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A note on inhomogeneous fractional Schrödinger equations

Tarek Saanouni[1], Salah Boulaaras[2*] and Congming Peng[3]

*Correspondence:
s.boularas@qu.edu.sa
[2]Department of Mathematics, College of Sciences and Arts, Qassim University, ArRass, Saudi Arabia
Full list of author information is available at the end of the article

Abstract

We study some energy well-posedness issues of the Schrödinger equation with an inhomogeneous mixed nonlinearity and radial data

$$i\dot{u} - (-\Delta)^s u \pm |x|^\rho |u|^{p-1} u \pm |u|^{q-1} u = 0, \quad 0 < s < 1, \rho \neq 0, p, q > 1.$$

Our aim is to treat the competition between the homogeneous term $|u|^{q-1}u$ and the inhomogeneous one $|x|^\rho |u|^{p-1}u$. We simultaneously treat two different regimes, $\rho > 0$ and $\rho < -2s$. We deal with three technical challenges at the same time: the absence of a scaling invariance, the presence of the singular decaying term $|\cdot|^\rho$, and the nonlocality of the fractional differential operator $(-\Delta)^s$. We give some sufficient conditions on the datum and the parameters N, s, ρ, p, q to have the global versus nonglobal existence of energy solutions. We use the associated ground states and some sharp Gagliardo–Nirenberg inequalities. Moreover, we investigate the L^2 concentration of the mass-critical blowing-up solutions. Finally, in the attractive regime, we prove the scattering of energy global solutions. Since there is a loss of regularity in Strichartz estimates for the fractional Schrödinger problem with nonradial data, in this work, we assume that $u_{|t=0}$ is spherically symmetric. The blowup results use ideas of the pioneering work by Boulenger el al. (J. Funct. Anal. 271:2569–2603, 2016).

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Keywords: Inhomogeneous mixed Schrödinger equation; Global/non-global existence; Partial differential equations

1 Introduction

We consider the fractional nonlinear Schrödinger (FNLS) equation

$$\begin{cases} i\dot{u} - (-\Delta)^s u + \lambda_1 |x|^\rho |u|^{p-1} u + \lambda_2 |u|^{q-1} u = 0, \\ u_{|t=0} = u_0. \end{cases} \tag{1.1}$$

Here and hereafter, $N \geq 2$, $\lambda_i = \pm 1$, $\rho \neq 0$, $p, q > 1$, and $u := u(t,x) : \mathbb{R} \times \mathbb{R}^N \to \mathbb{C}$. The fractional Laplacian operator is defined via the Fourier transform as follows:

$$\mathcal{F}\big[(-\Delta)^s \cdot\big] := |\cdot|^{2s} \mathcal{F}\cdot, \quad s \in (0,1).$$

Introduced by Laskin [18, 19], the fractional Schrödinger problem used the theory of functional measures caused by the Levy stochastic process with expansion of the Feynman path integral from the Brownian-like to the Levy-like quantum mechanical paths. The inhomogeneous nonlinear Schrödinger equation ($s = 1 = 1 - \lambda_2$) models the beam propagation [1, 12, 20, 23] in nonlinear optics and plasma physics.

The well-posedness issues of some particular cases of the above problem were investigated by many authors. Indeed, the fractional nonlinear Schrödinger equation [2, 3, 13, 14, 27, 32] corresponds to $\lambda_1 = 0$ in (1.1). If $\lambda_2 = 0$, then problem (1.1) fits the nonlinear Schrodinger equation, called NLS for short [26, 28]. Eventually, the Schrödinger equation with mixed power nonlinearity [8, 21, 22, 30] coincides with $\rho = 0$ and $s = 1$.

The FNLS with a mixed source term was investigated in [7, 9], where the questions of global/nonglobal existence and scattering of solutions were treated.

The aim of this note is to study the competition between the singular inhomogeneous local source term $|x|^\rho|u|^{p-1}u$ and the local homogeneous term $|u|^{q-1}u$. We try to generalize some results about the fractional Schrödinger problem with a mixed power source term to the inhomogeneous case. Indeed, we obtain a sharp threshold of global/nonglobal existence of energy solutions to problem (1.1). Moreover, we investigate the L^2 concentration of the mass-critical nonglobal solutions and obtain a scattering result in the defocusing regime, based on the Morawetz estimate and the decay in some Lebesgue spaces. There are at least three technical difficulties: the absence of scaling invariance, the presence of a singular inhomogeneous term, and a nonlocal fractional differential operator. The spherically symmetric assumption is due to the loss of regularity in Strichartz estimates in the nonradial regime [15]. The blowup results are based on the pioneering work [3], which partially resolves the open problem of nonglobal existence of solutions to FNLS using a localized variance identity. In the present work, we treat simultaneously the two different regimes $\rho > 0$ and $\rho < -2s$, contrarily to the most papers considering the inhomogeneous Schrödinger problem. A similar problem with a nonlocal source term of Hartree type was treated recently by the first author [29].

The note has the following plan. In Sect. 2, we derive the contribution and some standard estimates. Sections 3 and 4 are devoted to proving some localized variance-type identities. In Sect. 5, we give a nonglobal existence criterion. Section 6 deals with establishing the finite-time blowup of solutions with nonpositive energy. In Sects. 7 and 8, we investigate the L^2 concentration of the mass-critical solutions. In Sect. 9, we establish a threshold of global existence versus finite-time blowup of solutions. The scattering of defocusing global solutions in the energy space is proved in Sect. 10. Finally, a compact Sobolev embedding is given in the Appendix.

Let us denote the Lebesgue and Sobolev spaces and their classical norms:

$$L^r := L^r(\mathbb{R}^N), \qquad H^s := H^s(\mathbb{R}^N), \qquad H^s_{rd} := \{f \in H^s, f(\cdot) = f(|\cdot|)\};$$
$$\|\cdot\| := \|\cdot\|_2, \qquad \|\cdot\|_{H^s} := \left(\|\cdot\|^2 + \left\|(-\Delta\cdot)^{\frac{s}{2}}\right\|^2\right)^{\frac{1}{2}}.$$

Eventually, $x^\pm$ are two real numbers close to x satisfying $x^+ > x$ and $x^- < x$.

2 Main results and useful estimates

In this section, we collects the main results and some standard estimates.

2.1 Notations

Here and hereafter, we define the real numbers

$$\mathcal{I}_p := \mathcal{I}(N,p,\rho,s) := \frac{N(p-1)-2\rho}{2s}, \qquad \mathcal{J}_p := \mathcal{J}(N,p,\rho,s) := 1+p-\mathcal{I}_p;$$
$$\mathcal{J}_q := \mathcal{J}(N,q,0,s), \qquad \mathcal{I}_q := \mathcal{I}(N,q,0,s).$$

The energy critical exponents are

$$p^c := p^c(N,s,\rho) = 1 + \frac{2(2s+\rho)}{N-2s} \quad \text{and} \quad q^c := p^c(N,s,0).$$

The mass critical exponents are

$$p_c := p^c(N,s,\rho) = 1 + \frac{2(2s+\rho)}{N} \quad \text{and} \quad q_c := p^c(N,s,0).$$

In the spirit of [3], we denote $\zeta_R := R^2\zeta(\frac{\cdot}{R})$, where $\zeta \in C_0^\infty(\mathbb{R}^N)$ is a radial, and

$$\zeta : x \longmapsto \begin{cases} \frac{1}{2}|x|^2, & |x| \le 1, \\ 0, & |x| \ge 10, \end{cases} \quad \text{and} \quad \zeta'' \le 1.$$

By a direct calculus it follows that

$$\zeta_R'' \le 1, \qquad \zeta_R'(r) \le r, \quad \text{and} \quad \Delta\zeta_R \le N.$$

Define also

$$\zeta_2 := N - \Delta\zeta_R \ge 0, \qquad \zeta_1 := 1 - \zeta_R'' \ge 0, \quad \text{and} \quad \zeta_3 := 1 - \frac{x\cdot\nabla\zeta_R}{|x|^2} \ge 0.$$

Denote the localized virial

$$M_\zeta[u] := 2\Im\int_{\mathbb{R}^N} \bar{u}\nabla\zeta\nabla u\,dx = 2\Im\int_{\mathbb{R}^N} \bar{u}\partial_k\zeta\,\partial_k u\,dx.$$

Then $M_\zeta[u] = \langle u, \Gamma_\zeta u\rangle$, where $\Gamma_\zeta g := -i[\nabla\cdot(g\nabla\zeta) + \nabla\zeta\cdot\nabla g]$. Finally, we introduce the sequence of functions

$$u_n(t,\cdot) := \sqrt{\frac{\sin(\pi t)}{\pi}}\mathcal{F}^{-1}\left(\frac{\mathcal{F}u}{n+|\cdot|^2}\right).$$

2.2 Preliminaries

First, a sharp Gagliardo–Nirenberg-type inequality related to (1.1) was established in [26, 28].

Proposition 2.1 *Let* $\rho > -2s$ *and* $1 + \frac{2\rho\chi_{\{\rho>0\}}}{N-2s} < p < p^c$. *Then*:

1. *There is* (*a best constant*) $C(N,p,\rho,s) > 0$ *such that for all* $u \in H^s$ *if* $\rho \le 0$ *and all* $u \in H^s_{rd}$ *if* $\rho > 0$,

$$\int_{\mathbb{R}^N} |u|^{1+p}|x|^\rho\,dx \le C(N,p,\rho,s)\|u\|^{\mathcal{J}_p}\left\|(-\Delta)^{\frac{s}{2}}u\right\|^{\mathcal{I}_p}; \tag{2.1}$$

2. *Moreover,*

$$C(N,p,\rho,s) = \frac{1+p}{\mathcal{J}_p}\left(\frac{\mathcal{J}_p}{\mathcal{I}_p}\right)^{\frac{\mathcal{I}_p}{2}} \|Q_p\|^{-(p-1)},$$

where Q_p *resolves*

$$(-\Delta)^s Q_p + Q_p - |x|^\rho |Q_p|^{p-1} Q_p = 0, \quad Q_p \in H^s_{rd} - \{0\}; \tag{2.2}$$

3. *Furthermore, we have the Pohozaev identities*

$$\mathcal{J}_p \big\|(-\Delta)^{\frac{s}{2}} Q_p\big\|^2 = \mathcal{I}_p \|Q_p\|^2, \qquad \mathcal{J}_p \int_{\mathbb{R}^N} |Q_p|^{1+p}|x|^\rho \, dx = (1+p)\|Q_p\|^2.$$

Remark 2.2 For $A \subset \mathbb{R}$, the characteristic function $\chi_{\{A\}}$ is equal to one on A and to zero on the complement of A.

Second, problem (1.1) is locally well-posed in the energy space [26, 28].

Proposition 2.3 *Let* $\frac{N}{2N-1} < s < 1$, $\rho > -2s$, $1 + \frac{2\rho\chi_{\{\rho>0\}}}{N-2s} < p < p^c$, $1 < q < q^c$, *and* $u_0 \in H^s_{rd}$. *Then there is a unique local solution to* (1.1) *in the energy space*

$$C\big([0,T], H^s_{rd}\big).$$

Moreover, the following quantities, called respectively the mass and energy, are time invariant:

$$M\big(u(t)\big) := \big\|u(t)\big\|^2;$$

$$E\big(u(t)\big) := \big\|(-\Delta)^{\frac{s}{2}} u(t)\big\|^2 - \frac{2\lambda_1}{1+p}\int_{\mathbb{R}^N} \big|u(t)\big|^{1+p}|x|^\rho \, dx - \frac{2\lambda_2}{1+q}\int_{\mathbb{R}^N} \big|u(t)\big|^{1+q} \, dx.$$

Finally, we give a compact Sobolev embedding established in the Appendix.

Lemma 2.4 *Let* $s \in (0,1)$, $\rho > -2s$, *and* $1 + \frac{2\rho\chi_{\{\rho>0\}}}{N-1} < p < p^c$. *Then*:

$$H^s_{rd}\big(\mathbb{R}^N\big) \hookrightarrow\hookrightarrow L^{1+p}\big(|x|^\rho \, dx\big).$$

Now we give the contribution of this note.

2.3 Main results

To investigate the nonglobal existence of solutions, we need some variance-type estimates.

Theorem 2.5 *Let* $s \in (\frac{1}{2}, 1)$, $\rho > -2s$, $1 + \frac{2\rho\chi_{\{\rho>0\}}}{N-2s} < p < p^c$, *and* $1 < q < q^c$, *and let* $u \in C_T(H^s_{rd})$ *be a solution of* (1.1). *Then, for any* $R > 0$, $0 < \varepsilon_p < (1 - \frac{1}{2s})(p-1)$, *and* $0 < \varepsilon_q < (1 - \frac{1}{2s})(q-1)$, *on* $[0,T)$, *we have*

$$\begin{aligned} M'_{\zeta_R}[u] \leq{} & 4s\big\|(-\Delta)^{\frac{s}{2}} u\big\|^2 - \frac{4s\mathcal{I}_p\lambda_1}{1+p}\int_{\mathbb{R}^N} |u|^{1+p}|x|^\rho \, dx - \frac{4s\mathcal{I}_q\lambda_2}{1+q}\int_{\mathbb{R}^N} |u|^{1+q} \, dx \\ & + \frac{C}{R^{2s}} + \frac{C}{R^{\frac{(p-1)(N-1)}{2} - s\varepsilon_p - \rho}} \big\|(-\Delta)^{\frac{s}{2}} u\big\|^{\frac{p-1}{2s} + \varepsilon_p} + \frac{C}{R^{\frac{(q-1)(N-1)}{2} - s\varepsilon_q}} \big\|(-\Delta)^{\frac{s}{2}} u\big\|^{\frac{q-1}{2s} + \varepsilon_q}. \end{aligned}$$

In the mass-critical case, we have the following refined version.

Theorem 2.6 *Let $s \in (\frac{1}{2}, 1)$, $-2s < \rho < 0$, $1 < q < q^c$, and $1 < p < p^c$, and let $u \in C_T(H^s_{rd})$ be a solution of (1.1). Assume that $\mathcal{I}_p = 2$ or $\mathcal{I}_q = 2$. Then there exists $C := C(N, s, \rho) > 0$ such that, for all $\eta, R > 0$, on $[0, T)$, we have*

$$\frac{d}{dt} M_{\zeta_R}[u] \le 4sE[u_0] - 4 \int_0^\infty m^s \int_{\mathbb{R}^N} \bigl(\zeta_1 - C\eta\bigl(\zeta_2^{\frac{N}{\rho+2s}} + |\rho|\zeta_3^{\frac{N}{\rho+2s}} + \zeta_2^{\frac{2}{q-1}}\bigr)\bigr)|\nabla u_n|^2 \,dx\,dm + O\biggl(\frac{1}{R^{2s}} + \eta^{-\frac{\rho+2s}{N-\rho-2s}} \frac{1}{R^{2s}} + \eta\bigl(1 + R^{-2} + R^{-4}\bigr)\biggr).$$

Remark 2.7 The above localized variance estimates follow the idea of [3].

The next elementary result will be useful.

