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ORIGINAL PAPER

A remake of Bourgain–Brezis–Mironescu characterization of Sobolev spaces

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Abstract
We introduce a large class of concentrated p-Lévy integrable functions approximating the unity, which serves as the core tool from which we provide a nonlocal characterization of the Sobolev spaces and the space of functions of bounded variation via nonlocal energies forms. It turns out that this nonlocal characterization is a necessary and sufficient criterion to define Sobolev spaces on domains satisfying the extension property. We also examine the general case where the extension property does not necessarily hold. In the latter case we establish weak convergence of the nonlocal Radon measures involved to the local Radon measures induced by the distributional gradient.

Keywords Nonlocal energy forms · p-Lévy integrability · Sobolev spaces · Bounded variation spaces · Extension domains

Mathematics Subject Classification 26B30 · 46B45 · 46E27 · 46E30 · 46E35

1 Introduction

Let Ω be an open subset of $\mathbb{R}^d$, $d \geq 1$ and $1 \leq p < \infty$. We aim to provide a nonlocal characterization of first order Sobolev spaces on Ω using the following type nonlocal energy forms

$$\mathcal{E}^i_\Omega(u) = \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x) - u(y)|^p a^i_\Omega(x, y)\nu(x - y)\, \mathrm{d}y\, \mathrm{d}x, \quad i = 1, 2, 3, \tag{1.1}$$

where, $\nu : \mathbb{R}^d \setminus \{0\} \to [0, \infty)$ is measurable and satisfies the p-Lévy integrability condition

$$\int\limits_{\mathbb{R}^d} (1 \wedge |h|^p)\nu(h)\, \mathrm{d}h < \infty\,, \tag{1.2}$$

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and $a^1_\Omega(x,y)=\min(\mathbb{1}_\Omega(x),\mathbb{1}_\Omega(y))$, $a^2_\Omega(x,y)=\max(\mathbb{1}_\Omega(x),\mathbb{1}_\Omega(y))$ and $a^3_\Omega(x,y)=\frac{1}{2}(\mathbb{1}_\Omega(x)+\mathbb{1}_\Omega(y))$; where $\mathbb{1}_\Omega$ is the indicator function of Ω. Here and in what follows, the notation $a\wedge b$ stands for $\min(a,b)$, $a,b\in\mathbb{R}$. More explicitly, we can write the forms $\mathcal{E}^i_\Omega$, as follows

$$\mathcal{E}^1_\Omega(u)=\iint\limits_{\Omega\Omega}|u(x)-u(y)|^p\nu(x-y)\,\mathrm{d}y\,\mathrm{d}x,$$

$$\mathcal{E}^2_\Omega(u)=\iint\limits_{\mathcal{G}(\Omega)}|u(x)-u(y)|^p\nu(x-y)\,\mathrm{d}y\,\mathrm{d}x,\quad(\mathcal{G}(\Omega)=(\mathbb{R}^d\times\mathbb{R}^d)\setminus(\Omega^c\times\Omega^c)),$$

$$\mathcal{E}^3_\Omega(u)=\iint\limits_{\Omega\,\mathbb{R}^d}|u(x)-u(y)|^p\nu(x-y)\,\mathrm{d}y\,\mathrm{d}x.$$

Note in passing that $\mathcal{E}^1_\Omega\le\mathcal{E}^i_\Omega$, $\mathcal{E}^1_{\mathbb{R}^d}=\mathcal{E}^2_{\mathbb{R}^d}=\mathcal{E}^3_{\mathbb{R}^d}$ and $\frac{1}{2}\mathcal{E}^2_\Omega\le\mathcal{E}^3_\Omega\le\mathcal{E}^2_\Omega$ since $\frac{1}{2}a^2_\Omega\le a^3_\Omega\le a^2_\Omega$. The nonlocal forms $\mathcal{E}^i_\Omega$ are crucial in the study of Integro-Differential Equations (IDEs) involving nonlocal operators of p-Lévy types; see for instance the recent works [12, 16, 18]. For $p=2$, (1.2) is the well-known Lévy integrability condition. Actually, when ν is radial, the p-Lévy integrability (1.2) condition is consistent and self-generated in the sense that condition (1.2) holds true if and only if $\mathcal{E}^1_{\mathbb{R}^d}(u)<\infty$ for all $u\in C^\infty_c(\mathbb{R}^d)$; see Sect. 2.1 for the details. In addition, the p-Lévy integrability condition (1.2) indicates that ν is allowed to have a heavy singularity at the origin. For instance, $\nu(h)=|h|^{-d-sp}$ satisfies the condition (1.2) if and only if $s\in(0,1)$.

Next, to reach our goal, we need to introduce a general class of approximation of the unity by p-Lévy integrable functions. To be more precise, our standing approximation tool consists of a family of p-Lévy integrable functions, $(\nu_\varepsilon)_\varepsilon$ satisfying, for each $\varepsilon>0$ and every $\delta>0$,

$$\nu_\varepsilon\ge0\,\text{is radial},\ \int\limits_{\mathbb{R}^d}(1\wedge|h|^p)\nu_\varepsilon(h)\,\mathrm{d}h=1\quad\text{and}\quad\lim_{\varepsilon\to0}\int\limits_{|h|>\delta}(1\wedge|h|^p)\nu_\varepsilon(h)\,\mathrm{d}h=0.\qquad(1.3)$$

For instance, assume ν is radial and $\int_{\mathbb{R}^d}(1\wedge|h|^p)\nu(h)\,\mathrm{d}h=1$ then (see Proposition 2.2) one obtains a remarkable family $(\nu_\varepsilon)_\varepsilon$ satisfying (1.3) by the following rescaling

$$\nu_\varepsilon(h)=\begin{cases}\varepsilon^{-d-p}\nu\big(h/\varepsilon\big) & \text{if } |h|\le\varepsilon,\\ \varepsilon^{-d}|h|^{-p}\nu\big(h/\varepsilon\big) & \text{if } \varepsilon<|h|\le1,\\ \varepsilon^{-d}\nu\big(h/\varepsilon\big) & \text{if } |h|>1.\end{cases}$$

Another sub-class of $(\nu_\varepsilon)_\varepsilon$ satisfying (1.3) is obtained by putting $\nu_\varepsilon(h)=c_\varepsilon|h|^{-p}\rho_\varepsilon(h)$, where $(\rho_\varepsilon)_\varepsilon$ is a family of integrable functions approximating the unity, i.e., for each $\varepsilon>0$ and every $\delta>0$,

$$\rho_\varepsilon\ge0\ \text{is radial},\quad\int\limits_{\mathbb{R}^d}\rho_\varepsilon(h)\,\mathrm{d}h=1\quad\text{and}\quad\lim_{\varepsilon\to0}\int\limits_{|h|>\delta}\rho_\varepsilon(h)\,\mathrm{d}h=0,\qquad(1.4)$$

and $c_\varepsilon>0$ is a suitable norming constant for which the integrability condition in (1.3) is verified. From this perspective, the class of approximation of the unity $(\rho_\varepsilon)_\varepsilon$ satisfying (1.4) can be viewed as a subclass of $(\nu_\varepsilon)_\varepsilon$ satisfying (1.3). However, the converse is not warranted. In other words the class $(\nu_\varepsilon)_\varepsilon$ is more general than the class $(\rho_\varepsilon)_\varepsilon$. This is because the family $(\nu_\varepsilon)_\varepsilon$ also includes families of the forms $(c_\varepsilon|h|^{-p}\rho_\varepsilon(h))_\varepsilon$ for which $\rho_\varepsilon{}'s$ are not integrable. For a simple example, consider $\nu_\varepsilon(h)=a_{\varepsilon,d,p}|h|^{-d-(1-\varepsilon)p}$ (see Example 2.6) then $(\nu_\varepsilon)_\varepsilon$

satisfies (1.3) but there is no family $(\rho_\varepsilon)_\varepsilon$ satisfying (1.4) such that $\nu_\varepsilon(h) = c_\varepsilon|h|^{-p}\rho_\varepsilon(h)$. Viewed in the sense of the correspondence $(\rho_\varepsilon)_\varepsilon \mapsto (\nu_\varepsilon)_\varepsilon$ with $\nu_\varepsilon = c_\varepsilon|h|^{-p}\rho_\varepsilon$, the class of $(\nu_\varepsilon)_\varepsilon$ is therefore strictly lager than that of $(\rho_\varepsilon)_\varepsilon$.

