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RESEARCH ARTICLE

Final state observability estimates and cost-uniform approximate null-controllability for bi-continuous semigroups

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Abstract
We consider final state observability estimates for bi-continuous semigroups on Banach spaces, i.e. for every initial value, estimating the state at a final time $T > 0$ by taking into account the orbit of the initial value under the semigroup for $t \in [0, T]$, measured in a suitable norm. We state a sufficient criterion based on an uncertainty relation and a dissipation estimate and provide two examples of bi-continuous semigroups which share a final state observability estimate, namely the Gauß–Weierstraß semigroup and the Ornstein–Uhlenbeck semigroup on the space of bounded continuous functions. Moreover, we generalise the duality between cost-uniform approximate null-controllability and final state observability estimates to the setting of locally convex spaces for the case of bounded and continuous control functions, which seems to be new even for the case of Banach spaces.

Keywords Final state observability estimate · Bi-continuous semigroups · Cost-uniform approximate null-controllability · Saks space · Mixed topology

1 Introduction

Let X be a Banach space and $(S_t)_{t\geq 0}$ a semigroup on X, i.e. $S_0 = I$ and $S_{t+s} = S_t S_s$ for all $t, s \geq 0$. Moreover, let Y be another Banach space and $C \in \mathcal{L}(X; Y)$, a so-called observation operator, and $T > 0$. Then $(S_t)_{t\geq 0}$ satisfies a *final state observability estimate* w.r.t. some (Banach) space $\mathcal{Z}$ of functions on $[0, T]$ with values in Y, if there

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exists $C_{\mathrm{obs}} \geq 0$ such that

$$\|S_T x\|_X \leq C_{\mathrm{obs}} \left\| C S_{(\cdot)} x \right\|_{\mathcal{Z}} \qquad (x \in X).$$

Put differently, we can estimate the norm of the final state $S_T x$ by just taking into account the observations $C S_t x$ for $t \in [0, T]$. Typical applications stem from evolution equations on some function space over (a subset of) $\mathbb{R}^d$, where C is a restriction operator to a suitable subset Ω of $\mathbb{R}^d$ (or of the subset of $\mathbb{R}^d$ the functions are defined on) such that we want to control the final state on all of $\mathbb{R}^d$ by just measuring the evolution on the subset Ω. Final state observability estimates have been studied in various contexts due to its relation to null-controllability, see e.g. [1, 2, 4, 9, 10, 18, 21, 23, 25, 29] and references therein.

Classically, the space $\mathcal{Z}$ is some L_r-space with $r \in [1, \infty]$ (when working in Hilbert spaces, one usually chooses $r = 2$), and then the final state observability estimate yields the form

$$\|S_T x\|_X \leq \begin{cases} C_{\mathrm{obs}} \left(\int_0^T \|C S_t x\|_Y^r \, \mathrm{d}t \right)^{1/r} & \text{if } r \in [1, \infty), \\ C_{\mathrm{obs}} \operatorname{ess\,sup}_{t \in [0,T]} \|C S_t x\|_Y & \text{if } r = \infty, \end{cases} \qquad (x \in X).$$

Clearly, in order to formulate this final state observability estimate (i.e. to have a well-defined right-hand side) we need some regularity of the semigroup. Indeed, we require measurability of $t \mapsto \|C S_t x\|_Y$ for all $x \in X$. Of course, strong continuity of $(S_t)_{t \geq 0}$ yields continuity of these maps and is therefore sufficient, but also weaker regularities are suitable. In [1], dual semigroups of strongly continuous semigroups were considered which yield sufficient regularity.

In this paper we aim at two types of results. First, we consider final state observability estimates for so-called bi-continuous semigroups, see e.g. [16, 17], which are not strongly continuous for the norm-topology on X but only for a weaker topology. Note that dual semigroups are a special case of bi-continuous ones when considering the weak* topology. There are classical examples of bi-continuous semigroups such as the Gauß–Weierstraß semigroup on $C_b(\mathbb{R}^d)$, the space of bounded continuous functions (on $\mathbb{R}^d$), as well as the Ornstein–Uhlenbeck semigroup on $C_b(\mathbb{R}^d)$. Second, we relate cost-uniform approximate null-controllability of a control system sharing only weak continuity properties such as bi-continuity with a final state observability estimate for the dual system, thus generalising the well-known duality in Hilbert and Banach spaces [4, 6, 25, 29]. Since this demands to work in Hausdorff locally convex spaces, we here focus on continuous control functions which results in the space $\mathcal{Z}$ above being a space of vector measures.

The paper is organised as follows. In Sect. 2 we review bi-continuous semigroups and then turn to final-state observability estimates in Sect. 3 together with two examples in Sect. 4. Final state observability estimates are then related with cost-uniform approximate null-controllability via duality, which we will exploit in our context in Sect. 5.

2 Bi-continuous semigroups

In this short section we recall some notation and definitions from the theory of bi-continuous semigroups that we need in subsequent sections. For a vector space X over the field $\mathbb{R}$ or $\mathbb{C}$ with a Hausdorff locally convex topology τ_X we denote by $(X, \tau_X)'$ the topological linear dual space and just write $X' := (X, \tau_X)'$ if (X, τ_X) is a Banach space. We use the symbol $\mathcal{L}(X; Y) := \mathcal{L}((X, \|\cdot\|_X); (Y, \|\cdot\|_Y))$ for the space of continuous linear operators from a Banach space $(X, \|\cdot\|_X)$ to a Banach space $(Y, \|\cdot\|_Y)$ and denote by $\|\cdot\|_{\mathcal{L}(X;Y)}$ the operator norm on $\mathcal{L}(X; Y)$. If $X = Y$, we set $\mathcal{L}(X) := \mathcal{L}(X; X)$.

Definition 2.1 ([5, I.3.2 Definition]) Let $(X, \|\cdot\|_X)$ be a normed space and τ_X a Hausdorff locally convex topology on X.

(a) The triple $(X, \|\cdot\|_X, \tau_X)$ is called a *Saks space* if $(X, \|\cdot\|_X)$ is a Banach space, $\tau_X \subseteq \tau_{\|\cdot\|_X}$ and $(X, \tau_X)'$ is norming for X where $\tau_{\|\cdot\|_X}$ denotes the $\|\cdot\|_X$-topology.
(b) The *mixed topology* $\gamma_X := \gamma(\|\cdot\|_X, \tau_X)$ is the finest linear topology on X that coincides with τ_X on $\|\cdot\|_X$-bounded sets and such that $\tau_X \subseteq \gamma_X \subseteq \tau_{\|\cdot\|_X}$.

The mixed topology is actually Hausdorff locally convex and the definition given above is equivalent to the one from the literature [28, Section 2.1] by [28, Lemmas 2.2.1, 2.2.2].

Definition 2.2 We call a Saks space $(X, \|\cdot\|_X, \tau_X)$ *sequentially complete* if (X, γ_X) is sequentially complete.

Due to [28, Corollary 2.3.2] a Saks space $(X, \|\cdot\|_X, \tau_X)$ is sequentially complete if and only if (X, τ_X) is sequentially complete on $\|\cdot\|_X$-bounded sets, i.e. every $\|\cdot\|_X$-bounded τ_X-Cauchy sequence converges in X. In combination with [13, Remark 2.3 (c)] this yields that a triple $(X, \|\cdot\|_X, \tau_X)$ fulfils [17, Assumptions 1] if and only if it is a sequentially complete Saks space.

Definition 2.3 ([17, Definition 3]) Let $(X, \|\cdot\|_X, \tau_X)$ be a sequentially complete Saks space. Let $(S_t)_{t\geq 0}$ in $\mathcal{L}(X)$ be a semigroup on X. We say that $(S_t)_{t\geq 0}$ is *(locally) τ_X-bi-continuous* if

(a) it is exponentially bounded, i.e. there exist $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|S_t\|_{\mathcal{L}(X)} \leq Me^{\omega t}$ for all $t \geq 0$,
(b) $(S_t)_{t\geq 0}$ is a C_0-semigroup on (X, τ_X), i.e. for all $x \in X$ the map $[0, \infty) \ni t \mapsto T_t x \in (X, \tau_X)$ is continuous,
(c) it is (locally) bi-equicontinuous, i.e. for every $(x_n)_{n\in\mathbb{N}}$ in X and $x \in X$ with $\sup_{n\in\mathbb{N}} \|x_n\|_X < \infty$ and $\tau_X\text{-}\lim_{n\to\infty} x_n = x$ we have

$$\tau_X\text{-}\lim_{n\to\infty} S_t(x_n - x) = 0$$

(locally) uniformly for $t \in [0, \infty)$.

As in the case of C_0-semigroups on Banach spaces we can define generators for bi-continuous semigroups.

Definition 2.4 ([7, Definition 1.2.6]) Let $(X, \|\cdot\|_X, \tau_X)$ be a sequentially complete Saks space and $(S_t)_{t\geq 0}$ a locally τ_X-bi-continuous semigroup on X. The *generator* $(-A, D(A))$ is defined by

$$D(A) := \left\{ x \in X \mid \tau_X\text{-}\lim_{t\to 0+} \frac{S_t x - x}{t} \text{ exists in } X \text{ and } \sup_{t\in(0,1]} \frac{\|S_t x - x\|_X}{t} < \infty \right\},$$

$$-Ax := \tau_X\text{-}\lim_{t\to 0+} \frac{S_t x - x}{t}, \quad (x \in D(A)).$$

Remark 2.5 There is no common agreement whether to use the here presented definition of a generator or its negative. Throughout the entire paper we will stick to the definition made above, i.e. $-A$ is the generator.

3 Final state observability estimates for bi-continuous semigroups

The final state observability estimate rests on the following abstract theorem. It provides a sufficient criterion stating that an abstract uncertainty principle (also called spectral inequality), see (UP), together with a dissipation estimate, see (DISS), yields a final state observability estimate, and has its roots in [18], see also [1, 9, 21, 23].