Proposition 2.8 *Let $s \in (\frac{1}{2}, 1)$, $\rho > -2s$, $1 + \frac{2\rho\chi_{\{\rho>0\}}}{N-2s} < p < p^c$, and $1 < q < q^c$, and let $u \in C_{T^*}(H^s_{rd})$ be a maximal solution of (1.1). Assume that $E[u_0] \neq 0$ and there are $t_0, \delta > 0$ satisfying*

$$M_{\zeta_R}\bigl[u(t)\bigr] \le -\delta \int_{t_0}^t \bigl\|(-\Delta)^{\frac{s}{2}} u(\tau)\bigr\|^2 \,d\tau \quad \textit{for all } t \in [t_0, T^*).$$

Then $T^* < \infty$.

In the case of negative energy, we give a nonglobal existence result.

Proposition 2.9 *Let $s \in (\frac{1}{2}, 1)$, $-2s < \rho < 2s(N-2s)$, $1 + \frac{2\rho\chi_{\{\rho>0\}}}{N-2s} < p < \min\{1 + 4s, p^c\}$, and $1 < q < \min\{1 + 4s, q^c\}$. Then any maximal energy solution to (1.1) with negative energy is nonglobal if one of the following assumptions is satisfied*:

1. *$\mathcal{I}_q > 2$ and $(\mathcal{I}_q - \mathcal{I}_p)\lambda_1 \le 0$;*
2. *$\mathcal{I}_p > 2$ and $(\mathcal{I}_p - \mathcal{I}_q)\lambda_2 \le 0$.*

Remarks 2.10

1. The unnatural condition $\max\{p, q\} < 1 + 4s$, which seems to be technical, is due to the absence of a classical variance identity.
2. The contribution of the inhomogeneous term appears in the difference $\mathcal{I}_p - \mathcal{I}_q = \frac{N(p-q)-2\rho}{2s}$.

In the homogeneous mass-critical case, the situation reads as follows.

Proposition 2.11 *Let $s \in (\frac{N}{2N-1}, 1)$, $-2s < \rho < 0$, and $1 < p < p^c$, and let $u \in C_{T^*}(H^s_{rd})$ be a maximal solution to (1.1). Assume that $\lambda_1 = -1 = -\lambda_2$ and $\mathcal{I}_q = 2$. Then*:

1. *$T^* = \infty$ if $\|u_0\| < \|Q_q\|$;*
2. *If $s > \frac{1}{2}$, $\mathcal{I}_p < 2$, and $u_0 = c\rho^{\frac{N}{2}} Q_q(\rho\cdot)$, where $|c| > 1$ and $\rho > \bigl(\frac{2|c|^{-1+p} \int_{\mathbb{R}^N} |x|^\rho |Q_q|^{1+p}\,dx}{(1+p)(|c|^{q-1}-1)\|(-\Delta)^{\frac{s}{2}} Q_q\|^2}\bigr)^{\frac{1}{s(2-\mathcal{I}_p)}}$, then $T^* < \infty$, or there exist $C > 0$ and $t_* > 0$ such that*

$$\bigl\|(-\Delta)^{\frac{s}{2}} u(t)\bigr\| \ge Ct^s \quad \textit{for all } t \ge t_*;$$

3. *If $s > \frac{1}{2}$, $\mathcal{I}_p < 2$, $\|u_0\| = \|Q_q\|$, and $T^* < \infty$, then there is $\theta \in [0,2\pi]^{\mathbb{R}}$ such that*

$$\lim_{t\to T^*}\left\|\left(\frac{\|(-\Delta)^{\frac{s}{2}}Q_q\|}{\|(-\Delta)^{\frac{s}{2}}u(t)\|}\right)^{\frac{N}{2}}e^{i\theta(t)}u\left(t,\left(\frac{\|(-\Delta)^{\frac{s}{2}}Q_q\|}{\|(-\Delta)^{\frac{s}{2}}u(t)\|}\right)\cdot\right)-Q_q\right\|_{H^s}=0.$$

Remarks 2.12

1. The above result gives some sufficient conditions to get finite or infinite time blowup in the mass-critical homogeneous regime with small data. In the first case, the finite time blowup holds independently of the first component of the source term;
2. In the second case, which treats the complementary of the first one, there is a competition between the source term components;
3. The proof of the third case is omitted because it follows like and simpler than the last point in the next result.

Next, consider the case of mass-critical inhomogeneous regime. Let us take the open problem property, which is true for $\rho = 0$, see [10].

Assumption 1 There is a unique radial positive ground state to (2.2).

Proposition 2.13 *Let $s \in (\frac{N}{2N-1}, 1)$, $-2s < \rho < 0$, and $1 < q < q^c$, and let $u \in C_{T^*}(H^s_{rd})$ be a maximal solution to* (1.1). *Assume that $\lambda_1 = 1 = -\lambda_2$ and $\mathcal{I}_p = 2$. Then*:

1. *$T^* = \infty$ if $\|u_0\| < \|Q_p\|$;*
2. *If $s > \frac{1}{2}$, $\mathcal{I}_q < 2$, and $u_0 = c\rho^{\frac{N}{2}}Q_p(\rho\cdot)$, where $\rho > (\frac{2|c|^{-1+q}\int_{\mathbb{R}^N}|Q_p|^{1+q}\,dx}{(1+q)(|c|^{p-1}-1)\|(-\Delta)^{\frac{s}{2}}Q_p\|^2})^{\frac{1}{s(2-\mathcal{I}_q)}}$ and $|c| > 1$, then $T^* < \infty$, or there exist $C > 0$ and $t_* > 0$ such that*

   $$\left\|(-\Delta)^{\frac{s}{2}}u(t)\right\| \geq Ct^s \quad \text{for all } t \geq t_*;$$

3. *Under Assumption 1, if $s > \frac{1}{2}$, $\rho < 2s(N-1)$, $\mathcal{I}_q < 2$, $\|u_0\| = \|Q_p\|$, and $T^* < \infty$, then there is $\theta \in [0,2\pi]^{\mathbb{R}}$ such that*

$$\lim_{t\to T^*}\left\|\left(\frac{\|(-\Delta)^{\frac{s}{2}}Q_p\|}{\|(-\Delta)^{\frac{s}{2}}u\|}\right)^{\frac{N}{2}}e^{i\theta(t)}u\left(t,\left(\frac{\|(-\Delta)^{\frac{s}{2}}Q_p\|}{\|(-\Delta)^{\frac{s}{2}}u\|}\right)\cdot\right)-Q_p\right\|_{H^s}=0.$$

Remarks 2.14

1. The proofs of the first and second points are omitted because they follow the proof of Proposition 2.11;
2. In the last point, we need extra assumptions: Assumption 1 and $\rho < 2s(N-1)$.

Now we investigates the repulsive regime.

Theorem 2.15 *Let $\lambda_i = 1$, $\rho > -2s$, $1 + \frac{2\rho\chi_{\{\rho>0\}}}{N-2s} < p < p^c$, and $1 < q < q^c$, and let $u \in C_{T^*}(H^s_{rd})$ be a maximal solution to* (1.1).

1. *Assume that $\mathcal{I}_q = 2 < \mathcal{I}_p$, $\|u_0\| < \|Q_q\|$, and $E[u_0] < (1-\frac{2}{\mathcal{I}_p})(1-(\frac{\|u\|}{\|Q_q\|})^{\frac{4s}{N}})r_0^2$, where r_0 is defined in* (9.1). *If $\|(-\Delta)^{\frac{s}{2}}u_0\| < r_0$, then $T^* = \infty$. If $\|(-\Delta)^{\frac{s}{2}}u_0\| > r_0$, then $T^* < \infty$;*
2. *Assume that $2 < \mathcal{I}_p < \mathcal{I}_q$ and $E[u_0] < (1-\frac{2}{\mathcal{I}_p})r_1^2$, where r_1 is defined in* (9.2). *If $\|(-\Delta)^{\frac{s}{2}}u_0\| < r_1$, then $T^* = \infty$. If $\|(-\Delta)^{\frac{s}{2}}u_0\| > r_1$, then $T^* < \infty$;*

3. *Assume that* $\mathcal{I}_p > \mathcal{I}_q > 2$ *and* $E[u_0] < (1 - \frac{2}{\mathcal{I}_q})r_1^2$, *where* r_1 *is defined in* (9.2). *If* $\|(-\Delta)^{\frac{s}{2}}u_0\| < r_1$, *then* $T^* = \infty$. *If* $\|(-\Delta)^{\frac{s}{2}}u_0\| > r_1$, *then* $T^* < \infty$.

Remarks 2.16

1. The above result is in the spirit of the ground state threshold pioneered by Kenig and Merle [16] in the NLS case.
2. In the first case, where the source term contains a mass-critical component, an extra assumption is needed by comparison with the second and third cases, which are mass-supercritical.

Now let us investigate the case of an attractive and repulsive component in the source term.

Theorem 2.17 *Let* $\rho > -2s$, $1 + \frac{2\rho\chi_{\{\rho>0\}}}{N-2s} < p < p^c$, *and* $1 < q < q^c$, *and let* $u \in C_{T^*}(H^s_{rd})$ *be a maximal solution to* (1.1).

1. *Take* $\lambda_1 = 1 = -\lambda_2$, $\max\{2, \mathcal{I}_q\} < \mathcal{I}_p$, *and* $E[u_0] < (1 - \frac{2}{\mathcal{I}_p})x_p^2$, *where* x_p *is defined in* (10.1). *If* $\|(-\Delta)^{\frac{s}{2}}u_0\| < x_p$, *then* $T^* = \infty$. *If* $\|(-\Delta)^{\frac{s}{2}}u_0\| > x_p$, *then* $T^* < \infty$.
2. *Take* $\lambda_1 = -1 = -\lambda_2$, $\max\{2, \mathcal{I}_p\} < \mathcal{I}_q$, *and* $E[u_0] < (1 - \frac{2}{\mathcal{I}_q})x_q^2$, *where* x_q *is defined in* (10.1). *If* $\|(-\Delta)^{\frac{s}{2}}u_0\| < x_q$, *then* $T^* = \infty$. *If* $\|(-\Delta)^{\frac{s}{2}}u_0\| > x_q$, *then* $T^* < \infty$.

Remark 2.18 In the above result, where the components of nonlinearity have different kinds, the threshold depends of the term that has the higher exponent.

Finally, we consider the scattering of energy global solutions in the defocusing regime.

Theorem 2.19 *Take* $\lambda_1 = \lambda_2 = -1$. *Let* $\frac{N}{2N-1} \leq s < 1$, *let* $-2s < \rho < 0$, *or* $N > 6s$ *and* $0 < \rho < \min\{s, \frac{N}{2} - 3s\}$, *or* $N > 8s$ *and* $s < \rho < \frac{N}{2} - 3s$, *and let* $p_c < p < p^c$ *and* $q_c < q < q^c$. *Let* $u \in C(\mathbb{R}, H^s_{rd})$ *be a global solution to* (1.1). *Then there exist* $u_\pm \in H^s$ *satisfying*

$$\lim_{t\to\pm\infty} \left\| u(t) - e^{-it(-\Delta)^s}u_\pm \right\|_{H^s} = 0.$$

Remarks 2.20

1. In the case $\rho > 0$, some technical difficulties yield the restriction $N > 6s$, which gives $N \geq 4$ or $N > 8s$, which in turn gives $N \geq 5$ because $s \geq \frac{N}{2N-1}$.
2. The above result is based on a Morawetz estimate and a decay result in the spirit of [31];
3. The previous restrictions are not required in the decay result in Proposition 11.2.

2.4 Useful estimates

Let us give a fractional Strauss-type estimate [4].

Lemma 2.21 *Let* $N \geq 2$ *and* $\frac{1}{2} < s < \frac{N}{2}$. *Then*

$$\sup_{x\neq 0} |x|^{\frac{N}{2}-s}\left|u(x)\right| \leq C(N,s)\left\|(-\Delta)^{\frac{s}{2}}u\right\| \tag{2.3}$$

for every $u \in \dot{H}^s(\mathbb{R}^N)$, *where*

$$C(N,s) = \left(\frac{\Gamma(2s-1)\Gamma(\frac{N}{2}-s)\Gamma(\frac{N}{2})}{2^{2s}\pi^{\frac{N}{2}}\Gamma^2(s)\Gamma(\frac{N}{2}-1+s)}\right)^{\frac{1}{2}},$$

and Γ *is the gamma function.*

The next fractional chain rule [5] will be useful.

Lemma 2.22 *Let* $s \in (0,1]$, *and let* $1 < p, p_i, q_i < \infty$ *satisfy* $\frac{1}{p} = \frac{1}{p_i} + \frac{1}{q_i}$ *for* $1 \le i \le 2$. *Then*:

1. $\|(-\Delta)^{\frac{s}{2}} G(u)\|_p \lesssim \|(-\Delta)^{\frac{s}{2}} u\|_{q_1} \|G'(u)\|_{p_1}$ *for* $G \in C^1(\mathbb{C})$;
2. $\|(-\Delta)^{\frac{s}{2}}(uv)\|_p \lesssim \|(-\Delta)^{\frac{s}{2}} u\|_{p_1} \|v\|_{q_1} + \|(-\Delta)^{\frac{s}{2}} v\|_{p_2} \|u\|_{q_2}$.

The next result gives a vector-valued Leibniz rule for fractional derivatives [17].

Lemma 2.23 *Let* $s_1 + s_2 := s \in (0,1)$, $0 \le s_i \le s$, *and let* $1 < p, p_i, q, q_i < \infty$, $i \in \{1,2\}$, *satisfy* $\frac{1}{p} = \sum_{i=1}^2 \frac{1}{p_i}$ *and* $\frac{1}{q} = \sum_{i=1}^2 \frac{1}{q_i}$. *Then*

$$\left\|(-\Delta)^{\frac{s}{2}}(uv) - u(-\Delta)^{\frac{s}{2}} v - v(-\Delta)^{\frac{s}{2}} u\right\|_{L_T^q(L^p)} \lesssim \left\|(-\Delta)^{\frac{s_1}{2}} v\right\|_{L_T^{q_1}(L^{p_1})} \left\|(-\Delta)^{\frac{s_2}{2}} u\right\|_{L_T^{q_2}(L^{p_2})}.$$

Moreover, for $s_1 = 0$, *the value* $q_1 = \infty$ *is allowed.*

Let us recall a generalized Gagliardo–Nirenberg-type estimate [24].

Proposition 2.24 *Let* $1 \le q, p \le \infty$ *be such that*

$$\frac{1}{p} = \frac{\lambda}{q} + (1-\lambda)\left(\frac{1}{2} - \frac{s}{N}\right), \quad \lambda \in [0,1].$$

Then

$$\|\cdot\|_p \lesssim \|\cdot\|_q^{\lambda} \|\cdot\|_{\dot{H}^s}^{1-\lambda} \quad on\ \left(L^q \cap \dot{H}^s\right)(\mathbb{R}^N).$$

The following Gagliardo–Nirenberg-type estimate will be further useful.