We emphasize that our main goal is to characterize Sobolev spaces on an open set $\Omega \subset \mathbb{R}^d$ using a sequence $(\nu_\varepsilon)_\varepsilon$ satisfying (1.3). Let us recall that the Sobolev space $W^{1,p}(\Omega)$ is the Banach space of functions $u \in L^p(\Omega)$ whose first order distributional derivatives belong to $L^p(\Omega)$, with the norm $\|u\|_{W^{1,p}(\Omega)} = (\|u\|^p_{L^p(\Omega)} + \|\nabla u\|^p_{L^p(\Omega)})^{1/p}$. Another space of particular interest, that emerges naturally as a generalization of $W^{1,1}(\Omega)$ is the so called space of bounded variations $BV(\Omega)$. The space $BV(\Omega)$ consists in functions $u \in L^1(\Omega)$ with bounded variation, i.e., $|u|_{BV(\Omega)} < \infty$ where

$$|u|_{BV(\Omega)} := \sup\Big\{ \int_\Omega u(x)\mathrm{div}\phi(x)\,\mathrm{d}x \,:\, \phi \in C_c^\infty(\Omega, \mathbb{R}^d),\ \|\phi\|_{L^\infty(\Omega,\mathbb{R}^d)} \le 1\Big\}. \tag{1.5}$$

The space $BV(\Omega)$ is a Banach space under the norm $\|u\|_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + |u|_{BV(\Omega)}$. We denote the distributional derivative of a function $u \in BV(\Omega)$ by ∇u. Roughly speaking, $\nabla u = (\Lambda_1, \Lambda_2, \cdots, \Lambda_d)$ is a vector valued Radon measure on Ω such that

$$\int_\Omega u(x)\frac{\partial\varphi}{\partial x_i}(x)\,\mathrm{d}x = -\int_\Omega \varphi(x)\,\mathrm{d}\Lambda_i(x), \quad \text{for all} \quad \varphi \in C_c^\infty(\Omega),\ \ i = 1, \cdots, d.$$

The quantity $|\nabla u| = (\Lambda_1^2 + \cdots + \Lambda_d^2)^{1/2}$ is a positive Radon measure whose value on an open set $U \subset \Omega$ is $|\nabla u|(U) = |u|_{BV(U)}$. Conventionally, we put $\|\nabla u\|_{L^p(\Omega)} = \infty$ if $|\nabla u|$ is not in $L^p(\Omega)$ with $1 < p < \infty$ and for $p = 1$, $|u|_{BV(\Omega)} = \infty$ if the measure $|\nabla u|$ does not have a finite total variation. Note that, if $u \in W^{1,1}(\Omega)$ then $u \in BV(\Omega)$, $\partial_{x_i}u(x)\,\mathrm{d}x = \mathrm{d}\Lambda_i(x)$ and $|u|_{BV(\Omega)} = \|\nabla u\|_{L^1(\Omega)}$. Indeed, since $u \in W^{1,1}(\Omega)$, the integration by part implies

$$|u|_{BV(\Omega)} = \sup\Big\{ \int_\Omega \nabla u(x)\cdot\phi(x)\,\mathrm{d}x \,:\, \phi \in C_c^\infty(\Omega, \mathbb{R}^d),\ \|\phi\|_{L^\infty(\Omega,\mathbb{R}^d)} \le 1\Big\} \le \|\nabla u\|_{L^1(\Omega)}.$$

Conversely, since $\partial_{x_i}u \in L^1(\Omega)$, take a sequence $(\chi_n)_n \subset C_c^\infty(\Omega, \mathbb{R}^d)$, converging to ∇u in $L^1(\Omega, \mathbb{R}^d)$ and a.e. in Ω. Define $\chi_n^\varepsilon \in C_c^\infty(\Omega, \mathbb{R}^d)$, $\varepsilon > 0$ by $\chi_n^\varepsilon = \chi_n(|\chi_n|^2 + \varepsilon^2)^{-1/2}$, so that $\|\chi_n^\varepsilon\|_{L^\infty(\Omega,\mathbb{R}^d)} \le 1$. The convergence dominated theorem and the integration by parts imply that

$$\int_\Omega |\nabla u(x)|^2(|\nabla u(x)|^2 + \varepsilon^2)^{-1/2}\,\mathrm{d}x = \lim_{n\to\infty}\Big|\int_\Omega u(x) \div \chi_n^\varepsilon(x)\,\mathrm{d}x\Big| \le |u|_{BV(\Omega)}.$$

Whence Fatou's lemma implies

$$\int_\Omega |\nabla u(x)|\,\mathrm{d}x \le \liminf_{\varepsilon\to 0}\int_\Omega |\nabla u(x)|^2(|\nabla u(x)|^2 + \varepsilon^2)^{-1/2}\,\mathrm{d}x \le |u|_{BV(\Omega)}.$$

We are now in position to state our first result.

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^d$ be open and $u \in L^p(\Omega)$ such that*

$$A_p := \liminf_{\varepsilon\to 0}\iint_{\Omega\Omega} |u(x) - u(y)|^p\nu_\varepsilon(x - y)\,\mathrm{d}y\,\mathrm{d}x < \infty. \tag{1.6}$$

Then $u \in W^{1,p}(\Omega)$ *for* $1 < p < \infty$ *and* $u \in BV(\Omega)$ *for* $p = 1$. *Moreover, there hold the estimates*

$$\|\nabla u\|_{L^p(\Omega)} \leq d^2 \frac{A_p^{1/p}}{K_{d,1}} \qquad \text{and} \qquad |u|_{BV(\Omega)} \leq d^2 \frac{A_1}{K_{d,1}}. \tag{1.7}$$

The constant $K_{d,1}$ appearing in (1.7) is a universal constant independent of the geometry of Ω and is given by the following general mean value formula over the unit sphere

$$K_{d,p} = \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} |w \cdot e|^p \, \mathrm{d}\sigma_{d-1}(w) = \frac{\Gamma\big(\frac{d}{2}\big)\Gamma\big(\frac{p+1}{2}\big)}{\Gamma\big(\frac{d+p}{2}\big)\Gamma\big(\frac{1}{2}\big)}, \tag{1.8}$$

for any unit vector $e \in \mathbb{S}^{d-1}$; see Proposition 3.10 for the computation. The constant $K_{d,p}$ also appears in [5]. There is a similar constant in [22, Section 7] when studying nonlocal approximations of the p-Laplacian. Observe that in general, for every $z \in \mathbb{R}^d$, we have

$$\frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} |w \cdot z|^p \, \mathrm{d}\sigma_{d-1}(w) = |z|^p \frac{1}{|\mathbb{S}^{d-1}|} \int_{\mathbb{S}^{d-1}} |w \cdot e|^p \, \mathrm{d}\sigma_{d-1}(w) = |z|^p K_{d,p}. \tag{1.9}$$

Theorem 1.1 yields the following nonlocal characterization of constant functions; see also [7].

Theorem 1.2 *Assume* $\Omega \subset \mathbb{R}^d$ *is open and connected. If* $u \in L^p(\Omega)$, $1 \leq p < \infty$, *is such that* $A_p = 0$ *then* u *is almost everywhere constant on* Ω.

Let us now comment about Theorem 1.1. Observing that, $\mathcal{E}^1_\Omega(u) \leq \mathcal{E}^i_\Omega(u), i = 1, 2, 3$, Theorem 1.1 obviously remains true if the nonlocal forms of type $\mathcal{E}^1_\Omega$ are replaced with those of type $\mathcal{E}^2_\Omega$ or $\mathcal{E}^3_\Omega$. It is to be noted that, Theorem 1.1 is governed by two fundamental counter intuitive remarks. Firstly, the lack of reflexivity of $L^1(\Omega)$ implies that, in the case $p = 1$, the function belongs to $BV(\Omega)$ and not necessarily in $W^{1,1}(\Omega)$. In other words, assuming $A_1 < \infty$ is not enough to conclude that $u \in W^{1,1}(\Omega)$. We give here a mere counterexample in one dimension; see Counterexample 1 for the general case. For $d = 1$ and $p \geq 1$ we consider,

$$\Omega = (-1, 0) \cup (0, 1), \; u(x) = \frac{1}{2}\mathbb{1}_{[0,1)}(x) - \frac{1}{2}\mathbb{1}_{(-1,0)}(x) \text{ and } \nu_\varepsilon(h) = \frac{p\varepsilon(1-\varepsilon)}{2|h|^{1+(1-\varepsilon)p}}. \tag{1.10}$$

For $p = 1$, it is straightforwards to verify that $u \in BV(-1, 1)\backslash W^{1,1}(-1, 1)$ whereas we find that $A_1 = 1$. The second remark indicates that, the converse of Theorem 1.1 is not necessarily true in general. Indeed, by adopting the above example in (1.10), see also the counterexample 1, we find that $u \in W^{1,p}(\Omega)$ while $A_p = \infty$ for $p > 1$. A reasonable explanation to the latter matter is that, $\Omega = (-1, 0) \cup (0, 1)$ is not an extension $W^{1,p}$-domain. To put it another way, this situation in particular (and in general) occurs due to the lack of the regularity of the boundary $\partial\Omega$. Therefore, to investigate the converse of Theorem 1.1, we need some additional assumption on Ω such as the extension property. It is noteworthy to recall that $\Omega \subset \mathbb{R}^d$ is called to be a $W^{1,p}$-extension (resp. a BV-extension) domain if there exists a linear operator $E : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ (resp. $E : BV(\Omega) \to BV(\mathbb{R}^d)$) and a constant $C := C(d, p, \Omega)$ depending only on the domain Ω and the dimension d such that

$$Eu\,|_\Omega = u \quad \text{and} \quad \|Eu\|_{W^{1,p}(\mathbb{R}^d)} \leq C\|u\|_{W^{1,p}(\Omega)} \quad \text{for all} \quad u \in W^{1,p}(\Omega)$$

$$(\text{resp.} \quad Eu\,|_\Omega = u \quad \text{and} \quad \|Eu\|_{BV(\mathbb{R}^d)} \leq C\|u\|_{BV(\Omega)} \quad \text{for all} \quad u \in BV(\Omega)).$$

