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# Continuum limit of 2D fractional nonlinear Schrödinger equation

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*Abstract.* We prove that the solutions to the discrete nonlinear Schrödinger equation with non-local algebraically decaying coupling converge strongly in $L^2(\mathbb{R}^2)$ to those of the continuum fractional nonlinear Schrödinger equation, as the discretization parameter tends to zero. The proof relies on sharp dispersive estimates that yield the Strichartz estimates that are uniform in the discretization parameter. An explicit computation of the leading term of the oscillatory integral asymptotics is used to show that the best constants of a family of dispersive estimates blow up as the non-locality parameter $\alpha \in (1, 2)$ approaches the boundaries.

## 1. Introduction

The mathematical description of physical phenomena, in many instances, results in the formulation of partial differential equations (PDEs) describing state variables in continuum media. Despite the fact that it is highly unlikely to find exact solutions of many linear or nonlinear PDEs, advances in numerical analysis and scientific computing open the door to find approximate solutions to complex problems. In particular, numerical approximations based on finite difference schemes are constructed by discretizing spatial variables, leading to a system of coupled ordinary differential equations. In this line of research, the objective is then to determine how well the approximate solution evaluated in the grid approximates the solutions of the corresponding PDE.

On the other hand, there are well-known universal models that are *inherently discrete*. Generically referred to as coupled oscillator systems, they describe phenomena such as localization or synchronization, characteristic of its discrete nature. Best-known examples are the Fermi–Pasta–Ulam–Tsingou model, the discrete nonlinear Schrödinger equation and the Kuramoto model. The first two describe dynamics in a lattice with nearest neighbor interactions, whereas the Kuramoto model addresses synchronization for globally coupled oscillators. These and similar models continue to be studied given their applicability in photonics, lasers, and networks such as the power grid to name some. For such models, a suitable approximation named the long-wave

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approximation assumes a "smooth" variation of the state variable among neighbor lattices. Specifically in a one-dimensional lattice, this means $u_{n\pm1} \approx u_n$. In this regime, it is reasonable to consider continuum approximation. For a 1-d lattice model, the continuum approximation $u_{n\pm1} \to U(x \pm h)$, where $h > 0$ is small, with nearest neighbor coupling $C(u_{n+1} + u_{n-1})$ leads to a term proportional to $\frac{\partial^2 U}{\partial x^2}$ and in return, the system of ODEs is then approximated by a PDE.

Recently, there has been an increased interest in the models based on FNLSE. While most of the research deals with continuum models, including numerical computations of solutions in the nonlinear regime, less is known about discrete systems showing global coupling with algebraic decay on the coupling strength with respect to the distance between nodes in the lattice. This work considers such a case in a two-dimensional lattice and centers on the question of the validity of a suitable continuum approximation. This is not always a trivial task as, for instance, invariances and symmetries may arise or be lost. In contrast to the (continuum) nonlinear Schrödinger equation that admits the Galilean boost from which traveling wave solutions emerge, many lattice systems lack translational invariance. It is known that highly localized solutions in a lattice system do not propagate due to the presence of the Peirels–Nabarro potential [10,26]; for a recent work on FNLSE in this context, see [22]. All this is to point out the challenges and open problems that need to be studied by a combination of analytical and numerical tools. In this contribution, we report what we think are first analytic results on the underlying fundamental question of determining the continuum approximation on the FNLSE in more than one dimension.

## 2. Statement of the problem

This work concerns the continuum limit of the discrete fractional nonlinear Schrödinger equation (FNLSE)

$$i\dot{u}_h = (-\Delta_h)^{\frac{\alpha}{2}} u_h + \mu|u_h|^{p-1}u_h,\ u_h(x,0) = u_{0,h}(x), \tag{2.1}$$

to the continuum FNLS

$$i\partial_t u = (-\Delta)^{\frac{\alpha}{2}} u + \mu|u|^{p-1}u,\ u(x,0) = u_0(x), \tag{2.2}$$

as $h \to 0+$ where $\alpha \in (0,2]\backslash\{1\},\ p > 1,\ \mu = \pm 1$, and $u : \mathbb{R}^{2+1} \to \mathbb{C},\ u_h : h\mathbb{Z}^2 \times \mathbb{R} \to \mathbb{C}$. Let (2.1), (2.2) be well-posed in some Banach spaces $X, X_h$, respectively, where $0 < h \le 1$ denotes a discretization parameter. Suppose $u_{0,h} \in X_h$ is the discretized $u_0 \in X$. Given an interpolation operator $p_h : X_h \to X$ and $T > 0$ such that $u(t), u_h(t)$ denote the well-posed solutions on $[0,T]$, the main problem then reduces to identifying values of $\alpha, p$ that allows

$$\lim_{h\to 0} \sup_{t\in[0,T]} \|p_h u_h(t) - u(t)\|_X = 0.$$

The study of evolution equations on $\mathbb{R}$ with a general class of interaction kernel was done in [25] where the continuum limit was proved in the weak sense. By applying the analytic tools in [15] that yield dispersive estimates for the discrete Schrödinger evolution that are uniform in $h$, [16] extended the aforementioned weak convergence to strong convergence in the $L^2$-setting (with convergence rates) for $\alpha = 2$ in $\mathbb{R}^d$, $d = 1, 2, 3$ and $\alpha \in (0, 2) \setminus \{1\}$ on $\mathbb{R}$. The central perspective in [16], upon which we develop, that sharp dispersive estimates that are uniform in $h$ control the difference $p_h u_h - u$, at least in the scaling-subcritical regime, proved to be fruitful as can be illustrated in various works such as [11] that studied the case $\alpha = 2$ on $\mathbb{T}^2$ as the spatial domain, [12] that studied the large box limit for $\alpha = 2$ in $\mathbb{R}^d$, $d = 2, 3$, and [13] that showed the rigorous derivation of the KdV equation from the FPU system. Using a similar idea, the continuum limit of the space-time FNLS was investigated in [7]. Furthermore, see the works of Ignat and Zuazua [17–20,31] where novel approaches such as the Fourier filtering and the two-grid algorithm were used.

In practice, obtaining appropriate dispersive estimates reduces to oscillatory integral estimations, which is of central concern in our approach. Unlike the continuum case, the dispersion relation for the discrete evolution has degenerate critical points, which results in weaker dispersion than the continuum Schrödinger evolution. This in return admits weaker Strichartz estimates, which limits the class of nonlinearities that leads to the well-posedness of the corresponding nonlinear equation via the contraction mapping argument. To be more quantitative, let $U(t) = e^{-it(-\Delta)^{\frac{\alpha}{2}}}$, $U_h(t) = e^{-it(-\Delta_h)^{\frac{\alpha}{2}}}$ and $\|f\|_{L^p_h} := h^{\frac{d}{p}} (\sum_{x \in h\mathbb{Z}^2} |f(x)|^p)^{\frac{1}{p}}$ for $p < \infty$ with $\|f\|_{L^\infty_h} = \|f\|_{L^\infty(h\mathbb{Z}^2)}$; see Sect. 3 for notations. For $\alpha = 2$, [28, Theorem 1] established[1]

$$\|U_h(t)\|_{L^1_h \to L^\infty_h} \simeq_h |t|^{-\frac{d}{3}},$$

where the implicit constant blows up as $h \to 0+$, which contrasts with $\|U(t)\|_{L^1(\mathbb{R}^d) \to L^\infty(\mathbb{R}^d)} \simeq |t|^{-\frac{d}{2}}$. Our objective is to obtain Strichartz estimates for the discrete evolution that are uniform in $h$.

For $\alpha < 2$, [16, Proposition 3.1] obtained

$$\|U_h(t) P_N f\|_{L^\infty_h} \lesssim_\alpha \left(\frac{N}{h}\right)^{1-\frac{\alpha}{3}} |t|^{-\frac{1}{3}} \|f\|_{L^1_h}, \ \alpha \in (1, 2) \tag{2.3}$$

for all $N \in 2^{\mathbb{Z}}$ with $N \leq 1$ on $h\mathbb{Z}$ where the $P_N$ denotes the Littlewood–Paley operator. Our goal is to obtain a two-dimensional analog of (2.3). The proof in [16] cannot be directly generalized, since the set of degenerate critical points on $h\mathbb{Z}$ consists of isolated points whose corresponding oscillatory integrals cannot be estimated directly by the Van der Corput lemma. In higher dimensions, the set of degeneracy is geometrically more complicated. In fact, our analysis shows that the degenerate critical points define

[1]Denote $A \lesssim B$ when there exists a constant of non-interest $C > 0$ such that $A \leq CB$ and define $A \simeq B$ if $A \lesssim B$ and $B \lesssim A$.

a one-dimensional embedded smooth submanifold in the torus $[-\pi, \pi]^2$ where each singular point admits a unique direction along which the third derivative does not vanish (fold) except at four points (cusp) at which the fourth derivative does not vanish. This observation that a singular point is at worst a cusp is consistent with [1]. It is expected that more severe singularities exist in higher dimensions as the structure of the Hessian of the dispersion relation becomes more complicated. This dimension-dependent geometric complication is purely a remnant of non-locality since the linear evolution of classical Schrödinger operator on $h\mathbb{Z}^d$ splits as the $d$-fold tensor product on each dimension.

Consider the dispersion relation

$$w_{h,m}(\xi) = \left(m^2 + \frac{4}{h^2}\sum_{i=1}^{2} \sin^2\left(\frac{h\xi_i}{2}\right)\right)^{\frac{\alpha}{2}},$$

and the quantity of interest

$$\int_{\frac{\mathbb{T}^2}{h}} e^{i(x\cdot\xi - tw_{h,m}(\xi))}\eta(\xi)\mathrm{d}\xi,$$

where $\mathbb{T} = \frac{\mathbb{R}}{2\pi\mathbb{Z}} = [-\pi, \pi]$ and $\eta \in C_c^\infty(\frac{\mathbb{T}^2}{h})$; the dispersion relation of (2.2) is $w_{h,0}$. [27] showed that when $m = 0,\ \alpha = 1$, which corresponds to the dispersion relation of the discrete wave equation, then the quantity of interest decays as $O(t^{-\frac{2}{3}})$ in $d = 2$ and $O(t^{-\frac{7}{6}})$ in $d = 3$. When $m > 0,\ \alpha = 1$, which corresponds to the discrete Klein–Gordon equation, [1] showed that the quantity of interest decays as $O(t^{-\frac{3}{4}})$ in $d = 2$, and the result was extended to higher dimensions ($d = 3, 4$) in [5]. When $m = 0,\ \alpha = 2$, the time decay of the fundamental solution of the classical discrete Schrödinger equation was shown to be $O(t^{-\frac{d}{3}})$ in [28].

Our objective is to obtain the sharp time decay of the quantity of interest for $m = 0,\ \alpha \in (1, 2)$ in $d = 2$. In particular, it is shown that the oscillatory integral decays as $O(t^{-\frac{3}{4}})$. The main tool that we adopt is the analysis of Newton polyhedron generated by the Taylor expansion of the phase function $x\cdot\xi - tw_{h,0}(\xi)$ in an adapted coordinate system, a method pioneered in [30]. Furthermore, the asymptotics in both regimes $\alpha \to 1+$ (wave limit) and $\alpha \to 2-$ (Schrödinger limit) are studied. To our knowledge, the dependence on the non-local parameter has not been clearly investigated in previous works. To obtain the asymptotics of the leading term of $O(t^{-\frac{3}{4}})$ as a function of $\alpha$, we represent the phase function in a superadapted coordinate system to apply results of [8].

The relation of our work to the theory of stability of degenerate oscillatory integrals is subtle. A cursory observation might suggest that a degenerate integral (our quantity of interest) would be stable under a small perturbation in the non-local parameter. However, the phase fails to be smooth for $\alpha < 2$ and therefore becomes large in appropriate norm(s) as the support of $\eta$ becomes arbitrarily close to the origin. In our approach, it suffices to invoke the stability result [21] under linear perturbations in phase. For more general stability results under analytic or smooth perturbations, see [9,23]. For the support of $\eta$ close to the origin, $\sin z \sim z$ by the small angle approximation, after which one might wish to invoke [3] that obtained sharp dispersive estimates for radial dispersion relations. However, such approximation is not a linear perturbation and hence we handle that case by direct computation.

The paper is organized as follows. Notations and main results are presented in Sect. 3. Assuming the results hold, the desired continuum limit is shown in Sect. 4. The proof of our main proposition is in Sect. 5, followed by a concluding remark in Sect. 6.

## 3. Main results

To discuss continuum limit, the parameters that yield the well-posedness of (2.1), (2.2) must be identified. For the discrete equation, the linear operator

$$\Delta_h f(x) = \sum_{i=1}^{d} \frac{f(x+he_i)+f(x-he_i)-2f(x)}{h^2},\ x \in h\mathbb{Z}^d, \tag{3.1}$$

defines a bounded, nonnegative, self-adjoint operator on $L^2_h$, and so are its fractional powers given by functional calculus. Equivalently $(-\Delta_h)^{\frac{\alpha}{2}}$ is given by the Fourier multiplier

$$(-\Delta_h)^{\frac{\alpha}{2}} = \mathcal{F}_h^{-1} \left\{ \sum_{i=1}^{d} \frac{4}{h^2} \sin^2\left(\frac{h\xi_i}{2}\right) \right\}^{\frac{\alpha}{2}} \mathcal{F}_h, \tag{3.2}$$

where the discrete Fourier transform is defined as

$$\widehat{f}(\xi) = \mathcal{F}_h f(\xi) = h^d \sum_{x \in h\mathbb{Z}^d} f(x) e^{-ix\cdot\xi},\ f(x) = (2\pi)^{-d} \int_{\frac{\mathbb{T}^d}{h}} \widehat{f}(\xi) e^{ix\cdot\xi} \mathrm{d}\xi,$$

for $\xi \in \frac{\mathbb{T}^d}{h}$. Recall the Sobolev space on $h\mathbb{Z}^d$ for $s \in \mathbb{R},\ p \in (1,\infty)$ given by

$$\|f\|_{W_h^{s,p}} = \|\langle\nabla_h\rangle^s f\|_{L_h^p},\ \|f\|_{\dot{W}_h^{s,p}} = \||\nabla_h|^s f\|_{L_h^p},$$

where

$$\langle\nabla_h\rangle^s = \mathcal{F}_h^{-1}\langle\xi\rangle^s \mathcal{F}_h,\ |\nabla_h|^s = \mathcal{F}_h^{-1}|\xi|^s \mathcal{F}_h,$$

and $\langle\xi\rangle = (1+|\xi|^2)^{\frac{1}{2}}$ for $\xi \in \frac{\mathbb{T}^d}{h}$. The nonlinearity $u_h \mapsto |u_h|^{p-1}u_h$ is locally Lipschitz continuous due to $L^2_h \hookrightarrow L^\infty_h$, which yields an immediate well-posedness of (2.1) in $L^2_h$ via the contraction mapping argument. For the continuum case, consider the family of self-similar solutions

$$u(x,t) \to u_\lambda(x,t) := \lambda^{-\frac{\alpha}{p-1}} u\left(\frac{x}{\lambda}, \frac{t}{\lambda^\alpha}\right),$$

and observing that $\{u_\lambda(\cdot,t)\}_{\lambda>0}$ leaves $\dot{H}^{s_c}(\mathbb{R}^d)$ invariant for all $t$, one obtains the Sobolev-critical regularity

$$s_c = \frac{d}{2} - \frac{\alpha}{p-1}.$$

Our analysis is in the scaling-subcritical regime where the time of existence depends on the Sobolev norm of data. Moreover, suppose the power of nonlinearity is at least cubic.