Lemma 2.25 *Let* $Q_a(x_0)$ *be the square with center* x_0 *and edge length* a. *Then*

$$\|\cdot\|_{2+\frac{2s}{N}}^{2+\frac{2s}{N}} \lesssim \|\cdot\|_{\dot{H}^s}\left(\sup_{x\in\mathbb{R}^N} \|\cdot\|_{L^2(Q_1(x))}\right)^{1+\frac{2s}{N}} \quad on\ H^s. \tag{2.4}$$

Proof We cover $\mathbb{R}^N$ with disjoint $Q_1(x_j)$. Let $\sum_j \chi_j = 1$ be an associated positive unity partition. By Proposition 2.24,

$$\begin{aligned}\int_{\mathbb{R}^N} |u(t,x)|^{2+\frac{2s}{N}}\,dx &= \sum_j \int_{\mathbb{R}^N} |u(t,x)|^{2+\frac{2s}{N}} \chi_j\,dx \\ &\lesssim \sum_j \left\|\chi_j u(t)\right\|_{\dot{H}^s(Q_1(x_j))} \left\|\chi_j u(t)\right\|_{L^2(Q_1(x_j))}^{1+\frac{2s}{N}}\end{aligned}$$

$$\lesssim \|u(t)\|_{\dot{H}^s(\mathbb{R}^N)} \sup_{x\in\mathbb{R}^N}\left(\|u(t)\|^{1+\frac{2s}{N}}_{L^2(Q_1(x))}\right).$$

The proof is finished. □

We end this section by the following Bootstrap-type result [11].

Lemma 2.26 *Let $b > 0$, $\alpha > 1$, and $0 < a < (1-\frac{1}{\alpha})(\alpha b)^{\frac{1}{1-\alpha}}$. Take $f \in C([0,T],\mathbb{R}_+)$ satisfying $f(t) \le a + b(f(t))^\alpha$ for all $t \in [0,T]$ and $f(0) \le (\alpha b)^{\frac{1}{1-\alpha}}$. Then $f(t) \le \frac{\alpha}{\alpha-1}a$ for all $t \in [0,T]$.*

Finally, we recall some Strichartz estimates [15] for the fractional Schrödinger problem.

Definition 2.27 A couple of real numbers (q,r) such that $q,r \ge 2$ is said to be admissible if

$$\frac{4N+2}{2N-1} \le q \le \infty, \qquad \frac{2}{q} + \frac{2N-1}{r} \le N - \frac{1}{2},$$

or

$$2 \le q \le \frac{4N+2}{2N-1}, \qquad \frac{2}{q} + \frac{2N-1}{r} < N - \frac{1}{2}.$$

Proposition 2.28 *Let $N \ge 2$, $\mu \in \mathbb{R}$, $\frac{N}{2N-1} < s$, and $u_0 \in H^\mu_{rd}$. Then*

$$\|u\|_{L^q_t(L^r)\cap L^\infty_t(\dot{H}^\mu)} \lesssim \|u_0\|_{\dot{H}^\mu} + \left\|i\dot{u} - (-\Delta)^s u\right\|_{L^{\tilde{q}'}_t(L^{\tilde{r}'})}$$

if (q,r) and $(\tilde{q},\tilde{r})$ are admissible pairs such that $(\tilde{q},\tilde{r},N) \neq (2,\infty,2)$ or $(q,r,N) \neq (2,\infty,2)$ and satisfy the condition

$$\frac{2s}{q} + \mu = N\left(\frac{1}{2} - \frac{1}{r}\right), \qquad \frac{2s}{\tilde{q}} - \mu = N\left(\frac{1}{2} - \frac{1}{\tilde{r}}\right).$$

Remarks 2.29

1. For simplicity, we define the set

$$\Gamma^\mu := \left\{(q,r),\ \text{admissible}, (q,r,N) \neq (2,\infty,2) \text{ and } \frac{2s}{q} + \mu = N\left(\frac{1}{2} - \frac{1}{r}\right)\right\};$$
$$\Gamma := \Gamma^0.$$

2. If we take $\mu = 0$ in the previous inequality, then we obtain the classical Strichartz estimate.
3. In the non-radial case, there is a loss of regularity in Strichartz estimates [15].

3 Localized variance-type identity

This section is devoted to prove Theorem 2.5. Take $\lambda_1 = \lambda_2 = 1$ for simplicity and define the nonlinearity

$$\mathcal{G} := \mathcal{G}_p + \mathcal{G}_q := -|x|^\rho |u|^{p-1}u - |u|^{q-1}u.$$

Lemma 3.1

$$M'_\zeta\big[u(t)\big] = \int_0^\infty n^s \int_{\mathbb{R}^N} \big(4\overline{\partial_k u_n}\partial^2_{kl}\zeta\,\partial_l u_n - \Delta^2\zeta\,|u_n|^2\big)\,dx\,dn$$
$$+ \frac{4\rho}{1+p}\int_{\mathbb{R}^N} x\cdot\nabla\zeta\,|u|^{1+p}|x|^{\rho-2}\,dx - 2\frac{p-1}{1+p}\int_{\mathbb{R}^N}\Delta\zeta\,|u|^{1+p}|x|^\rho\,dx$$
$$- 2\frac{q-1}{1+q}\int_{\mathbb{R}^N}\Delta\zeta\,|u|^{1+q}\,dx.$$

Proof Using (1.1) and denoting $[A,B] := AB - BA$, we compute

$$M'_\zeta\big[u(t)\big] = \left\langle u(t), \left[-\frac{\mathcal{G}}{u}, i\Gamma_\zeta\right]u(t)\right\rangle + \big\langle u(t), \big[(-\Delta)^s, i\Gamma_\zeta\big]u(t)\big\rangle.$$

According to computation done in [3], we have

$$\big\langle u(t), \big[(-\Delta)^s, i\Gamma_\zeta\big]u(t)\big\rangle = \int_0^\infty n^s \int_{\mathbb{R}^N}\big(4\overline{\partial_k u_n}\partial^2_{kl}\zeta\,\partial_l u_n - \Delta^2\zeta\,|u_n|^2\big)\,dx\,dn.$$

Compute

$$\begin{aligned}(N_p) &:= \left\langle u, \left[-\frac{\mathcal{G}_p}{u}, i\Gamma_\zeta\right]u\right\rangle \\ &= \big\langle u, \big[-|u|^{p-1}|x|^\rho, i\Gamma_\zeta\big]u\big\rangle \\ &= \big\langle u, \big[-|u|^{p-1}|x|^\rho, div(\nabla\zeta\,\cdot) + \nabla\zeta\nabla\cdot\big]u\big\rangle \\ &= -\big\langle u, |x|^\rho|u|^{p-1}\big(div(\nabla\zeta u) + \nabla\zeta\nabla u\big)\big\rangle + \big\langle u, div\big(\nabla\zeta\,|x|^\rho|u|^{p-1}u\big) \\ &\quad + \nabla\zeta\nabla\big(|x|^\rho|u|^{p-1}u\big)\big\rangle.\end{aligned}$$

So

$$\begin{aligned}(N_p) &= -\big\langle u, |x|^\rho|u|^{p-1}(\Delta\zeta u + 2\nabla\zeta\nabla u)\big\rangle + \big\langle u, \Delta\zeta\,|x|^\rho|u|^{p-1}u + 2\nabla\zeta\nabla\big(|x|^\rho|u|^{p-1}u\big)\big\rangle \\ &= \big\langle u, \Delta\zeta\,|x|^\rho|u|^{p-1}u + 2\nabla\zeta\nabla\big(|x|^\rho|u|^{p-1}u\big) - |x|^\rho|u|^{p-1}(\Delta\zeta u + 2\nabla\zeta\nabla u)\big\rangle \\ &= 2\big\langle u, \nabla\zeta\nabla\big(|x|^\rho|u|^{p-1}u\big) - |x|^\rho|u|^{p-1}\nabla\zeta\nabla u\big\rangle \\ &= 2\big\langle u, \nabla\zeta\,\big(\nabla\big(|x|^\rho\big)|u|^{p-1}u + |x|^\rho\nabla\big(|u|^{p-1}\big)u\big)\big\rangle.\end{aligned}$$

By integration by parts we have

$$\begin{aligned}(N_p) &= 2\int_{\mathbb{R}^N}\nabla\zeta\nabla\big(|x|^\rho\big)|u|^{1+p}\,dx + 2\int_{\mathbb{R}^N}|x|^\rho\nabla\zeta\nabla\big(|u|^{p-1}\big)|u|^2\,dx \\ &= 2\int_{\mathbb{R}^N}\nabla\zeta\nabla\big(|x|^\rho\big)|u|^{1+p}\,dx + 2\frac{p-1}{1+p}\int_{\mathbb{R}^N}\nabla\zeta\nabla\big(|u|^{1+p}\big)|x|^\rho\,dx \\ &= 2\int_{\mathbb{R}^N}\nabla\zeta\nabla\big(|x|^\rho\big)|u|^{1+p}\,dx - 2\frac{p-1}{1+p}\int_{\mathbb{R}^N}|u|^{1+p}\big(\nabla\big(|x|^\rho\big)\nabla\zeta + |x|^\rho\Delta\zeta\big)\,dx \\ &= \frac{4}{1+p}\int_{\mathbb{R}^N}\nabla\zeta\nabla\big(|x|^\rho\big)|u|^{1+p}\,dx - 2\frac{p-1}{1+p}\int_{\mathbb{R}^N}\Delta\zeta\,|x|^\rho|u|^{1+p}\,dx\end{aligned}$$

$$= \frac{4\rho}{1+p}\int_{\mathbb{R}^N} x\cdot\nabla\zeta\,|u|^{1+p}|x|^{\rho-2}\,dx - 2\frac{p-1}{1+p}\int_{\mathbb{R}^N}\Delta\zeta\,|u|^{1+p}|x|^{\rho}\,dx.$$

Taking $\rho = 0$ in the previous calculation, we get the second term of the source term and finish the proof. □

Now we establish Theorem 2.5. Since $\Delta\zeta_R = N$ on $|x| < R$, we have

$$\begin{aligned}(N_p)^2 &:= -2\frac{p-1}{1+p}\int_{\mathbb{R}^N}\Delta\zeta_R|u|^{1+p}|x|^{\rho}\,dx \\ &= -2\frac{p-1}{1+p}\left(N\int_{\mathbb{R}^N}|u|^{1+p}|x|^{\rho}\,dx + \int_{|x|>R}(\Delta\zeta_R - N)|u|^{1+p}|x|^{\rho}\,dx\right).\end{aligned}$$

For $\frac{1}{2} < \alpha < s < \frac{N}{2}$, recall the interpolation inequality

$$\left\|(-\Delta)^{\frac{\alpha}{2}}\cdot\right\| \lesssim \|\cdot\|^{1-\frac{\alpha}{s}}\left\|(-\Delta)^{\frac{s}{2}}\cdot\right\|^{\frac{\alpha}{s}}.$$

So by (2.3) and properties of ζ_R we get for $0 < \epsilon \ll 1$ and $\alpha := \frac{1}{2} + \frac{s\epsilon}{p-1}$,

$$\begin{aligned}\int_{|x|>R}(\Delta\zeta_R - N)|u|^{1+p}|x|^{\rho}\,dx &\le \frac{C(\alpha,N)}{R^{(p-1)(\frac{N}{2}-\alpha)-\rho}}\left\|(-\Delta)^{\frac{\alpha}{2}}u\right\|^{p-1}\|u\|^2 \\ &\le \frac{C(\alpha,N)}{R^{(p-1)(\frac{N}{2}-\alpha)-\rho}}\left\|(-\Delta)^{\frac{s}{2}}u\right\|^{(p-1)\frac{\alpha}{s}} \\ &\le \frac{C(N,s,\epsilon)}{R^{\frac{(p-1)(N-1)}{2}-s\epsilon-\rho}}\left\|(-\Delta)^{\frac{s}{2}}u\right\|^{\frac{p-1}{2s}+\epsilon}.\end{aligned}$$

Since $\nabla\zeta_R(x) = x$ on $|x| < R$, we write

$$\begin{aligned}(N_p)^1 &:= \frac{4\rho}{1+p}\int_{\mathbb{R}^N} x\cdot\nabla\zeta_R|u|^{1+p}|x|^{\rho-2}\,dx \\ &= \frac{4\rho}{1+p}\left(\int_{\mathbb{R}^N}|u|^{1+p}|x|^{\rho}\,dx + \int_{|x|>R}\left(\frac{x\cdot\nabla\zeta_R}{|x|^2} - 1\right)|u|^{1+p}|x|^{\rho}\,dx\right) \\ &\le \frac{4\rho}{1+p}\int_{\mathbb{R}^N}|u|^{1+p}|x|^{\rho}\,dx + \frac{C(N,s,\epsilon)}{R^{\frac{(p-1)(N-1)}{2}-s\epsilon-\rho}}\left\|(-\Delta)^{\frac{s}{2}}u\right\|^{\frac{p-1}{2s}+\epsilon}.\end{aligned}$$

Thus, thanks to the estimate in [3],

$$\int_0^\infty n^s\int_{\mathbb{R}^N}\left(4\overline{\partial_k u_n}\partial_{kl}^2\zeta_R\partial_l u_n - \Delta^2\zeta_R|u_n|^2\right)dx\,dn \le 4s\left\|(-\Delta)^{\frac{s}{2}}u\right\|^2 + C\frac{1}{R^{2s}},$$

and by Lemma 3.1 we have

$$\begin{aligned}M'_{\zeta_R}[u(t)] = &\int_0^\infty n^s\int_{\mathbb{R}^N}\left(4\overline{\partial_k u_n}\partial_{kl}^2\zeta_R\partial_l u_n - \Delta^2\zeta_R|u_n|^2\right)dx\,dn \\ &+ \frac{4\rho}{1+p}\int_{\mathbb{R}^N} x\cdot\nabla\zeta_R|u|^{1+p}|x|^{\rho-2}\,dx - 2\frac{p-1}{1+p}\int_{\mathbb{R}^N}\Delta\zeta_R|u|^{1+p}|x|^{\rho}\,dx \\ &- 2\frac{q-1}{1+q}\int_{\mathbb{R}^N}\Delta\zeta_R|u|^{1+q}\,dx\end{aligned}$$

$$\le 4s\big\|(-\Delta)^{\frac{s}{2}}u\big\|^2 + C\frac{1}{R^{2s}} - \frac{4s\mathcal{I}_p}{1+p}\int_{\mathbb{R}^N}|u|^{1+p}|x|^\rho\,dx - \frac{4s\mathcal{I}_q}{1+q}\int_{\mathbb{R}^N}|u|^{1+q}\,dx$$
$$+ \frac{C}{R^{\frac{(p-1)(N-1)}{2}-s\epsilon_p-\rho}}\big\|(-\Delta)^{\frac{s}{2}}u\big\|^{\frac{p-1}{2s}+\epsilon_p} + \frac{C}{R^{\frac{(q-1)(N-1)}{2}-s\epsilon_q}}\big\|(-\Delta)^{\frac{s}{2}}u\big\|^{\frac{q-1}{2s}+\epsilon_q}.$$

The proof is completed.

