Journal of Applied Mathematics and Computing (2023) 69:1355–1376
https://doi.org/10.1007/s12190-022-01793-0

ORIGINAL RESEARCH

Check for updates

Qualitative analysis of second-order fuzzy difference equation with quadratic term

Qianhong Zhang[1] · **Miao Ouyang**[2] · **Bairong Pan**[1] · **Fubiao Lin**[1]

Received: 12 May 2022 / Revised: 8 September 2022 / Accepted: 12 September 2022 /
Published online: 28 September 2022

Abstract

In this paper, we explore the qualitative features of a second-order fuzzy difference equation with quadratic term

$$x_{n+1} = A + \frac{Bx_n}{x_{n-1}^2}, \quad n = 0, 1, \ldots .$$

Here the parameters $A, B \in \Re_F^+$ and the initial values $x_0, x_{-1} \in \Re_F^+$. Utilizing a generalization of division (g-division) of fuzzy numbers, we obtain some sufficient condition on the qualitative features including boundedness, persistence, and convergence of positive fuzzy solution of the model, Moreover two simulation examples are presented to verify our theoretical analysis.

Keywords Fuzzy difference equation · Persistence · Boundedness · Asymptotic behavior · G-division

Miao Ouyang, Bairong Pan and Fubiao Lin have contributed equally to this work.

✉ Qianhong Zhang
zqianhong68@163.com

Miao Ouyang
mouyang@xmut.edu.cn

Bairong Pan
1193602732@qq.com

Fubiao Lin
linfubiao0851@163.com

1 School of Mathematics and Statistics, Guizhou University of Finance and Economics, Huaxi University City, Guiyang 550025, Guizhou, China

2 School of Applied Mathematics, Xiamen University of Technology, Ligong Road, Xiamen 361024, Fujian, China

1 Introduction

Difference equation is one of the most important dynamical model. It is the analogue of corresponding differential equation and delay differential equation having an extensive applications in computer science, control engineering, chemistry, biology, economics, etc. (see [1–11]). Recently, many authors are interested in studying qualitative features of rational difference equation. For example, Bešo et al. [12] proposed a second-order rational difference equation with quadratic term

$$x_{n+1} = \gamma + \delta \frac{x_n}{x_{n-1}^2}, n = 0, 1, \ldots, \tag{1}$$

where $\gamma > 0$, $\delta > 0$, and the initial values $x_0 > 0$, $x_{-1} > 0$.

In 2017, Khyat et al. [13] explored a similar model with quadratic term in both the numerator and denominator

$$x_{n+1} = a + \frac{x_n^2}{x_{n-1}^2}, n = 0, 1, \ldots. \tag{2}$$

Furthermore, they obtained the globally asymptotically stability of (2) and the direction of the Neimark–Sacker bifurcation.

In the last few decades, there are many publications on the stability, oscillatory, periodicity, and boundedness of nonlinear rational difference equations. Moreover a lot of similar qualitative features also appear in nonlinear rational difference equation systems (see [14–17]).

Although these models are very simple in their forms, we can not understand fully the qualitative features of their solutions. In fact, these models inevitably implicit inherent uncertainty or vague. It is well known that fuzzy set is a powerful tool to cope with these uncertainties or subjective information in mathematical model. Therefore, it is a natural method to explore dynamical model with uncertainty or impression by establishing fuzzy difference equation (FDE) or fuzzy differential equation.

FDE is a special kind of difference equation whose coefficients and the initial condition are fuzzy numbers, and its' solution is a sequence of fuzzy number. Due to the advantage of FDE in dealing with inherent imprecision, the study on qualitative features of these models has become an important research topic both from theoretical viewpoint and in applications. Therefore, in the last decades, there has been an increasing results on the study of FDE (see [18–33]). It is found that fuzzy set theory has potential in the application of fuzzy differential equations and fuzzy time series (see [34–40]).

Inspired by previous works, in this paper, utilizing a generalization of division (g-division) of fuzzy numbers, we explore the qualitative features of positive fuzzy solution to the following FDE with quadratic term

$$x_{n+1} = A + \frac{Bx_n}{x_{n-1}^2}, \ n = 0, 1, \ldots, \tag{3}$$

where the initial condition x_{-1}, x_0, and the parameters A, B are positive fuzzy numbers.

The organization of this paper is arranged as follows. Section 2 gives some basic concepts of fuzzy numbers used throughout the paper. In Sect. 3, the qualitative features of the positive fuzzy solution to FDE (3) are obtained by virtue of g-division of fuzzy numbers. Section 4 presents two illustrative examples to verify our theoretic results. A general conclusion and discussion are drawn in Sect. 5.

2 Preliminary and definitions

In this section, we first review some basic concepts and definitions to be used in the sequel. These particular descriptions are found in many publications [20–22].

A function defined as $U : R \to [0, 1]$ is called a fuzzy number if it is normal, fuzzy convex, upper semi-continuous, and compactly support on R.

For $\alpha \in (0, 1]$, we denote the $\alpha-$cuts of U by $[U]_\alpha = \{x \in R : U(x) \geq \alpha\}$, and for $\alpha = 0$, the support of U is written as $\text{supp}U = [U]_0 = \overline{\{x \in R : U(x) > 0\}}$.

It is easy to see that the $[U]_\alpha$ is a closed interval. Provided that $\text{supp}U \subset (0, \infty)$, then U is said to be a positive fuzzy number. Particularly, if U is a positive real number, then it is said to be a trivial fuzzy number, i.e., $[U]_\alpha = [U, U], \alpha \in (0, 1]$.

Let U, V be fuzzy numbers, $[U]_\alpha = [U_{l,\alpha}, U_{r,\alpha}], [V]_\alpha = [V_{l,\alpha}, V_{r,\alpha}], \alpha \in [0, 1]$, $k > 0$. Addition and multiplication of fuzzy numbers are defined as follows.

$$[U + V]_\alpha = [U_{l,\alpha} + V_{l,\alpha}, U_{r,\alpha} + V_{r,\alpha}], \tag{4}$$

$$[kU]_\alpha = [kU_{l,\alpha}, kU_{r,\alpha}]. \tag{5}$$

The family of fuzzy numbers with addition and multiplication defined by Eqs. (4) and (5) is written as $\Re_F$. Particularly, the family of positive (resp. negative) fuzzy numbers is denoted by $\Re_F^+$(resp. $\Re_F^-$).

Definition 2.1 Let $U, V \in \Re_F$, the metric is defined as follows.

$$D(U, V) = \sup_{\alpha\in[0,1]} \max\{| U_{l,\alpha} - V_{l,\alpha} |, | U_{r,\alpha} - V_{r,\alpha} |\}. \tag{6}$$

Obviously, $(\Re_F, D)$ is a complete metric space.

Definition 2.2 [41] Let $U, V \in \Re_F$, $[U]_\alpha = [U_{l,\alpha}, U_{r,\alpha}], [V]_\alpha = [V_{l,\alpha}, V_{r,\alpha}]$, $0 \notin [V]_\alpha, \forall\alpha \in [0, 1]$. The g-division $\div_g$ of the fuzzy numbers is written as $W = U \div_g V$ with $\alpha-$cuts $[W]_\alpha = [W_{l,\alpha}, W_{r,\alpha}]$($[W]_\alpha^{-1} = [1/W_{r,\alpha}, 1/W_{l,\alpha}]$), where

$$[W]_\alpha = [U]_\alpha \div_g [V]_\alpha \Longleftrightarrow \begin{cases} (i) \;\; [U]_\alpha = [V]_\alpha[W]_\alpha, \\ \\ (ii) \; [V]_\alpha = [U]_\alpha[W]_\alpha^{-1}, \end{cases} \tag{7}$$

if W is a proper fuzzy number.

Remark 2.1 According to [41], Let $U, V \in \Re_F^+$, if $U \div_g V = W \in \Re_F^+$ exists, then there are two cases
Case (i). if $U_{l,\alpha} V_{r,\alpha} \leq U_{r,\alpha} V_{l,\alpha}, \forall \alpha \in [0, 1]$, then $W_{l,\alpha} = \frac{U_{l,\alpha}}{V_{l,\alpha}}$, $W_{r,\alpha} = \frac{U_{r,\alpha}}{Vr,\alpha}$,
Case (ii). if $U_{l,\alpha} V_{r,\alpha} \geq U_{r,\alpha} V_{l,\alpha}, \forall \alpha \in [0, 1]$, then $W_{l,\alpha} = \frac{U_{r,\alpha}}{V_{r,\alpha}}$, $W_{r,\alpha} = \frac{U_{l,\alpha}}{Vl,\alpha}$.

