sect. 3 of Chapter 12]. Also, recall that closedness of a nonnegative form $\mathfrak{s}$ means that the norm $\|x\|_\mathfrak{s} := (\mathfrak{s}[x,x]^2 + \langle x,x\rangle_H^2)^{1/2}$ defined on $\mathrm{Dom}(\mathfrak{s})$ is complete.

The construction of the Friedrichs extension now goes as follows. Let $S$ be as above and let $\mathfrak{s}_S[x,y] = \langle Sx, y\rangle_H$, $x,y \in \mathrm{Dom}(S)$, be the form associated with $S$. It is immediately seen that $\mathfrak{s}_S$ is densely defined Hermitian and nonnegative. However, more importantly, $\mathfrak{s}_S$ is closable; see [10, Lemma 10.16]. Let $\overline{\mathfrak{s}_S}$ be the closure of $\mathfrak{s}_S$. Although the completion procedure in the definition of $\overline{\mathfrak{s}_S}$ is abstract from its nature, it can be shown that $\overline{\mathfrak{s}_S}$ may be realized in $H$, which means, in particular, that $\mathrm{Dom}(\overline{\mathfrak{s}_S}) \subset H$. Then, $S_F$ is just $A_{\overline{\mathfrak{s}_S}}$, the operator associated with $\overline{\mathfrak{s}_S}$. See, for instance, [10, Definition 10.6].

Let $k \geqslant 1$. Define the form $\mathfrak{t}_\mathfrak{D}^{(k)}$ by

$$\mathfrak{t}_\mathfrak{D}^{(k)}[f,g] = \int_I \mathfrak{D}_{\mathrm{weak}}^{(k)} f(x) \overline{\mathfrak{D}_{\mathrm{weak}}^{(k)} g(x)} w(x)\,\mathrm{d}x, \qquad f,g \in H_\mathfrak{D}^k(I,w), \tag{4.1}$$

so that $\mathrm{Dom}(\mathfrak{t}_\mathfrak{D}^{(k)}) = H_\mathfrak{D}^k(I,w)$. The form $\mathfrak{t}_\mathfrak{D}^{(k)}$ restricted to $H_{\mathfrak{D},0}^k(I,w)$ will be denoted $\mathfrak{t}_{\mathfrak{D},0}^{(k)}$, and hence, $\mathrm{Dom}(\mathfrak{t}_{\mathfrak{D},0}^{(k)}) = H_{\mathfrak{D},0}^k(I,w)$. The form $\mathfrak{t}_\mathfrak{D}^{(k)}$ is Hermitian and nonnegative. Moreover, it is closed and this fact is just a consequence of the completeness of the norm $\|\cdot\|_{H^k(I,w)}$. The same is valid for $\mathfrak{t}_{\mathfrak{D},0}^{(k)}$. Let $\mathbb{L}_k := \mathbb{L}_{k,\mathfrak{D}}$ and $\mathbb{L}_{k,0} := \mathbb{L}_{k,\mathfrak{D},0}$ be the operators associated with the forms $\mathfrak{t}_\mathfrak{D}^{(k)}$ and $\mathfrak{t}_{\mathfrak{D},0}^{(k)}$, respectively. By the general theory, $\mathbb{L}_k$ and $\mathbb{L}_{k,0}$ are self-adjoint and nonnegative. It is worth mentioning that for $k=1$, the operators $\mathbb{L}_1$ and $\mathbb{L}_{1,0}$ should be thought off as the Neumann and Dirichlet 'Laplacians', two distinguished nonnegative self-adjoint extensions of the (minus) 'Laplacian' $-\mathfrak{D}^2$.

**Theorem 4.6.** *Let $k \geqslant 1$. The operators $\mathbb{L}_k$ and $\mathbb{L}_{k,0}$ are self-adjoint and nonnegative extensions of $(-1)^k\mathfrak{D}^{2k}$. Moreover, $\mathbb{L}_{k,0}$ is the Friedrichs extension of $(-1)^k\mathfrak{D}^{2k}$.*

*Proof.* We first check that both operators extend $(-1)^k\mathfrak{D}^{2k}$. It suffices to consider $\mathbb{L}_{k,0}$ only. From the general definition

$$\operatorname{Dom}(\mathbb{L}_{k,0}) = \{f \in H^k_{\mathfrak{D},0}(I,w)\colon \exists u_f \in L^2(w)\ \forall g \in H^k_{\mathfrak{D},0}(I,w)\ \ \mathfrak{t}^{(k)}_{\mathfrak{D},0}[f,g] = \langle u_f, g\rangle_{L^2(w)}\},$$

and $\mathbb{L}_{k,0}f = u_f$. We claim that $C_c^\infty(I) \subset \operatorname{Dom}(\mathbb{L}_{k,0})$ and $\mathbb{L}_{k,0}\varphi = (-1)^k\mathfrak{D}^{2k}\varphi$ for $\varphi \in C_c^\infty(I)$. For this purpose, we need to check that given $\varphi \in C_c^\infty(I)$, for every $g \in H^k_{\mathfrak{D},0}(I,w)$, it holds

$$\int_I \mathfrak{D}^k\varphi(x)\overline{\mathfrak{D}^{(k)}_{\text{weak}}g(x)}w(x)\,\mathrm{d}x = \langle(-1)^k\mathfrak{D}^{2k}\varphi, g\rangle_{L^2(w)}. \tag{4.2}$$