Theorem 3.1 ([1, Theorem A.1]) *Let X and Y be Banach spaces, $C \in \mathcal{L}(X;Y)$, $(S_t)_{t\geq 0}$ a semigroup on X, $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|S_t\|_{\mathcal{L}(X)} \leq M\mathrm{e}^{\omega t}$ for all $t \geq 0$, and assume that for all $x \in X$ the map $t \mapsto \|CS_t x\|_Y$ is measurable. Further, let $\lambda^* \geq 0$, $(P_\lambda)_{\lambda>\lambda^*}$ in $\mathcal{L}(X)$, $r \in [1,\infty]$, $d_0, d_1, d_3, \gamma_1, \gamma_2, \gamma_3, T > 0$ with $\gamma_1 < \gamma_2$, and $d_2 \geq 1$, and assume that*

$$\forall x \in X\ \forall \lambda > \lambda^*: \quad \|P_\lambda x\|_X \leq d_0 \mathrm{e}^{d_1 \lambda^{\gamma_1}} \|CP_\lambda x\|_Y \tag{UP}$$

and

$$\forall x \in X\ \forall \lambda > \lambda^*\ \forall t \in (0, T/2]: \quad \|(I - P_\lambda)S_t x\|_X \leq d_2 \mathrm{e}^{-d_3 \lambda^{\gamma_2} t^{\gamma_3}} \|x\|_X. \tag{DISS}$$

Then there exists $C_{\mathrm{obs}} \geq 0$ such that for all $x \in X$ we have

$$\|S_T x\|_X \leq \begin{cases} C_{\mathrm{obs}} \left(\int_0^T \|CS_t x\|_Y^r \,\mathrm{d}t \right)^{1/r} & \text{if } r \in [1,\infty), \\ C_{\mathrm{obs}} \operatorname{ess\,sup}_{t\in[0,T]} \|CS_t x\|_Y & \text{if } r = \infty. \end{cases}$$

Remark 3.2 The constant C_{obs} is explicit in all parameters and of the form

$$C_{\mathrm{obs}} = \frac{C_1}{T^{1/r}} \exp\left(\frac{C_2}{T^{\frac{\gamma_1\gamma_3}{\gamma_2-\gamma_1}}} + C_3 T \right),$$

with $T^{1/r} = 1$ if $r = \infty$, and suitable constants $C_1, C_2, C_3 \geq 0$ depending on the parameters; see [1, Theorem A.1] as well as [10, Theorem 2.1].

In order to obtain a version of this theorem for bi-continuous semigroups, we need to argue on measurability of $[0,\infty) \ni t \mapsto \|CS_t x\|_Y$ for all $x \in X$.

Lemma 3.3 *Let $(X, \|\cdot\|_X, \tau_X)$ be a sequentially complete Saks space, $(S_t)_{t\geq 0}$ a locally τ_X-bi-continuous semigroup on X, $(Y, \|\cdot\|_Y, \tau_Y)$ a Saks space, $C\colon X \to Y$ linear and sequentially τ_X-τ_Y-continuous on $\|\cdot\|_X$-bounded sets, and $x \in X$. Then $[0,\infty) \ni t \mapsto \|CS_t x\|_Y$ is measurable.*

Proof Since $[0,\infty) \ni t \mapsto \left|\langle y', CS_t x\rangle\right|$ is continuous for all $y' \in (Y,\tau_Y)'$ by the assumptions on C and the exponential boundedness of $(S_t)_{t\geq 0}$, and $\|CS_t x\|_Y = \sup\{\left|\langle y', CS_t x\rangle\right| \mid y' \in (Y,\tau_Y)', \left\|y'\right\|_{Y'} \leq 1\}$ for all $t \geq 0$, we obtain that $[0,\infty) \ni t \mapsto \|CS_t x\|_Y$ is lower semi-continuous and hence measurable. □

In view of Lemma 3.3, we can apply Theorem 3.1 to obtain the following.

Theorem 3.4 *Let $(X, \|\cdot\|_X, \tau_X)$ be a sequentially complete Saks space, $(S_t)_{t\geq 0}$ a locally τ_X-bi-continuous semigroup on X, $(Y, \|\cdot\|_Y, \tau_Y)$ a Saks space, $C \in \mathcal{L}(X;Y)$ such that C is also sequentially τ_X-τ_Y-continuous on $\|\cdot\|_X$-bounded sets. Further, let $\lambda^* \geq 0$, $(P_\lambda)_{\lambda>\lambda^*}$ a family in $\mathcal{L}(X)$, $r \in [1,\infty]$, $d_0, d_1, d_3, \gamma_1, \gamma_2, \gamma_3, T > 0$ with $\gamma_1 < \gamma_2$, and $d_2 \geq 1$, and assume that*

$$\forall x \in X\ \forall \lambda > \lambda^*\colon \quad \|P_\lambda x\|_X \leq d_0 \mathrm{e}^{d_1 \lambda^{\gamma_1}} \|CP_\lambda x\|_Y \tag{UP'}$$

and

$$\forall x \in X\ \forall \lambda > \lambda^*\ \forall t \in (0, T/2]\colon \quad \|(I-P_\lambda)S_t x\|_X \leq d_2 \mathrm{e}^{-d_3 \lambda^{\gamma_2} t^{\gamma_3}} \|x\|_X . \tag{DISS'}$$

Then there exists $C_{\mathrm{obs}} \geq 0$ such that for all $x \in X$ we have

$$\|S_T x\|_X \leq \begin{cases} C_{\mathrm{obs}} \left(\int_0^T \|CS_t x\|_Y^r \,\mathrm{d}t\right)^{1/r} & \text{if } r \in [1,\infty), \\ C_{\mathrm{obs}} \operatorname{ess\,sup}_{t\in[0,T]} \|CS_t x\|_Y & \text{if } r = \infty. \end{cases}$$

Remark 3.5 The statements in Theorem 3.1 and Theorem 3.4 can be generalised in the sense that one can obtain an estimate with an L_r-norm of $t \mapsto \|CS_t x\|_Y$ on a measurable subset $E \subseteq [0,T]$ with positive Lebesgue measure; cf. e.g. [2]. However, in this case the constant C_{obs} (cf. Remark 3.2) is not explicit anymore.

4 Two examples of bi-continuous semigroups

In this section we consider final state observability for two important examples: the Gauß–Weierstraß semigroup on $C_b(\mathbb{R}^d)$ and the Ornstein–Uhlenbeck semigroup on $C_b(\mathbb{R}^d)$. We begin with the study of restriction operators on $C_b(\mathbb{R}^d)$, restricting functions to suitable subsets, and relate this to an abstract uncertainty principle.

4.1 Restriction operators on $C_b(\mathbb{R}^d)$ and the uncertainty principle

Let $\Omega \subseteq \mathbb{R}^d$ be non-empty, $C\colon C_b(\mathbb{R}^d) \to C_b(\Omega)$ the restriction operator defined by $Cf := f|_\Omega$ for $f \in C_b(\mathbb{R}^d)$. Then $C \in \mathcal{L}(C_b(\mathbb{R}^d); C_b(\Omega))$. Let τ_{co} be the compact-open topology on $C_b(\mathbb{R}^d)$ (as well as on $C_b(\Omega)$). Then $(C_b(\Omega), \|\cdot\|_\infty, \tau_{\mathrm{co}})$ is a Saks space which is sequentially complete if Ω is locally compact (in particular if $\Omega = \mathbb{R}^d$).

Lemma 4.1 $C\colon (C_b(\mathbb{R}^d), \tau_{\mathrm{co}}) \to (C_b(\Omega), \tau_{\mathrm{co}})$ *is continuous.*

Proof The map C is clearly linear. Due to [22, Theorem 46.8] the compact-open topology τ_{co} on $C_b(Z)$ for a Hausdorff topological space Z is given by the system of seminorms

$$p_K^Z(f) := \sup_{x\in K} |f(x)| \quad (f \in C_b(Z))$$

for compact $K \subseteq Z$. Let $K \subseteq \Omega$ be compact in the relative topology. Then K is compact in $\mathbb{R}^d$ as well and

$$p_K^\Omega(Cf) = \sup_{x\in K} |f|_\Omega(x)| = \sup_{x\in K} |f(x)| = p_K^{\mathbb{R}^d}(f) \quad (f \in C_b(\mathbb{R}^d)),$$

which means that C is continuous. □

We now use the operator C to provide an uncertainty principle based on the well-known Logvinenko–Sereda theorem. Let $\eta \in C_c[0,\infty)$, $\mathbb{1}_{[0,1/2]} \leq \eta \leq \mathbb{1}_{[0,1]}$. For $\lambda > 0$ let $\chi_\lambda\colon \mathbb{R}^d \to \mathbb{R}$, $\chi_\lambda := \eta(|\cdot|/\lambda)$, and $P_\lambda \in \mathcal{L}(C_b(\mathbb{R}^d))$ be defined by $P_\lambda f := (\mathcal{F}^{-1}\chi_\lambda) * f$, where $\mathcal{F}$ denotes the Fourier transformation. By Young's inequality and scaling properties of the Fourier transformation, we have

$$\|P_\lambda\| \leq \left\|\mathcal{F}^{-1}\chi_\lambda\right\|_{L_1(\mathbb{R}^d)} = \left\|\mathcal{F}^{-1}\chi_1\right\|_{L_1(\mathbb{R}^d)} \quad (\lambda > 0).$$

Note that for all $f \in C_b(\mathbb{R}^d)$ and $\lambda > 0$ we have $\mathcal{F}P_\lambda f = \chi_\lambda \mathcal{F}f$ and therefore $\operatorname{spt} \mathcal{F}P_\lambda f \subseteq B[0,\lambda] \subseteq [-\lambda,\lambda]^d$, where $B[0,\lambda] := \{x \in \mathbb{R}^d \mid |x| \leq \lambda\}$ is the closed ball around 0 with radius λ.

Definition 4.2 Let $\Omega \subseteq \mathbb{R}^d$. Then Ω is called *thick* if Ω is measurable and there exist $L \in (0,\infty)^d$ and $\rho \in (0,1]$ such that

$$\lambda^d(\Omega \cap (x + (0,L))) \geq \rho\lambda^d((0,L)) \quad (x \in \mathbb{R}^d),$$

where λ^d denotes the d-dimensional Lebesgue measure, and $(0,L) := \prod_{j=1}^d (0,L_j)$ is the hypercube with sidelengths contained in L.

Thus, a measurable set $\Omega \subseteq \mathbb{R}^d$ is thick (with parameters L and ρ) provided the portion of Ω in every hypercube with sidelengths contained in L is at least ρ.

By the Logvinenko–Sereda theorem (see [12, 19]), if $\Omega \subseteq \mathbb{R}^d$ is a thick set, then there exist $d_0, d_1 > 0$ such that

$$\|P_\lambda f\|_{C_b(\mathbb{R}^d)} \leq d_0 \mathrm{e}^{d_1 \lambda} \|C P_\lambda f\|_{C_b(\Omega)} \quad (\lambda > 0,\, f \in C_b(\mathbb{R}^d)). \tag{LS}$$

Thus, (LS) yields an estimate of the form (UP').