**Lemma 3.1.** *FNLSE (2.2) is locally well-posed in $H^s(\mathbb{R}^2)$ for $s > s_c$ and $p \geq 3$ in the subcritical sense. For any $\alpha > 0$, $p > 1$, $d \in \mathbb{N}$, DNLSE (2.1) is globally well-posed in $L^2_h$. Moreover, they admit conserved mass and energy functionals given by*

$$M[u(t)] = \|u(t)\|^2_{L^2(\mathbb{R}^2)},\ E[u(t)] = \frac{1}{2}\int_{\mathbb{R}^2} ||\nabla|^{\frac{\alpha}{2}} u|^2 \mathrm{d}x + \frac{\mu}{p+1}\int_{\mathbb{R}^2} |u|^{p+1}\mathrm{d}x,$$

$$M_h[u_h(t)] = \|u_h(t)\|^2_{L^2_h},\ E_h[u_h(t)] = \frac{1}{2}\|(-\Delta_h)^{\frac{\alpha}{4}} u_h(t)\|^2_{L^2_h} + \frac{\mu}{p+1}\|u_h(t)\|^{p+1}_{L^{p+1}_h}.$$

*Proof.* See [14, Theorem 1.1] and [25, Proposition 4.1] for the first and second statement, respectively. □

More specifically, our setup is in the mass supercritical and energy subcritical regime, or equivalently,

$$\frac{2(p-1)}{p+1} < \alpha < 2, \tag{3.3}$$

in which every $u_0 \in H^{\frac{\alpha}{2}}(\mathbb{R}^2)$ has a local solution but not necessarily global; for blow-up criteria in the focusing mass supercritical case via localized virial estimates, see [2,6].

We specify the discretization described in the introduction. For $h > 0$, define $d_h : L^2(\mathbb{R}^d) \to L^2_h$ by

$$d_h f(x) = h^{-d}\int_{x+[0,h)^d} f(x')\mathrm{d}x'.$$

Conversely define $p_h : L^2_h \to L^2(\mathbb{R}^d)$ by

$$p_h f(x) = f(x') + D_h^+ f(x')\cdot(x-x'),\ x \in x' + [0,h)^d,\ x' \in h\mathbb{Z}^d$$

$$(D_h^+ f)_i(x') = \frac{f(x'+he_i) - f(x')}{h},\ i = 1,\ldots,d,$$

where $\{e_i\}_{i=1}^d$ generates $\mathbb{Z}^d$. The discretization converges to the continuum solution.

**Theorem 3.1.** *Let $p \geq 3$ and $\max(\frac{8}{7}, \frac{2(p-1)}{p+1}) < \alpha < 2$. For any arbitrary $u_0 \in H^{\frac{\alpha}{2}}(\mathbb{R}^2)$, let $u \in C([0,T]; H^{\frac{\alpha}{2}}(\mathbb{R}^2))$, $u_h \in C([0,T]; L^2_h)$ be the well-posed solutions* *from Lemma 3.1 where* $T = T(\|u_0\|_{H^{\frac{\alpha}{2}}}) > 0$. *Then there exists* $C_i = C_i(\|u_0\|_{H^{\frac{\alpha}{2}}}) > 0$, $i = 1, 2$ *independent of* $h > 0$ *such that*

$$\|p_h u_h(t) - u(t)\|_{L^2(\mathbb{R}^2)} \leq C_1 h^{\frac{\alpha}{2+\alpha}} (\|u_0\|_{H^{\frac{\alpha}{2}}} + \|u_0\|^p_{H^{\frac{\alpha}{2}}}) e^{C_2|t|}, \ t \in [0, T]. \quad (3.4)$$

*Remark 3.1.* To estimate the nonlinear part of $p_h u_h - u$ uniformly in $h$, we show that an appropriate space-time Lebesgue norm of $u_h$ is uniformly bounded in $[0, T(\|u_0\|_{H^{\frac{\alpha}{2}}})]$ (see Lemma 4.2). However, our proof is insufficient to conclude that a similar uniform bound holds in the energy-critical case, and therefore our method does not extend, at least directly, when $\alpha = \frac{2(p-1)}{p+1}$.

*Remark 3.2.* The result is local in time, and thus it is of interest to extend (3.4) such that the estimate holds for $t \in [0, T_e)$ where $0 < T_e \leq \infty$ is the maximal time of existence of (2.2). This extension is not straightforward due to the existence of finite-time blow-up solutions in the mass supercritical regime. For example if $T_e < \infty$, then $\lim_{t \to T_e -} \|u(t)\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^2)} = \infty$. Since $p_h u_h = O_h(1)$ by Lemma 4.1, for all $h > 0$ we have

$$\sup_{t \in [0, T_e)} \|p_h u_h(t) - u(t)\|_{H^{\frac{\alpha}{2}}(\mathbb{R}^2)} = \infty.$$

*Remark 3.3.* Suppose (2.2) were discretized by another means. Let $A_h$ be a self-adjoint linear operator on $L^2_h$ and let $v_h \in C([0, T]; L^2_h)$ be a solution of

$$i\dot{v}_h = A_h v_h + N(v_h), \ v_h(x, 0) = u_{0,h}(x). \quad (3.5)$$

Recall that $u_h(t), u(t)$ are well-posed $L^2$-solutions of (2.1), (2.2). If $\|u_h(t) - v_h(t)\|_{L^2_h} \lesssim h^\theta$, then $\|p_h v_h(t) - u(t)\|_{L^2(\mathbb{R}^2)} \lesssim h^{\theta'}$ for some $\theta, \theta' > 0$ by (3.4) and the triangle inequality.

It is expected that our approach would apply to a general class of discrete models governed by $\{A_h\}$. A priori, $A_h$ is assumed to act on $L^2_h$ and thus its extension to $L^2(\mathbb{R}^d)$ needs to be defined, after which, the limit of $A_h$ as $h \to 0$, if it exists, is considered. Let $m_h \in A\left(\frac{\mathbb{T}^d}{h}\right)$ and define $\mathcal{F}_h(A_h f)(\xi) = m_h(\xi)\mathcal{F}_h f(\xi)$ where

$$A\left(\frac{\mathbb{T}^d}{h}\right) = \{f \in L^\infty\left(\frac{\mathbb{T}^d}{h}\right) : \mathcal{F}_h^{-1} f \in L^1_h\}.$$

Denote $\nu_h = \mathcal{F}_h^{-1} m_h$. Since the Fourier coefficients are absolutely integrable, $\nu_h$ can be interpreted as a complex Borel measure on $\mathbb{R}^d$ given by

$$\nu_h(x) = \sum_{y \in h\mathbb{Z}^d} \mathcal{F}_h^{-1} m_h(y) \delta_y(x),$$

where $\delta_y$ is the Dirac mass at $y \in h\mathbb{Z}^d$. Then for $f \in L^2_h$, $x \in h\mathbb{Z}^d$,

$$A_h f(x) = \nu_h *_h f(x) := h^d \sum_{y \in h\mathbb{Z}^d} \mathcal{F}_h^{-1} m_h(y) f(x - y).$$

For $f \in C_c^\infty(\mathbb{R}^d)$, we have $f * \delta_y(x) = \int_{\mathbb{R}^d} f(x-y')d\delta_y(y') = f(x-y)$, and therefore

$$A_h f(x) = h^d \nu_h * f(x), \tag{3.6}$$

for $f \in \bigcup_{p\in[1,\infty]} L^p(\mathbb{R}^d)$ and

$$\|A_h f\|_{L^p(\mathbb{R}^d)} \leq h^d \|\nu_h\|_{TV} \|f\|_{L^p(\mathbb{R}^d)}, \tag{3.7}$$

where $\|\nu_h\|_{TV}$ measures the total variation.

**Proposition 3.1.** *Define $A_h$ as (3.6). Then, $A_h : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is bounded for all $p \in [1,\infty]$ with the operator norms satisfying*

$$\|A_h\|_{L^p\to L^p} \geq \|A_h\|_{L^2\to L^2} = \|m_h\|_{L^\infty\left(\frac{\mathbb{T}^d}{h}\right)}.$$

*Proof.* Since $A_h$ is a convolution against a finite measure with bounded symbol $m_h$, $A_h$ is a translation-invariant bounded linear operator on $L^p(\mathbb{R}^d)$ for all $p \in [1,\infty]$ that satisfies (3.7). Since $A_h$ is bounded on $L^p(\mathbb{R}^d)$, it is bounded on $L^{p'}(\mathbb{R}^d)$ by duality. By the Riesz–Thorin theorem, we have

$$\|A_h\|_{L^2\to L^2} \leq \|A_h\|_{L^p\to L^p}^{\frac{1}{2}} \|A_h\|_{L^{p'}\to L^{p'}}^{\frac{1}{2}} = \|A_h\|_{L^p\to L^p}.$$

The last equality is given by the fact that any translation-invariant bounded linear operator on $L^2(\mathbb{R}^d)$ is given by a bounded multiplier on the Fourier space. □

As an example, consider two classes of multipliers

$$\sigma_h(\xi) = \left(\frac{4}{h^2}\sum_{i=1}^d \sin^2\left(\frac{h\xi_i}{2}\right)\right)^{\frac{\alpha}{2}}, \quad m_h(\xi) = c_{d,\alpha} h^d \sum_{z\in h\mathbb{Z}^d\setminus\{0\}} \frac{1-\cos\xi\cdot z}{|z|^{d+\alpha}}, \tag{3.8}$$

where $c_{d,\alpha} = \frac{4^{\frac{\alpha}{2}}\Gamma(\frac{d+\alpha}{2})}{\pi^{\frac{d}{2}}|\Gamma(-\frac{\alpha}{2})|}$. It can be verified that $m_h$ defines

$$A_h f(x) = c_{d,\alpha} h^d \sum_{y\in h\mathbb{Z}^d\setminus\{x\}} \frac{f(x)-f(y)}{|x-y|^{d+\alpha}}. \tag{3.9}$$

**Proposition 3.2.** *Both $(-\Delta_h)^{\frac{\alpha}{2}}$ and $A_h$, where $A_h$ is as (3.9), are divergent as $h \to 0$ in the space of bounded linear operators on $L^2(\mathbb{R}^d)$ in the uniform operator topology. On the other hand, $(-\Delta_h)^{\frac{\alpha}{2}}$, $A_h$ converge to $(-\Delta)^{\frac{\alpha}{2}}$ strongly in $L^2$ in the Schwartz class.*

*Proof.* By direct computation,

$$\|\sigma_h\|_{L^\infty\left(\frac{\mathbb{T}^d}{h}\right)},\ \|m_h\|_{L^\infty\left(\frac{\mathbb{T}^d}{h}\right)} \gtrsim_d h^{-\alpha},$$

and hence the divergence by Proposition 3.1. The statements on strong convergence in $L^2$ follows as [25, Lemma 3.9], noting that

$$\sigma_h(\xi),\ m_h(\xi) \xrightarrow[h\to 0]{} |\xi|^\alpha,\ \forall \xi \in \mathbb{R}^d,$$

followed by the Dominated Convergence Theorem to interchange the limit as $h$ tends to zero and the integral in $\xi$, justified by

$$|\sigma_h(\xi)|,\ |m_h(\xi)| \lesssim |\xi|^\alpha,$$

independent of $h > 0$. □

A potential issue with $m_h$ in (3.8) is that the Euclidean metric $|\cdot|$ does not yield the physically relevant distance between two points on $h\mathbb{Z}^d$ when $d \geq 2$. Define $m_h^q(\xi) = h^d \sum_{z\in h\mathbb{Z}^d\setminus\{0\}} \frac{1-\cos\xi\cdot z}{|z|_q^{d+\alpha}}$ where $|\cdot|_q$ denotes the $l^q$ norm for $q \in [1,\infty]$.

**Proposition 3.3.** *For all $\xi \in \mathbb{R}^d$, $m_h^q(\xi) \xrightarrow[h\to 0]{} c|\xi|^\alpha$ for some nonzero constant $c$ if and only if $q = 2$. For all $q \in [1,\infty]\setminus\{2\}$, an operator $A_h^q$ defined by $m_h^q$ is nonnegative and self-adjoint on $L^2(\mathbb{R}^d)$, defined on the dense domain $H^\alpha(\mathbb{R}^d)$.*

*Proof.* Let $\xi = |\xi|\xi'$ where $\xi' \in S^{d-1}$, the unit sphere. Let $\rho(\xi') \in \mathbb{R}^{d\times d}$ be a rotation operator that takes the $d^{\text{th}}$ standard basis vector to $\xi'$, i.e., $\xi' = \rho(\xi')e_d$. Then, $\alpha < 2$ justifies the limit of Riemann sum, and by changing variables,

$$\lim_{h\to 0} m_h^q(\xi) = \int_{\mathbb{R}^d} \frac{1-\cos\xi\cdot z}{|z|_q^{d+\alpha}}\mathrm{d}z = |\xi|^\alpha \int \frac{1-\cos z_d}{|\rho(\xi')z|_q^{d+\alpha}}\mathrm{d}z.$$

A priori, the integral in the last expression, call it $I(\xi)$, reduces to a continuous function on $S^{d-1}$ that is constant if and only if $q = 2$, observing the norm-invariance under rotation if and only if $q = 2$. The domain of $A_q^h$ consists of $f \in L^2(\mathbb{R}^d)$ such that $|\xi|^\alpha I(\xi)\widehat{f} \in L^2$. Observing that $\inf_{\xi\in\mathbb{R}^d} |I(\xi)| > 0$ due to the norm equivalence of $\{|\cdot|_q\}$ and

$$I(\xi) \gtrsim \int \frac{1-\cos z_d}{|\rho(\xi')z|^{d+\alpha}}\mathrm{d}z = \int \frac{1-\cos z_d}{|z|^{d+\alpha}}\mathrm{d}z = c_{d,\alpha}^{-1} > 0,$$

it follows that $D(A_h^q) = H^\alpha(\mathbb{R}^d)$. That $A_h^q$ is nonnegative and self-adjoint follows from $I \geq 0$.

□

The continuum limit in higher dimensions, therefore, depends on the geometry of the underlying discrete model. This would potentially lead to complications on a spatial domain with an irregular lattice structure, which we leave as an open-ended thought. To this end, our analysis is restricted to $(-\Delta_h)^{\frac{\alpha}{2}}$.

To show the main result, linear dispersive estimates of the discrete evolution are developed. Let $\psi \in C_c^\infty((-2\pi, 2\pi); [0, 1])$ be an even function where $\psi = 1$ for $\xi \in [-\pi, \pi]$ and let $\eta(\xi) := \psi(|\xi|) - \psi(2|\xi|)$. For dyadic integers $N \leq 1$, define Littlewood–Paley projections given by

$$P_N = P_{N,h} := \mathcal{F}^{-1}\eta\left(\frac{h\xi}{N}\right)\mathcal{F},$$

where $\mathcal{F}$ is the Fourier transform on $\mathbb{R}^d$. Since $\xi \in \frac{\mathbb{T}^d}{h}$, $P_N$ is a smooth projector onto $\frac{\pi}{2}\frac{N}{h} \leq |\xi| \leq 2\pi\frac{N}{h}$ and altogether resolves the identity

$$\sum_{N\leq 1} P_N = Id.$$

The sum has an upper bound in $N$ since $h\xi = O_d(1)$.

Adopting the notations in [1], define subsets of $M := \mathbb{T}^2 \setminus \{0\}$ given by

$$K_3 = \left\{\left(\pm\frac{\pi}{2}, \pm\frac{\pi}{2}\right)\right\},\ K_2 = \{\xi \in M \setminus K_3 : \det D^2 w(\xi) = 0\},\ K_1 = M \setminus (K_2 \cup K_3),$$

where $w(\xi) = \left(\sum_{i=1}^{d} \sin^2(\frac{\xi_i}{2})\right)^{\frac{\alpha}{2}}$. The main proposition concerns a family of frequency-localized dispersive estimates with sharp time decay. Furthermore, the lower bounds of implicit constants blow up both in the wave and in the Schrödinger limit.