Examples of extension domains include bounded Lipschitz domains which are both $W^{1,p}$-extension and BV-extension domains. In particular euclidean balls and rectangles in $\mathbb{R}^d$ are extension domains. The upper half space $\mathbb{R}^d_+ = \{(x', x_d) \in \mathbb{R}^d \,:\, x_d > 0\}$ is a simple example of an unbounded extension domain. The geometric characterization of extension domains has been extensively studied in the last decades. The $W^{1,p}$-extension property of an open set Ω infers certain regularity of the boundary $\partial\Omega$. For instance, according to [21, Theorem 2] a $W^{1,p}$-extension domain $\Omega \subset \mathbb{R}^d$ is necessarily is a d-set, i.e., satisfies the volume density condition, viz., there exists a constant $c > 0$ such that $|\Omega \cap B(x,r)| \geq cr^d$ for all $x \in \partial\Omega$ and $0 < r < 1$. In virtue of the Lebesgue differentiation theorem one finds that a d-set Ω is a Jordan set [36], i.e., its boundary $\partial\Omega$ has Lebesgue measure zero, $|\partial\Omega| = 0$. Subsequently, for a $W^{1,p}$-extension domain Ω there holds

$$\int_{\partial\Omega} |\nabla Eu(x)|^p \,\mathrm{d}x = 0 \quad \text{for all} u \in W^{1,p}(\Omega). \tag{1.11}$$

To the best of our knowledge, the question whether the geometric characterization (1.11) remains true for a BV-extension domain is still unknown. However, thanks to [21, Lemma 2.4] or [17, Theorem 1.3] we know that every $W^{1,1}$-extension is a BV-extension domain. Throughout this article, we require a BV-extension domain Ω to satisfy the condition

$$|\nabla Eu|(\partial\Omega) = \int_{\mathbb{R}^d} \mathbb{1}_{\partial\Omega}(x) \,\mathrm{d}|\nabla Eu|(x) = 0 \quad \text{for all } u \in BV(\Omega). \tag{1.12}$$

It is to be noted that, in contrast to (1.11), having $|\partial\Omega| = 0$ does not necessarily imply (1.12). Indeed, it suffices to consider once more the example (1.10) where one gets $\nabla u = \delta_0$ (the Dirac measure at the origin), so that $|\nabla u|(\partial\Omega) = 1$. Some authors rather define a BV-extension domain together with the condition (1.12); see for instance [1, 17]. Extended discussions on BV-extension domains can be found in [23, 24]. Several references on extension domains for Sobolev spaces can be found in [35]. Our second main result, which is an improved converse of Theorem (1.1), reads as follows.

Theorem 1.3 *Assume $\Omega \subset \mathbb{R}^d$ is a $W^{1,p}$-extension domain. If $u \in L^p(\Omega)$ with $1 < p < \infty$ or $p = 1$ and $u \in W^{1,1}(\Omega)$ then we have*

$$\lim_{\varepsilon \to 0} \iint_{\Omega\Omega} |u(x) - u(y)|^p \nu_\varepsilon(x-y) \,\mathrm{d}y \,\mathrm{d}x = K_{d,p} \|\nabla u\|^p_{L^p(\Omega)}. \tag{1.13}$$

Moreover if $p = 1$ and Ω is a BV-extension domain then for $u \in L^1(\Omega)$ we have

$$\lim_{\varepsilon \to 0} \iint_{\Omega\Omega} |u(x) - u(y)| \nu_\varepsilon(x-y) \,\mathrm{d}y \,\mathrm{d}x = K_{d,1} |u|_{BV(\Omega)}. \tag{1.14}$$

We highlight that the counterexample 1 shows that the conclusion of Theorem 1.3 might be erroneous if Ω is not an extension domain. In one way of proving Theorem 1.3, we establish the following sharp version of the estimates in (1.7) (see Theorem 3.3)

$$\|\nabla u\|^p_{L^p(\Omega)} \leq \frac{A_p}{K_{d,p}} \qquad \text{and} \qquad |u|_{BV(\Omega)} \leq \frac{A_1}{K_{d,1}}. \tag{1.15}$$

Indeed, Theorem 1.3 shows that the estimates in (1.15) turn into equalities provided that Ω is an extension domain. As immediate consequences of Theorem 1.1 and Theorem 1.3 we have the following characterizations for the spaces $W^{1,p}(\Omega)$ and $BV(\Omega)$ when Ω is an extension domain.

Theorem 1.4 *Assume $\Omega \subset \mathbb{R}^d$ is a $W^{1,p}$-extension domain, $p > 1$ and let $u \in L^p(\Omega)$. Then $u \in W^{1,p}(\Omega)$ if and only if $A_p < \infty$. Moreover, we have*

$$\lim_{\varepsilon \to 0} \iint_{\Omega\Omega} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\, \mathrm{d}y\, \mathrm{d}x = K_{d,p} \|\nabla u\|^p_{L^p(\Omega)}.$$

Theorem 1.4′ *Assume $\Omega \subset \mathbb{R}^d$ is a BV-extension domain, $p = 1$, and let $u \in L^1(\Omega)$. Then $u \in BV(\Omega)$ if and only if $A_1 < \infty$. Moreover, we have*

$$\lim_{\varepsilon \to 0} \iint_{\Omega\Omega} |u(x) - u(y)| \nu_\varepsilon(x-y) \mathrm{d}y \mathrm{d}x = K_{d,1} |u|_{BV(\Omega)}.$$

In contrast to the forms of type $\mathcal{E}^1_\Omega$, the collapse phenomenon across $\partial\Omega$ occurs for the forms of type $\mathcal{E}^2_\Omega$ or $\mathcal{E}^3_\Omega$ in Theorem 1.3.

Theorem 1.5 *Assume that: (i) $\Omega \subset \mathbb{R}^d$ is a $W^{1,p}$-extension domain or (ii) $\mathbb{R}^d \setminus \overline{\Omega}$ is a $W^{1,p}$-extension domain and $\partial\Omega = \partial\overline{\Omega}$. For $u \in W^{1,p}(\mathbb{R}^d)$, $p \geq 1$, there hold the following limits:*

$$\lim_{\varepsilon \to 0} \iint_{\Omega\Omega^c} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\, \mathrm{d}y\, \mathrm{d}x = 0,$$

$$\lim_{\varepsilon \to 0} \iint_{\Omega\Omega} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\, \mathrm{d}y\, \mathrm{d}x = K_{d,p} \|\nabla u\|^p_{L^p(\Omega)},$$

$$\lim_{\varepsilon \to 0} \iint_{\Omega\,\mathbb{R}^d} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\, \mathrm{d}y\, \mathrm{d}x = K_{d,p} \|\nabla u\|^p_{L^p(\Omega)},$$

$$\lim_{\varepsilon \to 0} \iint_{(\Omega^c \times \Omega^c)^c} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\, \mathrm{d}y\, \mathrm{d}x = K_{d,p} \|\nabla u\|^p_{L^p(\Omega)}.$$

Moreover, for $u \in BV(\mathbb{R}^d)$ and $p = 1$ the above limits remain true provided that $|\nabla u|(\partial\Omega) = 0$.

Proof In fact, in both cases (i) and (ii) we have $|\partial\Omega| = |\partial\overline{\Omega}| = 0$. Thus, Theorem 3.5 yields the first limit. For the case (ii), the remaining limits follow from Theorem 1.3 since $\Omega \times \Omega = (\mathbb{R}^d \times \mathbb{R}^d) \setminus [\Omega^c \times \Omega \cup \Omega \times \Omega^c \cup \Omega^c \times \Omega^c]$, $\Omega \times \mathbb{R}^d = \Omega \times \Omega \cup \Omega \times \Omega^c$ and $(\Omega^c \times \Omega^c)^c = (\mathbb{R}^d \times \mathbb{R}^d) \setminus (\Omega^c \times \Omega^c)$. The case (i) is obtain by interchanging Ω and Ω^c. The situation $u \in BV(\mathbb{R}^d)$ is analogous. □

The next result, is an alternative to Theorem 1.3 if Ω is not an extension domain.

Theorem 1.6 *Let $\Omega \subset \mathbb{R}^d$ be open. Let $u \in W^{1,p}(\Omega)$ and define the Radon measures*

$$\mathrm{d}\mu_\varepsilon(x) = \int_\Omega |u(x) - u(y)|^p \nu_\varepsilon(x-y) \mathrm{d}y\, \mathrm{d}x.$$

The sequence $(\mu_\varepsilon)_\varepsilon$ converges weakly on Ω (in the sense of Radon measures) to the Radon measure $\mathrm{d}\mu(x) = K_{d,p} |\nabla u(x)|^p\, \mathrm{d}x$, i.e., $\mu_\varepsilon(E) \xrightarrow{\varepsilon \to 0} \mu(E)$ for every compact set $E \subset \Omega$. Moreover, if $p = 1$ and $u \in BV(\Omega)$ then $\mathrm{d}\mu(x) = K_{d,1}\, \mathrm{d}|\nabla u|(x)$.