4 Refined localized variance-type identity

In this section, we take $\lambda_1 = \lambda_2 = 1$ for simplicity and establish Theorem 2.6. Recall the identities [3]

$$\int_0^\infty n^s\int_{\mathbb{R}^N}\overline{\partial_k u_n}\partial_{kl}^2\zeta_R\partial_l u_n\,dx\,dn = \int_0^\infty n^s\int_{\mathbb{R}^N}\partial_r^2\zeta_R|\nabla u_n|^2\,dx\,dn;$$
$$s\big\|(-\Delta)^{\frac{s}{2}}u\big\|^2 = \int_0^\infty n^s\int_{\mathbb{R}^N}|\nabla u_n|^2\,dx\,dn.$$

Then Lemma 3.1 gives

$$\begin{aligned}M'_{\zeta_R}\big[u(t)\big] = {}& 4s\big\|(-\Delta)^{\frac{s}{2}}u\big\|^2 - 4\int_0^\infty n^s\int_{\mathbb{R}^N}\big(1-\partial_r^2\zeta_R\big)|\nabla u_n|^2\,dx\,dn\\
&- \int_0^\infty n^s\int_{\mathbb{R}^N}\Delta^2\zeta_R|u_n|^2\,dx\,dn\\
&- \frac{4s\mathcal{I}_p}{1+p}\int_{\mathbb{R}^N}|u|^{1+p}|x|^\rho\,dx - \frac{4s\mathcal{I}_q}{1+q}\int_{\mathbb{R}^N}|u|^{1+q}\,dx\\
&- 2\frac{p-1}{1+p}\int_{|x|>R}(\Delta\zeta_R - N)|u|^{1+p}|x|^\rho\,dx\\
&+ \frac{4\rho}{1+p}\int_{|x|>R}\left(\frac{x\cdot\nabla\zeta_R}{|x|^2}-1\right)|u|^{1+p}|x|^\rho\,dx\\
&- 2\frac{q-1}{1+q}\int_{|x|>R}(\Delta\zeta_R - N)|u|^{1+q}\,dx.\end{aligned}$$

Now, in view of the estimate [3]

$$\left|\int_0^\infty n^s\int_{\mathbb{R}^N}\Delta^2\zeta_R|u_n|^2\,dx\,dn\right| \le \frac{C}{R^{2s}},$$

we get

$$\begin{aligned}M'_{\zeta_R}\big[u(t)\big] = {}& 4sE[u_0] - 4\int_0^\infty n^s\int_{\mathbb{R}^N}\zeta_1|\nabla u_n|^2\,dx\,dn\\
&+ 4s\frac{2-\mathcal{I}_p}{1+p}\int_{\mathbb{R}^N}|u|^{1+p}|x|^\rho\,dx + 4s\frac{2-\mathcal{I}_q}{1+q}\int_{\mathbb{R}^N}|u|^{1+q}\,dx\\
&+ 2\frac{p-1}{1+p}\int_{|x|>R}\zeta_2|u|^{1+p}|x|^\rho\,dx - \frac{4\rho}{1+p}\int_{|x|>R}\zeta_3|x|^\rho|u|^{1+p}\,dx\\
&+ 2\frac{q-1}{1+q}\int_{|x|>R}\zeta_2|u|^{1+q}\,dx + O\left(\frac{1}{R^{2s}}\right).\end{aligned}$$

Now, since $\rho \le 0$ and $\mathcal{I}_p = 2$,

$$\begin{aligned}\int_{|x|>R} \zeta_2 |u|^{1+p}|x|^\rho\,dx &= \int_{|x|>R} \big(\zeta_2^{\frac{1}{p-1}}|u|\big)^{p-1}|u|^2|x|^\rho\,dx \\ &\le R^\rho \big\|\zeta_2^{\frac{1}{p-1}}u\big\|^{p-1}_{L^\infty(|x|>R)}\|u\|^2 \\ &\le CR^{b-(p-1)(\frac{N}{2}-s)}\big\|(-\Delta)^{\frac{s}{2}}\big(\zeta_2^{\frac{1}{p-1}}u\big)\big\|^{p-1} \\ &= CR^{b-(\rho+2s)(1-\frac{2s}{N})}\big\|(-\Delta)^{\frac{s}{2}}\big(\zeta_2^{\frac{1}{p-1}}u\big)\big\|^{\frac{2(\rho+2s)}{N}} \\ &\le \eta\big\|(-\Delta)^{\frac{s}{2}}\big(\zeta_2^{\frac{1}{p-1}}u\big)\big\|^2 + O\bigg(\eta^{-\frac{\rho+2s}{N-\rho-2s}}\frac{1}{R^{2s}}\bigg),\end{aligned}$$

where we usedthe Young inequality $ab \le \frac{\eta a^q}{q} + \frac{\eta^{1-q'}b^{q'}}{q'}$, for $q \ge 1$ and $\eta > 0$. Thus by the estimate [3]

$$s\big\|(-\Delta)^{\frac{s}{2}}\big(\zeta_2^{\frac{1}{p-1}}u\big)\big\|^2 = \int_0^\infty n^s\int_{\mathbb{R}^N}\zeta_2^{\frac{2}{p-1}}|\nabla u_n|^2\,dx\,dn + O\big(1+R^{-2}+R^{-4}\big)$$

we get

$$\begin{aligned}\int_{|x|>R}\zeta_2|u|^{1+p}|x|^\rho\,dx &\le \frac{\eta}{s}\int_0^\infty n^s\int_{\mathbb{R}^N}\zeta_2^{\frac{2}{p-1}}|\nabla u_n|^2\,dx\,dn \\ &\quad + O\bigg(\eta^{-\frac{\rho+2s}{N-\rho-2s}}\frac{1}{R^{2s}} + \eta\big(1+R^{-2}+R^{-4}\big)\bigg).\end{aligned}$$

Thus there exists $C := C(N,s,\rho) > 0$ such that

$$\begin{aligned}M'_{\zeta_R}\big[u(t)\big] &= 4sE[u_0] - 4\int_0^\infty n^s\int_{\mathbb{R}^N}\zeta_1|\nabla u_n|^2\,dx\,dn \\ &\quad + 4s\frac{2-\mathcal{I}_p}{1+p}\int_{\mathbb{R}^N}|u|^{1+p}|x|^\rho\,dx + 4s\frac{2-\mathcal{I}_q}{1+q}\int_{\mathbb{R}^N}|u|^{1+q}\,dx \\ &\quad + 2\frac{p-1}{1+p}\int_{|x|>R}\zeta_2|u|^{1+p}|x|^\rho\,dx - \frac{4\rho}{1+p}\int_{|x|>R}\zeta_3|u|^{1+p}|x|^\rho\,dx \\ &\quad + 2\frac{q-1}{1+q}\int_{|x|>R}\zeta_2|u|^{1+q}\,dx + O\bigg(\frac{1}{R^{2s}}\bigg) \\ &\le 4sE[u_0] - 4\int_0^\infty n^s\int_{\mathbb{R}^N}\Big(\zeta_1 - C\eta\big(\zeta_2^{\frac{N}{\rho+2s}} + |\rho|\zeta_3^{\frac{N}{\rho+2s}} + \zeta_2^{\frac{2}{q-1}}\big)\Big)|\nabla u_n|^2\,dx\,dn \\ &\quad + O\bigg(\frac{1}{R^{2s}} + \eta^{-\frac{\rho+2s}{N-\rho-2s}}\frac{1}{R^{2s}} + \eta\big(1+R^{-2}+R^{-4}\big)\bigg).\end{aligned}$$

This ends the proof.

5 Blowup criterion

In this section, we prove Proposition 2.8. Taking into account the conservation laws, the inequality $\|\cdot\|_{\dot H^{\frac12}} \le \|\cdot\|^{1-\frac{1}{2s}}\|\cdot\|^{\frac{1}{2s}}_{\dot H^s}$, and Lemma A.1 in [3], we get

$$\big|M_{\zeta_R}\big[u(t)\big]\big| \le C_R\big(\|u(t)\|^2_{\dot H^{\frac12}} + \|u(t)\|\,\|u(t)\|_{\dot H^{\frac12}}\big)$$

$$\leq C_R\big(\|u(t)\|_{\dot H^s}^{\frac{1}{s}} + \|u(t)\|_{\dot H^s}^{\frac{1}{2s}}\big).$$

Now we claim that

$$\inf_{t\geq 0}\|u(t)\|_{\dot H^s} \geq C > 0.$$

By contradiction, assume that there is $t_k \geq 0$ such that $\|u(t_k)\|_{\dot H^s} \to 0$. By the conservation laws we have

$$\int_{\mathbb{R}^N}\big|u(t_k)\big|^{1+p}|x|^\rho\,dx + \int_{\mathbb{R}^N}\big|u(t_k)\big|^{1+q}\,dx \to 0,$$

which gives the contradiction

$$0 \neq E[u_0] = E\big(u(t_k)\big) \to 0.$$

So

$$\big|M_{\zeta_R}\big[u(t)\big]\big| \leq C_R\|u(t)\|_{\dot H^s}^{\frac{1}{s}};$$
$$M_{\zeta_R}\big[u(t)\big] \leq -C_R\int_{t_0}^t \big|M_{\zeta_R}\big[u(\tau)\big]\big|^{2s}\,d\tau.$$

Then, for $s > \frac{1}{2}$ and finite $t_1 > 0$,

$$M_{\zeta_R}\big[u(t)\big] \leq -C_R|t - t_1|^{1-2s} \to -\infty \quad \text{as } t \to t_1.$$

Finally, $T^* < \infty$.

6 Blowup for negative energy

This section contains a proof of Proposition 2.9, which we do in two steps.

1. Case 1. Since $\lambda_1(\mathcal{I}_q - \mathcal{I}_p) \leq 0$, by Theorem 2.5 we have

$$\begin{aligned}
M'_{\zeta_R}[u] &\leq 2s\mathcal{I}_q E[u_0] - 2s(\mathcal{I}_q - 2)\big\|(-\Delta)^{\frac{s}{2}}u\big\|^2 + \frac{4s\lambda_1}{1+p}(\mathcal{I}_q - \mathcal{I}_p)\int_{\mathbb{R}^N}|u|^{1+p}|x|^\rho\,dx \\
&\quad + \frac{C}{R^{2s}} + \frac{C}{R^{\frac{(p-1)(N-1)}{2}-s\epsilon_p-\rho}}\big\|(-\Delta)^{\frac{s}{2}}u\big\|^{\frac{p-1}{2s}+\epsilon} + \frac{C}{R^{\frac{(q-1)(N-1)}{2}-s\epsilon_q}}\big\|(-\Delta)^{\frac{s}{2}}u\big\|^{\frac{q-1}{2s}+\epsilon} \\
&\leq 2s\mathcal{I}_q E[u_0] - 2s(\mathcal{I}_q - 2)\big\|(-\Delta)^{\frac{s}{2}}u\big\|^2 \\
&\quad + \frac{C}{R^{2s}} + \frac{C}{R^{\frac{(p-1)(N-1)}{2}-s\epsilon_p-\rho}}\big\|(-\Delta)^{\frac{s}{2}}u\big\|^{\frac{p-1}{2s}+\epsilon} \\
&\quad + \frac{C}{R^{\frac{(q-1)(N-1)}{2}-s\epsilon_q}}\big\|(-\Delta)^{\frac{s}{2}}u\big\|^{\frac{q-1}{2s}+\epsilon}.
\end{aligned}$$

Thanks to Young inequality, since $\max\{p,q\} < 1 + 4s$ and $E[u_0] < 0$, we get, for large $R > 0$,

$$M'_{\zeta_R}[u] \leq 2s\mathcal{I}_q E[u_0] - 2s(\mathcal{I}_q - 2)\big\|(-\Delta)^{\frac{s}{2}}u\big\|^2$$

$$+\frac{C}{R^{2s}}+\frac{C}{R^{\frac{(p-1)(N-1)}{2}-s\epsilon_p-\rho}}\left\|(-\Delta)^{\frac{s}{2}}u\right\|^{\frac{p-1}{2s}+\epsilon}+\frac{C}{R^{\frac{(q-1)(N-1)}{2}-s\epsilon_q}}\left\|(-\Delta)^{\frac{s}{2}}u\right\|^{\frac{q-1}{2s}+\epsilon}$$
$$\le s\mathcal{I}_q E[u_0]-(\mathcal{I}_q-2)\left\|(-\Delta)^{\frac{s}{2}}u\right\|^2. \tag{6.1}$$

Since $\mathcal{I}_q>2$, $M_{\zeta_R}[u(t)]\le M_{\zeta_R}[u_0]+2s\mathcal{I}_q E[u_0]t$. So there is $t_1>0$ such that $M_{\zeta_R}[u(t)]<0$ for all $t\ge t_1$. Now by integrating (6.1) it follows that

$$M_{\zeta_R}\big[u(t)\big]\le -(\mathcal{I}_q-2)\int_{t_1}^{t}\left\|(-\Delta)^{\frac{s}{2}}u(\tau)\right\|^2 d\tau \quad \text{for all } t\ge t_1.$$

We conclude by Proposition 2.8.

2. Case 2. Similar to case 1.

7 Mass-critical blowup

In this section, we prove Proposition 2.11.

7.1 Case 1

Proposition 2.1 gives

$$\begin{aligned}
E[u_0] &= \left\|(-\Delta)^{\frac{s}{2}}u(t)\right\|^2+\frac{2}{1+p}\int_{\mathbb{R}^N}\left|u(t)\right|^{1+p}|x|^\rho\,dx-\frac{2}{1+q}\int_{\mathbb{R}^N}\left|u(t)\right|^{1+q}dx\\
&\ge \left\|(-\Delta)^{\frac{s}{2}}u(t)\right\|^2\left(1-\frac{2C_{N,q,s}}{1+q}\|u\|^{A_q}\right)\\
&\ge \left\|(-\Delta)^{\frac{s}{2}}u(t)\right\|^2\left(1-\left[\frac{\|u\|}{\|Q_q\|}\right]^{\frac{4}{N}}\right).
\end{aligned}$$

This proves that $T^*=\infty$.

7.2 Case 2

Taking into account Theorem 2.6 and the Pohozaev identity, for any $\eta,R>0$, we have

$$\begin{aligned}
M'_{\zeta_R}\big[u(t)\big] &\le 4sE[u_0]-4\int_0^\infty n^s\int_{\mathbb{R}^N}\left(\zeta_1-C\eta\left(\zeta_2^{\frac{N}{\rho+2s}}+|\rho|\zeta_3^{\frac{N}{\rho+2s}}+\zeta_2^{\frac{2}{q-1}}\right)\right)|\nabla u_n|^2\,dx\,dn\\
&\quad+O\left(\frac{1}{R^{2s}}+\eta^{-\frac{\rho+2s}{N-\rho-2s}}\frac{1}{R^{2s}}+\eta\left(1+R^{-2}+R^{-4}\right)\right).
\end{aligned}$$

By taking the particular choice of ζ as in [26] there exists $0<\eta\ll 1$ such that

$$\zeta_1-C\eta\left(\zeta_2^{\frac{N}{\rho+2s}}+|\rho|\zeta_3^{\frac{N}{\rho+2s}}+\zeta_2^{\frac{2}{q-1}}\right)\ge 0\quad \forall R>0.$$

Thus, taking $R\ll 1$, we get

$$M'_{\zeta_R}\big[u(t)\big]<2sE[u_0]<0.$$

Indeed, with the assumptions, the next term is negative:

$$E[u_0]=\left|c\rho^s\right|^2\left\|(-\Delta)^{\frac{s}{2}}Q_q\right\|^2$$

$$\begin{aligned}
&- \frac{2|c|^{1+q}\rho^{2s}}{1+q}\int_{\mathbb{R}^N}|Q_q|^{1+q}\,dx + \frac{2|c|^{1+p}\rho^{s\mathcal{I}_p}}{1+p}\int_{\mathbb{R}^N}|x|^{\rho}|Q_q|^{1+p}\,dx \\
&= \left|c\rho^s\right|^2\left\|(-\Delta)^{\frac{s}{2}}Q_q\right\|^2 - |c|^{1+q}\rho^{2s}\left\|(-\Delta)^{\frac{s}{2}}Q_q\right\|^2\,dx \\
&\quad + \frac{2|c|^{1+p}\rho^{s\mathcal{I}_p}}{1+p}\int_{\mathbb{R}^N}|x|^{\rho}|Q_q|^{1+p}\,dx \\
&= \rho^{s\mathcal{I}_p}c^2\left[\left(1-|c|^{q-1}\right)\rho^{2s-s\mathcal{I}_p}\left\|(-\Delta)^{\frac{s}{2}}Q_q\right\|^2 + 2\frac{|c|^{-1+p}}{1+p}\int_{\mathbb{R}^N}|x|^{\rho}|Q_q|^{1+p}\,dx\right].
\end{aligned}$$

Assume that $T^* = \infty$. Then there exist $c > 0$ and $t_0 > 0$ such that

$$M_{\zeta_R}\left[u(t)\right] \le -ct \quad \forall t \ge t_0.$$

Lemma A.1 in [3] and the conservation laws, via the estimate $\|\cdot\|_{\dot{H}^{\frac{1}{2}}} \le \|\cdot\|^{1-\frac{1}{2s}}\|\cdot\|_{\dot{H}^s}^{\frac{1}{2s}}$, give

$$\begin{aligned}
\left|M_{\zeta_R}\left[u(t)\right]\right| &\le C_R\left(\left\|u(t)\right\|\left\|u(t)\right\|_{\dot{H}^{\frac{1}{2}}} + \left\|u(t)\right\|_{\dot{H}^{\frac{1}{2}}}^2\right) \\
&\le C_R\left(\left\|u(t)\right\|_{\dot{H}^{\frac{1}{2}}}^2 + 1\right) \\
&\le C_R\left(\left\|u(t)\right\|_{\dot{H}^s}^{\frac{1}{s}} + 1\right).
\end{aligned}$$

So

$$\left\|u(t)\right\|_{\dot{H}^s} \gtrsim t^s \quad \text{for all } t \ge t_1 > 0.$$

This ends the proof.