Definition 2.3 Let $x_n \in \Re_F^+, n = 1, 2, \ldots$, (x_n) is said to be persistence (resp. bounded) provided that there is $M > 0$ (resp. $N > 0$) satisfying

$$\text{supp } x_n \subset [M, \infty)(\text{resp. supp } x_n \subset (0, N]).$$

The sequence (x_n) is bounded and persistence if there are $M, N > 0$ satisfying

$$\text{supp } x_n \subset [M, N], n = 1, 2, \ldots.$$

The sequence (x_n) is unbounded if the norm of x_n, $\|x_n\|, n = 1, 2, \ldots,$ is unbounded.

Definition 2.4 If a sequence (x_n) of positive fuzzy numbers is satisfied with FDE (3), then x_n is said to be a positive solution of FDE (3). $x \in \Re_F^+$ is called a positive equilibrium of FDE (3) if

$$x = A + \frac{Bx}{x^2}.$$

Let $x_n, x \in \Re_F^+, n = 0, 1, 2, \ldots,$ we say x_n converges to x as $n \to \infty$ if $\lim_{n\to\infty} D(x_n, x) = 0$.

Lemma 2.1 *Let $g : R^+ \times R^+ \times R^+ \times R^+ \to R^+$ be continuous, $A, B, C, D \in \Re_F^+$. Then*

$$[g(A, B, C, D)]_\alpha = g([A]_\alpha, [B]_\alpha, [C]_\alpha, [D]_\alpha), \quad \alpha \in (0, 1]. \tag{8}$$

Lemma 2.2 [8] *Let I_x, I_y be some intervals of real numbers and let $f : I_x^2 \times I_y^2 \to I_x$,, $g : I_x^2 \times I_y^2 \to I_y$ be continuously differentiable functions. Then for every initial conditions $(x_i, y_i) \in I_x \times I_y (i = -1, 0)$, the following system of difference equations*

$$\begin{cases} x_{n+1} = f(x_n, x_{n-1}, y_n, y_{n-1}), \\ y_{n+1} = g(x_n, x_{n-1}, y_n, y_{n-1}). \end{cases} \quad n = 0, 1, 2, \ldots, \tag{9}$$

has a unique solution $\{(x_i, y_i)\}_{n=-1}^{+\infty}$.

Definition 2.5 [8] A point $(\overline{x}, \overline{y})$ is called an equilibrium point of system (9) if

$$\overline{x} = f(\overline{x}, \overline{x}, \overline{y}, \overline{y}), \quad \overline{y} = g(\overline{x}, \overline{x}, \overline{y}, \overline{y}).$$

That is, $(x_n, y_n) = (\overline{x}, \overline{y})$ for $n \geq 0$ is the solution of (9), or equivalently, $(\overline{x}, \overline{y})$ is a fixed point of the vector map (f, g).

Definition 2.6 [8] Let $(\overline{x}, \overline{y})$ be an equilibrium point of (9). (i) The equilibrium point $(\overline{x}, \overline{y})$ is called locally stable if for every $\varepsilon > 0$, there exists $\delta > 0$ such that for all $(x_i, y_i) \in I_x \times I_y (i = -1, 0)$, with $| x_{-1} - \overline{x} | + | x_0 - \overline{x} | < \delta, | y_{-1} - \overline{y} | + | y_0 - \overline{y} | < \delta$, we have $| x_n - \overline{x} | < \varepsilon, | y_n - \overline{y} | < \varepsilon$, for $n \geq 0$.
(ii) The equilibrium $(\overline{x}, \overline{y})$ of (9) is called locally asymptotically stable if it is locally stable, and if there exists $\gamma > 0$, such that for all $(x_i, y_i) \in I_x \times I_y (i = -1, 0)$, with $| x_{-1} - \overline{x} | + | x_0 - \overline{x} | < \gamma, | y_{-1} - \overline{y} | + | y_0 - \overline{y} | < \gamma$, we have

$$\lim_{n\to\infty} x_n = \overline{x}, \quad \lim_{n\to\infty} y_n = \overline{y}.$$

(iii) The equilibrium $(\overline{x}, \overline{y})$ of (9) is called a global attractor if for every $(x_i, y_i) \in I_x \times I_y (i = -1, 0)$, we have

$$\lim_{n\to\infty} x_n = \overline{x}, \quad \lim_{n\to\infty} y_n = \overline{y}.$$

(iv) The equilibrium $(\overline{x}, \overline{y})$ of (9) is called globally asymptotically stable if it is locally stable and a global attractor.
(v) The equilibrium $(\overline{x}, \overline{y})$ of (9) is called unstable if it is not stable.

3 Main results

3.1 Existence of positive fuzzy solution

In this section, we first discuss the existence of the positive fuzzy solution of FDE (3).

Theorem 3.1 *Consider FDE (3), where $A, B \in \Re_F^+$, then there is a unique positive fuzzy solution x_n of FDE (3), for $x_{-1}, x_0 \in \Re_F^+$.*

Proof The proof of Theorem is similar to those of Proposition 2.1 [19]. Assume that (x_n) is a sequence of fuzzy numbers and satisfies FDE (3) with initial values x_{-1}, x_0. Consider the $\alpha-$cuts, $\alpha \in (0, 1], n \in N^+$

$$\begin{cases} [x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \\ [A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}], \\ [B]_\alpha = [B_{l,\alpha}, B_{r,\alpha}]. \end{cases} \tag{10}$$

Applying Lemma 2.1, it follows from (3) and (10) that

$$\begin{aligned} [x_{n+1}]_\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] &= \left[A + \frac{Bx_n}{x_{n-1}^2} \right]_\alpha = [A]_\alpha + \frac{[B]_\alpha \times [x_n]_\alpha}{[x_{n-1}^2]_\alpha} \\ &= [A_{l,\alpha}, A_{r\alpha}] + \frac{[B_{l,\alpha} L_{n,\alpha}, B_{r,\alpha} R_{n,\alpha}]}{[L_{n-1,\alpha}^2, R_{n-1,\alpha}^2]}. \end{aligned}$$

Utilizing g-division of fuzzy numbers, one of the following two cases occurs.
Case (i)

$$[x_{n+1}]_\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[A_{l,\alpha} + \frac{B_{l,\alpha} L_{n,\alpha}}{L^2_{n-1,\alpha}}, A_{r,\alpha} + \frac{B_{r,\alpha} R_{n,\alpha}}{R^2_{n-1,\alpha}} \right]. \quad (11)$$

Case (ii)

$$[x_{n+1}]_\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}] = \left[A_{l,\alpha} + \frac{B_{r,\alpha} R_{n,\alpha}}{R^2_{n-1,\alpha}}, A_{r,\alpha} + \frac{B_{l,\alpha} L_{n,\alpha}}{L^2_{n-1,\alpha}} \right]. \quad (12)$$

If Case (i) holds true, from (11), one gets, for $\alpha \in (0, 1]$, $n \in \{0, 1, 2, \ldots\}$,

$$L_{n+1,\alpha} = A_{l,\alpha} + \frac{B_{l,\alpha} L_{n,\alpha}}{L^2_{n-1,\alpha}}, \quad R_{n+1,\alpha} = A_{r,\alpha} + \frac{B_{r,\alpha} R_{n,\alpha}}{R^2_{n-1,\alpha}}. \quad (13)$$

It is clear that there is a unique solution $(L_{n,\alpha}, R_{n,\alpha})$ for any initial value $(L_{k,\alpha}, R_{k,\alpha})$, $k = -1, 0$, $\alpha \in (0, 1]$. We only need to show that, for $\alpha \in (0, 1]$, $n \in N^+$, $[L_{n,\alpha}, R_{n,\alpha}]$ makes certain the positive fuzzy solution x_n of FDE (3) with the initial condition x_i, $i = 0, -1$, and

$$[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}]. \quad (14)$$

From [18], since $x_j \in \Re^+_F$, $j = -1, 0$, for $\alpha_i \in (0, 1]$, $i = 1, 2$, and $\alpha_1 \leq \alpha_2$, it has

$$0 < L_{j,\alpha_1} \leq L_{j,\alpha_2} \leq R_{j,\alpha_2} \leq R_{j,\alpha_1}, \; j = -1, 0. \quad (15)$$

We declare that, for $n = 0, 1, 2, \ldots,$

$$L_{n,\alpha_1} \leq L_{n,\alpha_2} \leq R_{n,\alpha_2} \leq R_{n,\alpha_1}. \quad (16)$$