**Proposition 3.4.** *For all* $t > 0,\ N \leq 1,\ 0 < h \leq 1,\ 1 < \alpha < 2$, *there exists* $0 < C_i(\alpha) < \infty,\ i = 1, 2, 3$ *such that*

$$\|U_h(t)P_N f\|_{L_h^\infty} \leq \begin{cases} C_3(\alpha)\left(\frac{N}{h}\right)^{2-\frac{3}{4}\alpha}|t|^{-\frac{3}{4}}\|f\|_{L_h^1}, & supp(\eta(\frac{\cdot}{N})) \cap K_3 \neq \emptyset \\ C_2(\alpha)\left(\frac{N}{h}\right)^{2-\frac{5}{6}\alpha}|t|^{-\frac{5}{6}}\|f\|_{L_h^1}, & supp(\eta(\frac{\cdot}{N})) \cap (K_2 \setminus K_3) \neq \emptyset \\ C_1(\alpha)\left(\frac{N}{h}\right)^{2-\alpha}|t|^{-1}\|f\|_{L_h^1}, & supp(\eta(\frac{\cdot}{N})) \cap (K_1 \setminus K_2 \cup K_3) \neq \emptyset, \end{cases} \tag{3.10}$$

*and*

$$C_3(\alpha) \gtrsim_\eta (2-\alpha)^{-\frac{1}{4}},\ C_2(\alpha) \gtrsim_\eta (\alpha-1)^{\frac{2}{3}-\frac{5\alpha}{12}},\ C_1(\alpha) \gtrsim_\eta (\alpha-1)^{-\frac{1}{2}}. \tag{3.11}$$

For more details on the domain of $N \in 2^{\mathbb{Z}}$ that satisfy (3.10), see (5.18). By interpolating the estimates in (3.10), one obtains

**Corollary 3.1.** *Assume the hypotheses of Proposition 3.4. Then,*

$$\|U_h(t)P_N f\|_{L_h^\infty} \lesssim_\alpha (\frac{N}{h})^{2-\frac{3}{4}\alpha}|t|^{-\frac{3}{4}}\|f\|_{L_h^1}.$$

*Remark 3.4.* Assuming Proposition 3.4, it is straightforward to obtain the Strichartz estimates for the linear evolution by averaging in $t$, $N$, which we briefly describe. Suppose $\|U_h(t)P_N\|_{L^1_h\to L^\infty_h} \lesssim_\alpha (\frac{N}{h})^\beta |t|^{-\sigma}$ for some $\beta,\sigma>0$. Define the Strichartz pair $(q,r)\in[2,\infty]^2$ by the relation

$$\frac{1}{q}+\frac{\sigma}{r}=\frac{\sigma}{2},\ (q,r,\sigma)\neq(2,\infty,1). \tag{3.12}$$

As in [3, p.1127], define $\tilde{U}(t)=P_N U_h\left((\frac{N}{h})^{\frac{\beta}{\sigma}}t\right)P_{\sim N}$ where $P_{\sim N}:=P_{N/2}+P_N+P_{2N}$. Then, $\{\tilde{U}(t)\}_{t\in\mathbb{R}}$ satisfies the hypotheses of [24, Theorem 1.2] from which follows

$$\|U_h(t)P_N f\|_{L^q_t L^r_h} \lesssim_{q,r} (\frac{N}{h})^{\beta(\frac{1}{2}-\frac{1}{r})}\|P_N f\|_{L^2_h} \simeq \|P_N|\nabla_h|^{\beta(\frac{1}{2}-\frac{1}{r})}f\|_{L^2_h}.$$

Squaring both sides and summing in $N$,

$$\begin{aligned}\|U_h(t)f\|_{L^q_t L^r_h} &\lesssim \left(\sum_{N\leq 1}\|U_h(t)P_N f\|^2_{L^q_t L^r_h}\right)^{\frac{1}{2}} \lesssim \left(\sum_{N\leq 1}\|P_N|\nabla_h|^{\beta(\frac{1}{2}-\frac{1}{r})}f\|^2_{L^2_h}\right)^{\frac{1}{2}}\\ &\simeq \||\nabla_h|^{\beta(\frac{1}{2}-\frac{1}{r})}f\|_{L^2_h},\end{aligned} \tag{3.13}$$

for $r\in[2,\infty)$ where the first inequality follows from the Littlewood–Paley inequality on the lattice [15, Theorem 4.2]. As an example, Corollary 3.1 asserts $(\beta,\sigma)=(2-\frac{3}{4}\alpha,\frac{3}{4})$ and hence by (3.13),

$$\|U_h(t)f\|_{L^q_t L^r_h} \lesssim_{q,r,\alpha} \||\nabla_h|^{\left(2-\frac{3}{4}\alpha\right)\left(\frac{1}{2}-\frac{1}{r}\right)}f\|_{L^2_h}.$$

The derivative loss occurs for $\alpha<\frac{8}{3}$ or $\alpha<2$ on $h\mathbb{Z}^2$ (for all $h>0$) or $\mathbb{R}^2$, respectively.

For $v\in\mathbb{R}^d$, define $\Phi_v(\xi)=v\cdot\xi-w(\xi)$ for $\xi\in\mathbb{R}^d$ and let $\zeta\in C^\infty_c(\mathbb{R}^d)$ be a test function. Consider

$$J=J_{\Phi_v,\zeta}(\tau):=\int_{\mathbb{R}^d}e^{i\tau\Phi_v(\xi)}\zeta(\xi)\mathrm{d}\xi,$$

where $\tau>0$ without loss of generality, for $\tau<0$ amounts to taking the complex conjugate of $J_{\Phi_v,\zeta}$. To show (3.10), observe that

$$\|U_h(t)P_N f\|_{L^\infty_h}=\|K_{t,N,h}*f\|_{L^\infty_h}\leq\|K_{t,N,h}\|_{L^\infty_h}\|f\|_{L^1_h}$$

by the Young's inequality applied to the convolution in $h\mathbb{Z}^d$ where

$$K_{t,N,h}(x)=(2\pi)^{-d}\int_{\mathbb{T}^d_h}e^{i\left\{x\cdot\xi-t\left(\frac{4}{h^2}\sum_{i=1}^d\sin^2(\frac{h\xi_i}{2})\right)^{\frac{\alpha}{2}}\right\}}\eta\left(\frac{h\xi}{N}\right)\mathrm{d}\xi.$$

Change variables $\xi \mapsto \frac{\xi}{h}$ and define $\tau = \frac{2^\alpha t}{h^\alpha},\ v = \frac{x}{h\tau}$ to obtain

$$K_{t,N,h}(x) = (2\pi h)^{-d} \int_{\mathbb{T}^d} e^{i\tau(v\cdot\xi - w(\xi))} \eta\left(\frac{\xi}{N}\right) \mathrm{d}\xi.$$

A priori since $x \in h\mathbb{Z}^d$, it follows that $v \in \tau^{-1}\mathbb{Z}^d$, which we consider as a subset of $\mathbb{R}^d$. If $\sigma_0 > 0$ is the sharp decay rate for $J_{\Phi_v,\zeta}$ in the sense that

$$\sup_{v\in\mathbb{R}^d} |J_{\Phi_v,\zeta}(\tau)| \leq C(\zeta)|\tau|^{-\sigma_0} \tag{3.14}$$

holds for all $\tau \in \mathbb{R} \setminus \{0\}$ for some $C(\zeta) > 0$ and no bigger $\sigma' > 0$ satisfies (3.14), then

$$\|K_{t,N,h}\|_{L_h^\infty} \leq (2\pi h)^{-d} C\left(\eta(\frac{\cdot}{N})\right)|\tau|^{-\sigma_0} = (2\pi)^{-d} 2^{-\sigma_0\alpha} \frac{C\left(\eta(\frac{\cdot}{N})\right)}{h^{d-\sigma_0\alpha}}|t|^{-\sigma_0}.$$

Hence, our goal reduces to obtaining (3.14) for a dyadic family of Littlewood–Paley functions. Outside of a neighborhood of the origin, $\Phi_v$ is analytic, and therefore the major contributions to $J$ are due to critical points $\xi \in \mathbb{R}^d$ that satisfy $\nabla\Phi_v(\xi) = 0$, or equivalently, $v = \nabla w(\xi) =: v_\xi$. In any arbitrary dimension,

$$\begin{aligned} \nabla w(\xi) &= \frac{\alpha}{4} w(\xi)^{-(\frac{2}{\alpha}-1)}(\sin\xi_1, \ldots, \sin\xi_d), \\ |\nabla w(\xi)|^2 &= \frac{\alpha^2}{16} \frac{\sum_{i=1}^d \sin^2\xi_i}{\left(\sum_{i=1}^d \sin^2\frac{\xi_i}{2}\right)^{2-\alpha}}. \end{aligned} \tag{3.15}$$

For any $\alpha > 1$, $|\nabla w(\xi)| \xrightarrow[\xi\to 0]{} 0$ and $\sup_{\xi\in\mathbb{T}^d} |\nabla w(\xi)| = C(\alpha, d) \in (0, \infty)$. With a slight abuse of notation, define $\nabla w : \mathbb{T}^d \to \mathbb{R}^d$ where $\nabla w(0) = 0$. Then, $\nabla w$ is continuous and its compact image defines a light cone. If $|v| \gg_{\alpha,d,h} 1$ (spacelike event), then $J$ decays faster than $\tau^{-n}$ for any $n \in \mathbb{N}$, by integration by parts, with the implicit constant dependent on $n$ and the distance between $v$ and the light cone (see Lemma 5.1). Inside the light cone (including the boundary), $J$ undergoes an algebraic decay due to critical points. For such $v$, it is generically true that the corresponding critical point(s) $\xi \in \mathbb{T}^d$ are non-degenerate, and therefore $J$ decays as $\tau^{-\frac{d}{2}}$. However, there exists a low-dimensional subset of $\mathbb{T}^d$ that retards even further the decay rate of $\frac{d}{2}$. We consider this problem of resolution of singularities for $d = 2$.

To systematically study the decay and asymptotics of $J$ as a function of $v$ and $\zeta$, consider the Taylor series expansion of $\Phi_v$. Let $\xi \in \mathbb{T}^2\backslash\{0\}$ and $v_\xi = \nabla w(\xi)$. Consider $\Phi_{v_\xi}$ so that $\nabla\Phi_{v_\xi}(\xi) = 0$. Pick $\zeta \in C_c^\infty$ around $\xi$ such that $\xi$ is the unique critical point in the support. Then, $J_{\Phi_{v_\xi},\zeta}$ has an asymptotic expansion

$$J_{\Phi_{v_\xi},\zeta} = d_0(\zeta)\tau^{-\sigma_0} + o(\tau^{-\sigma_0}), \tag{3.16}$$

as $\tau \to \infty$ where $\sigma_0$, or the *oscillatory index*, is chosen to be the minimal number such that for any neighborhood of $\xi$, say $U$, there exists $\zeta_U \in C_c^\infty(U)$ such that $d_0(\zeta_U) \neq 0$; in particular, $\sigma_0$ depends only on the phase, not the smooth bump function. Under some hypotheses, $\sigma_0, d_0$ are deduced from the higher order Taylor expansion of $\Phi_{v_\xi}$ (see Lemma 5.6), a process that we briefly describe.

Let $\Phi$ be a real-valued analytic function on a small neighborhood of the origin. Assume $\Phi(0) = 0,\ \nabla\Phi(0) = 0$ and therefore the Taylor expansion of $\Phi$ at the origin in the multi-index notation is

$$\Phi(x) = \sum_{|\alpha|\geq 2} c_\alpha x^\alpha = \sum_{|\alpha|\geq 2} \frac{\partial_\alpha \Phi(0)}{\alpha!} x^\alpha.$$

Define the Taylor support $\mathcal{T} = \{\alpha \in \mathbb{N}^d : c_\alpha \neq 0\}$ and assume that $\Phi$ is of finite type, i.e., $\mathcal{T} \neq \emptyset$. Define the *Newton polyhedron* of $\Phi$, call it $\mathcal{N}$, to be the convex hull of

$$\bigcup_{\alpha\in\mathcal{T}} \alpha + \mathbb{R}^d_+ = \bigcup_{\alpha\in\mathcal{T}} \alpha + \{x \in \mathbb{R}^d : x_i \geq 0\},$$

and the *Newton diagram* $\mathcal{N}_d$ to be the union of all compact faces of $\mathcal{N}$. Let $\mathcal{N}_{pr}$, the principal part of Newton diagram, be the subset of $\mathcal{N}_d$ that intersects the bisectrix $\{x_1 = x_2 = \cdots = x_d\}$. Define the principal part of $\Phi$ (or the normal form) as

$$\Phi_{pr}(x) = \sum_{|\alpha|\geq 2,\ \alpha\in\mathcal{N}_{pr}} c_\alpha x^\alpha.$$

Let $d = d(\Phi) = \inf\{t : (t, t, \dots, t) \in \mathcal{N}\}$ be the distance from the origin to $\mathcal{N}$. Since $\Phi$ is of finite type, $0 < d < \infty$. Note that $\mathcal{T}$ is not invariant under analytic coordinate transformations. Let $d_x$ be the distance computed in the $x$ coordinate system and define the *height* of $\Phi$ as $h(\Phi) = \sup_x d_x$ where the supremum is over all analytic coordinate systems. The coordinate system $(x)$ is *adapted* if $d_x = h$. In $\mathbb{R}^2$, see [30, Proposition 0.7,0.8] for sufficient conditions for $(x)$ to be adapted. An adapted system need not be unique. To obtain the asymptotics of oscillatory integrals, we work in a *superadapted* coordinate system defined specifically in dimension two in [8] as a coordinates system in which $\Phi_{pr}(x, \pm 1)$ have no real roots of order greater than or equal to $d_x(\Phi)$, possibly except $x = 0$. In particular, if $d(\Phi) > 1$ and $\Phi_{pr}(x, \pm 1)$ is a quadratic polynomial with no repeated roots, then $(x, y)$ is superadapted.

See the introductions of [1,8,21,30], from which this paper adopts all relevant terminologies, for a brief survey of the relationship between oscillatory integrals and Newton polyhedra. To illustrate these ideas, consider an example. Let $\Phi(x, y) = x^2 + y^2 + x^3$. Then,

$$\begin{aligned}
\mathcal{T} &= \{(2, 0), (0, 2), (3, 0)\}\\
\mathcal{N} &= \{(x, y) \in \mathbb{R}^2 : x + y \geq 2,\ x, y \geq 0\},\\
\mathcal{N}_d = \mathcal{N}_{pr} &= \{(x, y) \in \mathbb{R}^2 : x + y = 2,\ 0 \leq x \leq 2\}\\
\Phi_{pr}(x, y) &= x^2 + y^2,\ d_{(x,y)} = 1.
\end{aligned}$$

Since $\Phi_{pr}(x, \pm 1) = x^2 + 1$ has no real root, the given coordinates system is super-adapted.

## 4. Continuum limit

The proof of Theorem 3.1 is given.