Note in particular that, Theorem 1.6 implies that $(\mu_\varepsilon)_\varepsilon$ vaguely convergence to μ, i.e.

$$\lim_{\varepsilon\to 0}\int_\Omega \varphi(x)\,\mathrm{d}\mu_\varepsilon(x) = \int_\Omega \varphi(x)\,\mathrm{d}\mu(x), \quad \text{for every}\varphi \in C_c^\infty(\Omega).$$

Let us comment on Theorem 1.1 and Theorem 1.3 and related results in the literature. Bourgain–Brezis–Mironescu [5, Theorem 3' & Theorem 2] proved the characterization Theorem 1.1 under the stronger condition that $\Omega \subset \mathbb{R}^d$ is bounded Lipschitz and while considering the sub-class $\nu_\varepsilon(h) = c_\varepsilon |h|^{-p}\rho_\varepsilon(h)$ where $(\rho_\varepsilon)_\varepsilon$ satisfies (1.4). Beside this, with the same assumptions, Bourgain–Brezis–Mironescu in [5, Theorem 2] also established the relation (1.13). The case $\Omega = \mathbb{R}^d$ is also investigated by Brezis [7] while characterizing constant functions. The case $p = 1$, i.e., the relation (1.14), is also a natural subject of discussions in [5] wherein, the authors succeeded in the one dimensional setting when $\Omega = (0, 1)$, viz., they proved that

$$\int_0^1\int_0^1 \frac{|u(x)-u(y)|}{|x-y|}\rho_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x = K_{1,1}|u|_{BV(0,1)} \quad \text{for all } u \in BV(0,1).$$

The general case $d \geq 2$ was completed later in [9] when Ω is a bounded Lipschitz domain. In this perspective, [9, Lemma 2] also established a variant of Theorem 1.6 for the case $p = 1$. Clearly, our setting of Theorem 1.1 is more general as no restriction on Ω is required and, in the sense mentioned above, the class $(\nu_\varepsilon)_\varepsilon$ satisfying (1.3) is strictly lager than that of $(\rho_\varepsilon)_\varepsilon$ satisfying (1.4). In addition, in contrast to [5], Ω is not necessarily bounded in Theorem 1.3 and that the situation where Ω has a Lipschitz boundary appears as a particular case of Theorem 1.3. We point out that Theorem 1.3 is reminiscent of [19, Theorem 3.4] for $p = 2$. Ultimately, let us mention that, after the release of the first version of this work, the authors of [3] brought to our attention that they also established the relation (1.13) when $\nu_\varepsilon(h) = \varepsilon|h|^{-d-(1-\varepsilon)p}$ (fractional kernels) for $1 < p < \infty$. The case $p = 1$ is, however, not fully covered therein. Our approach in this paper extends the works from [5, 7, 9, 28]. In the wake of [5], several works regarding the characterization of Sobolev spaces and alike spaces have emerged in the recent years. For example [27, 29] for characterization of Sobolev spaces via families of anisotropic interacting kernels, [26, 30] for characterization of BV spaces, [33] for a study of asymptotic sharp fractional Sobolev inequality, [6] for characterization of Besov type spaces of higher order and [19] for the study of Mosco convergence of nonlocal quadratic forms.

This article is organized as follows. In the second section we address some examples of approximating sequence $(\nu_\varepsilon)_\varepsilon$ and some nonlocal spaces in connection with function of type ν_ε. The third section is devoted to the proofs of Theorem 1.1, Theorem 1.3 and Theorem 1.6.

Throughout this article, $\varepsilon > 0$ is a small quantity tending to 0. We frequently use the convex inequality $(a+b)^p \leq 2^{p-1}(a^p + b^p)$ for $a > 0, b > 0$, the Euclidean scalar product of $x = (x_1, x_2, \cdots, x_d) \in \mathbb{R}^d$ and $y = (y_1, y_2, \cdots, y_d) \in \mathbb{R}^d$ is $x\cdot y = x_1 x_1 + x_2 y_2 + \cdots + x_d y_d$ and denote the norm of x by $|x| = \sqrt{x\cdot x}$. The conjugate of $p \in [1,\infty)$ is denoted by p', i.e. $p + p' = pp'$ with the convention $1' = \infty$. Throughout, $|\mathbb{S}^{d-1}|$ denotes the area of the $d-1$-dimensional unit sphere, where we adopt the convention that $|\mathbb{S}^{d-1}| = 2$ if $d = 1$.

2 Preliminaries

2.1 *p*-Lévy integrability and approximation of Dirac measure

Definition 2.1 (i) A nonnegative Borel measure $\nu(\mathrm{d}h)$ on $\mathbb{R}^d$ is called a p-Lévy measure if $\nu(\{0\}) = 0$ and it satisfies the p-Lévy integrability condition; that is to say that

$$\int_{\mathbb{R}^d} (1 \wedge |h|^p)\nu(\mathrm{d}h) < \infty.$$

(ii) A family $(\nu_\varepsilon)_\varepsilon$ satisfying (1.3) is called *a Dirac approximation of p-Lévy measures.*

Patently, one recovers the usual definition of Lévy measures when $p = 2$. Such measures are paramount in the study of stochastic process of Lévy type; see for instance [2, 4, 32] for further details. We intentionally omit the dependence of ν and ν_ε on p. This dependence will be always clear from the context. The following result shows that by rescaling appropriately a radial p-Lévy integrable function $\nu(h)$ one obtains a family $(\nu_\varepsilon)_\varepsilon$ satisfying (1.3).

Proposition 2.2 *Let $\nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p)$ with $\nu \geq 0$. Define the rescaled family $(\nu_\varepsilon)_\varepsilon$, as*

$$\nu_\varepsilon(h) = \begin{cases} \varepsilon^{-d-p}\nu\big(h/\varepsilon\big) & \text{if } |h| \leq \varepsilon, \\ \varepsilon^{-d}|h|^{-p}\nu\big(h/\varepsilon\big) & \text{if } \varepsilon < |h| \leq 1, \\ \varepsilon^{-d}\nu\big(h/\varepsilon\big) & \text{if } |h| > 1. \end{cases} \tag{2.1}$$

Then for every $\delta > 0$, $\varepsilon \in (0, 1)$

$$\int_{\mathbb{R}^d} (1 \wedge |h|^p)\nu_\varepsilon(h)\,\mathrm{d}h = \int_{\mathbb{R}^d} (1 \wedge |h|^p)\nu(h)\,\mathrm{d}h \quad \text{and} \quad \lim_{\varepsilon \to 0} \int_{|h|>\delta} (1 \wedge |h|^p)\nu_\varepsilon(h)\,\mathrm{d}h = 0\,.$$

Proof Since $\nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p)$ the dominated convergence theorem yields

$$\lim_{\varepsilon \to 0} \int_{|h|>\delta} (1 \wedge |h|^p)\nu_\varepsilon(h)\,\mathrm{d}h = \lim_{\varepsilon \to 0} \int_{|h|>\delta/\varepsilon} (1 \wedge |h|^p)\nu(h)\,\mathrm{d}h = 0.$$

We omit the remaining details as it solely involves straightforward computations. □

The behavior of the rescaled family $(\nu_\varepsilon)_\varepsilon$ in (2.1) when $p = 2$ is governed by two keys observations. The first is that it gives rise to a family of Lévy measures with a concentration property at the origin. Secondly, from a probabilistic point of view one obtains a family of pure jumps Lévy processes $(X_\varepsilon)_\varepsilon$ each associated with the measure $\nu_\varepsilon(h)\mathrm{d}h$ from a Lévy process X associated with $\nu(h)\mathrm{d}h$. In fact, the family of stochastic processes $(X_\varepsilon)_\varepsilon$ converges in finite dimensional distributional sense (see [19]) to a Brownian motion provided that one in addition assumes that ν is radial. Proposition 2.5 (ii) below shows that the generator of the process X_ε denoted L_ε (see (2.4)), converges to $-\frac{1}{2d}\Delta$ which is the generator of a Brownian motion. In short, rescaling via (2.1) any isotropic pure jump Lévy process leads to a Brownian motion. This could be one more argument to back up the ubiquity of the Brownian motion. The convergence highlighted above is involved in a more significant context. For example in [19], the convergence in Mosco sense of the Dirichlet forms associated with process in play is established. Beside these observations, the works [16, 18] establish that if Ω is bounded with a Lipschitz boundary and u_ε satisfies in the weak sense nonlocal problems of the $L_\varepsilon u_\varepsilon = f$ in Ω augmented with Dirichlet condition $u_\varepsilon = 0$ on Ω^c (resp. Neumann condition $\mathcal{N} u_\varepsilon = 0$

on Ω^c) condition then $(u_\varepsilon)_\varepsilon$ converges in $L^2(\Omega)$ to some $u \in W^{1,2}(\Omega)$, where u is the weak solution to the local problem $-\frac{1}{2d}\Delta u = f$ in Ω augmented with Dirichlet boundary $u = 0$ on $\partial\Omega$ (resp. Neumann condition $\frac{\partial u}{\partial n} = 0$ on $\partial\Omega$). Here, L_ε is given by (2.4) and $\mathcal{N}_\varepsilon$ is defined by

$$\mathcal{N}_\varepsilon u(x) := \int_\Omega (u(x) - u(y))\nu_\varepsilon(x-y)\,\mathrm{d}y.$$