By mathematical induction. From (15), for $n = 0, 1$, it is true. Suppose that, for $n \leq k$, $k \in N^+$, (16) holds true, then it follows from (13) and (16) that, for $n = k + 1$,

$$\begin{aligned} L_{k+1,\alpha_1} &= A_{l,\alpha_1} + \frac{B_{l,\alpha_1} L_{k,\alpha_1}}{L^2_{k-1,\alpha_1}} \leq A_{l,\alpha_2} + \frac{B_{l,\alpha_2} L_{k,\alpha_2}}{L^2_{k-1,\alpha_2}} = L_{k+1,\alpha_2} \\ &= A_{l,\alpha_2} + \frac{B_{l,\alpha_2} L_{k,\alpha_2}}{L^2_{k-1,\alpha_2}} \leq A_{r,\alpha_2} + \frac{B_{r,\alpha_2} R_{k,\alpha_2}}{R^2_{k-1,\alpha_2}} = R_{k+1,\alpha_2} \\ &= A_{r,\alpha_2} + \frac{B_{r,\alpha_2} R_{k,\alpha_2}}{R^2_{k-1,\alpha_2}} \leq A_{r,\alpha_1} + \frac{B_{r,\alpha_1} R_{k,\alpha_1}}{R^2_{k-1,\alpha_1}} = R_{k+1,\alpha_1} \end{aligned}$$

So (16) holds true. Also, from (13), it has

$$L_{1,\alpha} = A_{l,\alpha} + \frac{B_{l,\alpha} L_{0,\alpha}}{L_{-1,\alpha}^2}, \quad R_{1,\alpha} = A_{r,\alpha} + \frac{B_{r,\alpha} R_{0,\alpha}}{R_{-1,\alpha}^2}, \quad \alpha \in (0, 1]. \tag{17}$$

Since $x_j \in \Re_F^+$, $j = -1, 0$, and $A, B \in \Re_F^+$, then $L_{i,\alpha}, R_{i,\alpha}, i = 0, -1$, are left continuous. So, one gets from (17) that $L_{1,\alpha}, R_{1,\alpha}$ are also left continuous. By inductively, we can show that, for $n \geq 1$, $L_{n,\alpha}$ and $R_{n,\alpha}$ are left continuous.

Secondly, we will prove that supp$x_n = \overline{\bigcup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}]}$ is compact. It need to show that $\bigcup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}]$ is bounded. Since $x_j \in \Re_F^+$, $j = -1, 0$, and $A, B \in \Re_F^+$, there exist $M_A > 0$, $N_A > 0$, $N_B > 0$, $M_B > 0$, $M_j > 0$, $N_j > 0$, $j = -1, 0$, such that, $\forall \alpha \in (0, 1]$,

$$\begin{cases} [A_{l,\alpha}, A_{r,\alpha}] \subset [M_A, N_A], \\ [B_{l,\alpha}, B_{r,\alpha}] \subset [M_B, N_B], \\ [L_{j,\alpha}, R_{j,\alpha}] \subset [M_j, N_j]. \end{cases} \tag{18}$$

Hence from (17) and (18), one can get, for $n = 1$,

$$[L_{1,\alpha}, R_{1,\alpha}] \subset \left[M_A + \frac{M_B M_0}{M_{-1}^2}, N_A + \frac{N_B N_0}{N_{-1}^2} \right], \alpha \in (0, 1]. \tag{19}$$

From which, it follows that

$$\bigcup_{\alpha \in (0,1]} [L_{1,\alpha}, R_{1,\alpha}] \subset \left[M_A + \frac{M_B M_0}{M_{-1}^2}, N_A + \frac{N_B N_0}{N_{-1}^2} \right], \quad \alpha \in (0, 1]. \tag{20}$$

Hence, $\overline{\bigcup_{\alpha \in (0,1]} [L_{1,\alpha}, R_{1,\alpha}]}$ is compact, and $\overline{\bigcup_{\alpha \in (0,1]} [L_{1,\alpha}, R_{1,\alpha}]} \subset (0, \infty)$. Deducing inductively, one can get that $\overline{\bigcup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}]}$ is compact, moreover, for $n = 1, 2, \ldots,$

$$\overline{\bigcup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}]} \subset (0, \infty). \tag{21}$$

And since $L_{n,\alpha}, R_{n,\alpha}$ are left continuous, from (16) and (21), we get that $[L_{n,\alpha}, R_{n,\alpha}]$ makes certain a positive sequence x_n satisfying (14).

Now we show that, for the initial conditions $x_i (i = 0, -1)$, x_n is the solution of (3). Since, $\forall \alpha \in (0, 1]$,

$$[x_{n+1}]_\alpha = [L_{n+1,\alpha}, R_{n+1,\alpha}]$$

$$= \left[A_{l,\alpha} + \frac{B_{l,\alpha} L_{n,\alpha}}{L^2_{n-1,\alpha}}, A_{r,\alpha} + \frac{B_{r,\alpha} R_{n,\alpha}}{R^2_{n-1,\alpha}} \right] = \left[A + \frac{B x_n}{x^2_{n-1}} \right]_\alpha .$$

Therefore, x_n is the solution of FDE (3) with initial conditions x_i, $i = 0, -1$.

Suppose that, for the initial values $x_i \in \Re^+_F$, $i = 0, -1$, $\overline{x}_n$ is another solution of (3). Then, deducing as above, it has, for $n \in N^+$

$$[\overline{x}_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \quad \alpha \in (0, 1]. \tag{22}$$

Then, (14) and (22) imply that $[x_n]_\alpha = [\overline{x}_n]_\alpha$, $\alpha \in (0, 1]$, $n \in N^+$, so $x_n = \overline{x}_n$.

If Case (ii) occurs, The proof is similar to the proof above. This completes the proof of Theorem 3.1. □

3.2 Dynamics of FDE (3)

In order to obtain results on qualitative features of the positive solutions, Case (i) and Case (ii) are considered respectively.

If Case (i) occurs, the following lemma is required.

Lemma 3.1 *Consider the following difference equations*

$$y_{n+1} = p + \frac{c y_n}{y^2_{n-1}}, \quad n = 0, 1, \ldots, \tag{23}$$

where $p^2 > c > 0$, $y_i \in (0, +\infty)$, $i = 0, -1$. *Then*

$$p \le y_n \le \frac{p^3}{p^2 - c} + y_3, \quad n \ge 4. \tag{24}$$

Proof It is clear that, for $n \ge 1$, $y_n > p$, $z_n > q$ from (23). Moreover, for $n \ge 4$,

$$y_n = p + \frac{c y_{n-1}}{y^2_{n-2}} \le p + \frac{c}{p^2} y_{n-1}, \tag{25}$$

By induction, it can get that, for $n - k \ge 3$

$$\begin{aligned} y_n &\le p + \frac{c}{p} + \frac{c}{p^4} y_{n-2} \le p + \frac{c}{p} + \frac{c^2}{p^3} + \frac{c^3}{p^6} y_{n-3} \le p + \frac{c}{p} + \frac{c^2}{p^3} + \frac{c^3}{p^5} + \frac{c^4}{p^8} y_{n-4} \\ &\le \cdots \le \sum_{i=1}^{k} \frac{c^{i-1}}{p^{2i-3}} + \frac{c^k}{p^{2k}} y_{n-k} = \frac{p}{1 - c/p^2} \left[1 - \left(\frac{c}{p^2} \right)^k \right] + \frac{c^k}{p^{2k}} y_{n-k} \\ &\le \frac{p^3}{p^2 - c} + y_{n-k} \end{aligned} \tag{26}$$

Noting $n - k \ge 3$ is equal to $k \le n - 3$. The proposition is true. □

Theorem 3.2 *Consider FDE (3), where $A, B \in \Re_F^+$, and $x_{-1}, x_0 \in \Re_F^+$. Suppose that there exist $M_A > 0, N_A > 0, M_B > 0, N_B > 0, \forall\alpha \in (0, 1]$, such that*

$$\begin{cases} M_A \le A_{l,\alpha} \le A_{r,\alpha} \le N_A, \\ M_B \le B_{l,\alpha} \le B_{r,\alpha} \le N_B, \\ M_A^2 > M_B, \quad N_A^2 > N_B. \end{cases} \tag{27}$$

Then every positive fuzzy solution x_n of FDE (3) is bounded and persists.

Proof If Case (i) occurs. Let x_n be a positive fuzzy solution of FDE (3). It is obvious from (11) that, for $n = 1, 2, \ldots, \quad \alpha \in (0, 1]$,

$$M_A \le L_{n,\alpha}, \quad M_A \le R_{n,\alpha}. \tag{28}$$

Then, from (21), Lemma 3.1, and $A_{l,\alpha} \ge M_A$, it has, for $n \ge 5$,

$$[L_{n,\alpha}, R_{n,\alpha}] \subset [M_A, T_\alpha], \tag{29}$$

where

$$T_\alpha = \frac{N_A^3}{N_A^2 - N_B} + R_{3,\alpha}.$$

Then since $x_n \in \Re_F^+$, there exists $T > 0$, satisfying,

$$T_\alpha \le T, \quad \forall\alpha \in (0, 1]. \tag{30}$$

Therefore, (29) and (30) imply that $[L_{n,\alpha}, R_{n,\alpha}] \subset [M_A, T]$, then, for $n \ge 4$, $\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}] \subset [M_A, T]$, so $\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha}, R_{n,\alpha}]} \subseteq [M_A, T]$. Thus the positive fuzzy solution x_n is bounded and persists.