Verification of (4.2) is based on the integration by parts formula for absolutely continuous functions on a closed finite interval. Let $J := J^- \cup J^+$, $J^\pm \subset I^\pm$, be the union of two closed intervals, such that the support of $\varphi$ included in $J$. We check (4.2) with $I$ replaced by $J$.

When $k = 1$ and $g \in H^1_0(I,w)$, (4.2) then becomes

$$\int_J \mathfrak{D}\varphi(x)\overline{\mathfrak{D}^{(1)}_{\text{weak}}g(x)}w(x)\,\mathrm{d}x = -\int_J \mathfrak{D}^2\varphi(x)\overline{g(x)}w(x)\,\mathrm{d}x.$$

This equality indeed holds, since, by Corollary 3.5, the weak derivative $g'_{\text{weak}}$ exists on $I$, and consequently, $g$ is absolutely continuous on $J$. Moreover, $\mathfrak{D}^{(1)}_{\text{weak}}g = pg' + qg_{\text{even}} - q^\dagger g_{\text{odd}}$ a.e. on $I$ and an application of the integration by parts formula plus a small calculation shows the required equality. The general case goes along the same lines using the general part of Corollary 3.5. Indeed, inserting

$$\mathfrak{D}^k_{\text{weak}}g(x) = \sum_{j=0}^k p^{[k]}_{k-j}g^{(k-j)} + \sum_{j=1}^k \left[q^{[k,1]}_{k-j}\left(g^{(k-j)}\right)_{\text{even}} + q^{[k,2]}_{k-j}\left(g^{(k-j)}\right)_{\text{odd}}\right]$$

into the left-hand side of (4.2) (with $I$ replaced by $J$) and then performing $k$-times integration by parts (recall that $g$ is $(k-1)$-times differentiable on $I$ and $g^{(k-1)}$ is absolutely continuous on $J$) lead to the right-hand side of (4.2).

It remains to prove that $\mathbb{L}_{k,0} = ((-1)^k\mathfrak{D}^{2k})_F$. We take the form $\mathfrak{s}^{(k)}[f,g] = \langle(-1)^k\mathfrak{D}^{2k}f, g\rangle_w$ on the domain $\operatorname{Dom}(\mathfrak{s}^{(k)}) = C_c^\infty(I)$ and consider its closure $\overline{\mathfrak{s}^{(k)}}$. We now claim that

$$\overline{\mathfrak{s}^{(k)}} = \mathfrak{t}^{(k)}_{\mathfrak{D},0}. \tag{4.3}$$

This is enough for our purpose since then, with the notation preceding Theorem 4.6, we have

$$\operatorname{Dom}\big(((-1)^k\mathfrak{D}^{2k})_F\big) = \operatorname{Dom}(A_{\overline{\mathfrak{s}^{(k)}}}) = \operatorname{Dom}(A_{\mathfrak{t}^{(k)}_{\mathfrak{D},0}}),$$

and as one immediately sees, the latter space coincides with $\operatorname{Dom}(\mathbb{L}_{k,0})$. Moreover, it follows that $((-1)^k\mathfrak{D}^k)_F f = \mathbb{L}_{k,0}f$ for $f$ from these joint domains. Returning to (4.3), we note that it is a consequence of the fact that $C_c^\infty(I)$ lies densely in $\operatorname{Dom}(\mathfrak{t}^{(k)}_{\mathfrak{D},0}) = H^k_{\mathfrak{D},0}(I,w)$ and $\mathfrak{t}^{(k)}_{\mathfrak{D},0}$ is closed. Here are details. Clearly, $\mathfrak{t}^{(k)}_{\mathfrak{D},0}$ extends $\mathfrak{s}^{(k)}$, and hence, the inclusion $\subset$ follows. For the opposite inclusion, let $f \in \operatorname{Dom}(\mathfrak{t}^{(k)}_{\mathfrak{D},0}) = H^k_{\mathfrak{D},0}(I,w)$ and take $\{\varphi_n\} \subset C_c^\infty(I)$, such that

$\varphi_n \to f$ in $H_0^1(I,w)$. Notably, this means that $\varphi_n \to f$ and $\mathfrak{D}^k\varphi_n \to \mathfrak{D}^{(k)}_{\rm weak} f$ in $L^2(w)$. We want to show that $f \in \mathrm{Dom}(\overline{\mathfrak{s}^{(k)}})$.