8 Inhomogeneous mass-critical blowup

In this section, we establish the third point of Proposition 2.13. Let us start with a compactness result [25].

Lemma 8.1 *Let a sequence of $v_n \in H^s_{rd}$ be such that $\sup_n \|v_n\|_{H^s} < \infty$. Assume that*

$$\limsup_n \left\|(-\Delta)^{\frac{s}{2}}v_n\right\| \le M, \qquad \limsup_n \int_{\mathbb{R}^N}|x|^{\rho}|v_n|^{1+p_c}\,dx \ge m^{1+p_c}.$$

Then a subsequence, denoted also by (v_n), satisfies

$$\begin{aligned}
&v_n \rightharpoonup V \quad \text{in } H^s_{rd}; \\
&\|V\| \ge \left(\frac{2}{(1+p)M^2}\right)^{\frac{N}{2(\rho+2s)}} m^{1+\frac{N}{\rho+2s}}\|Q_p\|.
\end{aligned}$$

As a consequence, we prove the next concentration result.

Lemma 8.2 *If $\lim_{t\to T^*} f(t)\|(-\Delta)^{\frac{s}{2}}u(t)\|^{\frac{1}{s}} = \infty$, then*

$$\liminf_{t\to T^*}\int_{|x|\le f(t)}\left|u(t,x)\right|^2\,dx \ge \|Q_p\|^2.$$

Proof Define the quantities

$$\alpha := \left(\frac{\|(-\Delta)^{\frac{s}{2}} Q_p\|}{\|(-\Delta)^{\frac{s}{2}} u\|}\right)^{\frac{1}{s}};$$

$$v := \alpha^{\frac{N}{2}} u(\cdot, \alpha \cdot);$$

$$t_n \to T^*, \qquad \alpha_n := \alpha(t_n), \qquad v_n := v(t_n).$$

Thus

$$\|v_n\|^2 = M(u);$$

$$\left\|(-\Delta)^{\frac{s}{2}} v_n\right\| = \left\|(-\Delta)^{\frac{s}{2}} Q_p\right\|.$$

Using the identity $\mathcal{I}_p = 2 > \mathcal{I}_q$ and Proposition 2.1, we have

$$\begin{aligned}
&\left\|(-\Delta)^{\frac{s}{2}} Q_p\right\|^2 - \frac{2}{1+p}\int_{\mathbb{R}^N} |v_n|^{1+p}|x|^\rho\, dx \\
&= \left\|(-\Delta)^{\frac{s}{2}} v_n\right\|^2 - \frac{2}{1+p}\int_{\mathbb{R}^N} |v_n|^{1+p}|x|^\rho\, dx \\
&= \alpha_n^{2s}\left(\left\|(-\Delta)^{\frac{s}{2}} u(t_n)\right\|^2 - \frac{2}{1+p}\int_{\mathbb{R}^N} \left|u(t_n)\right|^{1+p}|x|^\rho\, dx\right) \\
&= \alpha_n^{2s}\left(E(u) - \frac{2}{1+q}\int_{\mathbb{R}^N} \left|u(t_n)\right|^{1+q} dx\right) \\
&\le \alpha_n^{2s}\left(E(u) + C\left\|(-\Delta)^{\frac{s}{2}} u(t_n)\right\|^{\mathcal{I}_q}\right) \to 0.
\end{aligned}$$

Thus

$$\int_{\mathbb{R}^N} |v_n|^{1+p}|x|^\rho\, dx \to \frac{1+p}{2}\left\|(-\Delta)^{\frac{s}{2}} Q_p\right\|^2.$$

Denote

$$m^{1+p} := \frac{1+p}{2}\left\|(-\Delta)^{\frac{s}{2}} Q_p\right\|^2, \qquad M := \left\|(-\Delta)^{\frac{s}{2}} Q_p\right\|^2.$$

By Lemma 8.1 it follows that

$$v_n \rightharpoonup V \quad \text{in } H^s_{rd}, \qquad \|V\| \ge \|Q_p\|.$$

Moreover,

$$\frac{f(t_n)}{\alpha_n} = f(t_n)\left(\frac{\|(-\Delta)^{\frac{s}{2}} u_n\|}{\|(-\Delta)^{\frac{s}{2}} Q_p\|}\right)^{\frac{1}{s}} \to \infty.$$

So, for any $R > 0$, there is $n_0 \in \mathbb{N}$ such that $f(t_n) > R\alpha_n$ for $n > n_0$. Then

$$\liminf_n \int_{|x|\le f(t_n)} \left|u(t_n, x)\right|^2 dx \ge \liminf_n \int_{|x|\le R\alpha_n} \left|u(t_n, x)\right|^2 dx$$

$$\begin{aligned} &= \liminf_n \int_{|x|\leq R} \left|v(t_n,x)\right|^2 dx \\ &\geq \liminf_n \int_{|x|\leq R} \left|V(x)\right|^2 dx. \end{aligned}$$

Thus

$$\liminf_{T^*} \int_{|x|\leq f(t)} \left|u(t,x)\right|^2 dx \geq \|V\|^2 \geq \|Q_p\|^2.$$

This finishes the proof. □

Now we are ready to prove Proposition 2.13. Let us write

$$\|Q_p\| \leq \|V\| \leq \liminf_n \|v_n\| = \|u_0\| = \|Q_p\|.$$

Thus with the weak convergence, we have

$$\lim_n \|v_n - V\| = 0.$$

Moreover, by Lemma 2.4, via the assumption $\rho < 2s(N-1)$, we have

$$\lim_n \int_{\mathbb{R}^N} |x|^\rho |v_n - V|^{1+p} dx = 0.$$

Now taking into account the previous calculus and Proposition 2.1, we write

$$\begin{aligned} 0 &= \lim_n \left(\left\|(-\Delta)^{\frac{s}{2}} Q_p\right\|^2 - \frac{2}{1+p} \int_{\mathbb{R}^N} |v_n|^{1+p} |x|^\rho dx \right) \\ &= \left\|(-\Delta)^{\frac{s}{2}} Q_p\right\|^2 - \frac{2}{1+p} \int_{\mathbb{R}^N} |V|^{1+p} |x|^\rho dx \\ &\geq \left\|(-\Delta)^{\frac{s}{2}} Q_p\right\|^2 - \left(\frac{\|V\|}{\|Q_p\|} \right)^{p-1} \left\|(-\Delta)^{\frac{s}{2}} V\right\|^2 \\ &\geq \left\|(-\Delta)^{\frac{s}{2}} Q_p\right\|^2 - \left\|(-\Delta)^{\frac{s}{2}} V\right\|^2. \end{aligned}$$

Thus by the lower semicontinuity of $\|\cdot\|$ we have

$$\left\|(-\Delta)^{\frac{s}{2}} v_n\right\| = \left\|(-\Delta)^{\frac{s}{2}} Q_p\right\| = \left\|(-\Delta)^{\frac{s}{2}} V\right\|.$$

So we get

$$v_n \to V \quad \text{in } H^s.$$

So, by the Pohozaev identities,

$$\begin{aligned} &\left\|(-\Delta)^{\frac{s}{2}} Q_p\right\| = \left\|(-\Delta)^{\frac{s}{2}} V\right\|; \\ &\|Q_p\| = \|V\|; \end{aligned}$$

$$\int_{\mathbb{R}^N} |Q_p|^{1+p}|x|^\rho\,dx = \int_{\mathbb{R}^N} |V|^{1+p}|x|^\rho\,dx.$$

Then V is a minimizer of (2.1). Thus, using the Euler–Lagrange equation and a scaling, by Assumption 1 we have

$$V = aQ_p(b\cdot).$$

So $V = e^{i\theta}Q_p$, and

$$\alpha_n e^{-i\theta}u(t_n,\alpha_n\cdot) \to Q_p \quad \text{in } H^s.$$

This finishes the proof.

9 Global/nonglobal existence for $\lambda_1 = \lambda_2 = 1$

In this section, we prove Theorem 2.15. Thanks to Proposition 2.1,

$$\begin{aligned}
E[u_0] &= \big\|(-\Delta)^{\frac{s}{2}}u(t)\big\|^2 - \frac{2}{1+p}\int_{\mathbb{R}^N}\big|u(t)\big|^{1+p}|x|^\rho\,dx - \frac{2}{1+q}\int_{\mathbb{R}^N}\big|u(t)\big|^{1+q}\,dx \\
&\geq \big\|(-\Delta)^{\frac{s}{2}}u(t)\big\|^2 - \frac{2C_{N,p,\rho,s}}{1+p}\|u\|^{\mathcal{J}_p}\big\|(-\Delta)^{\frac{s}{2}}u(t)\big\|^{\mathcal{I}_p} \\
&\quad - \frac{2C_{N,q,s}}{1+q}\|u\|^{\mathcal{J}_q}\big\|(-\Delta)^{\frac{s}{2}}u(t)\big\|^{\mathcal{I}_q} \\
&= \big\|(-\Delta)^{\frac{s}{2}}u(t)\big\|^2 - \frac{2}{\mathcal{J}_p}\left(\frac{\mathcal{J}_p}{\mathcal{I}_p}\right)^{\frac{\mathcal{I}_p}{2}}\|Q_p\|^{-(p-1)}\|u\|^{\mathcal{J}_p}\big\|(-\Delta)^{\frac{s}{2}}u(t)\big\|^{\mathcal{I}_p} \\
&\quad - \frac{2}{\mathcal{J}_q}\left(\frac{\mathcal{J}_q}{\mathcal{I}_q}\right)^{\frac{\mathcal{I}_q}{2}}\|Q_q\|^{-(q-1)}\|u\|^{\mathcal{J}_q}\big\|(-\Delta)^{\frac{s}{2}}u(t)\big\|^{\mathcal{I}_q} \\
&:= g\big(\big\|(-\Delta)^{\frac{s}{2}}u(t)\big\|\big).
\end{aligned}$$

Let us consider three cases.

9.1 Case 1

In this case, $\mathcal{I}_q = 2$. Then

$$\begin{aligned}
g(X) &= \left(1 - \left(\frac{\|u\|}{\|Q_q\|}\right)^{\frac{4s}{N}}\right)X^2 - \frac{2}{\mathcal{J}_p}\left(\frac{\mathcal{J}_p}{\mathcal{I}_p}\right)^{\frac{\mathcal{I}_p}{2}}\|Q_p\|^{-(p-1)}\|u\|^{\mathcal{J}_p}X^{\mathcal{I}_p}; \\
g'(X) &= 2\left(1 - \left(\frac{\|u\|}{\|Q_q\|}\right)^{\frac{4s}{N}}\right)X - 2\left(\frac{\mathcal{J}_p}{\mathcal{I}_p}\right)^{\frac{\mathcal{I}_p}{2}-1}\|Q_p\|^{-(p-1)}\|u\|^{\mathcal{J}_p}X^{\mathcal{I}_p-1}.
\end{aligned}$$

Since $\|u_0\| < \|Q_q\|$, the unique positive root of g' is

$$g'(r_0) := g'\left[\left(\frac{1-(\frac{\|u\|}{\|Q_q\|})^{\frac{4s}{N}}}{(\frac{\mathcal{J}_p}{\mathcal{I}_p})^{\frac{\mathcal{I}_p}{2}-1}\|Q_p\|^{-(p-1)}\|u\|^{\mathcal{J}_p}}\right)^{\frac{1}{\mathcal{I}_p-2}}\right] = 0. \tag{9.1}$$

Thus

$$\max_{\mathbb{R}_+} g = g(r_0) = \frac{\mathcal{I}_p - 2}{\mathcal{I}_p}\left(1 - \left(\frac{\|u\|}{\|Q_q\|}\right)^{\frac{4s}{N}}\right) r_0^2.$$

Thus

$$g\left(\left\|(-\Delta)^{\frac{s}{2}} u(t)\right\|\right) \leq E[u_0] < g(r_0).$$

1. Subcase 1. Since $\|(-\Delta)^{\frac{s}{2}} u_0\| < r_0$, by a continuity argument we get $\sup_{\{t \geq 0\}} \|(-\Delta)^{\frac{s}{2}} u(t)\| < r_0$. So $T^* = \infty$.
2. Subcase 2. Now if $\|(-\Delta)^{\frac{s}{2}} u_0\| > r_0$, then, similarly, $\inf_{t \in [0,T^*)} \|(-\Delta)^{\frac{s}{2}} u(t)\| > r_0$. Thus the Gagliardo–Nirenberg inequality, Theorem 2.5, $2 = \mathcal{I}_q < \mathcal{I}_p$, and $\lambda_2 = 1$ give

$$\begin{aligned}
M'_{\zeta_R}[u] &\leq 2s\mathcal{I}_p E[u_0] - 2s(\mathcal{I}_p - 2)\left\|(-\Delta)^{\frac{s}{2}} u\right\|^2 + \frac{4s\lambda_2}{1+q}(\mathcal{I}_p - \mathcal{I}_q)\int_{\mathbb{R}^N} |u|^{1+q}\,dx \\
&\quad + \frac{C}{R^{2s}} + \frac{C}{R^{\frac{(p-1)(N-1)}{2} - s\epsilon_p - \rho}}\left\|(-\Delta)^{\frac{s}{2}} u\right\|^{\frac{p-1}{2s}+\epsilon} + \frac{C}{R^{\frac{(q-1)(N-1)}{2} - s\epsilon_q}}\left\|(-\Delta)^{\frac{s}{2}} u\right\|^{\frac{q-1}{2s}+\epsilon} \\
&\leq 2s\mathcal{I}_p E[u_0] - 2s(\mathcal{I}_p - 2)\left(1 - \left(\frac{\|u\|}{\|Q_q\|}\right)^{\frac{4s}{N}}\right)\left\|(-\Delta)^{\frac{s}{2}} u\right\|^2 \\
&\quad + \frac{C}{R^{2s}} + \frac{C}{R^{\frac{(p-1)(N-1)}{2} - s\epsilon_p - \rho}}\left\|(-\Delta)^{\frac{s}{2}} u\right\|^{\frac{p-1}{2s}+\epsilon_p} \\
&\quad + \frac{C}{R^{\frac{(q-1)(N-1)}{2} - s\epsilon_q}}\left\|(-\Delta)^{\frac{s}{2}} u\right\|^{\frac{q-1}{2s}+\epsilon_q}.
\end{aligned}$$