Remark 2.3 Assume the family $(\nu_\varepsilon)_\varepsilon$ satisfies (1.3). Let $\beta \in \mathbb{R}$, then for all $R > 0$ we have

$$\lim_{\varepsilon\to 0} \int_{|h|\le R} (1 \wedge |h|^\beta)\nu_\varepsilon(h)\mathrm{d}h = \begin{cases} 0 & \text{if} \quad \beta > p \\ 1 & \text{if} \quad \beta = p \\ \infty & \text{if} \quad \beta < p \end{cases}.$$

Indeed, for fixed $\delta > 0$, (1.3) implies

$$\lim_{\varepsilon\to 0} \int_{\delta<|h|\le R} (1 \wedge |h|^p)\nu_\varepsilon(h)\mathrm{d}h \le \lim_{\varepsilon\to 0} \int_{|h|>\delta} (1 \wedge |h|^p)\nu_\varepsilon(h)\mathrm{d}h = 0,$$

$$\lim_{\varepsilon\to 0} \int_{|h|\le\delta} (1 \wedge |h|^p)\nu_\varepsilon(h)\mathrm{d}h = 1 - \lim_{\varepsilon\to 0} \int_{|h|>\delta} (1 \wedge |h|^p)\nu_\varepsilon(h)\mathrm{d}h = 1.$$

Thus for $\beta > p$ we have

$$\lim_{\varepsilon\to 0} \int_{|h|\le R} (1 \wedge |h|^\beta)\nu_\varepsilon(h)\mathrm{d}h \le \lim_{\varepsilon\to 0} \Big(R^{\beta-p} \int_{\delta<|h|\le R} (1 \wedge |h|^p)\nu_\varepsilon(h)\mathrm{d}h + \delta^{\beta-p} \int_{|h|\le\delta} (1 \wedge |h|^p)\nu_\varepsilon(h)\mathrm{d}h \Big) = \delta^{\beta-p}.$$

Likewise for $\beta < p$ we have

$$\lim_{\varepsilon\to 0} \int_{|h|\le R} (1 \wedge |h|^\beta)\nu_\varepsilon(h)\mathrm{d}h \ge \lim_{\varepsilon\to 0} \Big(R^{\beta-p} \int_{\delta<|h|\le R} (1 \wedge |h|^p)\nu_\varepsilon(h)\mathrm{d}h + \delta^{\beta-p} \int_{|h|\le\delta} (1 \wedge |h|^p)\nu_\varepsilon(h)\mathrm{d}h \Big) = \delta^{\beta-p}.$$

In either case, letting $\delta \to 0$ provides the claim.

Remark 2.4 Assume the family $(\nu_\varepsilon)_\varepsilon$ satisfies (1.3). Note that the relation

$$\lim_{\varepsilon\to 0} \int_{|h|>\delta} (1 \wedge |h|^p)\nu_\varepsilon(h)\,\mathrm{d}h = 0, \tag{2.2}$$

is often known as the concentration property and is merely equivalent to

$$\lim_{\varepsilon\to 0} \int_{|h|>\delta} \nu_\varepsilon(h)\ \mathrm{d}h = 0, \quad \text{for all} \quad \delta > 0.$$

Indeed, for all $\delta > 0$ we have

$$\int\limits_{|h|>\delta} (1 \wedge |h|^p)\nu_\varepsilon(h)\ \mathrm{d}h \le \int\limits_{|h|>\delta} \nu_\varepsilon(h)\ \mathrm{d}h \le (1 \wedge \delta^p)^{-1} \int\limits_{|h|>\delta} (1 \wedge |h|^p)\nu_\varepsilon(h)\ \mathrm{d}h.$$

Consequently, for all $\delta > 0$ we also have

$$\lim_{\varepsilon\to 0} \int\limits_{|h|\le\delta} (1 \wedge |h|^p)\nu_\varepsilon(h)\ \mathrm{d}h = \lim_{\varepsilon\to 0} \int\limits_{|h|\le\delta} |h|^p \nu_\varepsilon(h)\ \mathrm{d}h = 1. \tag{2.3}$$

The next result infers certain some convergences of the family $(\nu_\varepsilon)_\varepsilon$ for the case $p = 1$ and $p = 2$.

Proposition 2.5 *Consider the family $(\nu_\varepsilon)_\varepsilon$ satisfying (1.3).*

(i) If $p = 1$ then we have $\langle \nu_\varepsilon, \varphi - \varphi(0)\rangle \xrightarrow{\varepsilon\to 0} 0$ for every $\varphi \in C_c^\infty(\mathbb{R}^d)$.
(ii) If $p = 2$ then for a bounded function $u : \mathbb{R}^d \to \mathbb{R}$ which is C^2 on a neighborhood of x,

$$\lim_{\varepsilon\to 0} L_\varepsilon u(x) = -\frac{1}{2d}\Delta u(x),$$

where Δ is the Laplace operator and L_ε is the integrodifferential operator

$$L_\varepsilon u(x) := -\frac{1}{2}\int\limits_{\mathbb{R}^d} (u(x+h) + u(x-h) - 2u(x))\nu_\varepsilon(h)\ \mathrm{d}h. \tag{2.4}$$

Proof (i) Let $\varphi \in C_c^\infty(\mathbb{R}^d)$ using the fundamental theorem of calculus we can write

$$\begin{aligned}
\langle \nu_\varepsilon, \varphi - \varphi(0)\rangle &= \int\limits_{\mathbb{R}^d} (\varphi(h) - \varphi(0))\, \nu_\varepsilon(h)\, \mathrm{d}h \\
&= \int\limits_{|h|\le 1} (\varphi(h) - \varphi(0) - \nabla\varphi(0)\cdot h)\, \nu_\varepsilon(h)\, \mathrm{d}h + \int\limits_{|h|\ge 1} (\varphi(h) - \varphi(0))\nu_\varepsilon(h)\, \mathrm{d}h \\
&= \int\limits_{|h|\le 1} \int_0^1 \int\limits_0^1 s((D^2\varphi(tsh)\cdot h)\cdot h)\, \nu_\varepsilon(h)\, \mathrm{d}s\, \mathrm{d}t\, \mathrm{d}h \\
&\quad + \int\limits_{|h|\ge 1} (\varphi(h) - \varphi(0))\nu_\varepsilon(h)\, \mathrm{d}h.
\end{aligned}$$

The conclusion clearly follows since

$$\Big| \int\limits_{|h|>1} (\varphi(h) - \varphi(0))\nu_\varepsilon(h)\, \mathrm{d}h \Big| \le 2\|\varphi\|_{L^\infty(\mathbb{R}^d)} \int\limits_{|h|>1} \nu_\varepsilon(h)\, \mathrm{d}h \xrightarrow{\varepsilon\to 0} 0$$

and by Remark 2.3 we have

$$\Big|\int\limits_{|h|\leq 1}\int\limits_0^1\int\limits_0^1 s((D^2\varphi(tsh)\cdot h)\cdot h)\,\nu_\varepsilon(h)\,\mathrm{d}s\,\mathrm{d}t\ \mathrm{d}h\Big|$$
$$\leq \|D^2\varphi\|_{L^\infty(\mathbb{R}^d)}\int\limits_{|h|\leq 1}|h|^2\nu_\varepsilon(h)\,\mathrm{d}h\xrightarrow{\varepsilon\to 0}0.$$

(ii) Note that D^2u is bounded in a neighborhood of x. Hence, for $0<\delta<1$ sufficiently small, for all $|h|<\delta$ we have the estimate

$$|u(x+h)+u(x-h)-2u(x)|\leq 4(\|u\|_{C_b(\mathbb{R}^d)}+\|D^2u\|_{C(B_{4\delta}(x))})(1\wedge|h|^2).$$

The boundedness of u implies that

$$\lim_{\varepsilon\to 0}\int\limits_{|h|>\delta}|u(x+h)+u(x-h)-2u(x)|\,\nu_\varepsilon(h)\,\mathrm{d}h=4\|u\|_{L^\infty(\mathbb{R}^d)}\lim_{\varepsilon\to 0}\int\limits_{|h|>\delta}\nu_\varepsilon(h)\,\mathrm{d}h=0.$$

Since the Hessian of u is continuous at x, given $\eta>0$ we have $|D^2(x+z)-D^2u(x)|<\eta$ for $|z|<4\delta$ with $\delta>0$ sufficiently small, Remark 2.3 implies

$$\lim_{\varepsilon\to 0}\frac{1}{2}\int\limits_0^1\int\limits_0^1 2t\int\limits_{|h|\leq\delta}|[((D^2u(x-th+2sth)-D^2u(x))\cdot h]\cdot h|\,\nu_\varepsilon(h)\,\mathrm{d}h\ \mathrm{d}s\,\mathrm{d}t$$
$$\leq\frac{\eta}{2}\lim_{\varepsilon\to 0}\int\limits_{|h|\leq\delta}(1\wedge|h|^2)\,\nu_\varepsilon(h)\,\mathrm{d}h=\frac{\eta}{2}.$$