To show that the convergence of positive fuzzy solution x_n, we give the following lemmas. □

Lemma 3.2 *Consider the following difference equation*

$$y_{n+1} = p + \frac{cy_n}{y_{n-1}^2}, \; n = 0, 1, 2, \ldots, \tag{31}$$

Assume $p^2 > \frac{4c}{3}$. Then the equilibrium $\overline{y}$ of (31) is locally asymptotically stable.

Proof Let $\overline{y}$ be an equilibrium of (31), we obtain that $\overline{y} = \frac{p+\sqrt{p^2+4c}}{2}$. The linearized equation of (31) at equilibrium $\overline{y}$ is

$$y_{n+1} - \frac{2c}{p^2 + 2c + p\sqrt{p^2 + 4c}} y_n + \frac{4c}{p^2 + 2c + p\sqrt{p^2 + 4c}} y_{n-1} = 0, \tag{32}$$

Since $p^2 > \frac{4c}{3}$, it can get

$$\frac{6c}{p^2 + 2c + p\sqrt{p^2 + 4c}} < 1.$$

By virtue of Theorem 1.3.7 [8], the equilibrium $\overline{y}$ of (31) is locally asymptotically stable. □

Lemma 3.3 *Consider Eq.* (23), *if* $p^2 > \frac{4c}{3}$. *Then every positive solution* (y_n) *of* (23) *tends to the positive equilibrium*

$$\overline{y} = \frac{p + \sqrt{p^2 + 4c}}{2}.$$

Proof From system (23), we obtain a unique equilibrium $\overline{y} = \frac{p+\sqrt{p^2+4c}}{2}$. If (y_n) is a positive solution of (23). Let

$$\Lambda_1 = \lim_{n\to\infty} \sup y_n, \quad \lambda_1 = \lim_{n\to\infty} \inf y_n.$$

Applying Lemma 3.1, we get $0 < p < \lambda_1 \leq \Lambda_1 < \infty$. From (23), it implies that

$$\Lambda_1 \leq p + \frac{c\Lambda_1}{\lambda_1^2}, \quad \lambda_1 \geq p + \frac{c\lambda_1}{\Lambda_1^2}.$$

This can derive that

$$\Lambda_1 \leq \lambda_1.$$

Thus $\Lambda_1 = \lambda_1$. Namely $\lim_{n\to\infty} y_n$ exists. Since $\overline{y}$ of (23) is the unique positive equilibrium, then $\lim_{n\to\infty} y_n = \overline{y}$.

Combining two lemmas above, we have the following theorem □

Theorem 3.3 *Consider Eq.* (23), *if* $p^2 > \frac{4c}{3}$, *then the unique positive equilibrium* $\overline{y}$ *of* (23) *is globally asymptotically stable.*

Theorem 3.4 *If* $A_{l,\alpha}^2 > \frac{4}{3} B_{l,\alpha}$, $A_{r,\alpha}^2 > \frac{4}{3} B_{r,\alpha}$, *for all* $\alpha \in (0, 1]$, *then every positive fuzzy solution* x_n *of FDE* (3) *converges to the fuzzy positive equilibrium* x *as* $n \to \infty$.

Proof Suppose that there is a fuzzy number x satisfying

$$x = A + \frac{Bx}{x^2}, \quad [x]_\alpha = [L_\alpha, R_\alpha], \quad \alpha \in (0, 1]. \tag{33}$$

in which $L_\alpha, R_\alpha \geq 0$. Then, from (33), one gets

$$L_\alpha = A_{l,\alpha} + \frac{B_{l,\alpha} L_\alpha}{L_\alpha^2}, \quad R_\alpha = A_{r,\alpha} + \frac{B_{r,\alpha} R_\alpha}{R_\alpha^2}. \tag{34}$$

Hence we have from (34) that

$$L_\alpha = \frac{A_{l,\alpha} + \sqrt{A_{l,\alpha}^2 + 4B_{l,\alpha}}}{2}, \quad R_\alpha = \frac{A_{r,\alpha} + \sqrt{A_{r,\alpha}^2 + 4B_{r,\alpha}}}{2}.$$

Let x_n be a positive fuzzy solution of FDE (3) satisfying (11). Since $A_{l,\alpha}^2 > \frac{4}{3}B_{l,\alpha}$, $A_{r,\alpha}^2 > \frac{4}{3}B_{r,\alpha}$, $\alpha \in (0, 1]$. Utilizing Lemma 3.3 to system (13), then

$$\lim_{n\to\infty} L_{n,\alpha} = L_\alpha, \quad \lim_{n\to\infty} R_{n,\alpha} = R_\alpha, \tag{35}$$

Then, from (35), it has

$$\lim_{n\to\infty} D(x_n, x) = \lim_{n\to\infty} \sup_{\alpha\in(0,1]} \{\max\{| L_{n,\alpha} - L_\alpha |, | R_{n,\alpha} - R_\alpha |\}\} = 0.$$

The proof of the theorem is completed.

Secondly, if Case (ii) occurs, i.e., for $n \in \{0, 1, 2, \ldots\}$,

$$L_{n+1,\alpha} = A_{l,\alpha} + \frac{B_{r,\alpha} R_{n,\alpha}}{R_{n-1,\alpha}^2}, \quad R_{n+1,\alpha} = A_{r,\alpha} + \frac{B_{l,\alpha} L_{n,\alpha}}{L_{n-1,\alpha}^2}, \quad \alpha \in (0, 1]. \tag{36}$$

The following lemmas are required. □

Lemma 3.4 *Consider the following difference equations system.*

$$y_{n+1} = p + \frac{dz_n}{z_{n-1}^2}, \quad z_{n+1} = q + \frac{cy_n}{y_{n-1}^2}, \quad n = 0, 1, \ldots, \tag{37}$$

where $y_i, z_i \in (0, +\infty)$, $i = -1, 0$. *If*

$$p > d > 1, \quad q > c > 1. \tag{38}$$

Then, for $n \geq 4$,

$$p \leq y_n \leq \frac{p(p^2q^2 + pqd)}{p^2q^2 - cd} + y_2, \quad q \leq z_n \leq \frac{q(p^2q^2 + pqc)}{p^2q^2 - cd} + z_2. \tag{39}$$

Proof Let $Y_n = \frac{y_n}{p}$, $Z_n = \frac{z_n}{q}$, then Eq. (37) can be transformed into the following systems

$$Y_{n+1} = 1 + \frac{d}{pq}\frac{Z_n}{Z_{n-1}^2}, \quad Z_{n+1} = 1 + \frac{c}{pq}\frac{Y_n}{Y_{n-1}^2}, \quad n = 0, 1, 2, \ldots, \tag{40}$$

From (40), we have $Y_n \geq 1$, $Z_n \geq 1$, for $n \geq 1$. And, for $n \geq 4$,

$$\begin{cases} Y_n \leq 1 + \frac{d}{pq} Z_{n-1} \leq 1 + \frac{d}{pq} + \frac{cd}{p^2q^2} Y_{n-2}, \\ Z_n \leq 1 + \frac{c}{pq} Y_{n-1} \leq 1 + \frac{c}{pq} + \frac{cd}{p^2q^2} Z_{n-2}. \end{cases} \tag{41}$$

Deducing inductively, it can conclude that, for $n - 2k \geq 2$,

$$\begin{aligned} Y_n &\leq 1 + \frac{d}{pq} + \frac{cd}{p^2q^2} Y_{n-2} \leq 1 + \frac{d}{pq} + \frac{cd}{p^2q^2} + \frac{cd^2}{p^3q^3} Z_{n-3} \\ &\leq 1 + \frac{d}{pq} + \frac{cd}{p^2q^2} + \frac{cd^2}{p^3q^3} + \frac{c^2d^2}{p^4q^4} Y_{n-4} \\ &\leq \cdots \leq 1 + \frac{d}{pq} + \frac{cd}{p^2q^2} + \frac{cd^2}{p^3q^3} + \frac{c^2d^2}{p^4q^4} + \cdots + \frac{c^{2k-1}d^{2k}}{p^{2k-1}q^{2k-1}} + \frac{c^{2k}d^{2k}}{p^{2k}q^{2k}} Y_{n-2k} \\ &= \frac{1}{1 - \frac{cd}{p^2q^2}} \left[1 - \left(\frac{cd}{p^2q^2} \right)^{k-1} \right] + \frac{\frac{d}{pq}}{1 - \frac{cd}{p^2q^2}} \left[1 - \left(\frac{cd}{p^2q^2} \right)^{k} \right] + \frac{c^{2k}d^{2k}}{p^{2k}q^{2k}} Y_{n-2k} \\ &\leq \frac{p^2q^2 + pqd}{p^2q^2 - cd} + Y_{n-2k}. \end{aligned} \tag{42}$$