**Lemma 4.1.** *Let $\beta \in [0, 1]$, $p > 1$, $d \in \mathbb{N}$. The implicit constants in the following estimates are independent of $h > 0$ and dependent only on $\beta, d$.*

1. $\|d_h f\|_{H_h^\beta} \lesssim \|f\|_{H^\beta(\mathbb{R}^d)}$.
2. $\|p_h f_h\|_{H^\beta(\mathbb{R}^d)} \lesssim \|f_h\|_{H_h^\beta}$.
3. $\|p_h d_h f - f\|_{L^2(\mathbb{R}^d)} \lesssim h^\beta \|f\|_{H^\beta(\mathbb{R}^d)}$.
4. $\|p_h U_h(t) f_h - U(t)u_0\|_{L^2(\mathbb{R}^d)} \lesssim h^{\frac{\beta}{1+\beta}} |t| (\|f_h\|_{H_h^\beta} + \|u_0\|_{H^\beta(\mathbb{R}^d)}) + \|p_h f_h - u_0\|_{L^2(\mathbb{R}^d)}$.
   *If $f_h$ is the discretization of $u_0$, i.e., $f_h = u_{0,h}$, then* $\|p_h U_h(t) u_{0,h} - U(t)u_0\|_{L^2(\mathbb{R}^d)} \lesssim \langle t \rangle h^{\frac{\beta}{1+\beta}} \|u_0\|_{H^\beta(\mathbb{R}^d)}$.
5. $\|p_h(|u_h|^{p-1}u_h) - |p_h u_h|^{p-1} p_h u_h\|_{L^2(\mathbb{R}^d)} \lesssim h^\beta \|u_h\|_{L_h^\infty}^{p-1} \|u_h\|_{H_h^\beta}$.

*Proof.* See [16,25]. □

**Lemma 4.2.** *Let $p \geq 3$ and $\max(\frac{8}{7}, \frac{2(p-1)}{p+1}) < \alpha < 2$. There exists $\delta = \delta(\alpha, p) > 0$ sufficiently small such that a Strichartz pair $(q, r)$, defined by (3.12) with $\sigma = \frac{3}{4}$ and $q := \frac{2\alpha}{2-\alpha+\delta}$, satisfies $p - 1 < q$ and yields the uniform $L_h^\infty$ estimate given by*

$$\|U_h(t) f_h\|_{L_t^q L_h^\infty} \lesssim \|f_h\|_{H_h^{\frac{\alpha}{2}}}.$$

*Proof.* Let $s = \frac{\alpha}{2} - (2 - \frac{3}{4}\alpha)(\frac{1}{2} - \frac{1}{r})$. Then, it can be verified by direct computation that $s > \frac{2}{r}$ using $\delta > 0$. Hence, by the Sobolev embedding and (3.13), respectively,

$$\|U_h(t) f_h\|_{L_t^q L_h^\infty} \lesssim \|U_h(t) f_h\|_{L_t^q W_h^{s,r}} \lesssim \|f_h\|_{H_h^{\frac{\alpha}{2}}}.$$

Furthermore, it can be directly verified that $(q, r) \in [2, \infty]^2$. From the Strichartz pair relation and the definition of $q$, we have $r \leq \infty$ iff $\alpha > \frac{8}{7}$. Lastly, by choosing $\delta > 0$ sufficiently small, $p - 1 < q$ is satisfied. □

*Remark 4.1.* From (3.3), we have

$$2 \leq p - 1 < \frac{2\alpha}{d - \alpha},$$

from which the definition of $q$ in Lemma 4.2 is motivated.

**Lemma 4.3.** *Assume the hypothesis of Theorem 3.1 and let q be given by Lemma 4.2. Given* $u_0 \in H^{\frac{\alpha}{2}}(\mathbb{R}^2)$*, let* $T \simeq \|u_0\|_{H^{\frac{\alpha}{2}}}^{-\frac{1}{\frac{1}{p-1}-\frac{1}{q}}}$*. By Lemma 3.1, let* $u, u_h$ *be the well-posed solutions corresponding to initial data* $u_0, u_{0,h}$*. Then,* $u, u_h$ *satisfy*

$$\begin{aligned} \|u\|_{L^\infty_{t\in[0,T]}H_x^{\frac{\alpha}{2}}} + \|u\|_{L^q_{t\in[0,T]}L^\infty_x} &\lesssim \|u_0\|_{H^{\frac{\alpha}{2}}}, \\ \|u_h\|_{L^\infty_{t\in[0,T]}H_h^{\frac{\alpha}{2}}} + \|u_h\|_{L^q_{t\in[0,T]}L^\infty_h} &\lesssim \|u_{0,h}\|_{H_h^{\frac{\alpha}{2}}}. \end{aligned} \tag{4.1}$$

*Proof.* The estimate for $u$ is derived from the proof of local well-posedness by the contraction mapping argument. Similarly the time of existence for the discrete evolution that ensures the estimate (4.1) is $T_h \sim \|u_{0,h}\|_{H_h^{\frac{\alpha}{2}}}^{-\frac{1}{\frac{1}{p-1}-\frac{1}{q}}}$. By Lemma 4.1, $T_h \gtrsim T$ uniformly in $h$, and therefore $u, u_h$ are well defined on $[0, T]$. □

*Proof of Theorem 3.1.* Let $u_0 \in H^{\frac{\alpha}{2}}(\mathbb{R}^2)$. There exists $T \sim \|u_0\|_{H^{\frac{\alpha}{2}}}^{-\frac{1}{\frac{1}{p-1}-\frac{1}{q}}}$ and $u \in C([0, T]; H^{\frac{\alpha}{2}}(\mathbb{R}^2))$, unique in a (smaller) Strichartz space (see [14, Theorem 1.1]), satisfying

$$u(t) = U(t)u_0 - i\mu \int_0^t U(t-s)(|u|^{p-1}u)(s)ds.$$

For $h > 0$, consider $u_{0,h} = d_h u_0 \in L^2_h$ and the global solution $u_h \in C^1([0,\infty); L^2_h)$ given by Lemma 3.1. Similarly as above,

$$p_h u_h(t) = p_h U_h(t)u_{0,h} - i\mu \int_0^t p_h U_h(t-s)(|u_h|^{p-1}u_h)(s)ds.$$

The difference $p_h u_h(t) - u(t)$ is given by

$$\begin{aligned} &= p_h U_h(t)u_{0,h} - U(t)u_0 \\ &\quad - i\mu \int_0^t (p_h U_h(t-s) - U(t-s)p_h)\,(|u_h|^{p-1}u_h)(s)ds \\ &\quad - i\mu \int_0^t U(t-s)\left(p_h(|u_h|^{p-1}u_h)(s) - |p_h u_h|^{p-1}p_h u_h(s)\right)ds \\ &\quad - i\mu \int_0^t U(t-s)\left(|p_h u_h|^{p-1}p_h u_h(s) - |u|^{p-1}u(s)\right)ds =: I + II + III + IV. \end{aligned}$$

Following the proof of [16, Theorem 1.1], we have

$$\begin{aligned} \|I\|_{L^2} &\lesssim h^{\frac{\alpha}{2+\alpha}}\langle t\rangle \|u_0\|_{H^{\frac{\alpha}{2}}} \\ \|II\|_{L^2},\ \|III\|_{L^2} &\lesssim h^{\frac{2}{2+\alpha}}\langle t\rangle^2 \|u_0\|^p_{H^{\frac{\alpha}{2}}} \\ \|IV\|_{L^2} &\lesssim \int_0^t (\|u_h(s)\|_{L^\infty_h} + \|u(s)\|_{L^\infty_x})^{p-1}\|p_h u_h(s) - u(s)\|_{L^2}ds. \end{aligned}$$

and altogether,

$$\|p_h u_h(t) - u(t)\|_{L^2} \lesssim h^{\frac{\alpha}{2+\alpha}} \langle t\rangle^2 (\|u_0\|_{H^{\frac{\alpha}{2}}} + \|u_0\|^p_{H^{\frac{\alpha}{2}}}) + \int_0^t (\|u_h(s)\|_{L^\infty_h} + \|u(s)\|_{L^\infty_x})^{p-1} \|p_h u_h(s) - u(s)\|_{L^2} ds.$$

By the Gronwall's inequality,

$$\|p_h u_h(t) - u(t)\|_{L^2} \lesssim h^{\frac{\alpha}{2+\alpha}} \langle t\rangle^2 (\|u_0\|_{H^{\frac{\alpha}{2}}} + \|u_0\|^p_{H^{\frac{\alpha}{2}}}) e^{\int_0^t (\|u_h(s)\|_{L^\infty_h} + \|u(s)\|_{L^\infty_x})^{p-1} ds}. \tag{4.2}$$

Applying (4.1) to (4.2), we obtain (3.4) where $C_1, C_2 > 0$ depend on various parameters including $\|u_0\|_{H^{\frac{\alpha}{2}}}$, but not $h$. This completes the proof. □

## 5. Proof of Proposition 3.4

Let $\alpha \in (1, 2)$ unless otherwise specified. For $\zeta \in C_c^\infty(M)$, the quantity $J_{\Phi_v,\zeta}$ is at worst a non-degenerate integral almost everywhere with respect to the Lebesgue measure on $\mathbb{R}^2_v$.

**Lemma 5.1.** *Let $\zeta \in C_c^\infty(U)$ where $U \subseteq \mathbb{R}^d \backslash \{0\}$. For $v \in \mathbb{R}^d$, suppose $\inf_{\xi \in U} |v - \nabla w(\xi)| \geq m > 0$ on $U$. Then, $|J_{\Phi_v,\zeta}| \lesssim_{n,m,\zeta} |\tau|^{-n}$ for all $\tau \in \mathbb{R}\backslash\{0\}$, $n \in \mathbb{N}$.*

Non-degenerate critical points are treated by the method of stationary phase. A well-known asymptotics [29, Chapter 8, Proposition 6] is given.

**Lemma 5.2.** *Let $\xi \in M$, $v \in \mathbb{R}^2$ satisfy $v = \nabla w(\xi)$ and* det $D^2 w(\xi) \neq 0$. *Then, there exists a small neighborhood around $\xi$ such that for all $\zeta \in C_c^\infty$ supported in the neighborhood,*

$$J_{\Phi_v,\zeta} = a_0 \tau^{-1} + o(\tau^{-1}),$$

*as $\tau \to \infty$ where*

$$a_0 = e^{i\frac{\pi}{4} sgn D^2 w(\xi)} e^{i\tau w(\xi)} \zeta(\xi) \sqrt{\frac{2\pi}{|\det D^2 w(\xi)|}}.$$

If $D^2 w(\xi)$ is singular, i.e., if $\xi \in M$ satisfies $H(\xi, \alpha) := \det D^2 w(\xi) = 0$, then the asymptotic formula of Lemma 5.2 is not applicable; $\xi = 0$ is not considered since the oscillatory integrals from Proposition 3.4 have test functions supported outside of the origin. Note that

$$D^2 w(\xi) = -\frac{\alpha}{16 w(\xi)^{\frac{4}{\alpha}-1}} \begin{pmatrix} \alpha\cos^2\xi_1 + 2(\cos\xi_2 - 2)\cos\xi_1 + 2 - \alpha & (2-\alpha)\sin\xi_1\sin\xi_2 \\ (2-\alpha)\sin\xi_1\sin\xi_2 & \alpha\cos^2\xi_2 + 2(\cos\xi_1 - 2)\cos\xi_2 + 2 - \alpha \end{pmatrix} \tag{5.1}$$

is well-defined for $\xi \in M$ and blows up (in the sense of determinant) as $\xi \to 0$. It can be shown directly that $D^2 w(\xi)$ is not the zero matrix for any $\xi \in M$. If $D^2 w(\xi)$ is of full rank, then the decay of $J_{\Phi_{v_\xi},\zeta}$ can be analyzed via Lemmas 5.1 and 5.2, and henceforth suppose $rank(D^2 w(\xi)) = 1$. Then, $H(\xi,\alpha) = \tilde{h}(\xi,\alpha)h(\xi,\alpha)$ where

$$\tilde{h}(\xi,\alpha) = \frac{\alpha^2}{128 w(\xi)^{\frac{8}{\alpha}-2}}(\cos\xi_1 + \cos\xi_2 - 2)$$
$$h(\xi,\alpha) = \alpha\cos\xi_1\cos\xi_2(\cos\xi_1+\cos\xi_2) - 4\cos\xi_1\cos\xi_2 + (2-\alpha)(\cos\xi_1+\cos\xi_2).$$

Since $h, \tilde{h}$ are symmetric under $\xi_1 \mapsto \pm\xi_1,\ \xi_2 \mapsto \pm\xi_2,\ (\xi_1,\xi_2) \mapsto (\xi_2,\xi_1)$, the domain of analysis could be restricted to the first quadrant either above or below the identity $\xi_1 = \xi_2$. By definition, $\tilde{h}$ is nonzero on $M$ and therefore the roots of $H$ correspond to those of $h$. Following the approach in [1], the representation of $h$ is in polynomials under the change of variables $a = \cos\xi_1,\ b = \cos\xi_2$, and is given by

$$h(a,b,\alpha) = \alpha ab(a+b) - 4ab + (2-\alpha)(a+b).$$

For brevity, let

$$E_\alpha = \{(a,b) \in [-1,1)^2 : h(a,b,\alpha) = 0\},$$
$$E = \bigcup_{\alpha\in(1,2)} E_\alpha.$$

For $\alpha \in [1,2)$, since $\nabla_{(a,b)}h \neq 0$ on $[-1,1)^2$, $E_\alpha$ is a smooth one-dimensional embedded submanifold by the implicit function theorem. This is false for $\alpha = 2$; a cusp $(a,b) = (0,0)$, which corresponds to $\xi = (\pm\frac{\pi}{2}, \pm\frac{\pi}{2})$, appears when $\alpha = 2$ since $\nabla_{(a,b)}h(0,0) = 0$. By direct computation, $E_{\alpha_1} \cap E_{\alpha_2} = \{(0,0)\}$ for all $\alpha_1, \alpha_2 \in [1,2]$. Hence, there exists a smooth map $E\backslash\{0\} \ni (a,b) \mapsto \alpha \in (1,2)$ satisfying $h(a,b,\alpha) = 0$ by the implicit function theorem, observing that $\partial_\alpha h(a,b,\alpha) = (a+b)(ab-1)$ is nonvanishing on $E\backslash\{0\}$. Observe that $E_\alpha$ consists of two connected components; one component, say $\Gamma^1_{(a,b)}(\alpha)$, passes through the origin whereas the other, say $\Gamma^2_{(a,b)}(\alpha)$, does not. As $\alpha \to 1+$, $\Gamma^2_{(a,b)}(\alpha)$ becomes arbitrarily close to $(a,b) = (1,1)$ whose corresponding point in $\mathbb{T}^2$, the origin, is not in $M$. In fact,

$$\bigcap_{\alpha_0\in(1,2]}\bigcup_{\alpha\in[1,\alpha_0)} E_\alpha = \Gamma^1_{(a,b)}(1), \quad \bigcap_{\alpha_0\in[1,2)}\bigcup_{\alpha\in(\alpha_0,2]} E_\alpha = \{ab = 0\}.$$

See Fig. 1 for the contour plots of $E_\alpha$.