Thus, the leftmost expression vanishes since $\eta>0$ is arbitrarily. Next, by symmetry we have $\int_{|h|\leq\delta}h_ih_j\nu_\varepsilon(h)\,\mathrm{d}h=0$ for $i\neq j$. The rotation invariance of the Lebesgue measure implies

$$\int\limits_{|h|\leq\delta}[D^2u(x)\cdot h]\cdot h\ \nu_\varepsilon(h)\,\mathrm{d}h=\sum_{\substack{i,j=1\\ i\neq j}}^d\int\limits_{|h|\leq\delta}\partial^2_{ij}u(x)h_ih_j\nu_\varepsilon(h)\,\mathrm{d}h+\sum_{i=1}^d\partial^2_{ii}u(x)\int\limits_{|h|\leq\delta}h_i^2\nu_\varepsilon(h)\,\mathrm{d}h$$
$$=\Delta u(x)\int\limits_{|h|\leq\delta}h_1^2\nu_\varepsilon(h)\,\mathrm{d}h=\frac{1}{d}\Delta u(x)\int\limits_{|h|<\delta}|h|^2\nu_\varepsilon(h)\,\mathrm{d}h$$
$$=\frac{1}{d}\Delta u(x)\int\limits_{|h|\leq\delta}(1\wedge|h|^2)\,\nu_\varepsilon(h)\,\mathrm{d}h\xrightarrow{\varepsilon\to 0}\frac{1}{d}\Delta u(x).$$

Finally, by the fundamental theorem of calculus we find that

$$-\frac{1}{2}\int\limits_{|h|\leq\delta}(u(x+h)+u(x-h)-2(x))\,\nu_\varepsilon(h)\,\mathrm{d}h$$
$$=-\frac{1}{2}\int\limits_{|h|\leq\delta}[D^2u(x)\cdot h]\cdot h\ \nu_\varepsilon(h)\,\mathrm{d}h$$
$$-\frac{1}{2}\int\limits_0^1\int\limits_0^1 2\int\limits_{|h|\leq\delta}[D^2u(x-th+2sth)\cdot h-D^2u(x)\cdot h]\cdot h\ \nu_\varepsilon(h)\,\mathrm{d}h\ \mathrm{d}s\,\mathrm{d}t\xrightarrow{\varepsilon\to 0}-\frac{1}{2d}\Delta u(x).$$

□

Let us give examples of ν_ε satisfying (1.3). The first example is related to fractional Sobolev spaces.

Example 2.6 The family $(\nu_\varepsilon)_\varepsilon$ of kernels defined for $h \neq 0$ by

$$\nu_\varepsilon(h) = a_{\varepsilon,d,p}|h|^{-d-(1-\varepsilon)p} \quad \text{with} \quad a_{\varepsilon,d,p} = \frac{p\varepsilon(1-\varepsilon)}{|\mathbb{S}^{d-1}|}.$$

The next class of examples is that of Proposition 2.2.

Example 2.7 Assume $\nu : \mathbb{R}^d \setminus \{0\} \to [0,\infty)$ is radial and consider the family $(\nu_\varepsilon)_\varepsilon$ such that each ν_ε is the rescaling of ν defined as in (2.1) provided that

$$\int_{\mathbb{R}^d} (1 \wedge |h|^p)\nu(h)\,\mathrm{d}h = 1.$$

A subclass is obtained if one considers an integrable radial function $\rho : \mathbb{R}^d \to [0,\infty)$ and defines $\nu(h) = c|h|^{-p}\rho(h)$ for a suitable normalizing constant $c > 0$.

Example 2.8 Assume $(\rho_\varepsilon)_\varepsilon$ is an approximation of the unity, i.e., satisfies (1.4). For instance, define $\rho_\varepsilon(h) = \varepsilon^{-d}\rho(h/\varepsilon)$ where $\rho \geq 0$ is radial and $\int_{\mathbb{R}^d}\rho(h)\,\mathrm{d}h = 1$. Define the family $(\nu_\varepsilon)_\varepsilon$ by $\nu_\varepsilon(h) = c_\varepsilon|h|^{-p}\rho_\varepsilon(h)$, where $c_\varepsilon > 0$ is a normalizing constant given by

$$c_\varepsilon^{-1} = \int_{|h|\leq 1} \rho_\varepsilon(h)\,\mathrm{d}h + \int_{|h|>1} |h|^{-p}\rho_\varepsilon(h)\,\mathrm{d}h,$$

for which the p-Lévy integrability condition in (1.3) holds. Note that $c_\varepsilon \to 1$ as $\varepsilon \to 0$.

Example 2.9 Let $0 < \varepsilon < 1$ and $\beta > -d$. We define

$$\nu_\varepsilon(h) = \frac{d+\beta}{|\mathbb{S}^{d-1}|\varepsilon^{d+\beta}}|h|^{\beta-p}\mathbb{1}_{B_\varepsilon}(h).$$

Some special cases are obtained for $\beta \in \{0, \varepsilon p - d, p\}$. For the limiting case $\beta = -d$, we put

$$\nu_\varepsilon(h) = \frac{1}{|\mathbb{S}^{d-1}|\log(\varepsilon_0/\varepsilon)}|h|^{-d-p}\mathbb{1}_{B_{\varepsilon_0}\setminus B_\varepsilon}(h).$$

Example 2.10 Let $0 < \varepsilon < \varepsilon_0 < 1$ and $\beta > -d$. Define

$$\nu_\varepsilon(h) = \frac{(|h|+\varepsilon)^\beta|h|^{-p}}{|\mathbb{S}^{d-1}|b_\varepsilon}\mathbb{1}_{B_{\varepsilon_0}}(h) \qquad \text{with} \quad b_\varepsilon = \varepsilon^{d+\beta}\int_{\frac{\varepsilon}{\varepsilon+\varepsilon_0}}^1 t^{-d-\beta-1}(1-t)^{d-1}\,\mathrm{d}t.$$

For the limiting case $\beta = -d$ consider

$$\nu_\varepsilon(h) = \frac{(|h|+\varepsilon)^{-d}|h|^{-p}}{|\mathbb{S}^{d-1}||\log\varepsilon|b_\varepsilon}\mathbb{1}_{B_{\varepsilon_0}(h)} \qquad \text{with} \quad b_\varepsilon = |\log\varepsilon|^{-1}\int_{\frac{\varepsilon}{\varepsilon+\varepsilon_0}}^1 t^{-1}(1-t)^{d-1}\,\mathrm{d}t.$$

In either case the constant $b_\varepsilon \to 1$ as $\varepsilon \to 0$ and is such that $\int_{\mathbb{R}^d}(1 \wedge |h|^p)\nu_\varepsilon(h)\,\mathrm{d}h = 1$. Another example familiar to the case $\beta = -d$ is

$$\nu_\varepsilon(h) = \frac{(|h|+\varepsilon)^{-d-p}}{|\mathbb{S}^{d-1}||\log\varepsilon|b_\varepsilon}\mathbb{1}_{B_\varepsilon(h)} \qquad \text{with} \quad b_\varepsilon = |\log\varepsilon|^{-1}\int_{\frac{\varepsilon}{\varepsilon+\varepsilon_0}}^1 t^{-1}(1-t)^{d+p-1}\,\mathrm{d}t.$$

2.2 Local and nonlocal spaces

Let $\nu : \mathbb{R}^d \setminus\{0\} \to [0,\infty)$ be p-Lévy integrable and $\mathcal{E}^i_\Omega(u),\ i=1,2,3$ be the forms defined in (1.1). The space $W^p_\nu(\Omega) = \big\{u \in L^p(\Omega)\ :|u|^p_{W^p_\nu(\Omega)} < \infty\big\}$ is a Banach space endowed with the norm $\|u\|_{W^p_\nu(\Omega)} = \big(\|u\|^p_{L^p(\Omega)} + |u|^p_{W^p_\nu(\Omega)}\big)^{1/p}$ with $|u|^p_{W^p_\nu(\Omega)} := \mathcal{E}^1_\Omega(u)$. For the standard example $\nu(h) = |h|^{-d-sp}$, $s \in (0,1)$, one recovers the Sobolev space of fractional order denoted $W^{s,p}(\Omega)$; see [10, 18] for more. If ν has full support, the space $W^p_\nu(\Omega|\,\mathbb{R}^d) = \big\{u : \mathbb{R}^d \to \mathbb{R}$ meas. : $u \in L^p(\Omega)$ and $|u|^p_{W^p_\nu(\Omega|\,\mathbb{R}^d)} < \infty\big\}$, $|u|^p_{W^p_\nu(\Omega|\,\mathbb{R}^d)} = \mathcal{E}^3_\Omega(u)$, is a Banach space with the norm $\|u\|_{W^p_\nu(\Omega|\,\mathbb{R}^d)} = \big(\|u\|^p_{L^p(\Omega)} + |u|^p_{W^p_\nu(\Omega|\,\mathbb{R}^d)}\big)^{1/p}$. See [15, 18, 19] for recent results involving this types of spaces.
We recall that, $\frac{1}{2}\mathcal{E}^2_\Omega(u) \le \mathcal{E}^3_\Omega(u) \le \mathcal{E}^2_\Omega(u)$. It is noteworthy to mention that, the space $\big(W^p_\nu(\Omega|\,\mathbb{R}^d), \|\cdot\|_{W^p_\nu(\Omega|\,\mathbb{R}^d)}\big)$ is the core energy space for a large class of nonlocal problems with Dirichlet, Neumann or Robin boundary conditions. See for instance [11, 12, 14, 16, 31]. If $\Omega \subset \mathbb{R}^d$ has a sufficiently regular boundary or $\Omega = \mathbb{R}^d$ then according to Theorem 1.3 and Theorem 1.5, it is legitimate to say that the nonlocal spaces $\big(W^p_{\nu_\varepsilon}(\Omega|\,\mathbb{R}^d), \|\cdot\|_{W^p_{\nu_\varepsilon}(\Omega|\,\mathbb{R}^d)}\big)_\varepsilon$ and $\big(W^p_{\nu_\varepsilon}(\Omega), \|\cdot\|_{W^p_{\nu_\varepsilon}(\Omega)}\big)_\varepsilon$ converge to the Sobolev space $\big(W^{1,p}(\Omega), \|\cdot\|^*_{W^{1,p}(\Omega)}\big)$ and $\big(BV(\Omega), \|\cdot\|^*_{BV(\Omega)}\big)$, where