$$\begin{aligned} Z_n &\leq 1 + \frac{c}{pq} + \frac{cd}{p^2q^2} Z_{n-2} \leq 1 + \frac{c}{pq} + \frac{cd}{p^2q^2} + \frac{c^2d}{p^3q^3} Y_{n-3} \\ &\leq 1 + \frac{c}{pq} + \frac{cd}{p^2q^2} + \frac{c^2d}{p^3q^3} + \frac{c^2d^2}{p^4q^4} Z_{n-4} \\ &\leq \cdots \leq 1 + \frac{c}{pq} + \frac{cd}{p^2q^2} + \frac{c^2d}{p^3q^3} + \frac{c^2d^2}{p^4q^4} + \cdots + \frac{c^{2k}d^{2k-1}}{p^{2k-1}q^{2k-1}} + \frac{c^{2k}d^{2k}}{p^{2k}q^{2k}} Z_{n-2k} \\ &= \frac{1}{1 - \frac{cd}{p^2q^2}} \left[1 - \left(\frac{cd}{p^2q^2} \right)^{k-1} \right] + \frac{\frac{c}{pq}}{1 - \frac{cd}{p^2q^2}} \left[1 - \left(\frac{cd}{p^2q^2} \right)^{k} \right] + \frac{c^{2k}d^{2k}}{p^{2k}q^{2k}} Z_{n-2k} \\ &\leq \frac{p^2q^2 + pqc}{p^2q^2 - cd} + Z_{n-2k}. \end{aligned} \tag{43}$$

Noting $n - 2k \geq 2$ is equal to $k \leq (n-2)/2$. The proposition is true. □

Lemma 3.5 *Consider Eq.* (37)*, if condition* (38) *holds true, then the unique positive equilibrium point* $(\overline{y}, \overline{z})$ *of* (37) *is locally asymptotically stable.*

Proof From (37), we obtain a positive equilibrium $(\overline{y}, \overline{z}) =$

$$\left(\frac{pq - d + c + \sqrt{(pq - d + c)^2 + 4pqd}}{2p}, \frac{pq - c + d + \sqrt{(pq - c + d)^2 + 4pqc}}{2q} \right).$$

The linearized equation of (37) at the equilibrium $(\overline{y}, \overline{z})$ is

$$\Psi_{n+1} = B\Psi_n, \tag{44}$$

here $\Psi_n = (y_n, y_{n-1}, z_n, z_{n-1})^T$,

$$B = \begin{pmatrix} 0 & 0 & \frac{d}{\overline{z}^2} & -\frac{2d}{\overline{z}^2} \\ 1 & 0 & 0 & 0 \\ \frac{c}{\overline{y}^2} & -\frac{2c}{\overline{y}^2} & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let $\lambda_i, i = 1, 2, 3, 4$ be the eigenvalues of matrix B, $L = \text{diag}(l_1, l_2, l_3, l_4)$ be a diagonal matrix, $l_1 = l_3 = 1, d_i = d_{2+i} = 1 - i\varepsilon (i = 2)$, and

$$0 < \varepsilon < \min\left\{\frac{1}{2} - \frac{3d}{2\overline{z}^2}, \frac{1}{2} - \frac{3c}{2\overline{y}^2}\right\}. \tag{45}$$

Clearly, L is an invertible matrix. Calculating LBL^{-1}, one has

$$LBL^{-1} = \begin{pmatrix} 0 & 0 & \frac{d}{\overline{z}^2} l_1 l_3^{-1} & -\frac{2d}{\overline{z}^2} l_1 l_4^{-1} \\ l_2 l_1^{-1} & 0 & 0 & 0 \\ \frac{c}{\overline{y}^2} l_3 l_1^{-1} & -\frac{2c}{\overline{y}^2} l_3 l_2^{-1} & 0 & 0 \\ 0 & 0 & l_4 l_3^{-1} & 0 \end{pmatrix}$$

From $l_1 > l_2 > 0, l_3 > l_4 > 0$, it implies that

$$l_2 l_1^{-1} < 1, l_4 l_3^{-1} < 1.$$

Furthermore, noting (45), we have

$$\frac{d}{\overline{z}^2} l_1 l_3^{-1} + \frac{2d}{\overline{z}^2} l_1 l_4^{-1} = \frac{d}{\overline{z}^2}\left(1 + \frac{2}{1 - 2\varepsilon}\right) < \frac{3d}{\overline{z}^2(1 - 2\varepsilon)} < 1,$$
$$\frac{c}{\overline{y}^2} l_3 l_1^{-1} + \frac{2c}{\overline{z}^2} l_3 l_2^{-1} = \frac{c}{\overline{y}^2}\left(1 + \frac{2}{1 - 2\varepsilon}\right) < \frac{3c}{\overline{y}^2(1 - 2\varepsilon)} < 1.$$

It is clear that B has the same eigenvalues as LBL^{-1}, and

$$\begin{aligned} \max_{1 \le i \le 4} | \lambda_i | &\le \|LBL^{-1}\|_\infty \\ &= \max\left\{l_2 l_1^{-1}, l_4 l_3^{-1}, \frac{d}{\overline{z}^2} l_1 l_3^{-1} + \frac{2d}{\overline{z}^2} l_1 l_4^{-1}, \frac{c}{\overline{y}^2} l_3 l_1^{-1} + \frac{2c}{\overline{y}^2} l_3 l_2^{-1}\right\} < 1. \end{aligned}$$

Therefore the equilibrium $(\overline{y}, \overline{z})$ of (37) is locally asymptotically stable. □

Lemma 3.6 *Consider Eq.* (37)*, if* (38) *hold true, then every positive solution* (y_n, z_n) *of* (37) *converges to the equilibrium point* $(\overline{y}, \overline{z})$.

Proof From (37), we obtain positive equilibrium $(\overline{y}, \overline{z}) =$

$$\left(\frac{pq-d+c+\sqrt{(pq-d+c)^2+4pqd}}{2p}, \frac{pq-c+d+\sqrt{(pq-c+d)^2+4pqc}}{2q}\right).$$

Suppose that (y_n, z_n) is an arbitrary positive solution of (37). Noting (35)–(37), one has

$$\lim_{n\to\infty}\sup y_n = H_1, \lim_{n\to\infty}\inf y_n = h_1, \quad \lim_{n\to\infty}\sup z_n = H_2, \quad \lim_{n\to\infty}\inf z_n = h_2. \tag{46}$$

where $h_i, H_i \in (0, +\infty)$, $i = 1, 2$. Then, noting (37) and (46), we have

$$H_1 \le p + \frac{dH_2}{h_2^2}, \quad h_1 \ge p + \frac{dh_2}{H_2^2}, \quad H_2 \le q + \frac{cH_1}{h_1^2}, \quad h_2 \ge q + \frac{ch_1}{H_1^2}.$$

From which we have

$$H_1 - h_1 \le d\left(\frac{h_2}{H_2^2} - \frac{H_2}{h_2^2}\right), \quad H_2 - h_2 \le c\left(\frac{h_1}{H_1^2} - \frac{H_1}{h_1^2}\right). \tag{47}$$

We claim that

$$H_1 = h_1, \quad H_2 = h_2. \tag{48}$$

Suppose contrarily that $H_1 > h_1$, then from the first inequality of (47), it can conclude $h_2 > H_2$, which is a contradiction. So $H_1 = h_1$. Similarly we can get $H_2 = h_2$. Noting (37) and (48), then $\lim_{n\to\infty} y_n = \overline{y}$, $\lim_{n\to\infty} z_n = \overline{z}$. The proof of Lemma 3.6 is completed.

Combining Lemma 3.5 with Lemma 3.6. We have the following theorem. □

Theorem 3.5 *Consider Eq. (37). If relation (38) holds true, then the unique positive equilibrium $(\overline{y}, \overline{z})$ is globally asymptotically stable.*

Theorem 3.6 *If*

$$A_{l,\alpha} > B_{r,\alpha} > 1, \quad A_{r,\alpha} > B_{l,\alpha} > 1, \quad \forall\alpha \in (0, 1]. \tag{49}$$

and, for $n = 0, 1, 2, \ldots,$

$$\frac{R_{n,\alpha}L_{n-1,\alpha}^2}{L_{n,\alpha}R_{n-1,\alpha}^2} \le \frac{B_{l,\alpha}}{B_r, \alpha}, \forall\alpha \in (0, 1]. \tag{50}$$

Then every positive fuzzy solution of FDE (3) converges to the positive fuzzy equilibrium x as $n \to +\infty$.