4.2 The Gauß–Weierstraß semigroup on $C_b(\mathbb{R}^d)$

Let $k\colon (0,\infty) \times \mathbb{R}^d \to \mathbb{R}$ be given by

$$k_t(x) := k(t,x) := \frac{1}{(4\pi t)^{d/2}} \mathrm{e}^{-|x|^2/(4t)} \quad (t > 0, x \in \mathbb{R}^d),$$

the so-called *Gauß–Weierstraß kernel*. For $t \geq 0$ we define $S_t \in \mathcal{L}(C_b(\mathbb{R}^d))$ by

$$S_t f := \begin{cases} f & t = 0, \\ k_t * f & t > 0. \end{cases}$$

Note that by Young's inequality and the fact that $\|k_t\|_{L_1(\mathbb{R}^d)} = 1$ for all $t > 0$ we have $\|S_t f\|_{C_b(\mathbb{R}^d)} \leq \|f\|_{C_b(\mathbb{R}^d)}$ for $f \in C_b(\mathbb{R}^d)$ and $t \geq 0$. It is easy to see that $(S_t)_{t\geq 0}$ is a semigroup, which is called the *Gauß–Weierstraß semigroup*. Let τ_{co} be the compact-open topology on $C_b(\mathbb{R}^d)$. Then $(S_t)_{t\geq 0}$ is locally τ_{co}-bi-continuous; see e.g. [17, Examples 6 (a)].

For $\lambda > 0$ let $P_\lambda \in \mathcal{L}(C_b(\mathbb{R}^d))$ be defined as in Subsect. 4.1. By [1, Proposition 3.2], there exist $d_2 \geq 1$ and $d_3 > 0$ such that

$$\|(I - P_\lambda) S_t f\|_{C_b(\mathbb{R}^d)} \leq d_2 \mathrm{e}^{-d_3 \lambda^2 t} \|f\|_{C_b(\mathbb{R}^d)} \quad (\lambda > 0, t \geq 0, f \in C_b(\mathbb{R}^d)), \tag{DISS(GW)}$$

i.e. a dissipation estimate (DISS') is fulfilled.

Thus, if $\Omega \subseteq \mathbb{R}^d$ is thick, then (LS) and (DISS(GW)) provide the estimates (UP') and (DISS') and so Theorem 3.4 yields a final state observability estimate for the Gauß–Weierstraß semigroup on $C_b(\mathbb{R}^d)$.

4.3 The Ornstein–Uhlenbeck semigroup on $C_b(\mathbb{R}^d)$

Let $M\colon (0,\infty) \times \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}$ be given by

$$M_t(x,y) := M(t,x,y) := \frac{1}{\pi^{d/2}(1-\mathrm{e}^{-2t})^{d/2}} \mathrm{e}^{-|y-\mathrm{e}^{-t}x|^2/(1-\mathrm{e}^{-2t})} \quad (t > 0, x, y \in \mathbb{R}^d),$$

the so-called *Mehler kernel*. For $t \geq 0$ we define $S_t \in \mathcal{L}(C_b(\mathbb{R}^d))$ by

$$S_t f := \begin{cases} f & t = 0, \\ \int_{\mathbb{R}^d} M_t(\cdot, y) f(y)\,\mathrm{d}y & t > 0. \end{cases}$$

Since $\int_{\mathbb{R}^d} M_t(\cdot, y)\,\mathrm{d}y = 1$ for all $t > 0$, we have $\|S_t f\|_{C_b(\mathbb{R}^d)} \leq \|f\|_{C_b(\mathbb{R}^d)}$ for $f \in C_b(\mathbb{R}^d)$ and $t \geq 0$. It is not difficult to see that $(S_t)_{t\geq 0}$ is a semigroup, which is called the *Ornstein–Uhlenbeck semigroup*. Let τ_{co} be the compact-open topology on $C_b(\mathbb{R}^d)$. Then $(S_t)_{t\geq 0}$ is locally τ_{co}-bi-continuous on $C_b(\mathbb{R}^d)$; see e.g. [16, Proposition 3.10].

Define $k\colon (0,1) \times \mathbb{R}^d \to \mathbb{R}$ by

$$k_s(x) := \frac{1}{\pi^{d/2}(1-s^2)^{d/2}} \mathrm{e}^{-|x|^2/(1-s^2)}.$$

Let $s \in (0, 1)$. Then we obtain

$$M_{\ln \frac{1}{s}}(\tfrac{1}{s}x, y) = \frac{1}{\pi^{d/2}(1-s^2)^{d/2}} \mathrm{e}^{-|y-x|^2/(1-s^2)} = k_s(x-y) \quad (x, y \in \mathbb{R}^d).$$

Hence,

$$\big(S_{\ln \frac{1}{s}} f\big)(\tfrac{1}{s}\,\cdot) = k_s * f \quad (f \in C_b(\mathbb{R}^d)).$$

For $\lambda > 0$ let $P_\lambda \in \mathcal{L}(C_b(\mathbb{R}^d))$ as in Subsect. 4.2. Since

$$\mathcal{F}k_s(\xi) = \mathrm{e}^{-(1-s^2)|\xi|^2/4} =: h_s(\xi) \quad (\xi \in \mathbb{R}^d),$$

for $\lambda > 0$ and $s \in (0, 1)$ we conclude that

$$\begin{aligned}\big((I - P_\lambda)S_{\ln \frac{1}{s}} f\big)(\tfrac{1}{s}\,\cdot) &= (I - P_{\lambda/s})\big(S_{\ln \frac{1}{s}} f(\tfrac{1}{s}\,\cdot)\big) \\ &= \mathcal{F}^{-1}\big((1-\chi_{\lambda/s})h_s\big) * f \quad (f \in C_b(\mathbb{R}^d)).\end{aligned}$$

Lemma 4.3 *There exist $d_2 \geq 1$ and $d_3 > 0$ such that for $\lambda > 0$ and $s \in (0, 1)$ we have*

$$\Big\|\mathcal{F}^{-1}\big((1-\chi_\lambda)h_s\big)\Big\|_{L_1(\mathbb{R}^d)} \leq d_2 \mathrm{e}^{-d_3\lambda^2(1-s^2)}.$$

Proof Let $\lambda > 0$, $s \in (0, 1)$, and define

$$\sigma_{s,\lambda} := (1 - \chi_{\sqrt{1-s^2}\lambda})h_s\Big(\tfrac{1}{\sqrt{1-s^2}}\,\cdot\Big) = (1 - \chi_{\sqrt{1-s^2}\lambda})\mathrm{e}^{-|\cdot|^2/4}.$$

Then by a linear substitution we obtain

$$\left\|\mathcal{F}^{-1}\big((1-\chi_\lambda)h_s\big)\right\|_{L_1(\mathbb{R}^d)} = \left\|\mathcal{F}^{-1}\sigma_{s,\lambda}\right\|_{L_1(\mathbb{R}^d)}.$$

Let $\alpha \in \mathbb{N}_0^d$, $|\alpha| \le d+1$. Then

$$\begin{aligned}\left|\partial^\alpha \sigma_{s,\lambda}\right| &\le \mathbb{1}_{\{|\cdot|\ge\sqrt{1-s^2}\lambda/2\}}\left|\partial^\alpha \mathrm{e}^{-|\cdot|^2/4}\right| \\ &\quad + \sum_{\beta\in\mathbb{N}_0^d,\beta<\alpha}\binom{\alpha}{\beta}\left|\partial^{\alpha-\beta}(1-\chi_{\sqrt{1-s^2}\lambda})\right|\left|\partial^\beta \mathrm{e}^{-|\cdot|^2/4}\right|.\end{aligned}$$

There exists $K \ge 0$ such that for all $\beta \in \mathbb{N}_0^d$ with $\beta \le \alpha$ and all $\xi \in \mathbb{R}^d$ we have

$$\left|\partial^\beta \mathrm{e}^{-|\cdot|^2/4}(\xi)\right| \le K(1+|\xi|)^{|\beta|}\mathrm{e}^{-|\xi|^2/4}.$$

Let

$$C_1 := \sup_{\beta\in\mathbb{N}_0^d,\beta\le\alpha,\xi\in\mathbb{R}^d} K(1+|\xi|)^{|\beta|}\mathrm{e}^{-|\xi|^2/16}.$$

Then, for $\beta \le \alpha$ and $|\xi| \ge \sqrt{1-s^2}\lambda/2$ we have

$$\left|\partial^\beta \mathrm{e}^{-|\cdot|^2/4}(\xi)\right| \le C_1\mathrm{e}^{-|\xi|^2/16}\mathrm{e}^{-(1-s^2)\lambda^2/32}.$$

Further, for $\beta < \alpha$ and $\xi \in \mathbb{R}^d$ we have

$$\begin{aligned}&\left|\partial^{\alpha-\beta}(1-\chi_{\sqrt{1-s^2}\lambda})(\xi)\right| \\ &\quad\le (\sqrt{1-s^2}\lambda)^{-|\alpha-\beta|}\left|\partial^{\alpha-\beta}\chi_1\Big(\tfrac{\xi}{\sqrt{1-s^2}\lambda}\Big)\right|\mathbb{1}_{\{\sqrt{1-s^2}\lambda/2\le|\cdot|\le\sqrt{1-s^2}\lambda\}}(\xi) \\ &\quad\le C_2(\sqrt{1-s^2}\lambda)^{-|\alpha-\beta|}\mathbb{1}_{\{\sqrt{1-s^2}\lambda/2\le|\cdot|\le\sqrt{1-s^2}\lambda\}}(\xi)\end{aligned}$$

where $C_2 := \max_{\beta<\alpha}\left\|\partial^{\alpha-\beta}\chi_1\right\|_{C_b(\mathbb{R}^d)}$. Hence, there exists $C \ge 0$ (which is independent of s and λ) such that if $\sqrt{1-s^2}\lambda \ge 1$, then for all $\xi \in \mathbb{R}^d$ we have

$$\left|\partial^\alpha \sigma_{s,\lambda}\right| \le C\mathrm{e}^{-|\xi|^2/16}\mathrm{e}^{-(1-s^2)\lambda^2/32}.$$

Therefore, increasing C, for all $x \in \mathbb{R}^d$ we obtain

$$\left|x^\alpha\mathcal{F}^{-1}\sigma_{s,\lambda}(x)\right| = \left|\mathcal{F}^{-1}(\partial^\alpha\sigma_{s,\lambda})(x)\right| \le C\mathrm{e}^{-(1-s^2)\lambda^2/32}. \tag{1}$$

By choosing $j \in \{1, \dots, d\}$ and $\alpha := (d+1)e_j$ for the j-th canonical unit vector e_j, we observe $\|x\|_\infty^{d+1} \left|\mathcal{F}^{-1}\sigma_{s,\lambda}(x)\right| \leq C\mathrm{e}^{-(1-s^2)\lambda^2/32}$ and hence

$$\left|\mathcal{F}^{-1}\sigma_{s,\lambda}(x)\right| \leq C\mathrm{e}^{-(1-s^2)\lambda^2/32}\,|x|^{-d-1} \tag{2}$$

for all $x \in \mathbb{R}^d \setminus \{0\}$, where we increased C.