Let $\{e_1, e_2\}$ be the standard basis of $\mathbb{R}^2$, and with a slight abuse of notation, consider $\{e_1, e_2\}$ as a global orthonormal frame of $\mathbb{T}^2$. For each $\xi \in M$, let $\{k_1(\xi), k_2(\xi)\}$ be an orthonormal basis of $T_\xi M$, the tangent space at $\xi$, with the coordinate system $(x, y)$, i.e., for all $v \in T_\xi M$, there exists unique $(x,y) \in \mathbb{R}^2$ such that $v = xk_1(\xi) + yk_2(\xi)$. Moreover, assume $\{k_1(\xi), k_2(\xi)\}$ diagonalizes $D^2 w(\xi)$ as

$$D^2 w(\xi) = \big(k_1(\xi)\ k_2(\xi)\big)\begin{pmatrix}\partial_{xx}w(\xi) & 0\\ 0 & \partial_{yy}w(\xi)\end{pmatrix}\big(k_1(\xi)\ k_2(\xi)\big)^{-1},$$

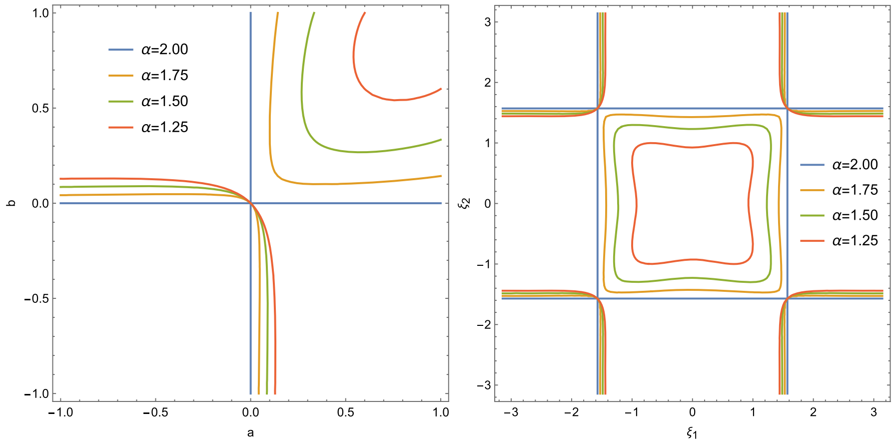


Figure 1. The contour plots of $E_\alpha$ and its correspondence in $(\xi_1, \xi_2) \in M$ are given for different values of $\alpha$

where $\partial_x = k_1(\xi) \cdot \nabla_\xi$, $\partial_y = k_2(\xi) \cdot \nabla_\xi$ are the directional derivatives. Suppose $\partial_{yy} w(\xi) = 0$ for all $\xi \in E$, or equivalently, $k_2(\xi)$ is the direction along which $D^2 w(\xi)$ is degenerate. Then, it follows that $\partial_{xx} w(\xi) \neq 0$ since $D^2 w(\xi)$ is not a zero matrix. Due to diagonalization, $\partial_{xy} w(\xi) = 0$.

To further investigate the higher-order directional derivatives, [1, Lemma 5.4] is extended by induction and the product rule for derivatives whose proof is immediate and hence omitted.

**Lemma 5.3.** *For $m \geq 2$, let $f \in C^{m+1}(U)$ where $\xi_0 \in U \subseteq \mathbb{R}^d$. Suppose $D^2 f(\xi_0)$ has rank $d-1$ and let $k_d$ be a normalized eigenvector corresponding to eigenvalue zero. Suppose $(k_d(\xi_0) \cdot \nabla_\xi)^j f(\xi_0) = 0$ for $2 \leq j \leq m$. Then, $(k_d(\xi_0) \cdot \nabla_\xi)^{m+1} f(\xi_0) = 0$ if and only if $(k_d(\xi_0) \cdot \nabla_\xi)^{m-1} \det D^2 f(\xi_0) = 0$.*

The inflection points of $\sin^2(\frac{\xi_i}{2})$ persist to exist as singular points of $D^2 w(\xi)$ even when $\alpha < 2$. By symmetry, the qualitative behavior of $J_{\Phi_{v,\zeta}}$ is the same near each point in $K_3$.

**Lemma 5.4.** *If $\xi \in E_\alpha \backslash K_3$, then $\partial_y^3 w(\xi) \neq 0$. Moreover, $\partial_y^3 w(P) = 0$ for all $P \in K_3$.*

*Proof.* For $\xi \in E_\alpha$, since $\nabla_\xi H(\xi, \alpha) = \tilde{h}(\xi) \nabla_\xi h(\xi, \alpha)$ where $\tilde{h}(\xi) \neq 0$, it suffices to show $v \cdot \nabla_\xi h(\xi, \alpha) = 0$ implies $\xi \in K_3$ where $v$ is any scalar multiple of $k_2(\xi)$. From (5.1), let

$$v = \begin{pmatrix} -(2-\alpha)\sin\xi_1 \sin\xi_2 \\ \alpha\cos^2\xi_1 + 2(\cos\xi_2 - 2)\cos\xi_1 + 2 - \alpha \end{pmatrix} \text{ or } \begin{pmatrix} \alpha\cos^2\xi_2 + 2(\cos\xi_1 - 2)\cos\xi_2 + 2 - \alpha \\ -(2-\alpha)\sin\xi_1 \sin\xi_2 \end{pmatrix} \tag{5.2}$$

Take the former; the proof for the latter is similar and thus is omitted. First assume $\sin\xi_2 \neq 0$.

In the $(a, b)$ coordinates, $\nabla_\xi h = -(\sin\xi_1 \partial_a h, \sin\xi_2 \partial_b h)$, and our task reduces to solving

$$\sin\xi_2 \left(-(2-\alpha)(1-a^2)\partial_a h + (\alpha a^2 + 2(b-2)a + 2 - \alpha)\partial_b h\right) = 0, \quad (5.3)$$

by applying Lemma 5.3 with $m = 2$. Modulo $\sin\xi_2$, (5.3) is a polynomial equation of degree 3 in $b$ (or $a$) and therefore can be solved explicitly. The intersection of $h(a, b, \alpha) = 0$ and (5.3) occurs at $(a, b) = (0, 0)$.

If $\xi$ lies at the intersection of $\sin\xi_2 = 0$ and $h(\xi, \alpha) = 0$, then it can be directly verified that the left vector of (5.2) is zero whereas the right vector is a scalar multiple of $\begin{pmatrix}1\\0\end{pmatrix}$. Then, the claim can be proved similarly as before. □

The higher-order derivatives at critical points determine the height of the phase function.

**Lemma 5.5.** *For all* $\xi \in M$,

$$h(\Phi_{v_\xi}) = \begin{cases} 1 & \text{if } \xi \notin E_\alpha, \\ \frac{6}{5} & \text{if } \xi \in E_\alpha \setminus K_3, \\ \frac{4}{3} & \text{if } \xi \in K_3. \end{cases}$$

*Proof.* Let $\xi \notin E_\alpha$. In any given coordinate system, say $(\tilde{x}, \tilde{y})$, if $\partial_{\tilde{x}\tilde{y}} w(\xi) \neq 0$, then $d_{(\tilde{x},\tilde{y})} = 1$. If $\partial_{\tilde{x}\tilde{y}} w(\xi) = 0$, then both $\partial^2_{\tilde{x}} w(\xi), \partial^2_{\tilde{y}} w(\xi) \neq 0$ due to non-degeneracy. In either case, $d_{(\tilde{x},\tilde{y})} = 1$. Taking supremum over all such coordinate systems, the first claim has been shown. The rest follows similarly as [1, Lemma 3.1] using Lemma 5.4. In particular, the Newton diagrams for $\xi \in E_\alpha \setminus K_3$ and $\xi \in K_3$ are given by

$$\mathcal{N}_d = \{3x + 2y = 6 : 0 \le x \le 2\},$$
$$\mathcal{N}_d = \{2x + y = 4 : 1 \le x \le 2\},$$

respectively. □

The computation of heights depends only on the nonzero Taylor coefficients of $\Phi_v$ and therefore does not reflect the variations on $\alpha$. However, the leading terms of asymptotics (3.16) depend on $\alpha$. Define

$$I_{\Phi_v,\zeta}(\epsilon) = \int_{\{0<\Phi_v<\epsilon\}} \zeta(\xi) \mathrm{d}\xi.$$

As in (3.16), $I$ has an asymptotic expansion as $\epsilon \to 0+$,

$$I_{\Phi_v,\zeta} \sim \sum_{j=0}^{\infty} (c_j(\zeta) + c'_j(\zeta)\log\epsilon)\epsilon^{r_j}, \; I_{-\Phi_v,\zeta} \sim \sum_{j=0}^{\infty} (C_j(\zeta) + C'_j(\zeta)\log\epsilon)\epsilon^{r_j},$$

where $\{r_j\}$ is an increasing arithmetic sequence of positive rational numbers such that the (minimal) $r_0$ is determined only by the phase function that renders at least one of

$c_0, c_0', C_0, C_0'$ nonzero. For $0 \leq m \leq \infty$, let $-\frac{1}{m}$ be the slope of the subset of $\mathcal{N}_d$ that the bisectrix intersects. Define $\Phi^+_{v_\xi,pr}(x,\pm 1) = \Phi_{v_\xi,pr}(x,\pm 1)$ if $\Phi_{v_\xi,pr}(x,\pm 1) > 0$ and zero otherwise. A summary of [8, Theorem 1.1,1.2] that applies to our case is given.

**Lemma 5.6.** *Let $\xi \in E_\alpha$. Then, $\Phi_{v_\xi} = \Phi_{v_\xi}(x,y)$ in the coordinate system defined by $\{k_1(\xi), k_2(\xi)\}$ is superadapted. The slowest decay of the asymptotics is given by $\sigma_0 = r_0 = \frac{1}{h(\Phi_{v_\xi})}$. The leading terms have vanishing logarithmic terms, i.e., $c_0' = C_0' = 0$, and moreover*

$$\begin{aligned} c_0 &= \lim_{\epsilon\to 0} \frac{I_{\Phi_{v_\xi},\zeta}(\epsilon)}{\epsilon^{\sigma_0}} = \frac{\zeta(0,0)}{m+1}\int_{\mathbb{R}} \Phi^+_{v_\xi,pr}(x,1)^{-\sigma_0} + \Phi^+_{v_\xi,pr}(x,-1)^{-\sigma_0}\mathrm{d}x \\ d_0 &= \lim_{\tau\to\infty} \frac{J_{\Phi_{v_\xi},\zeta}(\tau)}{\tau^{-\sigma_0}} = \sigma_0\Gamma(\sigma_0)\left(e^{i\frac{\pi\sigma_0}{2}}c_0 + e^{-i\frac{\pi\sigma_0}{2}}C_0\right), \end{aligned} \tag{5.4}$$

*where $C_0$ is computed similarly by replacing $\Phi_{v_\xi}$ by its negative.*

Since $\partial_y^3 w(P) = 0$ for all $P \in K_3$ (see Lemma 5.4), the decay of $J$ is the slowest on $K_3$. Recalling that $K_3 \subseteq \bigcap_{\alpha\in(1,2)} E_\alpha$, it is of interest to determine $d_0(\alpha)$ on $K_3$.

**Lemma 5.7.** *Let $\xi \in K_3$ and let $\zeta \in C_c^\infty$ be supported in a small neighborhood around $\xi$ in which $\xi$ is the unique critical point of $\Phi_{v_\xi}$. Then,*

$$d_0(\alpha) = c\cdot\zeta(\xi)\alpha^{-\frac{3}{4}}(2-\alpha)^{-\frac{1}{4}},$$

*where $c \in \mathbb{C}\setminus\{0\}$ is independent of $\zeta$ and $\alpha$.*

*Proof.* Without loss of generality, let $\xi = (\frac{\pi}{2},\frac{\pi}{2})$. Define $k_1 = \frac{e_1(\xi)+e_2(\xi)}{\sqrt{2}}$, $k_2 = \frac{-e_1(\xi)+e_2(\xi)}{\sqrt{2}}$ where the linear span of $\{k_1,k_2\}$ is coordinatized in $(x,y)$. Using $(\xi_1,\xi_2)$ as a coordinate for $\{e_1(0), e_2(0)\}$, we have

$$\xi_1 = \frac{\pi}{2} + \frac{x-y}{\sqrt{2}},\ \xi_2 = \frac{\pi}{2} + \frac{x+y}{\sqrt{2}}.$$

We first claim $(x,y)$ defines a superadapted system for $\Phi_{v_\xi}$ at $(x,y) = (0,0)$. Note that $\Phi_{v_\xi,pr} = -w_{pr}$ and

$$w(x,y) = \left\{\sin^2\left(\frac{1}{2}\left(\frac{\pi}{2} + \frac{x-y}{\sqrt{2}}\right)\right) + \sin^2\left(\frac{1}{2}\left(\frac{\pi}{2} + \frac{x+y}{\sqrt{2}}\right)\right)\right\}^{\frac{\alpha}{2}}.$$

The second order derivatives are

$$\partial_{xx}w(\xi) = -\frac{\alpha}{8}(2-\alpha),\ \partial_{xy}w(\xi) = \partial_{yy}w(\xi) = 0, \tag{5.5}$$

and the third order derivatives are

$$\partial_{yyy}w(\xi) = \partial_{xxy}w(\xi) = 0,\ \partial_{xyy}w(\xi) = -\frac{\alpha}{4\sqrt{2}},\ \partial_{xxx}w(\xi) = \frac{\alpha(\alpha^2-6\alpha+4)}{16\sqrt{2}}.$$

By direct computation, it can be verified that $\partial_y^j w(\xi) = 0$ for all $j \geq 2$. These derivatives determine the Newton polyhedron, Newton diagram, and $w_{pr}$ given by

$$\begin{aligned} \mathcal{N} &= \{2x + y \geq 4,\ x \geq 1,\ y \geq 0\},\ \mathcal{N}_d = \{2x + y = 4,\ 1 \leq x \leq 2\}, \\ \Phi_{v_\xi, pr} &= -w_{pr}(x, y) = \frac{\alpha(2-\alpha)}{16}x^2 + \frac{\alpha}{8\sqrt{2}}xy^2 =: Ax^2 + Bxy^2. \end{aligned} \tag{5.6}$$

The bisectrix intersects $\mathcal{N}_d$ at $x = y = \frac{4}{3} = d(\Phi_{v_\xi}) > 1$. Since $\Phi_{v_\xi, pr}(x, \pm 1) = Ax^2 + Bx$ has two real roots, $\{0, -\frac{B}{A}\}$, our first claim has been shown. It follows immediately from Lemma 5.6 that $r_0 = \sigma_0 = \frac{1}{d(\Phi_{v_\xi})} = \frac{3}{4}$. Moreover, $c_0' = C_0' = 0$ and

$$\begin{aligned} c_0 &= \frac{2}{3}\zeta(0,0)\int_{\mathbb{R}} \Phi^+_{v_\xi, pr}(x, 1)^{-\frac{3}{4}} + \Phi^+_{v_\xi, pr}(x, -1)^{-\frac{3}{4}}\mathrm{d}x \\ &= \frac{4}{3}\zeta(0,0)\left(\int_{-\infty}^{-\frac{B}{A}} (Ax^2 + Bx)^{-\frac{3}{4}}\mathrm{d}x + \int_0^\infty (Ax^2 + Bx)^{-\frac{3}{4}}\mathrm{d}x\right) \\ &= \frac{2^{\frac{27}{4}}\pi^{\frac{1}{2}}\Gamma(\frac{1}{4})}{3\Gamma(\frac{3}{4})}\zeta(0,0)\alpha^{-\frac{3}{4}}(2-\alpha)^{-\frac{1}{4}}, \end{aligned} \tag{5.7}$$

by using (5.6); the computation of $C_0$, which amounts to replacing $\Phi_{v_\xi}$ by $-\Phi_{v_\xi}$, is similar and thus is omitted. The conclusion of lemma follows immediately from the explicit form of $d_0$ in (5.4) and (5.7). □

*Remark 5.1.* If $\alpha = 2$, $\partial_{xx}w(\xi) = 0$ by (5.5). Consequently, the quadratic term of $\Phi_{v_\xi, pr}$ vanishes and the Newton diagram is given by

$$\mathcal{N}_d = \{x + y = 3,\ 1 \leq x \leq 3\}.$$

By analogy with the proof of Lemma 5.7, it is expected that the reciprocal of the distance of this diagram, $\frac{2}{3}$, yields the sharp decay rate of the corresponding oscillatory integral. Indeed, this expectation coincides with the result obtained in [28].