Since $\frac{\|u\|}{\|Q_q\|} < 1$, $\max\{p,q\} < 1 + 4s$, $E[u_0] < (1 - \frac{2}{\mathcal{I}_p})(1 - (\frac{\|u\|}{\|Q_q\|})^{\frac{4s}{N}})(1-\epsilon)r_0^2$ for some $\epsilon > 0$, and $C(R) \to 0$ as $R \to \infty$, we have

$$\begin{aligned}
M'_{\zeta_R}[u] &\leq 2s\mathcal{I}_p E[u_0] - 2s(\mathcal{I}_p - 2)\left(1 - \left(\frac{\|u\|}{\|Q_q\|}\right)^{\frac{4s}{N}}\right)\left\|(-\Delta)^{\frac{s}{2}} u\right\|^2 \\
&\quad + C(R)\left(1 + \left\|(-\Delta)^{\frac{s}{2}} u\right\|^2\right) \\
&\leq 2s(\mathcal{I}_p - 2)\left(1 - \left(\frac{\|u\|}{\|Q_q\|}\right)^{\frac{4s}{N}}\right)\left((1-\epsilon)r_0^2 - \left\|(-\Delta)^{\frac{s}{2}} u\right\|^2\right) \\
&\quad + C(R)\left(1 + \left\|(-\Delta)^{\frac{s}{2}} u\right\|^2\right) \\
&\leq -\epsilon s(\mathcal{I}_p - 2)\left(1 - \left(\frac{\|u\|}{\|Q_q\|}\right)^{\frac{4s}{N}}\right)\left\|(-\Delta)^{\frac{s}{2}} u\right\|^2,
\end{aligned}$$

where we used the inequality $\|(-\Delta)^{\frac{s}{2}} u\| > r_0 > 0$. This proof is achieved via Proposition 2.8.

9.2 Case 2

Compute

$$g(r) = r^2 - \frac{2}{\mathcal{J}_p}\left(\frac{\mathcal{J}_p}{\mathcal{I}_p}\right)^{\frac{\mathcal{I}_p}{2}} \|Q_p\|^{-(p-1)}\|u\|^{\mathcal{J}_p} r^{\mathcal{I}_p} - \frac{2}{\mathcal{J}_q}\left(\frac{\mathcal{J}_q}{\mathcal{I}_q}\right)^{\frac{\mathcal{I}_q}{2}} \|Q_q\|^{-(q-1)}\|u\|^{\mathcal{J}_q} r^{\mathcal{I}_q};$$

$$\begin{aligned} h(r) &:= \frac{g'(r)}{2r} \\ &= 1 - \left(\frac{\mathcal{J}_p}{\mathcal{I}_p}\right)^{\frac{\mathcal{I}_p}{2}-1} \|Q_p\|^{-(p-1)} \|u\|^{\mathcal{J}_p} r^{\mathcal{I}_p-2} - \left(\frac{\mathcal{J}_q}{\mathcal{I}_q}\right)^{\frac{\mathcal{I}_q}{2}-1} \|Q_q\|^{-(q-1)} \|u\|^{\mathcal{J}_q} r^{\mathcal{I}_q-2}. \end{aligned}$$

Thus h is nonincreasing on $\mathbb{R}_+$ and $h(0) = 1$. So it has a unique root $r_1 > 0$, that is,

$$h(r_1) = 0. \tag{9.2}$$

Thus $\sup_{r\geq 0} g(r) = g(r_1)$. So

$$\left(\frac{\mathcal{J}_p}{\mathcal{I}_p}\right)^{\frac{\mathcal{I}_p}{2}-1} \|Q_p\|^{-(p-1)} \|u\|^{\mathcal{J}_p} r_1^{\mathcal{I}_p-2} = 1 - \left(\frac{\mathcal{J}_q}{\mathcal{I}_q}\right)^{\frac{\mathcal{I}_q}{2}-1} \|Q_q\|^{-(q-1)} \|u\|^{\mathcal{J}_q} r_1^{\mathcal{I}_q-2}.$$

This gives

$$\begin{aligned} g(r_1) &= r_1^2\Bigg(1 - \frac{2}{\mathcal{I}_p}\left(\frac{\mathcal{J}_p}{\mathcal{I}_p}\right)^{\frac{\mathcal{I}_p}{2}-1} \|Q_p\|^{-(p-1)} \|u\|^{\mathcal{J}_p} r_1^{\mathcal{I}_p-2} \\ &\quad - \frac{2}{\mathcal{I}_q}\left(\frac{\mathcal{J}_q}{\mathcal{I}_q}\right)^{\frac{\mathcal{I}_q}{2}-1} \|Q_q\|^{-(q-1)} \|u\|^{\mathcal{J}_q} r_1^{\mathcal{I}_q-2}\Bigg) \\ &= r_1^2\Bigg(1 - \frac{2}{\mathcal{I}_p}\left(1 - \left(\frac{\mathcal{J}_q}{\mathcal{I}_q}\right)^{\frac{\mathcal{I}_q}{2}-1} \|Q_q\|^{-(q-1)} \|u\|^{\mathcal{J}_q} r_1^{\mathcal{I}_q-2}\right) \\ &\quad - \frac{2}{\mathcal{I}_q}\left(\frac{\mathcal{J}_q}{\mathcal{I}_q}\right)^{\frac{\mathcal{I}_q}{2}-1} \|Q_q\|^{-(q-1)} \|u\|^{\mathcal{J}_q} r_1^{\mathcal{I}_q-2}\Bigg) \\ &= r_1^2\left(1 - \frac{2}{\mathcal{I}_p}\right) + \left(\frac{2}{\mathcal{I}_p} - \frac{2}{\mathcal{I}_q}\right)\left(\frac{\mathcal{J}_q}{\mathcal{I}_q}\right)^{\frac{\mathcal{I}_q}{2}-1} \|Q_q\|^{-(q-1)} \|u\|^{\mathcal{J}_q} r_1^{\mathcal{I}_q}. \end{aligned}$$

By the assumptions $E[u_0] < (1 - \frac{2}{\mathcal{I}_p})r_1^2$ and $\mathcal{I}_p < \mathcal{I}_q$ we get

$$g\left(\left\|(-\Delta)^{\frac{s}{2}} u(t)\right\|\right) \leq E[u_0] < \left(1 - \frac{2}{\mathcal{I}_p}\right) r_1^2 < g(r_1).$$

1. Subcase 1. If $\|(-\Delta)^{\frac{s}{2}} u_0\| < r_1$, then, by the time continuity, $\sup_{t\geq 0} \|(-\Delta)^{\frac{s}{2}} u(t)\| < r_1$, and the solution is global.
2. Subcase 2. If $\|(-\Delta)^{\frac{s}{2}} u_0\| > r_1$, then $\inf_{\{t\in[0,T^*)\}} \|(-\Delta)^{\frac{s}{2}} u(t)\| > r_1$. Since $E[u_0] < (1 - \frac{2}{\mathcal{I}_p})(1-\epsilon) r_1^2$, $\epsilon > 0$, and $\mathcal{I}_p < \mathcal{I}_q$, by Theorem 2.5 it follows that for $C(R) \to 0$ as $R \to \infty$,

$$\begin{aligned} M'_{\zeta_R}[u] &\leq 2s\mathcal{I}_p E[u_0] - 2s(\mathcal{I}_p - 2)\left\|(-\Delta)^{\frac{s}{2}} u\right\|^2 + \frac{4s}{1+q}(\mathcal{I}_p - \mathcal{I}_q)\int_{\mathbb{R}^N} |u|^{1+q}\,dx \\ &\quad + \frac{C}{R^{2s}} + \frac{C}{R^{\frac{(p-1)(N-1)}{2} - s\epsilon_p - \rho}}\left\|(-\Delta)^{\frac{s}{2}} u\right\|^{\frac{p-1}{2s}+\epsilon} + \frac{C}{R^{\frac{(q-1)(N-1)}{2} - s\epsilon_q}}\left\|(-\Delta)^{\frac{s}{2}} u\right\|^{\frac{q-1}{2s}+\epsilon} \\ &< 2s(\mathcal{I}_p - 2)\left((1-\epsilon) r_1^2 - \left\|(-\Delta)^{\frac{s}{2}} u\right\|^2\right) + \left(1 + C(R)\right)\left\|(-\Delta)^{\frac{s}{2}} u\right\|^2 \\ &< -s(\mathcal{I}_p - 2)\epsilon\left\|(-\Delta)^{\frac{s}{2}} u\right\|^2, \end{aligned}$$

where we used $\|(-\Delta)^{\frac{s}{2}}u\|^2 > r_1 > 0$ via choosing $R \ll 1$. Proposition 2.8 closes the proof.

9.3 Case 3

It follows similarly to the case 2.

10 Global/nonglobal existence for $\lambda_1\lambda_2 = -1$

In this section, we prove Theorem 2.17.

10.1 Case 1

1. Subcase 1. Proposition 2.1 gives

$$\begin{aligned}
E[u_0] &= \left\|(-\Delta)^{\frac{s}{2}}u(t)\right\|^2 - \frac{2}{1+p}\int_{\mathbb{R}^N}\left|u(t)\right|^{1+p}|x|^\rho\,dx + \frac{2}{1+q}\int_{\mathbb{R}^N}\left|u(t)\right|^{1+q}dx \\
&\geq \left\|(-\Delta)^{\frac{s}{2}}u(t)\right\|^2 - \frac{2C_{N,p,\rho,s}}{1+p}\|u\|^{\mathcal{J}_p}\left\|(-\Delta)^{\frac{s}{2}}u(t)\right\|^{\mathcal{I}_p} \\
&\geq \left\|(-\Delta)^{\frac{s}{2}}u(t)\right\|^2 - \frac{2}{\mathcal{J}_p}\left(\frac{\mathcal{J}_p}{\mathcal{I}_p}\right)^{\frac{\mathcal{I}_p}{2}}\|Q_p\|^{-(p-1)}\|u\|^{\mathcal{J}_p}\left\|(-\Delta)^{\frac{s}{2}}u(t)\right\|^{\mathcal{I}_p} \\
&:= f_p\left(\left\|(-\Delta)^{\frac{s}{2}}u(t)\right\|\right).
\end{aligned}$$

Compute

$$f_p'(y) = 2y\left(1 - \left(\frac{\mathcal{J}_p}{\mathcal{I}_p}\right)^{\frac{\mathcal{I}_p}{2}-1}\|Q_p\|^{-(p-1)}\|u\|^{\mathcal{J}_p}y^{\mathcal{I}_p-2}\right).$$

So f' has the unique positive root such that $f_p(x_p) = (1-\frac{2}{\mathcal{I}_p})x_p^2$ and

$$x_p = \left(\left(\frac{\mathcal{J}_p}{\mathcal{I}_p}\right)^{\frac{\mathcal{I}_p}{2}-1}\|Q_p\|^{-(p-1)}\|u\|^{\mathcal{J}_p}\right)^{-\frac{1}{\mathcal{I}_p-2}}. \tag{10.1}$$

Then $f_p(\|(-\Delta)^{\frac{s}{2}}u(t)\|) \leq E[u_0] < f(x_p)$. So $\|(-\Delta)^{\frac{s}{2}}u_0\| < x_p$. Then, as previously, $\sup_{t\geq 0}\|(-\Delta)^{\frac{s}{2}}u(t)\| \leq x_p$, and the solution is global.

2. Subcase 2. Like before, $\inf_{t\geq 0}\|(-\Delta)^{\frac{s}{2}}u(t)\| > x_p$. Since $\lambda_2 = -1$, $\mathcal{I}_p > \max\{2, \mathcal{I}_q\}$, and $E[u_0] < (1-\frac{2}{\mathcal{I}_p})(1-\epsilon)x_p^2$, where $\epsilon > 0$, Theorem 2.5 gives for $R \ll 1$ and $C(R) \to 0$ as $R \to \infty$,

$$\begin{aligned}
M'_{\zeta_R}[u] &\leq 2s\mathcal{I}_pE[u_0] - 2s(\mathcal{I}_p-2)\left\|(-\Delta)^{\frac{s}{2}}u\right\|^2 + \frac{4s\lambda_2}{1+q}(\mathcal{I}_p-\mathcal{I}_q)\int_{\mathbb{R}^N}|u|^{1+q}dx \\
&\quad + \frac{C}{R^{2s}} + \frac{C}{R^{\frac{(p-1)(N-1)}{2}-s\epsilon_p-\rho}}\left\|(-\Delta)^{\frac{s}{2}}u\right\|^{\frac{p-1}{2s}+\epsilon} + \frac{C}{R^{\frac{(q-1)(N-1)}{2}-s\epsilon_q}}\left\|(-\Delta)^{\frac{s}{2}}u\right\|^{\frac{q-1}{2s}+\epsilon} \\
&\leq 2s(\mathcal{I}_p-2)\left((1-\epsilon)x_p^2 - \left\|(-\Delta)^{\frac{s}{2}}u\right\|^2\right) + \left(1 + C(R)\left\|(-\Delta)^{\frac{s}{2}}u\right\|^2\right) \\
&< -s\epsilon(\mathcal{I}_p-2)\left\|(-\Delta)^{\frac{s}{2}}u\right\|^2,
\end{aligned}$$

where one used $\|(-\Delta)^{\frac{s}{2}}u\|^2 > r_p > 0$. Proposition 2.8 finishes the proof.

10.2 Case 2

It follows as in case 1.

11 Scattering

In this section, we prove Theorem 2.19. In the rest of this section, $\lambda_1 = \lambda_2 = -1$. Let us start with a Morawetz identity.

11.1 Morawetz estimate

The next estimate is essential for proving the scattering.