Therefore, if $\sqrt{1-s^2}\lambda \geq 1$, we can conclude by (1) for $\alpha = 0$ and (2) that

$$\begin{aligned}\left\|\mathcal{F}^{-1}\big((1-\chi_\lambda)h_s\big)\right\|_{L_1(\mathbb{R}^d)} &= \left\|\mathcal{F}^{-1}\sigma_{s,\lambda}\right\|_{L_1(\mathbb{R}^d)}\\ &\leq C\mathrm{e}^{-(1-s^2)\lambda^2/32}\left(\int_{B[0,1]} 1\,\mathrm{d}x + \int_{\mathbb{R}^d\setminus B[0,1]} |x|^{-d-1}\,\mathrm{d}x\right)\\ &\leq C\mathrm{e}^{-(1-s^2)\lambda^2/32},\end{aligned}$$

where we increased C again.

It remains to prove the estimate for the case $\sqrt{1-s^2}\lambda < 1$. Note that

$$\begin{aligned}\left\|\mathcal{F}^{-1}(\chi_\lambda h_s)\right\|_{L_1(\mathbb{R}^d)} &= \left\|\mathcal{F}^{-1}\chi_\lambda * \mathcal{F}^{-1}h_s\right\|_{L_1(\mathbb{R}^d)}\\ &\leq \left\|\mathcal{F}^{-1}\chi_\lambda\right\|_{L_1(\mathbb{R}^d)}\|k_s\|_{L_1(\mathbb{R}^d)} = \left\|\mathcal{F}^{-1}\chi_1\right\|_{L_1(\mathbb{R}^d)},\end{aligned}$$

where the last equality follows form scaling properties of the Fourier transformation and the fact that k_s is normalised in $L_1(\mathbb{R}^d)$.

Thus, for $\sqrt{1-s^2}\lambda < 1$ we obtain

$$\left\|\mathcal{F}^{-1}(\chi_\lambda h_s)\right\|_{L_1(\mathbb{R}^d)} \leq \left\|\mathcal{F}^{-1}\chi_1\right\|_{L_1(\mathbb{R}^d)}\mathrm{e}^{1/32}\mathrm{e}^{-(1-s^2)\lambda^2/32},$$

which ends the proof. □

In view of Lemma 4.3, we obtain the dissipation estimate (DISS') as follows. Note that for $t \geq 0$ we have $\mathrm{e}^{2t} - 1 \geq 2t$. Let $t > 0$ and $\lambda > 0$, and set $s := \mathrm{e}^{-t} \in (0, 1)$. Then, for $f \in C_b(\mathbb{R}^d)$, Young's inequality and Lemma 4.3 yield

$$\begin{aligned}\|(I-P_\lambda)S_t f\|_{C_b(\mathbb{R}^d)} &= \left\|\big((I-P_\lambda)S_{\ln\frac{1}{s}}f\big)(\tfrac{1}{s}\,\cdot)\right\|_{C_b(\mathbb{R}^d)}\\ &\leq \left\|\mathcal{F}^{-1}\big((1-\chi_{\lambda/s})h_s\big)\right\|_{L_1(\mathbb{R}^d)}\|f\|_{C_b(\mathbb{R}^d)}\\ &\leq d_2\mathrm{e}^{-d_3\lambda^2 s^{-2}(1-s^2)}\|f\|_{C_b(\mathbb{R}^d)} = d_2\mathrm{e}^{-d_3\lambda^2(\mathrm{e}^{2t}-1)}\|f\|_{C_b(\mathbb{R}^d)}\\ &\leq d_2\mathrm{e}^{-2d_3\lambda^2 t}\|f\|_{C_b(\mathbb{R}^d)}. \qquad \text{(DISS(OU))}\end{aligned}$$

Thus, if $\Omega \subseteq \mathbb{R}^d$ is thick and $C\colon C_b(\mathbb{R}^d) \to C_b(\Omega)$ is the restriction map as in Subsect. 4.2, then (LS) and (DISS(OU)) provide the estimates (UP') and (DISS') and so Theorem 3.4 yields a final state observability estimate for the Ornstein–Uhlenbeck semigroup on $C_b(\mathbb{R}^d)$.

5 Cost-uniform approximate null-controllability and duality

In this section we want to show that cost-uniform approximate null-controllability is equivalent to final state observability of the dual system, which is known in the setting of norm-strongly continuous semigroups; see [4, 6, 25, 29]. In the bi-continuous setting this needs a bit of preparation so that we can formulate the corresponding definitions. Since we work in Hausdorff locally convex spaces, the choice of the "correct" integral may be delicate. We therefore provide the duality for continuous control functions, and thus relate it to a final state observability estimate of the dual system w.r.t. a space of vector measures.

Definition 5.1 Let $(X, \|\cdot\|_X, \tau_X)$ be a Saks space and $T > 0$. We set

$$C_{\tau,b}([0,T];X) := \{f \in C([0,T];(X,\tau_X)) \mid \|f\|_\infty := \sup_{t\in[0,T]} \|f(t)\|_X < \infty\}$$

where $C([0,T];(X,\tau_X))$ is the space of continuous functions from $[0,T]$ to (X,τ_X).

Remark 5.2 Let $(X, \|\cdot\|_X, \tau_X)$ be a Saks space and $T > 0$.

(a) Since the mixed topology γ_X coincides with τ_X on $\|\cdot\|_X$-bounded sets by [28, Lemma 2.2.1], and a subset of X is $\|\cdot\|_X$-bounded if and only if it is γ_X-bounded by [28, Corollary 2.4.1] we have

$$C_{\tau,b}([0,T];X) = C_b([0,T];(X,\gamma_X)) = C([0,T];(X,\gamma_X)).$$

We define two topologies on this space. First, the one given by the norm

$$\|f\|_\infty := \sup_{t\in[0,T]} \|f(t)\|_X \qquad (f \in C_{\tau,b}([0,T];X)).$$

Second, the Hausdorff locally convex topology γ_∞ induced by the directed system of seminorms given by

$$p_{\gamma_\infty}(f) := \sup_{t\in[0,T]} p_{\gamma_X}(f(t)) \qquad (f \in C_{\tau,b}([0,T];X))$$

for $p_{\gamma_X} \in \mathcal{P}_{\gamma_X}$ where $\mathcal{P}_{\gamma_X}$ is a directed system of seminorms that induces the mixed topology γ_X. Clearly, γ_∞ is coarser than the $\|\cdot\|_\infty$-topology. Further, $(C_{\tau,b}([0,T];X), \|\cdot\|_\infty)$ is a Banach space.

(b) We note that a subset $B \subseteq C_{\tau,b}([0,T];X)$ is $\|\cdot\|_\infty$-bounded if and only if it is γ_∞-bounded since a subset of X is $\|\cdot\|_X$-bounded if and only if it is γ_X-bounded by [28, 2.4.1 Corollary]. So $((C_{\tau,b}([0,T];X),\gamma_\infty)', \tau_b)$ is a topological subspace of $C_{\tau,b}([0,T];X)' = ((C_{\tau,b}([0,T];X), \|\cdot\|_\infty)', \|\cdot\|_{C_{\tau,b}([0,T];X)'})$ where τ_b denotes the topology of uniform convergence on γ_∞-bounded sets. In the following we use the notation $\|y'\|_{(C_{\tau,b}([0,T];X),\gamma_\infty)'} := \|y'\|_{C_{\tau,b}([0,T];X)'}$ for all $y' \in (C_{\tau,b}([0,T];X),\gamma_\infty)'$.

Proposition 5.3 *Let $(X, \|\cdot\|_X, \tau_X)$ be a sequentially complete Saks space, $(S_t)_{t\geq 0}$ a locally τ_X-bi-continuous semigroup on X and $T > 0$. Let $v \in C_{\tau,b}([0, T]; X)$ and set*

$$f: [0, T] \to X,\ f(t) := S_{T-t}v(t).$$

Then $f \in C_{\tau,b}([0, T]; X)$.

Proof We denote by $\mathcal{P}_{\tau_X}$ a system of directed seminorms that generates the topology τ_X on X. Let $(t_n)_{n\in\mathbb{N}}$ be a sequence in $[0, T]$ that converges to $t \in [0, T]$ and set $x_n := v(t_n) - v(t)$ for $n \in \mathbb{N}$. The sequence $(x_n)_{n\in\mathbb{N}}$ is $\|\cdot\|_X$-bounded and $\tau_X\text{-}\lim_{n\to\infty} x_n = 0$ due to our assumptions on v. We have for $p \in \mathcal{P}_{\tau_X}$ that

$$\begin{aligned} p(f(t_n) - f(t)) &= p(S_{T-t_n}v(t_n) - S_{T-t}v(t)) \\ &\leq p(S_{T-t_n}v(t_n) - S_{T-t_n}v(t)) + p(S_{T-t_n}v(t) - S_{T-t}v(t)) \\ &\leq p(S_{T-t_n}x_n) + p((S_{T-t_n} - S_{T-t})v(t)) \\ &\leq \sup_{s\in[0,T]} p(S_s x_n) + p((S_{T-t_n} - S_{T-t})v(t)). \end{aligned}$$

Combining our estimate above with the local bi-equicontinuity and τ_X-strong continuity of the semigroup, we deduce that $(f(t_n))_{n\in\mathbb{N}}$ converges to $f(t)$ in (X, τ_X). Hence, $f \in C([0, T]; (X, \tau_X))$. Furthermore, as the semigroup is exponentially bounded, there are $M \geq 1$ and $\omega \in \mathbb{R}$ such that for all $t \in [0, T]$

$$\begin{aligned} \|f(t)\|_X = \|S_{T-t}v(t)\|_X \leq \|S_{T-t}\|_{\mathcal{L}(X)} \|v(t)\|_X \leq Me^{\omega(T-t)} \|v(t)\|_X \\ \leq Me^{|\omega|T} \|v(t)\|_X, \end{aligned}$$

which yields that f is $\|\cdot\|_X$-bounded on $[0, T]$ because $v([0, T])$ is $\|\cdot\|_X$-bounded. □