**Lemma 5.8.** *For all $\xi \in E_\alpha \setminus K_3$, let $\zeta$ be as Lemma 5.7 such that its support does not intersect $K_3$. Then, we have*

$$d_0 = c \cdot \zeta(\xi)|\tilde{h}(\xi, \alpha)\partial_y h(\xi, \alpha)|^{-\frac{1}{3}}|TrD^2 w(\xi)|^{-\frac{1}{6}}, \tag{5.8}$$

*where $c \in \mathbb{C} \setminus \{0\}$ is independent of $\zeta, \alpha, \xi$.*

*Proof.* By taking $\partial_y = k_2(\xi) \cdot \nabla_\xi$ on $H$,

$$\partial_y H(\xi, \alpha) = \tilde{h}(\xi, \alpha)\partial_y h(\xi, \alpha) = \partial_x^2 w(\xi)\partial_y^3 w(\xi),$$

by $h(\xi, \alpha), \partial_y^2 w(\xi) = 0$. Since the trace of a matrix is the sum of eigenvalues, $TrD^2 w(\xi) = \partial_x^2 w(\xi)$, and therefore

$$\partial_y^3 w(\xi) = \frac{\tilde{h}(\xi, \alpha)\partial_y h(\xi, \alpha)}{TrD^2 w(\xi)}. \tag{5.9}$$

By Lemma 5.5, $\partial_x^2 w(\xi), \partial_y^3 w(\xi) \neq 0$, which yields $\sigma_0 = r_0 = \frac{5}{6}$ and

$$w_{pr}(x, y) = \frac{\partial_x^2 w(\xi)}{2} x^2 + \frac{\partial_y^3 w(\xi)}{6} y^3$$
$$\pm \Phi_{v_\xi, pr}(x, \pm 1) = \pm \frac{\partial_x^2 w(\xi)}{2} x^2 \pm \frac{\partial_y^3 w(\xi)}{6},$$

and therefore the coordinate system $(x, y)$ is superadapted. By Lemma 5.6, it suffices to compute $c_0$, given by

$$\begin{aligned} c_0 &= c \cdot \zeta(\xi) \int_{\mathbb{R}} \Phi^+_{v_\xi, pr}(x, 1)^{-\frac{5}{6}} + \Phi^+_{v_\xi, pr}(x, -1)^{-\frac{5}{6}} \mathrm{d}x \\ &= c \cdot \zeta(\xi) |\partial_x^2 w(\xi)|^{-\frac{1}{2}} |\partial_y^3 w(\xi)|^{-\frac{1}{3}} \\ &= c \cdot \zeta(\xi) |\tilde{h}(\xi, \alpha) \partial_y h(\xi, \alpha)|^{-\frac{1}{3}} |Tr D^2 w(\xi)|^{-\frac{1}{6}}, \end{aligned}$$

where the last equation is by (5.9). □

It is insightful to apply (5.8) to obtain the series expansion of $d_0$. For $\{(a, b) : h(a, b, \alpha) = 0\}$, it suffices to consider $a \geq b$ or $a \leq b$ by the symmetry of $h$ under $(a, b) \mapsto (b, a)$. Define the two roots of $h(a, b, \alpha) = 0$ in terms of $a, \alpha$ as

$$\begin{aligned} B_P(a, \alpha) &= \frac{4a - \alpha a^2 - (2 - \alpha) + \sqrt{(\alpha a^2 - 4a + 2 - \alpha)^2 - 4a^2 \alpha (2 - \alpha)}}{2a\alpha}, \quad a \in [-1, 0) \\ B(a, \alpha) &= \frac{4a - \alpha a^2 - (2 - \alpha) - \sqrt{(\alpha a^2 - 4a + 2 - \alpha)^2 - 4a^2 \alpha (2 - \alpha)}}{2a\alpha}, \quad a \in [\frac{2 - \alpha}{\alpha}, 1]. \end{aligned} \tag{5.10}$$

Several comments regarding $B_P$, $B$ are summarized below. The following lemma can be verified by direct computation using (5.10).

**Lemma 5.9.** *For all $\alpha \in (1, 2)$, the curve $a \mapsto B_P(a, \alpha)$ parametrizes $\Gamma^1_{(a,b)}(\alpha) \cap \{a \leq b\}$ and $a \mapsto B(a, \alpha)$ parametrizes $\Gamma^2_{(a,b)}(\alpha) \cap \{a \geq b\}$. The pointwise convergence $\lim_{a \to 0-} B_P(a, \alpha) = 0$, $\lim_{\alpha \to 2-} B_P(a, \alpha) = 0$ holds. Furthermore, $B(\cdot, \alpha)$ obtains the global maxima on $[\frac{2-\alpha}{\alpha}, 1]$ at the boundary where $B(\frac{2-\alpha}{\alpha}, \alpha) = B(1, \alpha) = \frac{2-\alpha}{\alpha}$. The global minimum is obtained at $a_m = (\frac{2-\alpha}{\alpha})^{\frac{1}{2}}$ and $B(a_m, \alpha) \geq 1 - (1 + \sqrt{2})(\alpha - 1)$.*

**Corollary 5.1.** *Consider $d_0 = d_0(a, B_P(a, \alpha), \alpha, \zeta)$ defined on $\Gamma^1_{(a,b)}(\alpha) \cap \{a \leq b\}$ and let $\zeta \in C_c^\infty$ be supported in a small neighborhood around $(a, B_P(a, \alpha))$ excluding $(a, b) = (0, 0)$. Then, for some $c \in \mathbb{C} \setminus \{0\}$ independent of $a, \alpha, \zeta$,*

$$d_0 = c \cdot \zeta(a, B_P(a, \alpha)) |a|^{-\frac{1}{3}} \sum_{j=0}^{\infty} a_j(\alpha) |a|^j \tag{5.11}$$

*holds where the series converges absolutely for all $a \in [-1, 0)$. The coefficients $\{a_j\}$ can be computed explicitly; for example, $a_0(\alpha) = c'(2 - \alpha)^{-\frac{1}{6}} \alpha^{-\frac{5}{6}}$ where $c'$ is a nonzero numerical constant independent of $\zeta, \alpha$.*

*Proof.* The series expansion (5.11) is shown by the general formula (5.8). The pointwise absolute convergence on $a \in [-1, 0)$ follows from the analyticity of the RHS of (5.8) on $\xi \in M$ corresponding to $\Gamma^1_{(a,b)}(\alpha) \cap \{a \leq b\}$. □

*Remark 5.2.* By direct computation, $a_j(\alpha)$ contains the term $(2-\alpha)^{-p_j}$ for all $j \geq 0$ for some $p_j > 0$, and therefore one obtains a singular behavior of the leading term of $J$ as $\alpha \to 2-$. Another interesting regime is when $\xi \to P$, or equivalently $a \to 0-$, along $E_\alpha$. A qualitative difference between a cusp ($\xi = P$) and a fold ($\xi \neq P$) is manifested quantitatively by the blow-up $|a|^{-\frac{1}{3}}$ as $a \to 0-$.

On the other hand, consider the case $\alpha \to 1+$. By symmetry, consider $\Gamma^2_{(a,b)}(\alpha) \cap \{a \geq b\}$. The asymptotic behavior of $d_0(a(\alpha), B(a(\alpha), \alpha), \alpha)$ where $a(\alpha) \in \left[\frac{2-\alpha}{\alpha}, 1\right]$ is computed as $\alpha \to 1+$.

**Corollary 5.2.** *Consider $d_0 = d_0(a(\alpha), B(a(\alpha), \alpha), \alpha, \zeta)$ and let $\zeta \in C_c^\infty$ be supported in a small neighborhood around $(a(\alpha), B(a(\alpha), \alpha))$. Then, there exists $\alpha_0 > 1$ such that*

$$|d_0| \simeq |\zeta(a, B(a, \alpha))|(\alpha - 1)^{\frac{2}{3} - \frac{5\alpha}{12}},$$

*whenever $\alpha \in (1, \alpha_0]$ and $a \in [\frac{2-\alpha}{\alpha}, 1]$. Furthermore, the implicit constants depend only on $\alpha_0$.*

*Proof.* Let $a = 1 - \tilde{a},\ b = 1 - \tilde{b}$. By $a \in [\frac{2-\alpha}{\alpha}, 1]$ and Lemma 5.9,

$$0 \leq \tilde{a} \leq \frac{2}{\alpha}(\alpha - 1),\ \frac{2}{\alpha}(\alpha - 1) \leq \tilde{b} \leq (1 + \sqrt{2})(\alpha - 1). \tag{5.12}$$

Observe that the image of $\Gamma^2_{(a,b)}$ under the inverse cosine lies in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]^2$, and it can be verified that for $z \in [0, \frac{\pi}{2}]$,

$$\frac{z}{2} \leq \sin z \leq z,\ 1 - \frac{z^2}{2} \leq \cos z \leq 1 - \frac{z^2}{4},\ \sqrt{2}z^{\frac{1}{2}} \leq \cos^{-1}(1 - z) \leq 2z^{\frac{1}{2}}. \tag{5.13}$$

Define $(\xi_1, \xi_2) \in M$ as the inverse cosine of $(a(\alpha), B(a(\alpha), \alpha))$, respectively; for definiteness, let $\xi_i \geq 0$. In the polar coordinate where $r^2 = \xi_1^2 + \xi_2^2$, (5.13) is used to obtain

$$|\tilde{h}(\xi, \alpha)| \simeq \frac{\alpha^2}{w(\xi)^{\frac{8}{\alpha}-2}}(\cos \xi_1 + \cos \xi_2 - 2) \simeq r^{-6+2\alpha}, \tag{5.14}$$

where we neglect any powers of $\alpha$ since they can be uniformly bounded for $\alpha \in (1, 2)$. To estimate $TrD^2w$, define $(\tilde{r}, \tilde{\theta})$ such that $\tilde{r}^2 = \tilde{a}^2 + \tilde{b}^2,\ \tan \tilde{\theta} = \frac{\tilde{b}}{\tilde{a}}$. Then,

$$\begin{aligned}|TrD^2w| &\simeq r^{\alpha-4}|\alpha(a^2 + b^2) + 2((b - 2)a + (a - 2)b) + 4 - 2\alpha| \\ &= r^{\alpha-4}| - 2\alpha(\tilde{a} + \tilde{b}) + \alpha(\tilde{a}^2 + \tilde{b}^2) + 4\tilde{a}\tilde{b}|\end{aligned}$$

We claim

$$-2\alpha(\tilde{a}+\tilde{b}) \leq -2\alpha(\tilde{a}+\tilde{b}) + \alpha(\tilde{a}^2+\tilde{b}^2) + 4\tilde{a}\tilde{b} \leq -\alpha(\tilde{a}+\tilde{b}).$$

The lower bound is trivial since $\tilde{a}, \tilde{b} \geq 0$. The upper bound is equivalent to

$$\frac{\tilde{a}^2+\tilde{b}^2}{\tilde{a}+\tilde{b}} = \frac{\tilde{r}}{\sin\tilde{\theta}+\cos\tilde{\theta}} \leq \frac{\alpha}{\alpha+2},$$

which holds uniformly on $\alpha \in (1,2)$ if $\tilde{r} \leq \frac{1}{3}$. Hence for all $\tilde{a}, \tilde{b}$ sufficiently small,

$$|TrD^2 w| \simeq r^{\alpha-4}(\tilde{a}+\tilde{b}) \simeq r^{\alpha-4}(\alpha-1), \tag{5.15}$$

by (5.12). Likewise for sufficiently small $\tilde{a}, \tilde{b}$

$$|\partial_y h| \simeq (\alpha-1)\tilde{b}^{\frac{1}{2}} \simeq (\alpha-1)^{\frac{3}{2}}. \tag{5.16}$$

For all $\epsilon > 0$, since $\Gamma^2_{(a,b)}(\alpha) \subseteq \{(a,b) \in [1-\epsilon, 1] \times [1-\epsilon, 1]\}$ whenever $\alpha \in (1, \alpha_0(\epsilon)]$ for some $\alpha_0(\epsilon) > 1$, there exists $\alpha_0 > 1$ sufficiently close to 1 such that all small angle approximations are justified (see (5.13)) and

$$\begin{aligned}\frac{|d_0|}{c \cdot \zeta(a(\alpha), B(a(\alpha), \alpha))} &= |\tilde{h}(\xi_1, \xi_2, \alpha)\partial_y h(\xi_1, \xi_2, \alpha)|^{-\frac{1}{3}} |TrD^2 w(\xi_1, \xi_2)|^{-\frac{1}{6}} \\ &\simeq (\alpha-1)^{-\frac{2}{3}} r^{\frac{8}{3}-\frac{5\alpha}{6}}.\end{aligned}$$

by (5.8), (5.14), (5.15), and (5.16). Combining with

$$r \simeq |\xi_1| + |\xi_2| = \cos^{-1}(1-\tilde{a}) + \cos^{-1}(1-\tilde{b}) \simeq (\alpha-1)^{\frac{1}{2}},$$

the proof is complete. □

*Remark 5.3.* As can be seen in Fig. 1, the trajectory $\alpha \mapsto \left(\frac{2-\alpha}{\alpha}, \frac{2-\alpha}{\alpha}\right)$ traces the intersection of $\Gamma^2_{(a,b)}$ and the bisectrix. For $a(\alpha) = \frac{2-\alpha}{\alpha}$, an explicit computation yields

$$d_0 = c \cdot \zeta(a(\alpha), a(\alpha)) 2^{-\frac{5\alpha}{12}} \alpha^{-(\frac{7}{6}-\frac{5\alpha}{12})} (\alpha-1)^{\frac{2}{3}-\frac{5\alpha}{12}} (2-\alpha)^{-\frac{1}{2}},$$

for some $c \in \mathbb{C} \setminus \{0\}$ independent of $\alpha, \zeta$. Suppose $supp(\zeta)$ is sufficiently small such that $\zeta(a(\alpha), a(\alpha)) = 1$. Then, note that $d_0 \xrightarrow[\alpha \to 2-]{} \infty$ as $\left(\frac{2-\alpha}{\alpha}, \frac{2-\alpha}{\alpha}\right)$ approaches the origin, which corresponds to the cusp $K_3$. Furthermore, $d_0 \xrightarrow[\alpha \to 1+]{} 0$ as $\left(\frac{2-\alpha}{\alpha}, \frac{2-\alpha}{\alpha}\right)$ approaches $(a,b) = (1,1)$, which corresponds to the origin of $\mathbb{T}^2$ where $w(\xi)$ blows up.

Another example of trajectory, given by $(a,b) = (1, \frac{2-\alpha}{\alpha})$ with the leading term

$$d_0 = c \cdot \zeta\left(1, \frac{2-\alpha}{\alpha}\right) \alpha^{-\frac{5}{12}(4-\alpha)} (\alpha-1)^{\frac{2}{3}-\frac{5\alpha}{12}}, \tag{5.17}$$

shows a different qualitative behavior as $\alpha \to 2-$.

*Proof of Proposition 3.4.* For all $\tau > 0$,

$$\sup_{v\in\mathbb{R}^2} |J_{\Phi_v,\eta(\frac{\cdot}{N})}| \leq \|\eta\|_{L^1(\mathbb{T}^2)} N^2,$$

by the triangle inequality. Hence $\tau \geq 1$.