$$\|u\|^*_{W^{1,p}(\Omega)} = \big(\|u\|^p_{L^p(\Omega)} + K_{d,p}\|\nabla u\|^p_{L^p(\Omega)}\big)^{1/p} \quad \text{and} \quad \|u\|^*_{BV(\Omega)} = \|u\|_{L^1(\Omega)} + K_{d,1}|u|_{BV(\Omega)}.$$

Let us recall the following standard approximation result for the space $BV(\Omega)$; see [13, p. 172], [25, Theorem 14.9] or [1, Theorem 3.9].

Theorem 2.11 *Let $\Omega \subset \mathbb{R}^d$ be open and $u \in BV(\Omega)$. There is a sequence $(u_n)_n$ in $BV(\Omega) \cap C^\infty(\Omega)$ such that $\|u_n - u\|_{L^1(\Omega)} \xrightarrow{n\to\infty} 0$ and $\|\nabla u_n\|_{L^1(\Omega)} \xrightarrow{n\to\infty} |u|_{BV(\Omega)}$.*

Warning: the above approximation theorem does not claim that $|u_n - u|_{BV(\Omega)} \xrightarrow{n\to\infty} 0$ but rather implies that $\|u_n\|_{W^{1,1}(\Omega)} \xrightarrow{n\to\infty} \|u\|_{BV(\Omega)}$. Strictly speaking, $BV(\Omega) \cap C^\infty(\Omega)$ is not necessarily dense in $BV(\Omega)$. Recall that, if a function $u \in L^1(\Omega)$ is regular enough, say, $u \in W^{1,1}(\Omega)$ then we have $u \in BV(\Omega)$. From this we find that $BV(\Omega) \cap C^\infty(\Omega) = W^{1,1}(\Omega) \cap C^\infty(\Omega)$.
Next, we establish some useful estimates. Note that for $h \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} |u(x+h) - u(x)|^p \,\mathrm{d}x \le 2^p \|u\|^p_{L^p(\mathbb{R}^d)}.$$

Furthermore, using the density of $C^\infty_c(\mathbb{R}^d)$ in $W^{1,p}(\mathbb{R}^d)$ we find that

$$\int_{\mathbb{R}^d} |u(x+h) - u(x)|^p \,\mathrm{d}x = \int_{\mathbb{R}^d} \Big|\int_0^1 \nabla u(x+th)\cdot h\Big|^p \,\mathrm{d}x \le |h|^p \|\nabla u\|^p_{L^p(\mathbb{R}^d)}.$$

Therefore, for every $u \in W^{1,p}(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$ we have

$$\int_{\mathbb{R}^d} |u(x+h) - u(x)|^p \,\mathrm{d}x \le 2^p(1 \wedge |h|^p)\|u\|^p_{W^{1,p}(\mathbb{R}^d)}. \tag{2.5}$$

By Theorem 2.11 the BV-norm of an element in $BV(\mathbb{R}^d)$ can be approximated by the $W^{1,1}$-norms of elements in $W^{1,1}(\mathbb{R}^d)$. Whence for $p=1$, (2.5) implies that, for $u \in BV(\mathbb{R}^d)$ and $h \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d} |u(x+h)-u(x)|\,\mathrm{d}x \le 2(1\wedge |h|)\|u\|_{BV(\mathbb{R}^d)}. \tag{2.6}$$

Lemma 2.12 *Assume $\nu : \mathbb{R}^d\setminus\{0\} \to [0,\infty)$ is p-Lévy integrable and $\Omega \subset \mathbb{R}^d$ is a $W^{1,p}$-extension domain (resp. BV-extension domain). There is $C = C(\Omega, d, p) > 0$ independent of ν such that*

$$\iint_{\Omega\Omega} |u(x)-u(y)|^p \nu(x-y)\mathrm{d}y\mathrm{d}x \le C\|u\|^p_{W^{1,p}(\Omega)}\|\nu\|_{L^1(\mathbb{R}^d, 1\wedge|h|^p)}, \quad \textit{for all } u \in W^{1,p}(\Omega)$$

$$(\textit{resp.} \iint_{\Omega\Omega} |u(x)-u(y)|\nu(x-y)\mathrm{d}y\mathrm{d}x \le C\|u\|_{BV(\Omega)}\|\nu\|_{L^1(\mathbb{R}^d, 1\wedge|h|)}, \quad \textit{for all} \quad u \in BV(\Omega)).$$

Proof Let $\overline{u}$ be a $W^{1,p}$-extension of u on $\mathbb{R}^d$. The estimate (2.5) implies

$$\begin{aligned}\iint_{\Omega\Omega} |u(x)-u(y)|^p \nu(x-y)\mathrm{d}y\mathrm{d}x &\le \iint_{\mathbb{R}^d\,\mathbb{R}^d} |\overline{u}(x+h)-\overline{u}(x)|^p\nu(h)\mathrm{d}h\mathrm{d}x\\ = \int_{\mathbb{R}^d}\nu(h)\mathrm{d}h\int_{\mathbb{R}^d}|\overline{u}(x+h)-\overline{u}(x)|^p\mathrm{d}x &\le C\|u\|^p_{W^{1,p}(\Omega)}\|\nu\|_{L^1(\mathbb{R}^d,1\wedge|h|^p)}.\end{aligned}$$

Likewise, if $p=1$ and $u \in BV(\Omega)$ one gets the other estimate from the estimate (2.6). □

An immediate consequence of Lemma 2.12 is the following embedding result.

Theorem 2.13 *Assume $\nu \in L^1(\mathbb{R}^d, 1\wedge|h|^p)$ with $p \ge 1$ and $\Omega \subset \mathbb{R}^d$ is a $W^{1,p}$-extension domain. There holds that the embedding $W^{1,p}(\Omega) \hookrightarrow W^p_\nu(\Omega)$ is continuous. Furthermore, for $p=1$ and if Ω is a BV-extension domain then the embedding $BV(\Omega) \hookrightarrow W^1_\nu(\Omega)$ is also continuous.*

It is worth emphasizing that the above embeddings may fail if Ω is not an extension domain (see the counterexample 1). Another straightforward consequence of Lemma 2.12 is the following.

Theorem 2.14 *Let Ω be a $W^{1,p}$-extension domain, $p \ge 1$. There is $C = (\Omega, d, p) > 0$ such that*

$$\limsup_{\varepsilon\to 0}\iint_{\Omega\Omega}|u(x)-u(y)|^p\nu_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y \le C\|u\|^p_{W^{1,p}(\Omega)} \qquad \textit{for all} u \in W^{1,p}(\Omega).$$

If $p=1$ and Ω is a BV-extension domain we also have,

$$\limsup_{\varepsilon\to 0}\iint_{\Omega\Omega}|u(x)-u(y)|\nu_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y \le C\|u\|_{BV(\Omega)} \qquad \textit{for all } u \in BV(\Omega).$$

The next proposition shows that the p-Lévy integrability condition is consistent and optimal in the sense that it draws a borderline for which a space of type $W^p_\nu(\Omega)$ is trivial or not.

Proposition 2.15 *Let $\nu : \mathbb{R}^d \to [0,\infty]$ be symmetric. The following assertions are true.*

(i) *If* $\nu \in L^1(\mathbb{R}^d)$ *then* $W^p_\nu(\Omega) = L^p(\Omega)$ *and* $W^p_\nu(\Omega|\,\mathbb{R}^d) \cap L^p(\mathbb{R}^d) = L^p(\mathbb{R}^d)$.