Proposition 5.4 *Let $(X, \|\cdot\|_X, \tau_X)$ be a sequentially complete Saks space, $(S_t)_{t\geq 0}$ a locally τ_X-bi-continuous semigroup on X, $(U, \|\cdot\|_U, \tau_U)$ a Saks space, and $B \in \mathcal{L}(U; X)$ such that B is also sequentially τ_U-τ_X-continuous on $\|\cdot\|_U$-bounded sets. Let $T > 0$, $u \in C_{\tau,b}([0, T]; U)$ and set $f: [0, T] \to X$, $f(t) := S_{T-t}Bu(t)$. Then f is τ_X-Pettis integrable and γ_X-Pettis integrable and both integrals coincide.*

Proof The statement follows from [14, 2.5 Proposition (a)] and Proposition 5.3 because the map $v: t \mapsto Bu(t)$ belongs to $C_{\tau,b}([0, T]; X)$. □

Now, we have everything at hand to formulate the definition of cost-uniform approximate null-controllability in the bi-continuous setting. Let $(X, \|\cdot\|_X, \tau_X)$ be a sequentially complete Saks space, $(U, \|\cdot\|_U, \tau_U)$ a Saks space, $(S_t)_{t\geq 0}$ a locally τ_X-bi-continuous semigroup on X with generator $(-A, D(A))$, and $B \in \mathcal{L}(U; X)$ such that B is also sequentially τ_U-τ_X-continuous on $\|\cdot\|_U$-bounded sets, and $T > 0$. We consider the linear control system

$$\begin{aligned} \dot{x}(t) &= -Ax(t) + Bu(t) \quad (t > 0), \\ x(0) &= x_0 \in X, \end{aligned} \tag{ConSys}$$

where $u \in C_{\tau,b}([0,T];U)$. The function x is called *state function* and u is called *control function*. The unique mild solution of (ConSys) is given by Duhamel's formula

$$x(t) = S_t x_0 + \int_0^t S_{t-r} B u(r) \mathrm{d}r \qquad (t \in [0,T])$$

due to [17, Proposition 11 (a)] and Proposition 5.4. Let $\mathcal{P}_{\tau_X}$ be a directed system of seminorms that induces the topology τ_X.

Definition 5.5 We say that (ConSys) is *cost-uniform approximately τ_X-null-controllable in time T via $C_{\tau,b}([0,T];U)$* if there exists $C \geq 0$ such that for all $x_0 \in X$, $\varepsilon > 0$ and $p_{\tau_X} \in \mathcal{P}_{\tau_X}$ there exists $u \in C_{\tau,b}([0,T];U)$ with $\|u\|_\infty \leq C\|x_0\|_X$ such that $p_{\tau_X}(x(T)) \leq \varepsilon$.

We note that this definition of cost-uniform approximate τ_X-null-controllability does not depend on the choice of $\mathcal{P}_{\tau_X}$.

Remark 5.6 We can analogously define the notion of cost-uniform approximate γ_X-null-controllability in time T via $C_{\tau,b}([0,T];U)$ by using $p_{\gamma_X} \in \mathcal{P}_{\gamma_X}$ instead of $p_{\tau_X} \in \mathcal{P}_{\tau_X}$. In view of Proposition 5.13 and Remark 5.14 these two notions are equivalent.

Next, we prepare the definition of final state observability of the dual system where we need to clarify which kind of duality we have to use.

Definition 5.7 Let $(X, \|\cdot\|_X, \tau_X)$ be a Saks space and $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ the scalar field of X.

(a) We call $(X, \|\cdot\|_X, \tau_X)$ *C-sequential* if (X, γ_X) is C-sequential, i.e. every convex sequentially open subset of (X, γ_X) is already open (see [24, p. 273]).
(b) We call $(X, \|\cdot\|_X, \tau_X)$ a *Mazur space* if (X, γ_X) is a Mazur space, i.e.

$$X'_\gamma := (X, \gamma_X)' = \{x' \colon X \to \mathbb{K} \mid x' \text{ linear and } \gamma_X\text{-sequentially continuous}\}$$

(see [27, p. 40]).

Examples of C-sequential Saks spaces can be found in [13, Example 2.4, Remarks 3.19, 3.20, Corollary 3.23].

Remark 5.8 Let $(X, \|\cdot\|_X, \tau_X)$ be a Saks space.

(a) If $(X, \|\cdot\|_X, \tau_X)$ is C-sequential, then it is a Mazur space by [27, Theorem 7.4] (cf. [14, 3.6 Proposition (b)]).
(b) The space

$$X^\circ := \{x' \in X' \mid x' \ \tau_X\text{-sequentially continuous on } \|\cdot\|_X\text{-bounded sets}\}$$

is a closed linear subspace of the norm dual X' and hence a Banach space with norm given by $\|x^\circ\|_{X^\circ} := \|x^\circ\|_{X'}$ for $x^\circ \in X^\circ$ due to [8, Proposition 2.1] (note that the proof of [8, Proposition 2.1] does not use [8, Hypothesis A (ii)] which is the sequential completeness of $(X, \|\cdot\|_X, \tau_X)$). We have $X^\circ = X'_\gamma$ if and only if (X, γ_X) is a Mazur space by [14, 3.5 Remark].

Let $(X, \|\cdot\|_X)$ and $(U, \|\cdot\|_U)$ be Banach spaces. We recall that the dual operator B' of an element $B \in \mathcal{L}(U; X)$ is defined by $\langle B'x', u\rangle := \langle x', Bu\rangle$ for $x' \in X'$ and $u \in U$.

Proposition 5.9 *Let $(X, \|\cdot\|_X, \tau_X)$ be a sequentially complete C-sequential Saks space and $(S_t)_{t\geq 0}$ a locally τ_X-bi-continuous semigroup on X. Then the operators given by $S_t^\circ x^\circ := S_t' x^\circ$ for $t \geq 0$ and $x^\circ \in X^\circ$ belong to $\mathcal{L}(X^\circ)$ and form a $\tau_c(X^\circ, (X, \|\cdot\|_X))$-strongly continuous, exponentially bounded semigroup on X° where $\tau_c(X^\circ, (X, \|\cdot\|_X))$ denotes the topology of uniform convergence on $\|\cdot\|_X$-compact sets.*

Proof Since (X, γ_X) is C-sequential, in particular Mazur by Remark 5.8 (a), $(S_t)_{t\geq 0}$ is quasi-γ_X-equicontinuous and $X^\circ = X'_\gamma$ by [13, Theorem 3.17 (a)] and Remark 5.8 (b). In particular, S_t° is the γ_X-dual map of S_t for all $t \geq 0$. Furthermore, $(S_t^\circ)_{t\geq 0}$ is exponentially bounded (w.r.t. $\|\cdot\|_{\mathcal{L}(X^\circ)}$) because $(S_t)_{t\geq 0}$ is exponentially bounded. It follows that $S_t^\circ \in \mathcal{L}(X^\circ)$ for all $t \geq 0$ and $(S_t^\circ)_{t\geq 0}$ is a $\sigma(X^\circ, X)$-strongly continuous semigroup. As $\sigma(X^\circ, X)$ and the mixed topology $\gamma^\circ := \gamma(\|\cdot\|_{X^\circ}, \sigma(X^\circ, X))$ coincide on $\|\cdot\|_{X^\circ}$-bounded sets by Definition 2.1 (b), $(S_t^\circ)_{t\geq 0}$ is also γ°-strongly continuous. Due to [13, Proposition 3.22 (a)] we have $\gamma^\circ = \tau_c(X^\circ, (X, \|\cdot\|_X))$. □

The semigroup $(S_t^\circ)_{t\geq 0}$ in the setting of Proposition 5.9 resembles a bi-continuous semigroup. For instance, we note that the generator $(-A^\circ, D(A^\circ))$ of $(S_t^\circ)_{t\geq 0}$ from Proposition 5.9 is given by

$$\begin{aligned} D(A^\circ) &= \left\{x^\circ \in X^\circ \mid \tau_c(X^\circ, (X, \|\cdot\|_X))\text{-}\lim_{t\to 0+} \frac{S_t^\circ x^\circ - x^\circ}{t} \text{ exists in } X^\circ\right\}, \\ -A^\circ x &= \tau_c(X^\circ, (X, \|\cdot\|_X))\text{-}\lim_{t\to 0+} \frac{S_t^\circ x^\circ - x^\circ}{t} \qquad (x^\circ \in D(A^\circ)) \end{aligned}$$

and fulfils

$$\begin{aligned} D(A^\circ) = \Big\{x^\circ \in X^\circ \mid \sigma(X^\circ, X)\text{-}\lim_{t\to 0+} \frac{S_t^\circ x^\circ - x^\circ}{t} \text{ ex. in } X^\circ, \\ \sup_{t\in(0,1]} \frac{\left\|S_t^\circ x^\circ - x^\circ\right\|_{X^\circ}}{t} < \infty\Big\}, \\ -A^\circ x = \sigma(X^\circ, X)\text{-}\lim_{t\to 0+} \frac{S_t^\circ x^\circ - x^\circ}{t} \qquad (x^\circ \in D(A^\circ)) \end{aligned}$$

by [5, I.1.10 Proposition] for the mixed topology $\gamma^\circ = \tau_c(X^\circ, (X, \|\cdot\|_X))$ and the exponential boundedness of $(S_t^\circ)_{t\geq 0}$ (cf. [14, p. 6] in the bi-continuous setting). What is missing for bi-continuity are sequential completeness of the corresponding Saks space $(X^\circ, \|\cdot\|_{X^\circ}, \sigma(X^\circ, X))$ and (local) bi-equicontinuity.