Lemma 11.1 *Let $-2s < \rho < 1$, $1 + \frac{2\rho\chi_{\{\rho>0\}}}{N-2s} < p < p^c$, and $1 < q < q^c$, and let $u \in C(\mathbb{R}, H^s_{rd})$ be a global solution to (1.1). Then*

$$\int_{\mathbb{R}}\int_{\mathbb{R}^N} |u|^{1+p-\frac{2(\rho-1)}{N-2s}}\,dx\,dt + \int_{\mathbb{R}}\int_{\mathbb{R}^N} |u|^{1+q+\frac{2}{N-2s}}\,dx\,dt \lesssim E[u_0].$$

Proof Using Lemma 3.1, we have

$$\begin{aligned} M'_\zeta\big[u(t)\big] &= \int_0^\infty n^s \int_{\mathbb{R}^N} \big(4\overline{\partial_k u_n}\partial^2_{kl}\zeta\,\partial_l u_n - \Delta^2\zeta\,|u_n|^2\big)\,dx\,dn \\ &\quad - \frac{4\rho}{1+p}\int_{\mathbb{R}^N} x\cdot\nabla\zeta\,|u|^{1+p}|x|^{\rho-2}\,dx + 2\frac{p-1}{1+p}\int_{\mathbb{R}^N}\Delta\zeta\,|u|^{1+p}|x|^\rho\,dx \\ &\quad + 2\frac{q-1}{1+q}\int_{\mathbb{R}^N}\Delta\zeta\,|u|^{1+q}\,dx. \end{aligned}$$

We pick $\zeta := |\cdot|$ and compute $\nabla\zeta = \frac{\cdot}{|\cdot|}$, $\Delta\zeta = \frac{N-1}{|\cdot|}$, and

$$\Delta^2\zeta = \begin{cases} -4\pi\,\delta(x-y) & \text{if } N = 3, \\ -(N-1)(N-3)|x-y|^{-3} & \text{if } N \geq 4. \end{cases}$$

So

$$\begin{aligned} \frac{d}{dt}M_{|\cdot|}\big[u(t)\big] &\geq -\frac{4\rho}{1+p}\int_{\mathbb{R}^N}|u|^{1+p}|x|^{\rho-1}\,dx + 2\frac{(p-1)(N-1)}{1+p}\int_{\mathbb{R}^N}|u|^{1+p}|x|^{\rho-1}\,dx \\ &\quad + 2\frac{(q-1)(N-1)}{1+q}\int_{\mathbb{R}^N}\frac{|u|^{1+q}}{|x|}\,dx \\ &\geq 2\bigg(\frac{(p-1)(N-1)-2\rho}{1+p}\bigg)\int_{\mathbb{R}^N}|u|^{1+p}|x|^{\rho-1}\,dx \\ &\quad + 2\frac{(q-1)(N-1)}{1+q}\int_{\mathbb{R}^N}\frac{|u|^{1+q}}{|x|}\,dx. \end{aligned}$$

Now by Strauss inequality (2.3), if $\rho - 1 < 0$ and $(p-1)(N-1) - 2\rho > 0$, then

$$\begin{aligned} \frac{d}{dt}M_{|\cdot|}\big[u(t)\big] &\geq 2\bigg(\frac{(p-1)(N-1)-2\rho}{1+p}\bigg)\int_{\mathbb{R}^N}|u|^{1+p}|x|^{\rho-1}\,dx \\ &\quad + 2\frac{(q-1)(N-1)}{1+q}\int_{\mathbb{R}^N}\frac{|u|^{1+q}}{|x|}\,dx \end{aligned}$$

$$\gtrsim \int_{\mathbb{R}^N} |u|^{1+p-\frac{2(\rho-1)}{N-2s}}\,dx + \int_{\mathbb{R}^N} |u|^{1+q+\frac{2}{N-2s}}\,dx.$$

Lemma 2.3 in [6] gives $|M_{|\cdot|}[u]| \lesssim \|u\|^2_{\dot H^{\frac{1}{2}}} \lesssim \|u\|^2_{H^s}$. Thus

$$\begin{aligned}\|u(t)\|^2_{H^s} &\gtrsim M_{|\cdot|}\big[u(t)\big]\\ &\gtrsim \int_{\mathbb{R}}\int_{\mathbb{R}^N} |u|^{1+p-\frac{2(\rho-1)}{N-2s}}\,dx\,dt + \int_{\mathbb{R}}\int_{\mathbb{R}^N} |u|^{1+q+\frac{2}{N-2s}}\,dx\,dt.\end{aligned}$$ □

11.2 Decay of global solutions

Let us establish the extinction of global solutions.

Proposition 11.2 *Let* $\frac{N}{2N-1} \le s < 1$, $-2s < \rho < 1$, $1 + \frac{2\rho\chi_{\{\rho>0\}}}{N-2s} < p < p^c$, *and* $1 < q < q^c$. *Let* $u \in C(\mathbb{R}, H^s_{rd})$ *be a global solution to* (1.1). *Then*

$$\lim_{t\to\infty} \|u(t)\|_r = 0 \quad \text{for all } 2 < r < \frac{2N}{N-2s}.$$

The proof is based on the next result.

Lemma 11.3 *Let* $\chi \in C_0^\infty(\mathbb{R}^N)$. *Take a sequence of functions such that*

$$\sup_n \|\varphi_n\|_{H^s} < \infty, \qquad \varphi_n \rightharpoonup \varphi \quad \text{in } H^s.$$

Let u_n, u *be solutions to* (1.1) *such that* $u_n(0,\cdot) = \varphi_n$ *and* $u(0,\cdot) = \varphi$. *Then for any* $\varepsilon > 0$, *there are* $T_\varepsilon > 0$ *and* $n_\varepsilon \in \mathbb{N}$ *satisfying*

$$\|\chi(u_n - u)\|_{L^\infty_{T_\varepsilon}(L^2(\mathbb{R}^N))} < \varepsilon \quad \forall n > n_\varepsilon.$$

Proof of Lemma 11.3 Denote $v := \chi u$, $v_n := \chi u_n$, $w_n := v_n - v$, and $z_n := u_n - u$. Then

$$\begin{aligned} i\dot v - (-\Delta)^s v &= \chi\big(i\dot u - (-\Delta)^s u\big) + \chi(-\Delta)^s u - (-\Delta)^s v\\ &= \chi|x|^\rho |u|^{p-1}u + |u|^{q-1}u + \chi(-\Delta)^s u - (-\Delta)^s v.\end{aligned}$$

By the Strichartz estimate, for some admissible couples $(\alpha,\beta), (\alpha_1,\beta_1) \in \Gamma$, we get

$$\begin{aligned}\|w_n\|_{L^\infty_T(L^2)\cap L^\alpha_T(L^\beta)} &\lesssim \|\chi(\varphi_n - \varphi)\| + \|\chi(|u_n|^{q-1}u_n - |u|^{q-1}u)\|_{L^{\alpha'}_T(L^{\beta'})}\\ &\quad + \||x|^\rho\chi(|u_n|^{p-1}u_n - |u|^{p-1}u)\|_{L^{\alpha_1'}_T(L^{\beta_1'})}\\ &\quad + \|\chi(-\Delta)^s z_n - (-\Delta)^s w_n\|_{L^1_T(L^2)}.\end{aligned} \tag{11.1}$$

The Rellich theorem gives, for a subsequence,

$$\epsilon_n := \|(\varphi_n - \varphi)\chi\| \to 0 \quad \text{as } n \to \infty. \tag{11.2}$$

By Lemma 2.23 we write

$$\|\chi(-\Delta)^s z_n - (-\Delta)^s w_n\|_{L^1_T(L^2)} \lesssim \|z_n(-\Delta)^s\chi\|_{L^1_T(L^2)} + \|(-\Delta)^s\chi\|_{L^1_T(L^{\frac{N}{s}})}\|z_n\|_{L^\infty_T(L^{\frac{2N}{N-2s}})}$$

$$
\begin{aligned}
&\lesssim \|z_n\|_{L^1_T(L^2)} + T\|z_n\|_{L^\infty_T(\dot H^s)}\\
&\lesssim \Big(\|u\|_{L^\infty(\mathbb{R},H^s)} + \sup_n \|u_n\|_{L^\infty(\mathbb{R},H^s)}\Big)T\\
&:= NT. \qquad (11.3)
\end{aligned}
$$

Take $(\alpha,\beta) := (\frac{8s}{(q-1)(N-2s)}, \frac{4N}{2N-(q-1)(N-2s)})$. Since $\frac{\beta(q-1)}{\beta-2} = \frac{2N}{N-2s}$ and $q < q^c$, by Sobolev embeddings via the Hölder estimate we get

$$
\begin{aligned}
\big\|\chi\big(|u_n|^{q-1}u_n - |u|^{q-1}u\big)\big\|_{L^{\alpha'}_T(L^{\beta'})} &\lesssim \big(\|u_n\|^{q-1}_{L^\infty_T L^{\frac{\beta(q-1)}{\beta-2}}} + \|u\|^{q-1}_{L^\infty_T (L^{\frac{\beta(q-1)}{\beta-2}})}\big)\|w_n\|_{L^{\alpha'}_T(L^\beta)}\\
&\lesssim T^{\frac{\alpha-2}{\alpha}}\big(\|u_n\|^{q-1}_{L^\infty_T(H^s)} + \|u\|^{q-1}_{L^\infty_T(H^s)}\big)\|w_n\|_{L^\alpha_T(L^\beta)}\\
&\lesssim T^{\frac{\alpha-2}{\alpha}} N^{q-1}\|w_n\|_{L^\alpha_T(L^\beta)}. \qquad (11.4)
\end{aligned}
$$

Let us estimate the term $\||x|^\rho\chi(|u_n|^{p-1}u_n - |u|^{p-1}u)\|_{L^{\alpha_1'}_T(L^{\beta_1'})}$. If $\rho \geq 0$, then the estimation follows like for the homogeneous term. Assume that $-2s < \rho < 0$. Without loss of generality, assume also that $supp(\chi) \subset B(0,1)$. Take $\gamma := (\frac{N}{|\rho|})^- := \frac{N}{|\rho|} - \epsilon$ for some $\epsilon > 0$ close to zero and $\beta_1 := \frac{1+p}{1-\frac{1}{\gamma}}$. Since $p < p^c$, we can take $0 < \epsilon \ll 1$ such that $\beta_1 \in (2, \frac{2N}{N-2s})$ and $\alpha_1 > 2$. Thus by the Hölder and Sobolev inequalities,

$$
\begin{aligned}
\big\||x|^\rho\chi\big(|u_n|^{p-1}u_n - |u|^{p-1}u\big)\big\|_{L^{\alpha_1'}_T(L^{\beta_1'})} &\lesssim \big\||x|^\rho\big\|_\gamma \big\|\big(\|u_n\|^{p-1}_{\beta_1} + \|u\|^{p-1}_{\beta_1}\big)\|w_n\|_{\beta_1}\big\|_{L^{\alpha_1'}_T}\\
&\lesssim T^{\frac{\alpha_1-2}{\alpha_1}}\big(\|u_n\|^{q-1}_{L^\infty_T(H^s)} + \|u\|^{q-1}_{L^\infty_T(H^s)}\big)\|w_n\|_{L^{\alpha_1}_T(L^{\beta_1})}\\
&\lesssim T^{\frac{\alpha_1-2}{\alpha_1}} N^{q-1}\|w_n\|_{L^{\alpha_1}_T(L^{\beta_1})}. \qquad (11.5)
\end{aligned}
$$

As a consequence of (11.2)–(11.5), we get

$$
\begin{aligned}
\|w_n\|_{L^\infty_T(L^2)\cap L^\alpha_T(L^\beta)\cap L^{\alpha_1}_T(L^{\beta_1})} &\lesssim \epsilon_n + NT + \big(T^{\frac{\alpha-2}{\alpha}}N^{p-1} + T^{\frac{\alpha_1-2}{\alpha_1}}N^{q-1}\big)\|w_n\|_{L^\alpha_T(L^\beta)\cap L^{\alpha_1}_T(L^{\beta_1})}\\
&\lesssim \frac{\epsilon_n + NT}{1 - T^{\frac{\alpha-2}{\alpha}}N^{p-1} - T^{\frac{\alpha_1-2}{\alpha_1}}N^{q-1}}.
\end{aligned}
$$

The proof is achieved. □

Now we prove the decay of global solutions following the method of [31].

Proof of Proposition 11.2 We prove the decay in $L^{2+\frac{2s}{N}}$ and conclude with the conservation laws.

Suppose that there exist $t_n \to \infty$ and $\varepsilon > 0$ satisfying

$$
\big\|u(t_n)\big\|_{L^{2+\frac{2s}{N}}(\mathbb{R}^N)} > \varepsilon \quad \forall n \in \mathbb{N}.
$$

Thus by (2.4) there are $\varepsilon > 0$ and a sequence $x_n \in \mathbb{R}^N$ satisfying

$$
\big\|u(t_n)\big\|_{L^2(Q_1(x_n))} \geq \varepsilon \quad \forall n \in \mathbb{N}.
$$

Arguing as in [31], we get, for all $t \in [t_n, t_n + T]$ and $n \geq n_\varepsilon$,

$$\|u(t)\|_{L^2(Q_2(x_n))} \geq \frac{\varepsilon}{4}.$$

Since $\lim_{n\to\infty} t_n = \infty$, assume that $t_{n+1} - t_n > T$ for $n \geq n_\varepsilon$. Thus Proposition 11.1 gives

$$\begin{aligned} E[u_0] &\gtrsim \int_{\mathbb{R}} \int_{\mathbb{R}^N} |u(t,x)|^{1+p-\frac{2(\rho-1)}{N-2s}} \, dx \, dt + \int_{\mathbb{R}} \int_{\mathbb{R}^N} |u(t,x)|^{1+q+\frac{2}{N-2s}} \, dx \, dt \\ &\gtrsim \sum_n \int_{t_n}^{t_n+T} \int_{Q_2(x_n)} |u(t,x)|^{q+\frac{2}{N-2s}+1} \, dx \, dt \\ &\gtrsim \sum_n \varepsilon^{1+q+\frac{2}{N-2s}} T = \infty. \end{aligned}$$

This completes the proof of Proposition 11.2. □

11.3 Proof of Theorem 2.19

We define

$$S(I) := \bigcap_{(q,r)\in\Gamma} L^q(I, L^r) \quad \text{and} \quad \langle (-\Delta)^{\frac{s}{2}} \rangle \cdot := \cdot + (-\Delta)^{\frac{s}{2}} \cdot .$$

We start with some nonlinear estimates.