For this, it suffices to ensure existence of $\{\varphi_n\} \subset C_c^\infty(I)$ convergent to $f$ in $L^2(w)$, such that $\mathfrak{s}^{(k)}[\varphi_n - \varphi_m, \varphi_n - \varphi_m] \to 0$ as $n, m \to \infty$; see [10, p. 224]. However

$$\begin{aligned}\mathfrak{s}^{(k)}[\varphi_n - \varphi_m, \varphi_n - \varphi_m] &= \langle (-1)^k \mathfrak{D}^{2k}(\varphi_n - \varphi_m), \varphi_n - \varphi_m\rangle_{L^2(w)} \\ &= \langle \mathfrak{D}^k(\varphi_n - \varphi_m), \mathfrak{D}^k(\varphi_n - \varphi_m)\rangle_{L^2(w)},\end{aligned}$$

and the latter required convergence to 0 follows, since $\mathfrak{D}^k\varphi_n$ being convergent in $L^2(w)$ is a Cauchy sequence there. □

The end of this section is devoted to discussion of relations between the Friedrichs extensions of $(-1)^k\mathfrak{D}^{2k}$ and $(\mathfrak{d}^\dagger\mathfrak{d})^k$ and $(\mathfrak{d}\mathfrak{d}^\dagger)^k$. Consider the operators

$$T_{\rm e/o}\colon L^2_{\rm even/odd}(I,w) \to L^2(I^+,w), \qquad T_{\rm e/o} f = f^+.$$

Then, up to the multiplicative constant $\sqrt{2}$, $T_{\rm e/o}$ are unitary isomorphisms. The isomorphism $T_{\rm e}$ identifies $\mathfrak{D}$ and $\mathfrak{d}$, and $(-1)^k\mathfrak{D}^{2k}$ and $(\mathfrak{d}^\dagger\mathfrak{d})^k$, $k \geqslant 1$. Similarly, the isomorphism $T_{\rm o}$ identifies $\mathfrak{D}$ and $-\mathfrak{d}^\dagger$, and $(-1)^k\mathfrak{D}^{2k}$ and $(\mathfrak{d}\mathfrak{d}^\dagger)^k$. (Recall that the domains of considered operators are either $C_c^\infty(I)$ or $C_c^\infty(I^+)$.)

From now on, to fix the attention, we consider only $T_{\rm e}$; $T_{\rm o}$ is treated analogously. By Proposition 4.4, the pairs of Sobolev spaces $H^1_{\mathfrak{D},\rm even}(I,w)$ and $H_{\mathfrak{d}}(I^+,w)$, $H^{2k}_{\mathfrak{D},\rm even}(I,w)$ and $H_{(\mathfrak{d}^\dagger\mathfrak{d})^k}(I^+,w)$, and their counterparts with the 0 affix, are also identified through $T_{\rm e}$. This gives an impact to compare the Friedrichs extensions. Recall that the Friedrichs extension of $(-1)^k\mathfrak{D}^{2k}$ we denoted as $\mathbb{L}_{k,0}$. The Friedrichs extensions of $(\mathfrak{d}^\dagger\mathfrak{d})^k$ and $(\mathfrak{d}\mathfrak{d}^\dagger)^k$ we shall denote by $\mathbb{L}_{(\mathfrak{d}^\dagger\mathfrak{d})^k}$ and $\mathbb{L}_{(\mathfrak{d}\mathfrak{d}^\dagger)^k}$, respectively. These extensions were described in [11, Theorem 5.2].

**Proposition 4.7.** *We have*

$$\mathbb{L}_{(\mathfrak{d}^\dagger\mathfrak{d})^k} = (\mathbb{L}_{k,0})^+_{\rm e} \quad \textit{and} \quad \mathbb{L}_{(\mathfrak{d}\mathfrak{d}^\dagger)^k} = (\mathbb{L}_{k,0})^+_{\rm o},$$

*in the sense that* $\mathrm{Dom}(\mathbb{L}_{(\mathfrak{d}^\dagger\mathfrak{d})^k}) = \mathrm{Dom}(\mathbb{L}_{k,0})^+_{\rm e}$ *and* $\mathrm{Dom}(\mathbb{L}_{(\mathfrak{d}\mathfrak{d}^\dagger)^k}) = \mathrm{Dom}(\mathbb{L}_{k,0})^+_{\rm o}$, *and* $\mathbb{L}_{(\mathfrak{d}^\dagger\mathfrak{d})^k}(f^+) = (\mathbb{L}_{k,0}f)^+$ *for* $f \in \mathrm{Dom}(\mathbb{L}_{k,0})_{\rm e}$, *and analogously in the second case.*