Proof The proof is similar to those of Theorem 3.4. Assume that there is a positive fuzzy number x satisfying (33). From (33), condition (49) and (50), we can get

$$L_\alpha = A_{l,\alpha} + \frac{B_{r,\alpha} R_\alpha}{R_\alpha^2}, \quad R_\alpha = A_{r,\alpha} + \frac{B_{l,\alpha} L_\alpha}{L_\alpha^2}. \tag{51}$$

Hence we can from (51) have that

$$\begin{cases} L_\alpha = \frac{A_{l,\alpha}A_{r,\alpha}+B_{r,\alpha}-B_{l,\alpha}+\sqrt{(A_{l,\alpha}A_{r,\alpha}+B_{r,\alpha}-B_{l,\alpha})^2+4A_{l,\alpha}A_{r,\alpha}B_{l,\alpha}}}{2A_{r,\alpha}}, \\ R_\alpha = \frac{A_{l,\alpha}A_{r,\alpha}+B_{l,\alpha}-B_{r,\alpha}+\sqrt{(A_{l,\alpha}A_{r,\alpha}+B_{l,\alpha}-B_{r,\alpha})^2+4A_{l,\alpha}A_{r,\alpha}B_{r,\alpha}}}{2A_{l,\alpha}}. \end{cases} \tag{52}$$

Suppose that x_n is a positive fuzzy solution of FDE (3) such that (10) holds. Noting (50) and (51), utilizing Lemma 3.5 and Lemma 3.6 to (36), we have

$$\lim_{n\to\infty} L_{n,\alpha} = L_\alpha, \quad \lim_{n\to\infty} R_{n,\alpha} = R_\alpha, \tag{53}$$

Then

$$\lim_{n\to\infty} D(x_n, x) = \lim_{n\to\infty} \sup_{\alpha\in(0,1]} \{\max\{| L_{n,\alpha} - L_\alpha |, | R_{n,\alpha} - R_\alpha |\}\} = 0.$$

The proof of the theorem is completed. □

Remark 3.1 In fuzzy discrete dynamical systems. To find qualitative behavior of solutions for discrete fuzzy difference equation, it is very vital to utilize which operations such as addition, scalar multiplication, division of fuzzy numbers. In [19–22, 25, 26, 28–31], Using Zadeh extension principle, the authors obtained dynamical behaviors of some fuzzy difference equations. However, utilizing g-division of fuzzy numbers, Zhang et al. [23, 24] studied the dynamical behaviors of some nonlinear fuzzy difference equations. Compared with the former, The advantage is that the support sets of positive fuzzy solution of latter is smaller than those of former. In fact, it is obvious that the degree of fuzzy uncertainty is reduced by virtue of g-division of fuzzy numbers. Based on this fact above, Therefore, we consider the qualitative behaviors of the fuzzy difference equation with quadratic term by virtue of g-division of fuzzy numbers.

4 Numerical examples

In this section, two illustrative examples are presented to verify the effectiveness of theoretic results.

Example 4.1 Consider the following fuzzy difference equation

$$x_{n+1} = A + \frac{Bx_n}{x_{n-1}^2}, \ n = 0, 1, \ldots, \tag{54}$$

where $A, B \in \Re_F^+$ and the initial conditions $x_i \in \Re_F^+$, $i = 0, -1$ are as follows

$$A(x) = \begin{cases} x-3, & 3 \le x \le 4 \\ -x+5, & 4 \le x \le 5 \end{cases}, \quad B(x) = \begin{cases} x-4, & 4 \le x \le 5 \\ -x+6, & 5 \le x \le 6 \end{cases} \tag{55}$$

$$x_{-1}(x) = \begin{cases} 2x-4, & 2 \le x \le 2.5 \\ -2x+6, & 2.5 \le x \le 3 \end{cases}, \quad x_0(x) = \begin{cases} x-1, & 1 \le x \le 2 \\ -x+3, & 2 \le x \le 3 \end{cases} \tag{56}$$

From (55), one has

$$[A]_\alpha = [3+\alpha, 5-\alpha], \ [B]_\alpha = [4+\alpha, 6-\alpha], \ \alpha \in (0, 1]. \tag{57}$$

From (56), one has

$$[x_{-1}]_\alpha = [2 + \frac{1}{2}\alpha, 3 - \frac{1}{2}\alpha], \ \ [x_0]_\alpha = [1+\alpha, 3-\alpha], \ \alpha \in (0, 1]. \tag{58}$$

Therefore, it follows that

$$\begin{cases} \overline{\bigcup_{\alpha\in(0,1]}[A]_\alpha} = [3, 5], \ \ \overline{\bigcup_{\alpha\in(0,1]}[B]_\alpha} = [4, 6], \\ \overline{\bigcup_{\alpha\in(0,1]}[x_{-1}]_\alpha} = [2, 3], \ \overline{\bigcup_{\alpha\in(0,1]}[x_0]_\alpha} = [1, 3]. \end{cases} \tag{59}$$

It is clear that Case (i) occurs, so Eq. (54) can results in a difference equation system with parameter α,

$$L_{n+1,\alpha} = A_{l,\alpha} + \frac{B_{l,\alpha} L_{n,\alpha}}{L_{n-1,\alpha}^2}, \ \ R_{n+1,\alpha} = A_{r,\alpha} + \frac{B_{r,\alpha} R_{n,\alpha}}{R_{n-1,\alpha}^2}, \ \alpha \in (0, 1]. \tag{60}$$

Therefore, $A_{l,\alpha}^2 > \frac{4}{3} B_{l,\alpha}, A_{r,\alpha}^2 > \frac{4}{3} B_{r,\alpha} \forall\alpha \in (0, 1]$, and the initial value $x_i \in \Re_F^+$, $(i = 0, -1)$, so applying Theorem 3.2, then every positive solution x_n of Eq. (54) is bounded and persistent.

In addition, from Theorem 3.4, there is a unique positive equilibrium $\overline{x} = (4, 5, 6)$. Moreover every positive fuzzy solution x_n of Eq. (54) converges to $\overline{x}$ with respect to D as $n \to \infty$ (see Figs. 1, 2, 3).

Example 4.2 Consider Eq. (54), where $A, B \in \Re_F^+$ and the initial value $x_i \in \Re_F^+$, $i = 0, -1$ are as follows

$$A(x) = \begin{cases} x-6, & 6 \le x \le 7 \\ -2x+15, & 7 \le x \le 7.5 \end{cases}, \quad B(x) = \begin{cases} 2x-3, & 1.5 \le x \le 2 \\ -x+3, & 2 \le x \le 3 \end{cases} \tag{61}$$

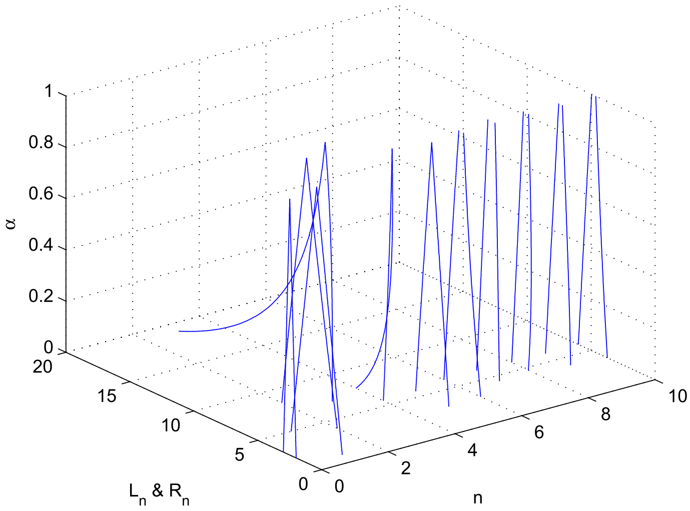

Fig. 1 The dynamics of system (60)

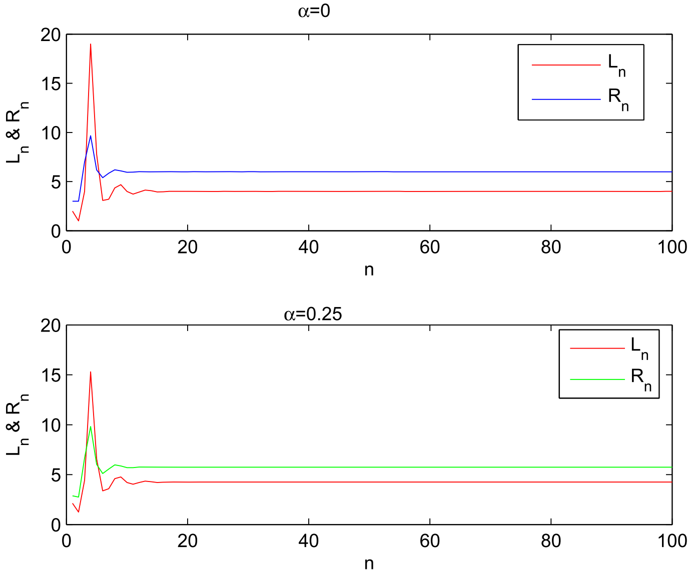

Fig. 2 The solution of system (60) at $\alpha = 0$ and $\alpha = 0.25$

$$x_{-1}(x) = \begin{cases} 2x - 4, & 2 \leq x \leq 2.5 \\ -2x + 6, & 2.5 \leq x \leq 3 \end{cases}, \; x_0(x) = \begin{cases} x - 1, & 1 \leq x \leq 2 \\ -x + 3, & 2 \leq x \leq 3 \end{cases} \tag{62}$$