Considering $supp\left(\eta(\frac{\cdot}{N})\right) = \{|\xi| \in [\frac{\pi}{2}N, 2\pi N]\}$ and

$$\min_{\xi\in K_2\cup K_3} |\xi| = \cos^{-1}\left(\frac{2-\alpha}{\alpha}\right), \quad \max_{\xi\in\Gamma^2_{(\xi_1,\xi_2)}(\alpha)} |\xi| = \sqrt{2}\cos^{-1}\left(\frac{2-\alpha}{\alpha}\right)$$

obtained at

$$\left\{(\xi_1,\xi_2) : \left(\pm\cos^{-1}\left(\frac{2-\alpha}{\alpha}\right), 0\right), \left(0, \pm\cos^{-1}\left(\frac{2-\alpha}{\alpha}\right)\right)\right\},$$
$$\left\{(\xi_1,\xi_2) : \pm\cos^{-1}\left(\frac{2-\alpha}{\alpha}\right), \pm\cos^{-1}\left(\frac{2-\alpha}{\alpha}\right)\right\} \subseteq M,$$

respectively, define $N_\alpha \in 2^{\mathbb{Z}}$ to be the largest number satisfying

$$2\pi N_\alpha < r_\alpha := \cos^{-1}\left(\frac{2-\alpha}{\alpha}\right).$$

Note that $N_\alpha$ increases as $\alpha$ increases with $\lim_{\alpha\to 1+} N_\alpha = 0$ and $N_2 = 2^{-3}$. Using the support condition of $\eta(\frac{\cdot}{N})$, the set of $N \in 2^{\mathbb{Z}}$ that satisfies the RHS of (3.10) is given by

$$\begin{aligned}
supp\left(\eta\left(\frac{\cdot}{N}\right)\right) \cap K_3 \neq \emptyset &\iff N \in S_3 := \{2^0, 2^{-1}\}\\
supp\left(\eta\left(\frac{\cdot}{N}\right)\right) \cap (K_2 \setminus K_3) \neq \emptyset &\iff N \in S_2 := \left[\frac{1}{2\pi}r_\alpha, \frac{2\sqrt{2}}{\pi}r_\alpha\right] \setminus S_3\\
supp\left(\eta\left(\frac{\cdot}{N}\right)\right) \cap (K_1 \setminus (K_2\cup K_3)) \neq \emptyset &\iff N \in S_1 := \left(\left(\frac{2\sqrt{2}}{\pi}r_\alpha, 2^{-2}\right] \cup (0, N_\alpha]\right) \setminus (S_2\cup S_3).
\end{aligned} \tag{5.18}$$

Suppose $N > N_\alpha$. For every $\xi \in \mathbb{T}^2 \backslash B(0, \frac{r_\alpha}{2})$, there exists a neighborhood $\Omega_\xi(\alpha)$ containing $\xi$ and a constant $C_\xi(\alpha) > 0$ such that for all $\zeta \in C_c^\infty(\Omega_\xi)$,

$$\sup_{v\in\mathbb{R}^2} |J_{\Phi_v,\zeta}| \leq C_\xi \|\zeta\|_{C^3(\mathbb{R}^2)} \tau^{-\frac{1}{h(\Phi_{v_\xi})}}, \tag{5.19}$$

by [21, Theorem 1.1]. By compactness, an open cover $\{\Omega_\xi\}$ of $\mathbb{T}^2 \backslash B(0, \frac{r_\alpha}{2})$ reduces to a finite subcover $\{\Omega_{\xi_j}\}_{j=1}^{n_0}$, where $n_0 = n_0(\alpha) \in \mathbb{N}$, and let $\{\phi_j\}_{j=1}^{n_0}$ be a ($\alpha$-dependent) partition of unity subordinate to the finite subcover. Note that if $\xi \in K_1$, then $U_\xi \cap K_2 = \emptyset$, and if $\xi \in K_2$, then $U_\xi \cap K_3 = \emptyset$ since the oscillatory indices of $\Phi_{v_\xi}$ at $\xi$ are distinct on each $K_j$. Consequently each $\xi \in K_3$ contributes to the finite subcover for all $\alpha \in (1, 2)$.

Let $\eta_{N,j}(\cdot) = \eta(\frac{\cdot}{N})\phi_j(\cdot)$. Then, $\|\eta_{N,j}\|_{C^3} \lesssim N^{-3}$ where the implicit constant depends on the given partition of unity. Since $N > N_\alpha$, we have $N^{-3} \le C(\alpha)N^{2-\frac{3}{4}\alpha}$ where $C(\alpha) = N_\alpha^{-5+\frac{3}{4}\alpha}$.

By Lemma 5.5 and $\tau \ge 1$, the slowest decay occurs on $K_3$ with $h(\Phi_v) = \frac{4}{3}$. By Lemma 5.5, (5.19), and the triangle inequality,

$$\begin{aligned}\sup_{v\in\mathbb{R}^2} |J_{\Phi_v,\eta(\frac{\cdot}{N})}| &\le \sup_{v\in\mathbb{R}^2} \sum_{j=1}^{n_0} |J_{\Phi_v,\eta_{N,j}}| \\ &\le \left(\sum_{j=1}^{n_0} C_{\xi_j}\|\eta_{N,j}\|_{C^3}\right)\tau^{-\frac{3}{4}} \lesssim_\alpha \sum_{j=1}^{n_0} C_{\xi_j}\cdot N^{2-\frac{3}{4}\alpha}\tau^{-\frac{3}{4}}. \quad (5.20)\end{aligned}$$

For $N > N_\alpha$, a similar argument using the partition of unity and (5.19) yields (3.10) with sharp decay rates $\sigma_0 \in \{\frac{3}{4}, \frac{5}{6}, 1\}$.

Suppose $N \le N_\alpha \le \frac{1}{8}$. Recalling that $v = \frac{x}{h\tau}$, do a change of variable $\xi \mapsto N\xi$ to obtain

$$\sup_{v\in\mathbb{R}^2} |J_{\Phi_v,\eta(\frac{\cdot}{N})}(\tau)| = N^2 \sup_{x\in\mathbb{R}^2} \left|\int_{\mathbb{R}^2} e^{i(x\cdot\xi - \tau w(N\xi))}\eta(\xi)\mathrm{d}\xi\right|.$$

By adopting the proof of [3, Proposition 1], we obtain sharp dispersive estimates of a free solution governed by a non-smooth, non-homogeneous dispersion relation. A change of variables

$$z_i = \frac{2}{N}\sin\left(\frac{N\xi_i}{2}\right),\ \xi_i = \frac{2}{N}\sin^{-1}\left(\frac{Nz_i}{2}\right),\ \tau \mapsto 2^\alpha\tau,$$

with $J_c(z) = \left(\left(1-\left(\frac{Nz_1}{2}\right)^2\right)\left(1-\left(\frac{Nz_2}{2}\right)^2\right)\right)^{-\frac{1}{2}}$, yields

$$\begin{aligned}N^2\int_{\mathbb{R}^2} e^{i(x\cdot\xi-\tau w(N\xi))}\eta(\xi)\mathrm{d}\xi &= N^2\int_{\mathbb{R}^2} e^{i(x\cdot\xi(z)-\tau N^\alpha\rho^\alpha)}\eta(\xi(z))J_c(z)\mathrm{d}z \\ &= N^2\int_0^\infty e^{-i\tau N^\alpha\rho^\alpha}G(\rho,x,N)\rho\mathrm{d}\rho := I,\end{aligned}$$

where we denote $(r,\theta)$ and $(\rho,\phi)$ as the polar coordinates for $x = (x_1,x_2)$ and $z = (z_1,z_2)$, respectively, and

$$\begin{aligned}G(\rho,x,N) = G(\rho) &= \int_{S^1} e^{ix\cdot\xi(z)}\eta(\xi(z))J_c(z)\mathrm{d}\phi(z) \\ &= \int_0^{2\pi} e^{i\lambda\Phi_G(\phi)}\eta(\xi(\rho,\phi))J_c(z(\rho,\phi))\mathrm{d}\phi,\end{aligned}$$

where $\lambda = \rho r$ and

$$\Phi_G(\phi) = \frac{2}{\rho N}\left(\cos\theta\sin^{-1}\left(\frac{N\rho\cos\phi}{2}\right) + \sin\theta\sin^{-1}\left(\frac{N\rho\sin\phi}{2}\right)\right). \quad (5.21)$$

By the support condition of $\eta$,

$$\left(\frac{N\pi}{4}\right)^2 \leq \sin^{-1}\left(\frac{Nz_1}{2}\right)^2 + \sin^{-1}\left(\frac{Nz_2}{2}\right)^2 \leq (N\pi)^2,$$

and the small angle approximation $z \leq \sin^{-1} z \leq 2z$ on $z \in [0, \frac{1}{\sqrt{2}}]$, one obtains

$$\frac{\pi}{4} \leq \rho \leq 2\pi,$$

since $\frac{N|z_i|}{2} = |\sin(\frac{N\xi_i}{2})| \leq \frac{1}{\sqrt{2}}$. When clear in context, we use the same symbols $\eta$, $J_c$ for the representations in different variables. We prove

$$|I| \lesssim (\alpha - 1)^{-\frac{1}{2}} N^{2-\alpha}\tau^{-1}, \tag{5.22}$$

from which the proof is complete by interpolating with the trivial bound $|I| \lesssim N^2$.

Let $r_0 > 0$, independent of $N$, to be specified later and suppose $r \leq r_0$. Integration by parts yields

$$I = \frac{N^{2-\alpha}}{i\alpha\tau} \int e^{-i\tau N^\alpha \rho^\alpha} \partial_\rho(G(\rho)\rho^{2-\alpha})\mathrm{d}\rho. \tag{5.23}$$

Since $|G| \lesssim 1$ and the domain of integration is supported away from the origin,

$$|G(\rho)\partial_\rho(\rho^{2-\alpha})| \lesssim 1.$$

By the chain rule $\partial_\rho = \cos\phi\partial_{z_1} + \sin\phi\partial_{z_2}$ and the estimate,

$$|\partial_{z_i} e^{ix\cdot\xi(z)}| = |\partial_{z_i} e^{i\frac{2x_i}{N}\sin^{-1}(\frac{Nz_i}{2})}| = \frac{|x_i|}{\sqrt{1-(\frac{Nz_i}{2})^2}} \leq \sqrt{2}|x_i| \leq \sqrt{2}r_0,$$

one obtains

$$\sup_{N\leq N_\alpha} \left| \int_{S^1} \partial_\rho \left(e^{ix\cdot\xi(z)}\right) \eta(\xi(z)) J_c(z)\mathrm{d}\phi(z) \right| \lesssim r_0.$$

By repeated applications of the product and chain rule,

$$\sup_{N\leq N_\alpha} |\partial_\rho^k \eta(\xi(z))| \lesssim_{k,\eta} 1, \quad \sup_{N\leq N_\alpha} |\partial_\rho^k J_c(z)| \lesssim_k 1,$$

for all $k \geq 0$ and therefore

$$|\partial_\rho(G(\rho)\rho^{2-\alpha})| \lesssim 1 + r_0,$$

altogether implying

$$|I| \lesssim_{r_0,\alpha} N^{2-\alpha}\tau^{-1}.$$

Suppose $r > r_0$. From (5.21),

$$\partial_\phi \Phi_G(\rho,\phi) = -\frac{\cos\theta\sin\phi}{\left(1-\left(\frac{N\rho\cos\phi}{2}\right)^2\right)^{\frac{1}{2}}} + \frac{\sin\theta\cos\phi}{\left(1-\left(\frac{N\rho\sin\phi}{2}\right)^2\right)^{\frac{1}{2}}}$$

$$\partial_\phi^2 \Phi_G(\rho,\phi) = -\left(1-\left(\frac{N\rho}{2}\right)^2\right)\left(\frac{\cos\theta\cos\phi}{(1-(\frac{N\rho\cos\phi}{2})^2)^{\frac{3}{2}}} + \frac{\sin\theta\sin\phi}{\left(1-\left(\frac{N\rho\sin\phi}{2}\right)^2\right)^{\frac{3}{2}}}\right).$$

For a fixed $\rho > 0$, denote $\Phi_G(\phi) = \Phi_G(\rho,\phi)$. The critical points correspond to the roots of $\partial_\phi \Phi_G$, and are the solutions to

$$g(\rho,\phi)\tan\phi = \tan\theta,\ g(\rho,\phi) := \left(\frac{1-\left(\frac{N\rho\sin\phi}{2}\right)^2}{1-\left(\frac{N\rho\cos\phi}{2}\right)^2}\right)^{\frac{1}{2}}. \tag{5.24}$$

By direct computation, $g(\rho,\cdot)$ is a strictly positive $\pi$-periodic even function such that $g(\rho, \frac{\pi}{4}+\frac{k\pi}{2}) = 1$ for all $k \in \mathbb{Z}$. Furthermore, $g(\rho,\phi) > 1$ if $\phi \in [0, \frac{\pi}{4})$, $g(\rho,\phi) < 1$ if $\phi \in (\frac{\pi}{4}, \frac{\pi}{2}]$, and

$$\partial_\phi|_{\phi=\frac{\pi}{4}} (g(\rho,\cdot)\tan(\cdot)) = 4 - \frac{16}{8-N^2\rho^2} > 0.$$

Therefore, $\tan\phi < g(\rho,\phi)\tan\phi < 1$ on $(0,\frac{\pi}{4})$ and $g(\rho,\phi)\tan\phi < \tan\phi$ on $(\frac{\pi}{4},\frac{\pi}{2})$. By symmetry, suppose $x_i \geq 0$ for $i = 1, 2$, and therefore $0 \leq \tan\theta \leq \infty$. By graphing $\phi \mapsto g(\rho,\phi)\tan\phi$ on $[0, 2\pi)$, there exist two solutions, $\phi_\pm(\rho)$, where $\phi_+ \in [0, \frac{\pi}{2}]$ and $\phi_- = \phi_+ + \pi$. A crude estimate $|\phi_\pm - \theta_\pm| \leq \frac{\pi}{4}$ follows from a further inspection of the graph where $\theta_+ = \theta,\ \theta_- = \theta + \pi$.

The critical points are non-degenerate. If $\phi$ satisfies $\partial_\phi^2 \Phi_G = 0$, then $g(\rho,\phi)^3\cot\phi = -\tan\theta$. Assuming $\partial_\phi^2\Phi_G(\phi_\pm) = 0$ and substituting (5.24), we have $g(\rho,\phi_\pm)^2 + \tan^2\phi_\pm = 0$, a contradiction. Since $\phi_\pm$ and $\theta_\pm$ are in the same quadrants, we have $\cos\theta_\pm\cos\phi_\pm,\ \sin\theta_\pm\sin\phi_\pm \geq 0$, and therefore

$$|\partial_\phi^2\Phi_G(\phi_\pm)| \simeq \cos(\phi_\pm - \theta_\pm) \simeq 1, \tag{5.25}$$

independent of $N, \rho$.

Construct $\chi_\pm \in C_c^\infty(\mathbb{R}_\phi)$ given by $\chi_+ = 1$ on $[0, \frac{\pi}{2}]$, supported in $(-\frac{\pi}{8}, \frac{5\pi}{8})$, $\chi_- = 1$ on $[\pi, \frac{3\pi}{2}]$, supported in $(\frac{7\pi}{8}, \frac{13\pi}{8})$, and define $\chi_0 = 1 - (\chi_+ + \chi_-)$. Since $|\phi_\pm - \theta_\pm| \leq \frac{\pi}{4}$, we have $\chi_\pm(\phi_\pm) = 1$ for all $N \leq N_\alpha$. Let $\tilde{\chi}_\pm := \eta J_c \chi_\pm$ and $\tilde{\chi}_0 := \eta J_c \chi_0$. Note that since

$$|\partial_\phi^k \eta|, |\partial_\phi^k J_c|, |\partial_\phi^k \chi_\pm| \lesssim_k 1, \tag{5.26}$$

for all $k \geq 0$ independent of $N, \rho$, so are the higher-order partial derivatives (in $\phi$) of $\tilde{\chi}_{\pm}$. Define

$$G_{\pm}(\lambda) = \int_0^{2\pi} e^{i\lambda\Phi_G(\phi)}\tilde{\chi}_{\pm}(\phi)\mathrm{d}\phi,$$

and similarly for $G_0$, and hence $G = G_+ + G_- + G_0$. By [29, Chapter VIII, Proposition 3], $G_{\pm}$ has the asymptotics as $\lambda \to \infty$ given by

$$G_{\pm}(\lambda) = \sqrt{\frac{2\pi}{|\partial_\phi^2 \Phi_G(\phi_{\pm})|}} e^{i(\lambda\Phi_G(\phi_{\pm}) - \frac{\pi}{4})}\tilde{\chi}_{\pm}(\phi_{\pm})\lambda^{-\frac{1}{2}} + \tilde{G}_{\pm}(\lambda). \tag{5.27}$$

More precisely, for all $k \in \mathbb{N} \cup \{0\}$, there exists $\lambda_0(k),\ C(k) > 0$ such that

$$|\partial_\lambda^k \tilde{G}_{\pm}| \leq C\lambda^{-(\frac{3}{2}+k)}, \tag{5.28}$$

for all $\lambda \geq \lambda_0$. Since the estimates (5.25), (5.26) are uniform with respect to $N, \rho$, the constants $\lambda_0, C$ can be chosen to be independent of $N, \rho$. Since $\rho \leq 2\pi$, let $r_0 \geq \frac{\max(\lambda_0(0), \lambda_0(1))}{2\pi}$.