Remark 5.10 Let $(X, \|\cdot\|_X, \tau_X)$ be a sequentially complete Saks space and $(S_t)_{t\geq 0}$ a τ_X-bi-continuous semigroup on X with generator $(-A, D(A))$. If

(i) $X^\circ \cap \{x' \in X' \mid \|x'\|_{X'} \leq 1\}$ is sequentially complete w.r.t. $\sigma(X^\circ, X)$,

(ii) every $\|\cdot\|_{X'}$-bounded $\sigma(X^\circ, X)$-null sequence in X° is τ_X-equicontinuous on $\|\cdot\|_X$-bounded sets,

(see [8, Hypothesis B and C]), then $(X^\circ, \|\cdot\|_{X^\circ}, \sigma(X^\circ, X))$ is a sequentially complete Saks space and $(S_t^\circ)_{t\geq 0}$ is a locally $\sigma(X^\circ, X)$-bi-continuous semigroup on X° by [8, Proposition 2.4] with generator $(-A^\circ, D(A^\circ))$ fulfilling

$$\begin{aligned} &D(A^\circ) = \{x^\circ \in X^\circ \mid \exists y^\circ \in X^\circ \, \forall x \in D(A) : \langle -Ax, x^\circ\rangle = \langle x, y^\circ\rangle\}, \\ &-A^\circ x^\circ = y^\circ \quad (x^\circ \in D(A^\circ)), \end{aligned}$$

by [3, Lemma 1].

We refer to [14, 3.9 Example] for examples of sequentially complete Saks spaces satisfying (i) and (ii) of Remark 5.10.

Proposition 5.11 *Let $(X, \|\cdot\|_X, \tau_X)$ and $(U, \|\cdot\|_U, \tau_U)$ be Saks spaces, and $B \in \mathcal{L}(U; X)$ such that B is also sequentially τ_U-τ_X-continuous on $\|\cdot\|_U$-bounded sets. Then $B^\circ := B'|_{X^\circ} \in \mathcal{L}(X^\circ; U^\circ)$ and is also $\sigma(X^\circ, X)$-$\sigma(U^\circ, U)$-continuous.*

Proof Let $x^\circ \in X^\circ$, $u \in U$ and $(u_n)_{n\in\mathbb{N}}$ a $\|\cdot\|_U$-bounded sequence in U that τ_U-converges to u. Since $B \in \mathcal{L}(U; X)$ and B is also sequentially τ_U-τ_X-continuous on $\|\cdot\|_U$-bounded sets, we have that $(Bu_n)_{n\in\mathbb{N}}$ is $\|\cdot\|_X$-bounded and τ_X-convergent to Bu. This implies

$$\langle B^\circ x^\circ, u\rangle = \langle x^\circ, Bu\rangle = \lim_{n\to\infty} \langle x^\circ, Bu_n\rangle,$$

yielding $B^\circ x^\circ \in U^\circ$ and the $\sigma(X^\circ, X)$-$\sigma(U^\circ, U)$-continuity of B°. Furthermore, we note that

$$\begin{aligned} \left\|B^\circ x^\circ\right\|_{U^\circ} &= \left\|B^\circ x^\circ\right\|_{U'} = \sup_{\|u\|_U\leq 1} |\langle B^\circ x^\circ, u\rangle| \\ &= \sup_{\|u\|_U\leq 1} |\langle x^\circ, Bu\rangle| \leq \sup_{\|u\|_U\leq 1} \|Bu\|_X \left\|x^\circ\right\|_{X^\circ} \\ &= \|B\|_{\mathcal{L}(U;X)} \left\|x^\circ\right\|_{X^\circ} \end{aligned}$$

and thus $B^\circ \in \mathcal{L}(X^\circ; U^\circ)$. □

Remark 5.12 Let $(X, \|\cdot\|_X, \tau_X)$ and $(U, \|\cdot\|_U, \tau_U)$ be Saks spaces, $B \in \mathcal{L}(U; X)$. Consider the following assertions:

(a) B is τ_U-τ_X-continuous on $\|\cdot\|_U$-bounded sets.
(b) B is sequentially τ_U-τ_X-continuous on $\|\cdot\|_U$-bounded sets.

Then (a)⇒(b) holds. Moreover, if $(U, \|\cdot\|_U, \tau_U)$ is C-sequential, then (b)⇒(a) holds.

Proof The implication (a)⇒(b) is obviously true. Let assertion (b) hold. It follows from [28, Theorem 2.3.1] that B is sequentially γ_U-τ_X-continuous. Hence, (a) holds by [27, Theorem 7.4] if (U, γ_U) is C-sequential. □

Proposition 5.13 *Let* $(V, \|\cdot\|_V)$, $(W, \|\cdot\|_W)$ *be Banach spaces,* $(Z, \|\cdot\|_Z, \tau_Z)$ *a Saks space,* $\mathcal{P}_{\gamma_Z}$ *a directed system of seminorms that induces the topology* γ_Z, $F \in \mathcal{L}(V; Z)$ *and* $G \in \mathcal{L}(W; Z)$. *Then the following assertions are equivalent:*

(a) There exists $c_1 \geq 0$ *such that* $\|F'z'\|_{V'} \leq c_1 \|G'z'\|_{W'}$ *for all* $z' \in Z'_\gamma$.
(b) There exists $c_2 \geq 0$ *such that*

$$\{Fv \mid v \in V,\ \|v\|_V \leq 1\} \subseteq \overline{\{Gw \mid w \in W,\ \|w\|_W \leq c_2\}}^{\gamma_Z}.$$

(c) There exists $c_3 \geq 0$ *such that for all* $v \in V$, $\varepsilon > 0$ *and* $p_{\gamma_Z} \in \mathcal{P}_{\gamma_Z}$ *there is* $w \in W$ *with* $\|w\|_W \leq c_3 \|v\|_V$ *such that* $p_{\gamma_Z}(Fv + Gw) \leq \varepsilon$.

Moreover, we can choose $c_1 = c_2 = c_3$.

Proof First, we note that if M and N are convex sets in Z, then we have $N \subseteq \overline{M}^{\gamma_Z}$ if and only if

$$\sup_{z \in M} \mathrm{Re}\langle z', z\rangle \leq \sup_{z \in N} \mathrm{Re}\langle z', z\rangle \tag{3}$$

for all $z' \in Z'_\gamma$ by [4, p. 220]. Second, let $z' \in Z'_\gamma$. For every $z \in Z$ there is $\lambda_z \in \mathbb{C}$ with $|\lambda_z| \leq 1$ such that $|\langle z', z\rangle| = \mathrm{Re}\langle z', \lambda_z z\rangle$. Thus, if M and N are additionally circled sets, then (3) is equivalent to

$$\sup_{z \in M} |\langle z', z\rangle| \leq \sup_{z \in N} |\langle z', z\rangle|$$

for all $z' \in Z'_\gamma$. Hence, setting $N := \{Fv \mid v \in V,\ \|v\|_V \leq 1\}$, $M := \{Gw \mid w \in W,\ \|w\|_W \leq c_2\}$ and observing that $N, M \subseteq Z$ are convex, circled sets, we obtain the equivalence of (a) and (b), and that we can choose $c_1 = c_2$.

Let us turn to the equivalence of (b) and (c). For $\varepsilon > 0$ and $p_{\gamma_Z} \in \mathcal{P}_{\gamma_Z}$ we set $U_{\varepsilon, p_{\gamma_Z}} := \{z \in Z \mid p_{\gamma_Z}(z) \leq \varepsilon\}$. For any $M \subseteq Z$ we have

$$\overline{M}^{\gamma_Z} = \bigcap_{\varepsilon > 0,\ p_{\gamma_Z} \in \mathcal{P}_{\gamma_Z}} M + U_{\varepsilon, p_{\gamma_Z}} \tag{4}$$

(see e.g. [11, 2.1.4 Proposition]). Let assertion (b) hold, $v \in V$ with $v \neq 0$, $\varepsilon > 0$ and $p_{\gamma_Z} \in \mathcal{P}_{\gamma_Z}$. Then there are $\widetilde{w} \in W$ with $\|\widetilde{w}\|_W \leq c_2$ and $z \in Z$ with $p_{\gamma_Z}(z) \leq \frac{\varepsilon}{\|v\|_V}$ such that $F\left(-\frac{v}{\|v\|_V}\right) = G\widetilde{w} + z$ by (b) and (4). From writing

$$-Fv = \|v\|_V\, F\left(\frac{-v}{\|v\|_V}\right) = \|v\|_V\, (G\widetilde{w} + z) = G(\|v\|_V\, \widetilde{w}) + \|v\|_V\, z,$$

setting $w := \|v\|_V\, \widetilde{w}$, and using $\|w\|_W = \|v\|_V\, \|\widetilde{w}\|_W \leq c_2 \|v\|_V$ and

$$p_{\gamma_Z}(Fv + Gw) = p_{\gamma_Z}(-\|v\|_V\, z) = \|v\|_V\, p_{\gamma_Z}(z) \leq \|v\|_V \frac{\varepsilon}{\|v\|_V} = \varepsilon,$$

we conclude that (c) holds (the case $v = 0$ is obvious).

Now, let assertion (c) hold. Let $v \in V$ with $\|v\|_V \leq 1$, $\varepsilon > 0$ and $p_{\gamma_Z} \in \mathcal{P}_{\gamma_Z}$. Then there is $\widetilde{w} \in W$ with $\|\widetilde{w}\|_W \leq c_3$ such that $p_{\gamma_Z}(Fv + G\widetilde{w}) \leq \varepsilon$. Setting $w := -\widetilde{w}$, using $\|w\| = \|\widetilde{w}\|_W \leq c_3$ and $Fv = Gw + Fv + G\widetilde{w}$, we see that (b) holds due to (4). The proof of the equivalence of (b) and (c) also shows that we can choose $c_2 = c_3$. □

The proof of the equivalence of (a) and (b) is just an adaptation of the proof of [4, Theorem 2.2, (iii)⇔(iv)].

Remark 5.14 Let $(Z, \|\cdot\|_Z, \tau_Z)$ be a Saks space. Since the mixed topology γ_Z coincides with τ_Z on $\|\cdot\|_Z$-bounded sets, we may equivalently replace the γ_Z-closure by the τ_Z-closure in Proposition 5.13 (b) and thus $\mathcal{P}_{\gamma_Z}$ in (c) by a directed system of seminorms $\mathcal{P}_{\tau_Z}$ that induces the topology τ_Z, too.