*Proof.* We focus on considering the first pair only; for the second pair, one argues analogously. The proof relies on observing that the constructions of the Friedrichs extensions of $(-1)^k\mathfrak{D}^{2k}$ and $(\mathfrak{d}^\dagger\mathfrak{d})^k$ agree on the level of forms. Recall that for the construction of $\mathbb{L}_{k,0}$, the form $\mathfrak{t}^{(k)}_{\mathfrak{D}}$ defined in (4.1) with domain restricted to $H^k_{\mathfrak{D},0}(I,w)$ was used. On the other hand, as explained in the proof of [11, Theorem 5.2] (with slightly different notation), the form

$$\mathfrak{t}^{(k)}_{\mathfrak{d}}[f,g] = \int_{I^+} \mathfrak{d}^{(k)}_{\rm weak} f(x)\overline{\mathfrak{d}^{(k)}_{\rm weak} g(x)}w(x)\,\mathrm{d}x,$$

on the domain $H_{(\mathfrak{d}^\dagger\mathfrak{d})^k,0}(I^+,w)$ was used to define $\mathbb{L}_{(\mathfrak{d}^\dagger\mathfrak{d})^k}$. It is immediately seen that if the domain of the form $\mathfrak{t}^{(k)}_{\mathfrak{D}}$ is further restricted to $H^k_{\mathfrak{D},0}(I,w)_{\rm e}$, then these two forms are identified through $T_e$. Consequently, the resulting self-adjoint extensions coincide. □

**Author contributions** The entire manuscript was prepared solely by myself.

**Funding** None.

**Availability of Data and Materials** Not applicable.

**Declarations**

**Ethical Approval** Not applicable.

**Conflict of Interest** The authors declare that there is no conflict of interest.



## References

[1] Ben Salem, N., Samaali, T.: Hilbert transform and related topics associated with Jacobi–Dunkl operators of compact and noncompact types. Adv. Pure Appl. Math. **2**, 367–388 (2011)

[2] Ben Salem, N., Samaali, T.: Hilbert transform and related topics associated with the differential Jacobi operator on $(0, +\infty)$. Positivity **15**, 221–240 (2011)

[3] Chouchane, F.: Harmonic analysis associated with the Jacobi–Dunkl operator on $]-\frac{\pi}{2}, \frac{\pi}{2}[$. J. Comput. Appl. Math. **178**, 75–89 (2005)

[4] Chouchene, F., Mili, M., Trimèche, K.: Positivity of the intertwining operator and harmonic analysis associated with the Jacobi–Dunkl operator on $\mathbb{R}$. Anal. Appl. **1**, 387–412 (2003)

[5] Langowski, B.: Harmonic analysis operators related to symmetrized Jacobi expansions. Acta Math. Hungar. **140**, 248–292 (2013)

[6] Langowski, B.: Sobolev spaces associated with Jacobi expansions. J. Math. Anal. Math. **420**, 1533–1551 (2014)

[7] Langowski, B.: Potential and Sobolev spaces related to symmetrized Jacobi expansions. SIGMA Symmetry Integr. Geom. Methods Appl. **11**, 073 (17 pages) (2015)

[8] Nowak, A., Stempak, K.: A symmetrized conjugacy scheme for orthogonal expansions. Proc. R. Soc. Edinb. **143A**, 427–443 (2013)

[9] Nowak, A., Stempak, K., Szarek, T.Z.: On harmonic analysis operators in Laguerre–Dunkl and Laguerre-symmetrized settings. SIGMA Symmetry Integr. Geom Methods Appl. **12**, 096 (39 pages) (2016)

[10] Schmüdgen, K.: Unbounded Self-Adjoint Operators on Hilbert Space. Graduate Texts in Mathematics, vol. 265. Springer, Berlin (2012)

[11] Stempak, K.: Spectral properties of ordinary differential operators admitting special decompositions. Commun. Pure Appl. Anal. **20**, 1961–1986 (2021)

[12] Stempak, K.: Harmonic analysis associated with the Jacobi–Dunkl operator on $(-\pi, \pi)$: exotic cases. J. Complex Anal. Oper. Theory **178**, 75–89 (2021)

[13] Yao, S., Sun, J., Zettl, A.: The Sturm–Liouville Friedrichs extension. Appl. Math. **60**, 299–320 (2015)

Krzysztof Stempak
Wrocław
Poland
e-mail: Krzysztof.Stempak@pwr.edu.pl

Received: October 21, 2022.
Revised: February 26, 2023.
Accepted: March 6, 2023.