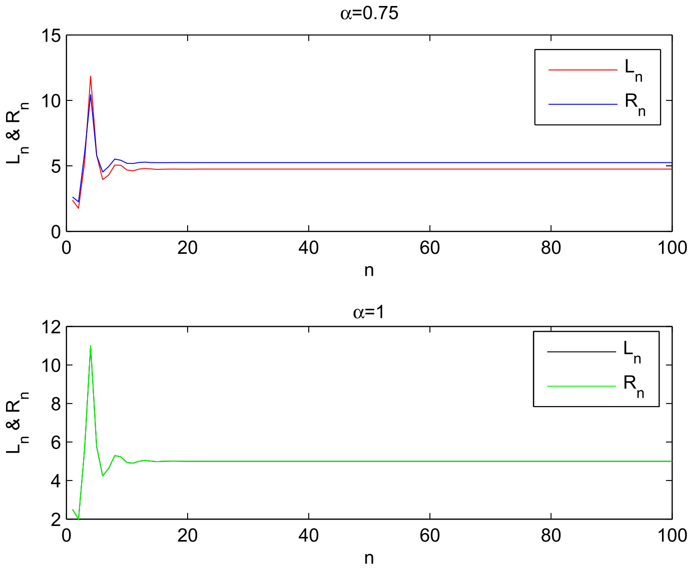

Fig. 3 The solution of system (60) at $\alpha = 0.75$ and $\alpha = 1$

From (61), one has

$$[A]_\alpha = \left[6 + \alpha, 7.5 - \frac{1}{2}\alpha\right], \ [B]_\alpha = \left[1.5 + \frac{1}{2}\alpha, 3 - \alpha\right], \ \alpha \in (0, 1]. \tag{63}$$

From (62), one has

$$[x_{-1}]_\alpha = \left[2 + \frac{1}{2}\alpha, 3 - \frac{1}{2}\alpha\right], \ \ [x_0]_\alpha = [1 + \alpha, 3 - \alpha], \ \alpha \in (0, 1]. \tag{64}$$

Therefore, it follows that

$$\overline{\bigcup_{\alpha\in(0,1]} [A]_\alpha} = [6, 7.5], \ \overline{\bigcup_{\alpha\in(0,1]} [B]_\alpha} = [1.5, 3], \ \overline{\bigcup_{\alpha\in(0,1]} [x_{-1}]_\alpha} = [2, 3],$$
$$\overline{\bigcup_{\alpha\in(0,1]} [x_0]_\alpha} = [1, 3]. \tag{65}$$

It is clear that Case (ii) occurs, so Eq. (54) can result in a difference equation systems with parameter α,

$$L_{n+1,\alpha} = A_{l,\alpha} + \frac{B_{r,\alpha} R_{n,\alpha}}{R^2_{n-1,\alpha}}, \ \ R_{n+1,\alpha} = A_{r,\alpha} + \frac{B_{l,\alpha} L_{n,\alpha}}{L^2_{n-1,\alpha}}, \ \alpha \in (0, 1]. \tag{66}$$

It is clear that the initial value $x_i \in \Re^+_F (i = 0, -1)$, and (47) is satisfied, so applying Theorem 3.6, there is a unique positive equilibrium $\overline{x} = (6.3879, 7.2749, 7.7348)$.

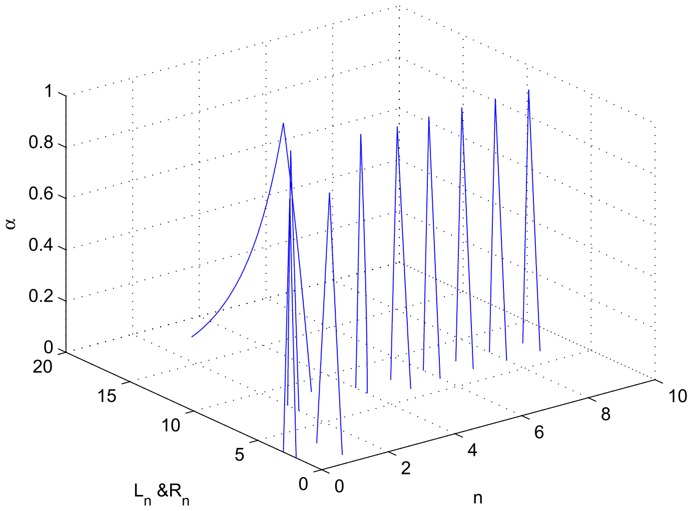

Fig. 4 The Dynamics of system (66)

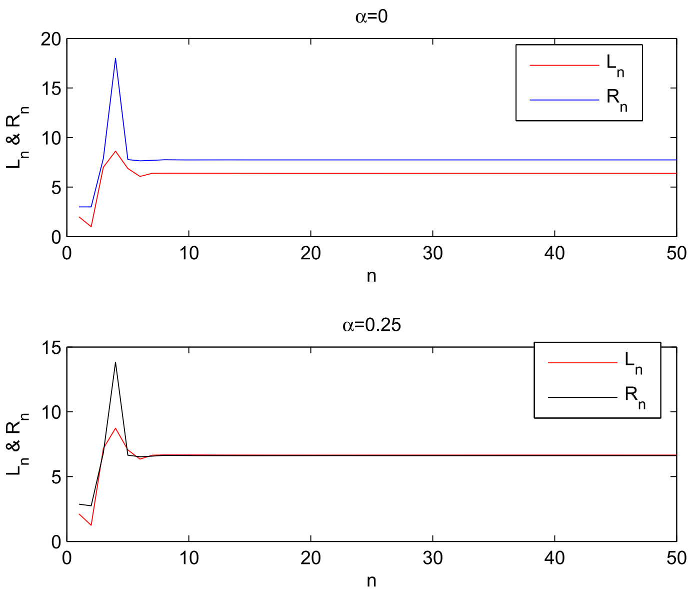

Fig. 5 The solution of system (66) at $\alpha = 0$ and $\alpha = 0.25$

Moreover every positive fuzzy solution x_n of Eq. (54) converges $\overline{x}$ with respect to D as $n \to \infty$. (see Figs. 4, 5, 6)

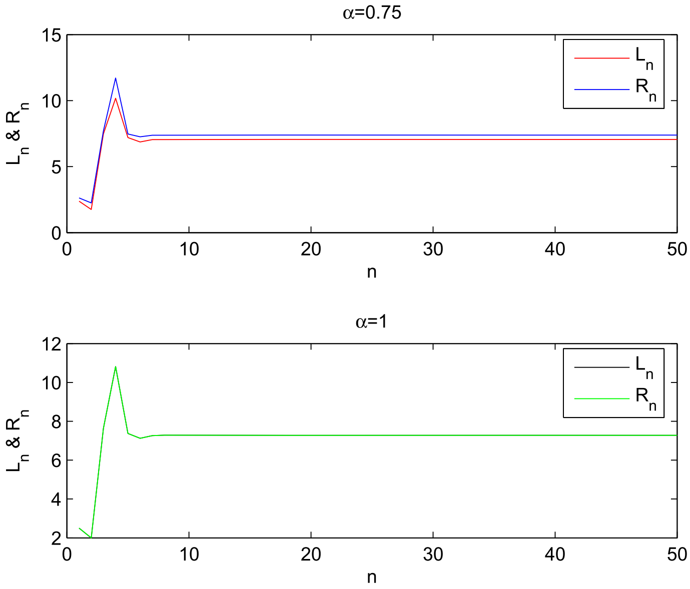

Fig. 6 The solution of system (66) at $\alpha = 0.75$ and $\alpha = 1$

5 Conclusion

In this work, utilizing g-division of fuzzy numbers, the qualitative features of second-order FDE $x_{n+1} = A + \frac{Bx_n}{x_{n-1}^2}$ are studied. The main theoretic results are as follows.

(i) The existence and uniqueness of positive fuzzy solution of FDE (3) are obtained under the positive initial values x_{-1}, x_0.

(ii) The positive fuzzy solution is bounded and persistence either Case (i) or Case (ii) occurs. Moreover, every positive fuzzy solution x_n tend to the unique equilibrium x as $n \to \infty$, if $A_{l,\alpha}^2 > \frac{4}{3}B_{l,\alpha}$, $A_{r,\alpha}^2 > \frac{4}{3}B_{r,\alpha}$, $\alpha \in (0, 1]$. And also, every positive fuzzy solution x_n of FDE (3) converges to the unique equilibrium x as $n \to \infty$. if $A_{l,\alpha} > B_{r,\alpha} > 1$, $A_{r,\alpha} > B_{1,\alpha} > 1$ and $\frac{R_{n,\alpha}L_{n-1,\alpha}^2}{L_{n,\alpha}R_{n-1,\alpha}^2} \leq \frac{B_{l,\alpha}}{B_{r,\alpha}}$, $\alpha \in (0, 1]$, $n = 0, 1, 2, \ldots$.