Proposition 5.15 *Let $(X, \|\cdot\|_X, \tau_X)$ be a sequentially complete C-sequential Saks space, $(U, \|\cdot\|_U, \tau_U)$ a Saks space, $(S_t)_{t\geq 0}$ a locally τ_X-bi-continuous semigroup on X and $B \in \mathcal{L}(U; X)$ such that B is also τ_U-τ_X-continuous on $\|\cdot\|_U$-bounded sets.*

(a) $(S_t B)_{t\geq 0}$ is quasi-γ_U-γ_X-equicontinuous.
(b) Let $T > 0$ and $\mathcal{B}^T : C_{\tau,b}([0, T]; U) \to X$ be given by

$$\mathcal{B}^T u := \int_0^T S_{T-t} Bu(t)\mathrm{d}t.$$

Then $\mathcal{B}^T \in \mathcal{L}(C_{\tau,b}([0, T]; U); X) = \mathcal{L}((C_{\tau,b}([0, T]; U), \|\cdot\|_\infty); (X, \|\cdot\|_X))$ and $\mathcal{B}^T$ is also γ_∞–γ_X-continuous.
(c) Let $\mathcal{B}^{T\circ} x^\circ := \mathcal{B}^{T\prime} x^\circ$ for $x^\circ \in X^\circ$. Then $\mathcal{B}^{T\circ} x^\circ \in (C_{\tau,b}([0, T]; U), \gamma_\infty)'$ for all $x^\circ \in X^\circ$.

Proof (a) Let $M \subseteq U$ be a $\|\cdot\|_U$-bounded set. Then the restriction $B|_M : M \to X$ of B to M is $\tau_U|_M$-τ_X-continuous. Since $B \in \mathcal{L}(U; X)$, the set $B(M)$ is $\|\cdot\|_X$-bounded. As the mixed topology γ_X coincides with τ_X on $\|\cdot\|_X$-bounded sets by Definition 2.1 (b), it follows that $B|_M$ is $\tau_U|_M$-γ_X-continuous, yielding that B is γ_U-γ_X-continuous by [5, I.1.7 Corollary]. Due to [13, Theorem 3.17 (a)] $(S_t B)_{t\geq 0}$ is quasi-γ_U-γ_X-equicontinuous, proving part (a).

(b) Let $\mathcal{P}_{\gamma_X}$ and $\mathcal{P}_{\gamma_U}$ be directed systems of seminorms that induce the mixed topologies γ_X and γ_U, respectively. For $p_{\gamma_X} \in \mathcal{P}_{\gamma_X}$ we set $V_{p_{\gamma_X}} := \{x \in X \mid p_{\gamma_X}(x) < 1\}$ and denote its polar set by $V^\circ_{p_{\gamma_X}} := \{x' \in X'_\gamma \mid \forall\ x \in V_{p_{\gamma_X}} :\ |x'(x)| \leq 1\}$. It follows from part (a) that there are $C \geq 0$ and $p_{\gamma_U} \in \mathcal{P}_{\gamma_U}$ such that for all $u \in C_{\tau,b}([0, T]; U)$ we have

$$\begin{aligned} p_{\gamma_X}(\mathcal{B}^T u) &= \sup_{x' \in V^\circ_{p_{\gamma_X}}} \left|\langle x', \int_0^T S_{T-t} Bu(t)\mathrm{d}t\rangle\right| \leq \sup_{x' \in V^\circ_{p_{\gamma_X}}} \int_0^T |\langle x', S_{T-t} Bu(t)\rangle|\mathrm{d}t \\ &\leq T \sup_{x' \in V^\circ_{p_{\gamma_X}}} \sup_{t\in[0,T]} |\langle x', S_{T-t} Bu(t)\rangle| = T \sup_{t\in[0,T]} p_{\gamma_X}(S_{T-t} Bu(t)) \\ &\underset{(a)}{\leq} CT \sup_{t\in[0,T]} p_{\gamma_U}(u(t)) \end{aligned} \tag{5}$$

where we used [20, Proposition 22.14] in the first and second to last equation to get from p_{γ_X} to $\sup_{x' \in V^\circ_{p_{\gamma_X}}}$ and back. Thus, $\mathcal{B}^T$ is γ_∞-γ_X-continuous. Furthermore, since X'_γ is norming for X by [15, Lemma 5.5 (a)], we may choose $\mathcal{P}_{\gamma_X}$ such that $\|x\|_X = \sup_{p_{\gamma_X} \in \mathcal{P}_{\gamma_X}} p_{\gamma_X}(x)$ for all $x \in X$ by [13, Remark 2.2 (c)]. Due to our previous estimates and the exponential boundedness of $(S_t)_{t \geq 0}$ we obtain

$$\begin{aligned} \left\| \mathcal{B}^T u \right\|_X &= \sup_{p_{\gamma_X} \in \mathcal{P}_{\gamma_X}} p_{\gamma_X}(\mathcal{B}^T u) \underset{(5)}{\leq} T \sup_{p_{\gamma_X} \in \mathcal{P}_{\gamma_X}} \sup_{t \in [0,T]} p_{\gamma_X}(S_{T-t} Bu(t)) \\ &= T \sup_{t \in [0,T]} \|S_{T-t} Bu(t)\|_X \leq T \sup_{t \in [0,T]} \|S_{T-t}\|_{\mathcal{L}(X)} \|Bu(t)\|_X \\ &\leq T M e^{|\omega| T} \|B\|_{\mathcal{L}(U;X)} \sup_{t \in [0,T]} \|u(t)\|_U , \end{aligned}$$

yielding $\mathcal{B}^T \in \mathcal{L}(C_{\tau,b}([0,T];U);X)$.

(c) It follows from $X^\circ = X'_\gamma$ by Remark 5.8 (a) and part (b) that $\mathcal{B}^{T\circ}$ is the dual map of the γ_∞-γ_X-continuous map $\mathcal{B}^T$ and hence $\mathcal{B}^{T\circ} x^\circ \in (C_{\tau,b}([0,T];U), \gamma_\infty)'$ for all $x^\circ \in X^\circ$. □

Now, we are ready to write down the dual system of (ConSys) and to phrase the kind of final state observability of this dual system we are seeking for. Let $(X, \|\cdot\|_X, \tau_X)$ be a sequentially complete C-sequential Saks space, $(U, \|\cdot\|_U, \tau_U)$ a Saks space, $(S_t)_{t \geq 0}$ a locally τ_X-bi-continuous semigroup on X with generator $(-A, D(A))$, and $B \in \mathcal{L}(U;X)$ such that B is τ_U-τ_X-continuous on $\|\cdot\|_U$-bounded sets, and $T > 0$. Using Proposition 5.9 and Proposition 5.11, the dual system of (ConSys) is given by

$$\begin{aligned} \dot{x}(t) &= -A^\circ x(t) \quad (t > 0), \\ y(t) &= B^\circ x(t) \quad (t \geq 0), \qquad\qquad \text{(ObsSys)} \\ x(0) &= x_0 \in X^\circ. \end{aligned}$$

Definition 5.16 We say that (ObsSys) satisfies a *final state observability estimate in* $(C_{\tau,b}([0,T];U), \gamma_\infty)'$ if there exists $C_{\mathrm{obs}} \geq 0$ such that

$$\left\| S_T^\circ x^\circ \right\|_{X^\circ} \leq C_{\mathrm{obs}} \left\| \mathcal{B}^{T\circ} x^\circ \right\|_{(C_{\tau,b}([0,T];U),\gamma_\infty)'}$$

for all $x^\circ \in X^\circ$.

We spend the remaining part of this section with proving that cost-uniform approximate τ_X-null-controllability in time T via $C_{\tau,b}([0,T];U)$ of (ConSys) is equivalent to a final state observability estimate of (ObsSys) in $(C_{\tau,b}([0,T];U), \gamma_\infty)'$, and that the latter space is actually a certain space of vector measures.

Let Ω be a Hausdorff locally compact space, (U, ϑ_U) a Hausdorff locally convex space and $\mathcal{P}_{\vartheta_U}$ a directed system of seminorms that induces ϑ_U. We denote by $\mathscr{B}(\Omega)$ the Borel σ-algebra on Ω, by $M(\Omega)$ the space of all bounded complex (or real) Borel measures on Ω, and by $M(\Omega; (U, \vartheta_U)')$ the space of all finitely additive vector measures $\nu\colon \mathscr{B}(\Omega) \to (U, \vartheta_U)'$, i.e. $\nu(N_1 \cup N_2) = \nu(N_1) + \nu(N_2)$ for all disjoint $N_1, N_2 \in \mathscr{B}(\Omega)$, such that

(i) $\nu(\cdot)u \in M(\Omega)$ for all $u \in U$, and
(ii) there exist $p \in \mathcal{P}_{\vartheta_U}$ and $C \geq 0$ such that

$$\sup_{(\mathcal{N},\mathcal{U}_p)} \Big| \sum_{(N,u)\in(\mathcal{N},\mathcal{U}_p)} \nu(N)u \Big| \leq C$$

where the supremum is taken over all finite partitions $\mathcal{N}$ of Ω into disjoint Borel sets and all finite sets $\mathcal{U}_p$ in U such that $p(u) \leq 1$ for all $u \in \mathcal{U}_p$.

Let $(U, \|\cdot\|_U, \tau_U)$ be a Saks space and $T > 0$. By [26, Theorem 1], the compactness of $[0, T]$ and Remark 5.2 (a) the map

$$\Theta_\gamma\colon M([0,T]; U'_\gamma) \to (C_{\tau,b}([0,T]; U), \gamma_\infty)', \ \Theta_\gamma(\nu)(u) := \int_0^T u(t)\mathrm{d}\nu,$$

is a linear isomorphism. By the same theorem in combination with [26, Lemma 4] the map

$$\Theta_{\|\cdot\|}\colon M([0,T]; U') \to (C([0,T]; (U, \|\cdot\|_U)), \|\cdot\|_\infty)', \ \Theta_{\|\cdot\|}(\nu)(u) := \int_0^T u(t)\mathrm{d}\nu,$$

is a topological isomorphism w.r.t. the semivariation norm on $M([0, T]; U')$ and the dual norm on $(C([0, T]; (U, \|\cdot\|_U)), \|\cdot\|_\infty)'$ and

$$\big\|\Theta_{\|\cdot\|}(\nu)\big\|_{(C([0,T];(U,\|\cdot\|_U)),\|\cdot\|_\infty)'} = \|\nu\|_{var} \qquad (\nu \in M([0,T]; U'))$$

where the semivariation norm is given by

$$\|\nu\|_{var} := \sup_{(\mathcal{N},\mathcal{U}_{\|\cdot\|_U})} \Big| \sum_{(N,u)\in(\mathcal{N},\mathcal{U}_{\|\cdot\|_U})} \nu(N)u \Big| \qquad (\nu \in M([0,T]; U'))$$

and the supremum is taken over all $(\mathcal{N}, \mathcal{U}_{\|\cdot\|_U})$ as in (ii) above with p replaced by $\|\cdot\|_U$. We note that it follows from U'_γ being a topological subspace of U' (see [5, I.1.18 Proposition]) and γ_U being coarser than the $\|\cdot\|_U$-topology, that $M([0, T]; U'_\gamma)$ is a topological subspace of $M([0, T]; U')$ (if equipped with the relative topology).