Lemma 11.4 *Let $\frac{N}{2N-1} \leq s < 1$, $-2s < \rho < 0$, or $N > 6s$ and $0 < \rho < \min\{s, \frac{N}{2} - 3s\}$, or $N > 8s$ and $s < \rho < \frac{N}{2} - 3s$, $p_c < p < p^c$, and $q_c < q < q^c$. Let $u \in C(\mathbb{R}, H^s_{rd})$ be a solution to (1.1). Then there are real numbers $2 < r_1, r_2, r_3 < \frac{2N}{N-2s}$, $0 < \theta_1 < q-1$, and $0 < \theta_2, \theta_3 < p-1$ such that*

$$\begin{aligned} &\|\langle (-\Delta)^{\frac{s}{2}} \rangle (u - e^{i.\Delta} u_0)\|_{S(0,T)} \\ &\quad \lesssim \|u\|^{q-1-\theta_1}_{L^\infty_T(L^{r_1})} \|\langle (-\Delta)^{\frac{s}{2}} \rangle u\|^{1+\theta_1}_{S(0,T)} + \|u\|^{p-1-\theta_2}_{L^\infty_T(L^{r_2})} \|\langle (-\Delta)^{\frac{s}{2}} \rangle u\|^{1+\theta 2}_{S(0,T)} \\ &\qquad + \|u\|^{p-1-\theta_3}_{L^\infty_T(L^{r_3})} \|\langle (-\Delta)^{\frac{s}{2}} \rangle u\|^{1+\theta_3}_{S(0,T)}. \end{aligned}$$

Proof of Lemma 11.4 Using the Strichartz estimate, we write

$$\|\langle (-\Delta)^{\frac{s}{2}} \rangle (u(t) - e^{-it(-\Delta)^s} u(t_0))\|_{S(I)} \lesssim \|\langle (-\Delta)^{\frac{s}{2}} \rangle |u|^{q-1} u\|_{S'(I)} + \|\langle (-\Delta)^{\frac{s}{2}} \rangle |x|^\rho |u|^{p-1} u\|_{S'(I)}.$$

Denote $I := (0, T)$ and take the admissible couple $(q_1, r_1) := (\frac{4s(1+q)}{N(q-1)}, 1 + q)$ and $\theta_1 := q_1 - 1 \in (1, q)$. Then, because $q_c < q < q^c$, we get

$$2 < r_1 < \frac{2N}{N-2s} \quad \text{and} \quad \theta_1 \in (1, q).$$

Thanks to the Hölder estimate, write

$$\begin{aligned} \|u^q\|_{L^{\frac{4s(1+q)}{4s(1+q)-N(q-1)}}(I, L^{\frac{1+q}{q}})} &= \big\| \|u\|^{q-\theta_1}_{L^{1+q}} \|u\|^{\theta_1}_{L^{1+q}} \big\|_{L^{\frac{4s(1+q)}{4s(1+q)-N(q-1)}}(I)} \\ &\lesssim \|u\|^{q-\theta_1}_{L^\infty(I, L^{1+q})} \big\| \|u\|^{\theta_1}_{L^{1+q}} \big\|_{L^{\frac{4s(1+q)}{4s(1+q)-N(q-1)}}(I)} \end{aligned}$$

$$\lesssim \|u\|^{q-\theta_1}_{L^\infty(I,L^{r_1})}\|u\|^{\theta_1}_{L^{q_1}(I,L^{r_1})}.$$

By Lemma 2.22 we get

$$\begin{aligned}
(\mathcal{I}) &:= \big\|(-\Delta)^s\big(|u|^{q-1}u\big)\big\|_{L^{\frac{4s(1+q)}{4s(1+q)-N(q-1)}}(I,L^{\frac{1+q}{q}})} \\
&\lesssim \Big\|\big\|(-\Delta)^s u\big\|_{L^{1+q}}\|u\|^{q-1}_{L^{1+q}}\Big\|_{L^{\frac{4s(1+q)}{4s(1+q)-N(q-1)}}(I)} \\
&\lesssim \big\|(-\Delta)^s u\big\|_{L^{\frac{4s(1+q)}{N(q-1)}}(I,L^{1+q})}\|u\|^{q-\theta_1}_{L^\infty(I,L^{1+q})}\|u\|^{\theta_1-1}_{L^{\frac{2s(\theta_1-1)(1+q)}{2s(1+q)-N(q-1)}}(I,L^{1+q})} \\
&\lesssim \big\|(-\Delta)^s u\big\|_{L^{\frac{4s(1+q)}{N(q-1)}}(I,L^{1+q})}\|u\|^{q-\theta_1}_{L^\infty(I,L^{1+q})}\|u\|^{\theta_1-1}_{L^{\frac{4s(1+q)}{N(q-1)}}(I,L^{1+q})} \\
&\lesssim \|u\|^{\theta_1}_{L^{q_1}(I,W^{s,r_1})}\|u\|^{q-\theta_1}_{L^\infty(I,L^{r_1})}.
\end{aligned}$$

To estimate the inhomogeneous term, we discuss three cases.

1. Case 1: $-2s < \rho < 0$.

   Take $\gamma := (\frac{N}{|\rho|})^-$, $r_2 := \frac{1+p}{1-\frac{1}{\gamma}}$, the admissible couple (q_2, r_2), and $\theta_2 := q_2 - 1$. Since $p_c < p < p^c$, we get $2 < r_2 < \frac{2N}{N-2s}$ and $2 < q_2 = \frac{4s(1+p)}{N(p-1)+2|\rho|^+} < 1 + p$. Thus $\theta_2 \in (1, p)$, and by the Hölder estimate,

$$\begin{aligned}
\big\||\cdot|^\rho u^p\big\|_{L^{q_2'}(I,L^{r_2'})} &\le \Big\|\big\||\cdot|^\rho\big\|_\gamma \|u\|^{p-\theta_2}_{r_2}\|u\|^{\theta_2}_{r_2}\Big\|_{L^{q_2'}(I)} \\
&\lesssim \Big\|\|u\|^{p-\theta_2}_{r_2}\|u\|^{\theta_2}_{r_2}\Big\|_{L^{q_2'}(I)} \\
&\lesssim \|u\|^{p-\theta_2}_{L^\infty(I,r_2)}\|u\|^{\theta_2}_{L^{q_2}(I,L^{r_2})}.
\end{aligned}$$

   Now, using the identity on $\mathbb{R}^N$, we have

$$\big|(-\Delta)^{\frac{s}{2}}\big(|\cdot|^\rho|u|^{p-1}u\big)\big| \lesssim |\cdot|^{\rho-s}|u|^p + |\cdot|^\rho|u|^{p-1}\big|(-\Delta)^{\frac{s}{2}}u\big|.$$

   It is sufficient to estimate, for $(q_3, r_3) \in \Gamma$, the term $\||\cdot|^{\rho-s}u^p\|_{L^{q_3'}(I,L^{r_3'})}$. Take $\mu := (\frac{N}{|\rho-s|})^-$, $r_3 := \frac{1+p}{1-\frac{1}{\mu}+\frac{s}{N}}$, the admissible couple $(q_3, r_3) \in \Gamma$, and $\theta_3 := q_3 - 2$. Because $p_c < p < p^c$, we have

$$2 < r_3 < \frac{2N}{N-2s} \quad \text{and} \quad \theta_3 := q_3 - 2 \in (0, p-1).$$

   Using the Hölder and Sobolev estimates, write

$$\begin{aligned}
\big\||\cdot|^{\rho-s}u^p\big\|_{L^{q_3'}(I,L^{r_3'})} &\le \Big\|\big\||\cdot|^{\rho-s}\big\|_\mu \|u\|^{p-1-\theta_3}_{r_3}\|u\|^{\theta_3}_{r_3}\|u\|_{\frac{Nr_3}{N-sr_3}}\Big\|_{L^{q_3'}(I)} \\
&\lesssim \Big\|\|u\|^{p-1-\theta_3}_{r_3}\|u\|^{\theta_3}_{r_3}\|u\|_{\frac{Nr_3}{N-sr_3}}\Big\|_{L^{q_3'}(I)} \\
&\lesssim \|u\|^{p-1-\theta_3}_{L^\infty(I,L^{r_3})}\|u\|_{L^{q_3}(I,\dot W^{s,r_3})}\|u\|^{\theta_3}_{L^{q_3}(I,L^{r_3})} \\
&\lesssim \|u\|^{p-1-\theta_3}_{L^\infty(I,L^{r_3})}\|u\|^{1+\theta_3}_{L^{q_3}(I,W^{s,r_3})}.
\end{aligned}$$

   The proof of the first case is achieved by regrouping the previous computation.

2. Case 2: $0 < \rho < \min\{s, \frac{N}{2} - 3s\}$.

   Take $(q,r) \in \Gamma$, $\theta := q - 1$, and $a > 1$ satisfying $\frac{1}{r'} = \frac{p-1}{r} + \frac{1}{a}$. This necessarily gives $r > p$ and $a = \frac{r}{r-p}$. Using the Gagliardo-Nirenberg inequality in Proposition 2.1, we get

$$\begin{aligned}\big\| |\cdot|^{\rho} u^p \big\|_{L^{q'}(I,L^{r'})} &\le \big\| \big\| |\cdot|^{\rho} u \big\|_a \|u\|_r^{p-1-\theta} \|u\|_r^{\theta} \big\|_{L^{q'}(I)} \\ &\lesssim \|u\|_{L^\infty(I,H^s)} \big\| \|u\|_r^{p-1-\theta} \|u\|_r^{\theta} \big\|_{L^{q'}(I)} \\ &\lesssim \|u\|_{L^\infty(I,H^s)} \|u\|_{L^\infty(L^r)}^{p-1-\theta} \|u\|_{L^q(I,L^r)}^{\theta}.\end{aligned}$$

   Here we need the assumptions

$$\begin{aligned}&2 + \frac{2a\rho}{N-2s} < a < 2 + \frac{2(2s+a\rho)}{N-2s} \quad \Leftrightarrow \quad \frac{2(N-2s)}{N-2s-2\rho} < a < \frac{2N}{N-2s-2\rho}; \\ &2 < r < \frac{2N}{N-2s} \quad \Leftrightarrow \quad \frac{2N}{N+2s-2\rho} < a < \frac{2(N-2s)}{N-6s-2\rho}; \\ &1 < \theta < p-1 \quad \Leftrightarrow \quad \frac{2N}{N+2s-2\rho} < a < \frac{2N}{N+4s-2N\frac{\rho+2s}{N-2s}}.\end{aligned}$$

   A direct computation gives no contradiction in the previous inequalities. The second one gives $N > 6s$ and $\rho < \frac{N}{2} - 3s$. The third one gives

$$(N+4s)(N-2s) > 2N(\rho+2s) \quad \text{and} \quad N + 2s - 2\rho > N + 4s - 2N\frac{\rho+2s}{N-2s}.$$

   This is equivalent to

$$(N+4s)(N-2s) > 2N(\rho+2s) > 2(N-2s)(\rho+s).$$

   So $N^2 - 2(\rho+s)N - 8s^2 > 0$, which is satisfied if $N > 4s$ and $\rho < \frac{N}{2} - s - \frac{4s^2}{N}$. This is weaker than the condition $0 < \rho < \frac{N}{2} - 3s$. The quantity $\| |\cdot|^{\rho} (-\Delta)^{\frac{s}{2}} (|u|^{p-1}u)\|_{L^{q'}(I,L^{r'})}$ can be controlled similarly. Moreover, since $\rho < s$, by the previous calculations,

$$\big\| |\cdot|^{\rho-s} u^p \big\|_{L^{q_3'}(I,L^{r_3'})} \lesssim \|u\|_{L^\infty(I,L^{r_3})}^{p-1-\theta_3} \|u\|_{L^{q_3}(I,W^{s,r_3})}^{1+\theta_3}.$$

3. Third case: $s < \rho < \frac{N}{2} - 3s$.

   The term $\| |\cdot|^{\rho} u^p \|_{L^{q'}(I,L^{r'})}$ can be controlled as in the second case. To estimate $\| |\cdot|^{\rho-s} u^p \|$, we have the three previous assumptions for $\rho - s$ rather than for ρ. Thus the necessary condition is $0 < \rho - s < \frac{N}{2} - 3s$ and $0 < \rho < \frac{N}{2} - 3s$. Thus $s < \rho < \frac{N}{2} - 3s$. □

Now we prove the scattering.

Proof of Theorem 2.19 Using Lemma 11.4, Proposition 11.2, and Lemma 2.26, it follows that $\langle (-\Delta)^{\frac{s}{2}} \rangle u \in S(\mathbb{R})$. This implies that

$$\begin{aligned}&\big\langle (-\Delta)^{\frac{s}{2}} \big\rangle \big(u - e^{i.\Delta} u_0\big) \in S'(\mathbb{R}); \\ &\big\langle (-\Delta)^{\frac{s}{2}} \big\rangle \big(|\cdot|^{\rho} |u|^{p-1} u + |u|^{q-1} u \big) \in S'(\mathbb{R}).\end{aligned}$$

Then, as $t,t'\to\infty$, we have

$$\left\|e^{it(-\Delta)^s}u(t)-e^{it'(-\Delta)^s}u(t')\right\|_{H^s}\lesssim\left\|\langle(-\Delta)^{\frac{s}{2}}\rangle\left(|\cdot|^\rho|u|^{p-1}u+|u|^{q-1}u\right)\right\|_{S'(t,t')}\to 0.$$

Taking $u_\pm:=\lim_{t\to\pm\infty}e^{it(-\Delta)^s}u(t)$ in H^s, we get

$$\lim_{t\to+\infty}\left\|u(t)-e^{-it(-\Delta)^s}u_+\right\|_{H^s}=0.$$

The scattering is proved. □

Appendix

In this section, we prove Lemma 2.4. Let $\epsilon>0$ and $1+\frac{2\rho\chi_{\{\rho>0\}}}{N-2s}<p<p^c$. Take a bounded sequence (u_n) in H^s. Without loss of generality, we assume that $u_n\rightharpoonup 0$ in H^s. The purpose is to prove that $\int_{\mathbb{R}^N}|u_n|^{1+p}|x|^\rho\,dx\to 0$.

A. Case 1: $-2s<\rho<0$. Take $R>\epsilon^\rho$. By the Hölder inequality, if (q,q') satisfies $q'|\rho|<N$, then we get

$$\begin{aligned}\int_{|x|\le R}|u_n|^{1+p}|x|^\rho\,dx&\le\|u_n\|^{1+p}_{L^{(1+p)q}(|x|<R)}\left\||x|^\rho\right\|_{L^{q'}(|x|\le R)}\\&\le C\|u_n\|^{1+p}_{L^{(1+p)q}(|x|<R)}\int_0^R\frac{dt}{t^{-q'\rho-N+1}}\\&\le C\|u_n\|^{1+p}_{L^{(1+p)q}(|x|<R)}R^{N+q'\rho}.\end{aligned}$$

Now, since $1<p<p^c$, we take $1<q<\frac{N}{\rho+N}$ and get $2<(1+p)q<\frac{2N}{N-2s}$. Thus by compact Sobolev embeddings it follows that

$$\int_{|x|\le R}|u_n|^{1+p}|x|^\rho\,dx\le C\|u_n\|^{1+p}_{L^{(1+p)q}(|x|<R)}R^{N+q'\rho}\to 0.$$

Furthermore, by Sobolev embeddings,

$$\int_{|x|\ge R}|u_n|^{1+p}|x|^\rho\,dx\le R^\rho\|u_n\|^{1+p}_{1+p}\le C\epsilon.$$

The proof is ended.

B. Case 2: $\rho\ge 0$. Using Compact Sobolev embeddings, write

$$\int_{|x|\le\epsilon}|u_n|^{1+p}|x|^\rho\,dx\le C\|u_n\|^{1+p}_{1+p}\to 0.$$

Furthermore, by the Rellich theorem and Strauss inequality we have

$$\int_{\epsilon\le|x|\le\frac{1}{\epsilon}}|u_n|^{1+p}|x|^\rho\,dx\le C\|u_n\|^{p-1}_{L^\infty(\epsilon\le|x|\le\frac{1}{\epsilon})}\int_{\epsilon\le|x|\le\frac{1}{\epsilon}}|u_n|^2\,dx\to 0.$$

Now, by the Strauss inequality,

$$\int_{|x|\ge\frac{1}{\epsilon}}|u_n|^{1+p}|x|^\rho\,dx=\int_{|x|\ge\frac{1}{\epsilon}}|x|^{\rho-(p-1)\frac{N-2s}{2}}\left(|x|^{\frac{N-2s}{2}}|u_n|\right)^{p-1}|u_n|^2\,dx$$

$$\leq C\int_{|x|\geq\frac{1}{\epsilon}}|x|^{\rho-(p-1)\frac{N-2s}{2}}|u_n|^2\,dx$$
$$\leq C\epsilon^{(p-1)\frac{N-2s}{2}-\rho}.$$

Since $\rho-(p-1)\frac{N-2s}{2}<0$, the proof is ended.

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Author details
[1]Departement of Mathematics, College of Science and Arts in Uglat Asugour, Qassim University, Buraydah, Kingdom of Saudi Arabia. [2]Department of Mathematics, College of Sciences and Arts, Qassim University, ArRass, Saudi Arabia. [3]School of Mathematics and Statistics, Tianshui Normal University, Tianshui 741000, China.

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