Away from the critical points, the integral in $\phi$ yields a rapid decay in $\lambda$. We claim

$$|\partial_\lambda G_0| \lesssim_k \lambda^{-k}, \tag{5.29}$$

for all $\lambda > 0$ and $k \geq 1$ uniformly in $N, \rho$. Since

$$\partial_\lambda G_0 = i\int_0^{2\pi} e^{i\lambda\Phi_G(\phi)}\Phi_G(\phi)\tilde{\chi}_0(\phi)\mathrm{d}\phi = -\frac{1}{\lambda}\int_0^{2\pi} e^{i\lambda\Phi_G(\phi)}\partial_\phi\left(\frac{\Phi_G\tilde{\chi}_0}{\partial_\phi\Phi_G}\right)\mathrm{d}\phi,$$

and $|\partial_\phi\Phi_G| \geq |\sin(\phi - \theta)| \geq \sin(\frac{\pi}{8})$ for all $\phi \in [\frac{5\pi}{8}, \frac{7\pi}{8}] \cup [\frac{13\pi}{8}, \frac{15\pi}{8}]$ and $\theta \in [0, \frac{\pi}{2}] \cup [\pi, \frac{3\pi}{2}]$, (5.29) is shown for $k = 1$ by the triangle inequality; for $k \geq 2$, (5.29) is shown by repeated use of integration by parts.

For $r \geq r_0$, consider the integral in (5.23) with $G$ replaced by $G_0$. Since by (5.29),

$$|\partial_\rho G_0| = r|\partial_\lambda G_0| \lesssim \frac{r}{\lambda} = \frac{1}{\rho},$$

the integration by parts yields an estimate consistent with (5.22). By replacing $G$ by $\tilde{G}_{\pm}$ in the same integral, the bound (5.22) follows by (5.28).

It remains to show that $I$ with $G$ replaced by the leading term of $G_+$ in (5.27) satisfies (5.22); the analysis on $G_-$ is similar and therefore is omitted. Consider

$$II := N^2 r^{-\frac{1}{2}} \int_0^\infty e^{i\tau\Psi(\rho)}a(\rho)\mathrm{d}\rho$$

where

$$a(\rho) := \eta(\rho, \phi_+)J_c(\rho, \phi_+)\chi_+(\phi_+)|\partial_\phi^2\Phi_G(\phi_+)|^{-\frac{1}{2}}\rho^{-\frac{1}{2}}$$
$$\Psi(\rho) = -N^\alpha\rho^\alpha + \Phi_G(\phi_+)\frac{r\rho}{\tau}.$$

The region of integration

$$\{\rho \in [\frac{\pi}{4}, 2\pi] : \alpha N^{\alpha}\rho^{\alpha-1} < \frac{1}{2}|\partial_\rho(\Phi_G(\phi_+)\rho)|\frac{r}{\tau} \text{ or } \alpha N^{\alpha}\rho^{\alpha-1} > 2|\partial_\rho(\Phi_G(\phi_+)\rho)|\frac{r}{\tau}\}$$

is included in the case

$$\frac{r}{\tau} \gg N|\nabla w(N\xi)| \text{ or } \frac{r}{\tau} \ll N|\nabla w(N\xi)| \tag{5.30}$$

since $|\partial_\rho(\Phi_G(\phi_+)\rho)| \simeq 1$ uniformly in $N, \rho$ as can be observed in (5.31). For $r, \xi$ satisfying (5.30), the lower bound of the phase function on $supp(\eta) \subseteq \mathbb{R}^2_\xi$ is

$$|\nabla_\xi(\frac{x}{\tau}\cdot\xi - w(N\xi))| \geq \frac{N}{2}|\nabla w(N\xi)| \simeq_\alpha N^{\alpha}.$$

Let $E_i = supp(\eta)\cap\{|\partial_{\xi_i}(\frac{x}{\tau}\cdot\xi - w(N\xi))| \gtrsim N^{\alpha}\}$ for $i = 1, 2$. By direct computation,

$$|\partial^2_{\xi_i}(\frac{x}{\tau}\cdot\xi - w(N\xi))| \lesssim N^{\alpha},$$

and thus by integration by parts,

$$\begin{aligned} &N^2\left|\int_{\mathbb{R}^2} e^{i(x\cdot\xi-\tau w(N\xi))}\eta(\xi)\mathrm{d}\xi\right| \\ &\leq N^2\left(\left|\int_{E_1} e^{i(x\cdot\xi-\tau w(N\xi))}\eta(\xi)\mathrm{d}\xi\right| + \left|\int_{E_2} e^{i(x\cdot\xi-\tau w(N\xi))}\eta(\xi)\mathrm{d}\xi\right|\right) \\ &\lesssim N^{2-\alpha}\tau^{-1}. \end{aligned}$$

It suffices to assume $\frac{r}{\tau} \simeq N|\nabla w(N\xi)| \simeq N^{\alpha}(\xi_1^2+\xi_2^2)^{\frac{\alpha-1}{2}} \simeq N^{\alpha}$. By direct computation,

$$\begin{aligned} \partial_\rho(\Phi_G(\phi_+)\rho) &= \frac{\cos\theta\cos\phi_+}{\left(1-\left(\frac{N\rho\cos\phi_+}{2}\right)^2\right)^{\frac{1}{2}}} + \frac{\sin\theta\sin\phi_+}{\left(1-\left(\frac{N\rho\sin\phi_+}{2}\right)^2\right)^{\frac{1}{2}}} \\ \partial^2_\rho(\Phi_G(\phi_+)\rho) &= \frac{N^2\rho}{4}\left(\frac{\cos\theta\cos^3\phi_+}{\left(1-\left(\frac{N\rho\cos\phi_+}{2}\right)^2\right)^{\frac{3}{2}}} + \frac{\sin\theta\sin^3\phi_+}{\left(1-\left(\frac{N\rho\sin\phi_+}{2}\right)^2\right)^{\frac{3}{2}}}\right) \\ &\quad - \left(\frac{\cos\theta\sin\phi_+}{\left(1-\left(\frac{N\rho\cos\phi_+}{2}\right)^2\right)^{\frac{3}{2}}} - \frac{\sin\theta\cos\phi_+}{\left(1-\left(\frac{N\rho\sin\phi_+}{2}\right)^2\right)^{\frac{3}{2}}}\right)\partial_\rho\phi_+ =: III + IV. \end{aligned} \tag{5.31}$$

We claim $|\partial^2_\rho(\Phi_G(\phi_+)\rho)| \lesssim N^2$. Since $N\rho \leq \frac{\pi}{4}$ and $\theta, \phi_+ \in [0, \frac{\pi}{2}]$,

$$\sup_{\rho\in[\frac{\pi}{4},2\pi]} |III| \lesssim N^2.$$

Since $|IV| \lesssim |\partial_\rho \phi_+|$, it suffices to show

$$\sup_{\rho\in[\frac{\pi}{4},2\pi]} |\partial_\rho \phi_+| \lesssim N^2, \tag{5.32}$$

which follows from implicitly differentiating (5.24), thereby obtaining

$$\begin{aligned}\partial_\rho \phi_+(\rho) &= -\frac{\partial_\rho g(\rho,\phi_+)\cdot \tan\phi_+}{\partial_\phi g(\rho,\phi_+)\cdot\tan\phi_+ + g\sec^2\phi_+}\\ &= \frac{N^2\rho\sin(4\phi_+)}{(1-(\frac{N\rho}{2})^2)(16-N^2\rho^2(1-\cos(4\phi)))}.\end{aligned}$$

By the triangle inequality,

$$\begin{aligned}|\partial_\rho^2\Psi| &\geq \alpha(\alpha-1)N^\alpha\rho^{\alpha-2} - |\partial_\rho^2(\Phi_G(\phi_+)\rho)|\frac{r}{\tau}\\ &\gtrsim (\alpha-1)N^\alpha - N^2\frac{r}{\tau} \simeq (\alpha-1-N^2)N^\alpha\\ &\gtrsim (\alpha-1)N^\alpha,\end{aligned}$$

where the last inequality follows from $N \leq N_\alpha$. By the Van der Corput lemma [29, Chapter VIII],

$$|II| \lesssim (\alpha-1)^{-\frac{1}{2}}N^{2-\alpha}\tau^{-1}\left(\|a\|_{L^\infty([\frac{\pi}{4},2\pi])} + \|\partial_\rho a\|_{L^1([\frac{\pi}{4},2\pi])}\right), \tag{5.33}$$

since $\frac{r}{\tau} \simeq N^\alpha$. By (5.25), $a \in L^\infty([\frac{\pi}{4}, 2\pi])$ uniformly in $N$. To estimate $\partial_\rho a$, the term that needs most care is $\partial_\rho|\partial_\phi^2\Phi_G(\phi_+)|^{-\frac{1}{2}}$. Since $\phi_+, \theta \in [0, \frac{\pi}{2}]$, we have $\partial_\phi^2\Phi_G(\phi_+) \leq 0$. By (5.25), (5.32), the chain rule

$$\partial_\rho\left(\partial_\phi^2\Phi_G(\phi_+)\right) = \partial_{\rho\phi\phi}\Phi_G(\phi_+) + \partial_\phi^3\Phi_G(\phi_+)\cdot\partial_\rho\phi_+,$$

and the uniform bound

$$\sup_{N\leq N_\alpha}|\partial_\rho^{k_1}\partial_\phi^{k_2}\Phi_G| \lesssim_{k_1,k_2} 1,$$

we have $\partial_\rho a \in L^\infty([\frac{\pi}{4}, 2\pi])$ uniformly in $N$.

Lastly, we show (3.11). Let $C_3(\alpha) > 0$ satisfy

$$C_3(\alpha) = \inf\left\{C > 0 : \sup_{v\in\mathbb{R}^2}|J_{\Phi_v,\eta(\frac{\cdot}{N})}| \leq CN^{2-\frac{3}{4}\alpha}\tau^{-\frac{3}{4}},\ \forall\tau > 0, N \in S_3\right\},$$

and define $C_i(\alpha)$ similarly for $i = 1, 2$. By (5.33), (5.20), we have $\max_{1\leq i\leq 3} C_i(\alpha) < \infty$. For $\sigma_0 \in \{\frac{3}{4}, \frac{5}{6}, 1\}$ and $\xi \in supp(\eta(\frac{\cdot}{N}))$, we have

$$\lim_{\tau\to\infty}|J_{\Phi_{v_\xi},\eta(\frac{\cdot}{N})}|\tau^{\sigma_0} \leq C_i(\alpha). \tag{5.34}$$

The limit above is a constant multiple of the nonzero leading terms given by (3.16) due to the set of critical points of $\Phi_{v_\xi}$ whose cardinality is uniformly bounded above for all $\alpha \in (1, 2)$ by observing (3.15).

For $i = 3$, $\sigma_0 = \frac{3}{4}$, the nonzero contributions to the limit are due to the cusps in $K_3$. Let $N \in S_3$. By Lemma 5.7,

$$c \cdot \alpha^{-\frac{3}{4}}(2-\alpha)^{-\frac{1}{4}} \leq C_3(\alpha), \tag{5.35}$$

where $c > 0$ depends only on $\eta$.

For $i = 2$, $\sigma_0 = \frac{5}{6}$, let $N \in S_2$. For $\alpha$ sufficiently close to 2, we have $N = 2^{-2}$. Since $\eta(\frac{\xi(\alpha)}{2^{-2}}) \xrightarrow[\alpha\to 2-]{} 0$ for $\xi = (0, r_\alpha)$, we may replace $\eta(\frac{\cdot}{2^{-2}})$ by another smooth bump function $\tilde{\eta}$ supported in $\{|\xi| \in [\frac{\pi}{8}, \frac{\pi}{2} + \epsilon_0]\}$ where $\epsilon_0 > 0$ is sufficiently small so that $supp(\tilde{\eta}) \cap K_3 = \emptyset$. Arguing as (5.35) by using (5.17), one obtains

$$(\alpha-1)^{\frac{2}{3}-\frac{5\alpha}{12}} \lesssim_{\tilde{\eta}} C_2(\alpha).$$

On the contrary, suppose $\alpha > 1$ is not close to 2 such that $N \in S_2$ satisfies $N < 2^{-2}$. Then, there exists $N^{(\alpha)} \in S_2$ such that $\frac{2}{3\pi} r_\alpha \leq N^{(\alpha)} \leq \frac{4}{3\pi} r_\alpha$. Then, $|\eta(\frac{r_\alpha}{N^{(\alpha)}})| \geq c > 0$ where $c$ is independent of $\alpha$. Using the same example (5.17), one obtains

$$(\alpha-1)^{\frac{2}{3}-\frac{5\alpha}{12}} \lesssim_{\eta} C_2(\alpha).$$

Lastly, let $N \in S_1$ and $\sigma_0 = 1$. Pick $\xi \in \mathbb{T}^2$ such that $|\xi| = N\pi$. Arguing as (5.34) and invoking Lemma 5.2, we have

$$C_1(\alpha) \geq \sqrt{2\pi}|\eta(\frac{\xi}{N})| \cdot |H(\xi,\alpha)|^{-\frac{1}{2}} N^{\alpha-2} \gtrsim (\alpha-1)^{-\frac{1}{2}},$$

where the last inequality follows from using the small angle approximation (see (5.13), (5.12)) to obtain

$$|H(\xi,\alpha)| \simeq (\alpha-1)N^{-4+2\alpha}.$$

## 6. Conclusion and future work

We have shown, with a convergence rate, the continuum limit of DNLSE on $h\mathbb{Z}^2$ to the FNLSE on $\mathbb{R}^2$ as $h \to 0$ in the energy subcritical regime for finite time. Our proof employs sharp dispersive estimates that are obtained by studying appropriate degenerate oscillatory integrals. It is of interest to compare the sharp decay rate of $\sigma_0 = \frac{3}{4}$ to that in the discrete classical Schrödinger equation ($\sigma_0 = \frac{2}{3}$) and the discrete wave equation ($\sigma_0 = \frac{2}{3}$) at the cost of the best constants blowing up as $\alpha \to 1+, 2-$. As for future work, it is of interest to extend to the case of mixed fractional derivatives [4] where (3.2), in dimension two, is replaced by an appropriate discrete analog of

$$\left(-\frac{\partial^2}{\partial x_1^2}\right)^{\frac{\alpha_1}{2}} + \left(-\frac{\partial^2}{\partial x_2^2}\right)^{\frac{\alpha_2}{2}}.$$

By numerical and asymptotic techniques, we will explore the conditions of highly localized states in the discrete models that may relate to finite-time blow-up solutions in the continuum limit. □


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## Declarations

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