Author Contributions All authors contributed equally in this article. They read and approved the final manuscript.

Funding This work was financially supported by Guizhou Scientific and Technological Platform Talents ([2022]020-1), Scientific Research Foundation of Guizhou Provincial Department of Science and Technology ([2020]1Y008, [2022]021, [2022]026), and Scientific Climbing Programme of Xiamen University of Technology (XPDKQ20021).

Data Availability Data sharing is not applicable to this article as no datasets were generated or analyzed during the current study.

Declarations

Conflict of interest The authors declare that they have no conflict of interest.

References

1. DeVault, R., Ladas, G., Schultz, S.W.: On the recursive sequence $x_{n+1} = A/x_n + 1/x_{n-2}$. Proc. Am. Math. Soc. **126**(11), 3257–3261 (1998)
2. Abu-Saris, R.M., DeVault, R.: Global stability of $y_{n+1} = A + \frac{y_n}{y_{n-k}}$. Appl. Math. Lett. **16**, 173–178 (2003)
3. Amleh, A.M., Grove, E.A., Ladas, G., Georgiou, D.A.: On the recursive sequence $x_{n+1} = A + \frac{x_{n-1}}{x_n}$. J. Math. Anal. Appl. **233**, 790–798 (1999)
4. He, W.S., Li, W.T., Yan, X.X.: Global attractivity of the difference equation $x_{n+1} = a + \frac{x_{n-k}}{x_n}$. Appl. Math. Comput. **151**, 879–885 (2004)
5. DeVault, R., Ladas, G., Schultz, S.W.: Necessary and sufficient conditions the boundedness of $x_{n+1} = A/x_n^p + B/x_{n-1}^q$. J. Differ. Equ. Appl. **3**, 259–266 (1998)
6. Agarwal, R.P., Li, W.T., Pang, P.Y.H.: Asymptotic behavior of a class of nonlinear delay difference equations. J. Differ. Equ. Appl. **8**, 719–728 (2002)
7. Kocić, V.L., Ladas, G.: Global Behavior of Nonlinear Difference Equations of Higher Order with Application. Kluwer Academic Publishers, Dordrecht (1993)
8. Kulenonvić, M.R.S., Ladas, G.: Dynamics of Second Order Rational Difference Equations with Open Problems and Conjectures. Chapaman & Hall, Boca Raton (2002)
9. Li, W.T., Sun, H.R.: Dynamic of a rational difference equation. Appl. Math. Comput. **163**, 577–591 (2005)
10. Su, Y.H., Li, W.T.: Global attractivity of a higher order nonlinear difference equation. J. Differ. Equ. Appl. **11**, 947–958 (2005)
11. Hu, L.X., Li, W.T.: Global stability of a rational difference equation. Appl. Math. Comput. **190**, 1322–1327 (2007)
12. Bešo, E., Kalabušić, S., Mujić, N., Pilav, E.: Boundedness of solutions and stability of certain second-order difference equation with quadratic term. Adv. Differ. Equ. **2020**, 1–22 (2020)
13. Khyat, T., Kulenović, M.R.S., Pilav, E.: The Neimark–Sacker bifurcation and symptotic approximation of the invariant curve of a certain difference equation. J. Comput. Anal. Appl. **23**(8), 1335–1346 (2017)
14. Papaschinopoulos, G., Schinas, C.J.: On a system of two nonlinear difference equations. J. Math. Anal. Appl. **219**, 415–426 (1998)
15. Yang, X.: On the system of rational difference equations $x_n = A + y_{n-1}/x_{n-p}y_{n-q}$, $y_n = A + x_{n-1}/x_{n-r}y_{n-s}$. J. Math. Anal. Appl. **307**, 305–311 (2005)
16. Zhang, Q., Yang, L., Liu, J.: Dynamics of a system of rational third-order difference equation. Adv. Differ. Equ. **2012**(136), 1–8 (2012)
17. Ibrahim, T.F., Zhang, Q.: Stability of an anti-competitive system of rational difference equations. Arch. Sci. **66**(5), 44–58 (2013)
18. Deeba, E.Y., De Korvin, A.: Analysis by fuzzy difference equations of a model of CO_2 level in the blood. Appl. Math. Lett. **12**, 33–40 (1999)
19. Deeba, E.Y., De Korvin, A., Koh, E.L.: A fuzzy difference equation with an application. J. Differ. Equ. Appl. **2**, 365–374 (1996)
20. Papaschinopoulos, G., Schinas, C.J.: On the fuzzy difference equation $x_{n+1} = \sum_{i=0}^{k-1} A_i/x_{n-i}^{p_i} + 1/x_{n-k}^{p_k}$. J. Differ. Equ. Appl. **6**, 75–89 (2000)
21. Papaschinopoulos, G., Papadopoulos, B.K.: On the fuzzy difference equation $x_{n+1} = A + B/x_n$. Soft. Comput. **6**, 456–461 (2002)
22. Papaschinopoulos, G., Papadopoulos, B.K.: On the fuzzy difference equation $x_{n+1} = A + x_n/x_{n-m}$. Fuzzy Sets Syst. **129**, 73–81 (2002)

23. Zhang, Q., Lin, F., Zhong, X.: On discrete time Beverton–Holt population model with fuzzy environment. Math. Biosci. Eng. **16**, 1471–1488 (2019)
24. Zhang, Q., Zhang, W., Lin, F., Li, D.: On dynamic behavior of second-order exponential-type fuzzy difference equation. Fuzzy Sets Syst. **419**, 169–187 (2021)
25. Stefanidou, G., Papaschinopoulos, G., Schinas, C.J.: On an exponential-type fuzzy difference equation. Adv. Differ. Equ. (2010). https://doi.org/10.1155/2010/196920
26. Zhang, Q., Yang, L., Liao, D.: Behavior of solutions to a fuzzy nonlinear difference equation. Iran. J. Fuzzy Syst. **9**, 1–12 (2012)
27. Chrysafis, K.A., Papadopoulos, B.K., Papaschinopoulos, G.: On the fuzzy difference equations of finance. Fuzzy Sets Syst. **159**(24), 3259–3270 (2008)
28. Zhang, Q., Yang, L., Liao, D.: On first order fuzzy Ricatti difference equation. Inf. Sci. **270**, 226–236 (2014)
29. Wang, C., Su, X., Liu, P., Hu, X., Li, R.: On the dynamics of a five-order fuzzy difference equation. J. Nonlinear Sci. Appl. **10**, 3303–3319 (2017)
30. Wang, C., Li, J., Jia, L.: Dynamics of a high-order nonlinear fuzzy difference equation. J. Appl. Anal. Comput. **11**(1), 404–421 (2021)
31. Jia, L.: Dynamic behaviors of a class of high-order fuzzy difference equations. J. Math. **4**, 1–13 (2020)
32. Sun, T., Su, G., Qin, B.: On the fuzzy difference equation $x_n = F(x_{n-1}, x_{n-k})$. Fuzzy Sets Syst. **387**, 81–88 (2020)
33. Wang, C., Li, J.: Periodic solution for a max-type fuzzy difference equation. J. Math. **3**, 1–12 (2020)
34. Kocak, C.: First-order ARMA type fuzzy time series method based on fuzzy logic relation tables. Math. Probl. Eng. **2013**, 12 (2013)
35. Ivaz, K., Khastan, A., Nieto, J.J.: A numerical method of fuzzy differential equations and hybrid fuzzy differential equations. Abstr. Appl. Anal. **2013**, 10 (2013)
36. Malinowski, M.T.: On a new set-valued stochastic integral with respect to semi- martingales and its applications. J. Math. Anal. Appl. **408**, 669–680 (2013)
37. Malinowski, M.T.: Some properties of strong solutions to stochastic fuzzy differential equations. Inf. Sci. **252**, 62–80 (2013)
38. Malinowski, M.T.: Approximation schemes for fuzzy stochastic integral equations. Appl. Math. Comput. **219**, 11278–11290 (2013)
39. Hua, M., Cheng, P., Fei, J., Zhang, J., Chen, J.: Robust H1-ltering for uncertain discrete-time fuzzy stochastic systems with sensor nonlinearities and time-varying delay. J. Appl. Math. **2012**, 25 (2012)
40. Bera, S., Khajanchi, S., Roy, T.K.: Stability analysis of fuzzy HTLV-I infection model: a dynamic approach. J. Appl. Math. Comput. (2022). https://doi.org/10.1007/s12190-022-01741-y
41. Stefanini, L.: A generalization of Hukuhara difference and division for interval and fuzzy arithmetic. Fuzzy Sets Syst. **161**, 1564–1584 (2010)