Theorem 5.17 *Let $(X, \|\cdot\|_X, \tau_X)$ be a sequentially complete C-sequential Saks space, $(U, \|\cdot\|_U, \tau_U)$ a Saks space, $(S_t)_{t\geq 0}$ a locally τ_X-bi-continuous semigroup on X and $B \in \mathcal{L}(U; X)$ such that B is also τ_U-τ_X-continuous on $\|\cdot\|_U$-bounded sets. For $T > 0$ we have $B^\circ S^\circ_{(\cdot)} x^\circ \odot \lambda \in M([0, T]; U'_\gamma)$ and $\mathcal{B}^{T\circ} x^\circ = \Theta_\gamma(B^\circ S^\circ_{T-(\cdot)} x^\circ \odot \lambda)$ for all $x^\circ \in X^\circ$ as well as*

$$\Big\|\mathcal{B}^{T\circ} x^\circ\Big\|_{(C([0,T];(U,\|\cdot\|_U)),\|\cdot\|_\infty)'} = \Big\|B^\circ S^\circ_{(\cdot)} x^\circ \odot \lambda\Big\|_{var} \qquad (x^\circ \in X^\circ)$$

where

$$(B^\circ S^\circ_{(\cdot)} x^\circ \odot \lambda)(N)u := \int_N \langle B^\circ S^\circ_t x^\circ, u\rangle \mathrm{d}t \qquad (N \in \mathscr{B}([0,T]),\ u \in U)$$

for the Lebesgue measure λ.

Proof. First, we recall that $X^\circ = X'_\gamma$ by Remark 5.8 (a) as the C-sequential space (X, γ_X) is Mazur. Let $x^\circ \in X^\circ$. Due to Proposition 5.15 (b) and Proposition 5.9 the map $\mathcal{B}^T$ is γ_∞-γ_X-continuous and

$$\langle \mathcal{B}^{T\circ} x^\circ, u\rangle = \int_0^T \langle x^\circ, S_{T-t}Bu(t)\rangle \mathrm{d}t = \int_0^T \langle B^\circ S^\circ_{T-t} x^\circ, u(t)\rangle \mathrm{d}t$$

for all $u \in C_{\tau,b}([0,T]; U)$.

For $N \in \mathscr{B}([0,T])$ and $u \in U$ we define $(B^\circ S^\circ_{(\cdot)} x^\circ \odot \lambda)(N)u := \int_N \langle B^\circ S^\circ_t x^\circ, u\rangle \mathrm{d}t$ and show that $B^\circ S^\circ_{T-(\cdot)} x^\circ \odot \lambda \in M([0,T]; U'_\gamma)$. By the proof of Proposition 5.15 (b) there are $C \geq 0$ and $p_{\gamma_U} \in \mathcal{P}_{\gamma_U}$ such that

$$|\langle B^\circ S^\circ_{T-t} x^\circ, u\rangle| = |\langle x^\circ, S_{T-t}Bu\rangle| \leq C p_{\gamma_U}(u)$$

for all $t \in [0,T]$ and all $u \in U$. In combination with the continuity of the map $t \mapsto \langle B^\circ S^\circ_{T-t} x^\circ, u\rangle$ on $[0,T]$ by Proposition 5.9 and Proposition 5.11 this implies that $B^\circ S^\circ_{T-(\cdot)} x^\circ \odot \lambda \colon \mathscr{B}([0,T]) \to U'_\gamma$ is a well-defined finitely additive vector measure and that

$$(B^\circ S^\circ_{T-(\cdot)} x^\circ \odot \lambda)(\cdot)u = \int_{(\cdot)} \langle B^\circ S^\circ_{T-t} x^\circ, u\rangle \mathrm{d}t \qquad (u \in U)$$

belongs to $M(\Omega)$. Let $\mathcal{N}$ be a finite partition of $[0,T]$ into disjoint Borel sets and $\mathcal{U}_{p_{\gamma_U}}$ a finite subset of U such that $p_{\gamma_U}(u) \leq 1$ for all $u \in \mathcal{U}_{p_{\gamma_U}}$. Then we have

$$\begin{aligned} \Big|\sum_{(N,u)\in(\mathcal{N},\mathcal{U}_{p_{\gamma_U}})} (B^\circ S^\circ_{T-(\cdot)} x^\circ \odot \lambda)(N)u\Big| &= \Big|\sum_{(N,u)\in(\mathcal{N},\mathcal{U}_{p_{\gamma_U}})} \int_N \langle B^\circ S^\circ_{T-t} x^\circ, u\rangle \mathrm{d}t\Big| \\ &\leq \sum_{(N,u)\in(\mathcal{N},\mathcal{U}_{p_{\gamma_U}})} \lambda(N) C p_{\gamma_U}(u) \\ &\leq C \sum_{N\in\mathcal{N}} \lambda(N) = C\lambda([0,T]) = CT, \end{aligned}$$

yielding $B^\circ S^\circ_{T-(\cdot)} x^\circ \odot \lambda \in M([0,T]; U'_\gamma)$. Analogously, $B^\circ S^\circ_{(\cdot)} x^\circ \odot \lambda \in M([0,T]; U'_\gamma)$.

Finally, since

$$\begin{aligned} \langle \mathcal{B}^{T\circ} x^\circ, u\rangle &= \int_0^T \langle B^\circ S^\circ_{T-t} x^\circ, u(t)\rangle \mathrm{d}t = \int_0^T u(t)\mathrm{d}(B^\circ S^\circ_{T-(\cdot)} x^\circ \odot \lambda) \\ &= \Theta_\gamma(B^\circ S^\circ_{T-(\cdot)} x^\circ \odot \lambda)(u) \end{aligned}$$

for all $u \in C_{\tau,b}([0,T];U)$ and

$$\langle \mathcal{B}^{T\circ} x^\circ, u\rangle = \Theta_\gamma(B^\circ S^\circ_{T-(\cdot)} x^\circ \odot \lambda)(u) = \Theta_{\|\cdot\|}(B^\circ S^\circ_{T-(\cdot)} x^\circ \odot \lambda)(u)$$

for all $u \in C([0,T];(U,\|\cdot\|_U))$, it holds that

$$\begin{aligned}\left\|\mathcal{B}^{T\circ} x^\circ\right\|_{(C([0,T];(U,\|\cdot\|_U)),\|\cdot\|_\infty)'} &= \left\|\Theta_{\|\cdot\|}(B^\circ S^\circ_{T-(\cdot)} x^\circ \odot \lambda)\right\|_{(C([0,T];(U,\|\cdot\|_U)),\|\cdot\|_\infty)'} \\ &= \left\|B^\circ S^\circ_{T-(\cdot)} x^\circ \odot \lambda\right\|_{var}.\end{aligned}$$

Let $\mathcal{N}$ be a finite partition of $[0,T]$ and $\mathcal{U}_{\|\cdot\|_U}$ a finite set in U such that $\|u\|_U \le 1$ for all $u \in \mathcal{U}_{\|\cdot\|_U}$. Then $T-\mathcal{N} := \{T-N \mid N \in \mathcal{N}\}$ is also a finite partition of $[0,T]$, and

$$\begin{aligned}\sum_{(N,u)\in(\mathcal{N},\mathcal{U}_{\|\cdot\|_U})} (B^\circ S^\circ_{T-(\cdot)} x^\circ \odot \lambda)(N)u &= \sum_{(N,u)\in(\mathcal{N},\mathcal{U}_{\|\cdot\|_U})} \int_N \langle B^\circ S^\circ_{T-t} x^\circ, u\rangle \mathrm{d}t \\ &= -\sum_{(T-N,u)\in(T-\mathcal{N},\mathcal{U}_{\|\cdot\|_U})} \int_{T-N} \langle B^\circ S^\circ_t x^\circ, u\rangle \mathrm{d}t.\end{aligned}$$

Thus, we obtain

$$\left\|\mathcal{B}^{T\circ} x^\circ\right\|_{(C([0,T];(U,\|\cdot\|_U)),\|\cdot\|_\infty)'} = \left\|B^\circ S^\circ_{T-(\cdot)} x^\circ \odot \lambda\right\|_{var} = \left\|B^\circ S^\circ_{(\cdot)} x^\circ \odot \lambda\right\|_{var}. \quad \square$$

Theorem 5.18 *Let $(X, \|\cdot\|_X, \tau_X)$ be a sequentially complete C-sequential Saks space, $(U, \|\cdot\|_U, \tau_U)$ a Saks space, $(S_t)_{t\ge 0}$ a locally τ_X-bi-continuous semigroup on X and $B \in \mathcal{L}(U;X)$ such that B is also τ_U-τ_X-continuous on $\|\cdot\|_U$-bounded sets, and $T > 0$. Then the following assertions are equivalent:*

(a) The system in (ConSys) is cost-uniform approximately τ_X-null-controllable in time T via $C_{\tau,b}([0,T];U)$.
(b) The system in (ObsSys) satisfies a final state observability estimate in $(C_{\tau,b}([0,T];U),\gamma_\infty)'$.

If additionally $\tau_U = \tau_{\|\cdot\|_U}$, then each of the preceding assertions is equivalent to:

(c) There exists $C_{\mathrm{obs}} \ge 0$ such that

$$\forall\, x^\circ \in X^\circ: \quad \left\|S^\circ_T x^\circ\right\|_{X^\circ} \le C_{\mathrm{obs}} \left\|B^\circ S^\circ_{(\cdot)} x^\circ \odot \lambda\right\|_{var}.$$

Proof This statement follows from the equivalence of (a) and (c) in Proposition 5.13 with $V := Z := X$, $W := C_{\tau,b}([0,T];U)$ equipped with $\|\cdot\|_\infty$, $F := S_T$ and $G := \mathcal{B}^T$ in combination with Theorem 5.17, Remark 5.14 and $X^\circ = X'_\gamma$. $\square$

Even in the setting of Banach spaces, i.e. $\tau_X = \tau_{\|\cdot\|_X}$, $\tau_U = \tau_{\|\cdot\|_U}$, where we have $C_{\tau,b}([0,T];U) = C([0,T];(U,\|\cdot\|_U))$ and

$$(C_{\tau,b}([0,T];U),\gamma_\infty)' = (C([0,T];(U,\|\cdot\|_U)),\|\cdot\|_\infty)' = M([0,T];U')$$

as well as $X^\circ = X'$, the results of Theorem 5.18 seem to be new.

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