(ii) *If* $\nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p)$ *then* $W^{1,p}(\mathbb{R}^d) \subset W^p_\nu(\mathbb{R}^d)$, *hence the spaces* $W^p_\nu(\Omega)$ *and* $W^p_\nu(\Omega|\,\mathbb{R}^d)$ *contain* $C_c^\infty(\mathbb{R}^d)$. *Moreover, if* Ω *is bounded, then the spaces* $W^p_\nu(\Omega)$ *and* $W^p_\nu(\Omega|\,\mathbb{R}^d)$ *also contain bounded Lipschitz functions.*

(iii) *Assume* ν *is radial,* Ω *is connected and put* $C_\delta = \int_{B_\delta(0)} |h|^p \nu(h)\,\mathrm{d}h$. *If* $C_\delta = \infty$ *for all* $\delta > 0$, *then any function* $u \in W^{1,p}(\Omega) \cap W^p_\nu(\Omega)$ *or* $u \in C^1(\Omega) \cap W^p_\nu(\Omega)$ *is a constant function.*

(iv) *Assume* $\nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p)$ *and* ν *is radial. Given* $u \in W^{1,p}(\mathbb{R}^d)$ *there is* $\delta = \delta(u) > 0$ *so that*

$$2^{-p} K_{d,p} C_\delta \|\nabla u\|^p_{L^p(\mathbb{R}^d)} \le |u|^p_{W^p_\nu(\mathbb{R}^d)} \le 2^p \|\nu\|_{L^1(\mathbb{R}^d, 1\wedge|h|^p\,\mathrm{d}h)} \|u\|^p_{W^{1,p}(\mathbb{R}^d)}. \tag{2.7}$$

Proof (i) is obvious. (ii) For a bounded Lipschitz function u, we have $|u(x) - u(y)|^p \le C(1 \wedge |x-y|^p)$ for some $C > 0$. Hence if Ω is bounded by integrating both sides, it follows that $u \in W^p_\nu(\Omega)$ and $u \in W^p_\nu(\Omega|\,\mathbb{R}^d)$. The inclusion $W^{1,p}(\mathbb{R}^d) \subset W^p_\nu(\mathbb{R}^d)$ follows from Lemma 2.12 or from the estimate (2.5). (iii) Let $u \in W^{1,p}(\Omega) \cap W^p_\nu(\Omega)$ or $u \in C^1(\Omega) \cap W^p_\nu(\Omega)$ and let $K \subset \Omega$ be a compact set. Since $|\nabla u| \in L^p(K)$, for arbitrary $\eta > 0$ there is $0 < \delta = \delta(\eta, K) < \operatorname{dist}(K, \partial\Omega)$ such that,

$$\|\nabla u(\cdot + h) - \nabla u\|_{L^p(K)} < \eta \quad \text{for all} |h| \le \delta.$$

Minkowski's inequality implies

$$\Big(\int_K \int_{B_\delta(0)} |\nabla u(x) \cdot h|^p \nu(h)\,\mathrm{d}h\,\mathrm{d}x \Big)^{1/p} \le \Big(\int_K \int_{B_\delta(0)} \Big| \int_0^1 \nabla u(x + th) \cdot h\,\mathrm{d}t \Big|^p \nu(h)\,\mathrm{d}h\,\mathrm{d}x \Big)^{1/p} + \eta C_\delta^{1/p}.$$

The choice $0 < \delta < \operatorname{dist}(K, \partial\Omega)$ ensures that $B_\delta(x) \subset \Omega$ for all $x \in K$. From the foregoing, using the fundamental theorem of calculus, polar coordinates and the formula (1.9) yield

$$\begin{aligned} |u|_{W^p_\nu(\Omega)} &\ge \Big(\int_K \int_{B_\delta(0)} \Big| \int_0^1 \nabla u(x + th) \cdot h\,\mathrm{d}t \Big|^p \nu(h)\,\mathrm{d}h\;\mathrm{d}x \Big)^{1/p} \\ &\ge \Big(\int_K \int_{\mathbb{S}^{d-1}} |\nabla u(x) \cdot w|^p \mathrm{d}\sigma_{d-1}(w) \int_0^\delta r^{p+d-1} \nu(r)\,\mathrm{d}r \Big)^{1/p} - \eta \Big(\int_{B_\delta(0)} |h|^p \nu(h)\,\mathrm{d}h \Big)^{1/p} \\ &= \Big(K_{d,p}^{1/p} \|\nabla u\|_{L^p(K)} - \eta \Big) \Big(\int_{B_\delta(0)} |h|^p \nu(h)\,\mathrm{d}h \Big)^{1/p}. \end{aligned}$$

Therefore, for each $\eta > 0$ and each compact set $K \subset \Omega$ we have

$$|u|_{W^p_\nu(\Omega)} \ge C_\delta^{1/p} \big(K_{d,p}^{1/p} \|\nabla u\|_{L^p(K)} - \eta \big). \tag{2.8}$$

Since $\|u\|_{W^p_\nu(\Omega)} < \infty$ and $C_\delta = \infty$, this is possible only if $\|\nabla u\|^p_{L^p(K)} = 0$. As the compact set $K \subset \Omega$ is arbitrary, we find that $\nabla u = 0$ a.e. on Ω. Thus u is a constant since Ω is connected. (iv) The upper inequality clearly follows from (2.5). Proceeding as for the estimate (2.8) by taking $\Omega = \mathbb{R}^d$ and $K = \mathbb{R}^d$ also yields that, for all $\eta > 0$ there is $\delta = \delta(\eta) > 0$ such that

$$|u|_{W^p_\nu(\mathbb{R}^d)} \ge C_\delta^{1/p} \big(K_{d,p}^{1/p} \|\nabla u\|_{L^p(\mathbb{R}^d)} - \eta \big). \tag{2.9}$$

If $\|\nabla u\|_{L^p(\mathbb{R}^d)} \neq 0$, taking $\eta = \frac{1}{2}K_{d,p}^{1/p}\|\nabla u\|_{L^p(\mathbb{R}^d)}$ yields $|u|^p_{W^p_\nu(\mathbb{R}^d)} \geq 2^{-p}K_{d,p}C_\delta \|\nabla u\|^p_{L^p(\mathbb{R}^d)}$. This estimate remains true for any $\delta > 0$, if $\|\nabla u\|_{L^p(\mathbb{R}^d)} = 0$. □

The next theorem provides a characterization of the p-Lévy integrability condition.

Theorem 2.16 *Assume $\nu : \mathbb{R}^d \to [0, \infty]$ is radial. The following assertions are equivalent.*

(i) The p-Lévy integrability condition in (1.2) holds.
(ii) The embedding $W^{1,p}(\mathbb{R}^d) \hookrightarrow W^p_\nu(\mathbb{R}^d)$ is continuous.
(iii) $\mathcal{E}^1_{\mathbb{R}^d}(u) < \infty$ for all $u \in W^{1,p}(\mathbb{R}^d)$.
(iv) $\mathcal{E}^1_{\mathbb{R}^d}(u) < \infty$ for all $u \in C^\infty_c(\mathbb{R}^d)$.
(v) There exists $u \in C^\infty_c(B_1(0)) \setminus \{0\}$ such that $\mathcal{E}^1_{\mathbb{R}^d}(u_n) < \infty$ for all $n \geq 1$, $u_n(x) = n^d u(nx)$.

This remains true when $p = 1$ with $BV(\mathbb{R}^d)$ in place of $W^{1,1}(\mathbb{R}^d)$.

Proof $(i) \implies (ii)$. The right hand side of the estimate (2.7) implies the continuity of the embedding $W^{1,p}(\mathbb{R}^d) \hookrightarrow W^p_\nu(\mathbb{R}^d)$. The implications $(ii) \implies (iii)$, $(iii) \implies (iv)$ and $(iv) \implies (v)$ are straightforward. Let us prove that $(v) \implies (i)$. Given that $u \in C^\infty_c(B_1(0) \setminus \{0\}$ we have $\|\nabla u\|_{L^p(\mathbb{R}^d)} \neq 0$. By Proposition 2.15 (iv) there exists $\delta = \delta(u) > 0$ (see the estimate (2.7)) such that $\mathcal{E}^1_{\mathbb{R}^d}(u) \geq 2^{-p}K_{d,p}C_\delta\|\nabla u\|^p_{L^p(\mathbb{R}^d)}$ and hence $C_\delta = \int_{B_\delta(0)} |h|^p \nu(h)\,\mathrm{d}h < \infty$. Next, we fix $n \geq 1$ such that $\delta > \frac{2}{n}$ so that $\operatorname{supp} u_n \subset B_{\delta/2}(0)$. Since $B_{\delta/2}(x) \subset B_\delta(0)$ for all $x \in B_{\delta/2}(0)$ we have

$$\infty > \mathcal{E}^1_{\mathbb{R}^d}(u_n) \geq 2 \int\limits_{B_{\delta/2}(0)} |u_n(x)|^p \int\limits_{\mathbb{R}^d \setminus B_{\delta/2}(0)} \nu(x-y)\,\mathrm{d}y\,\mathrm{d}x \geq 2\|u_n\|^p_{L^p(\mathbb{R}^d)} \int\limits_{\mathbb{R}^d \setminus B_\delta(0)} \nu(h)\,\mathrm{d}h.$$

Thus $\int_{|h|\geq\delta} \nu(h)\,\mathrm{d}h < \infty$. Accordingly $\nu \in L^1(\mathbb{R}^d, 1 \wedge |h|^p)$. The case $p = 1$ follows analogously. □

3 Main results

First and foremost, the proof of Theorem 1.3 in the case $\Omega = \mathbb{R}^d$ is much simpler. Indeed, by the estimates (2.9) and (3.10) below, for sufficiently small $\eta > 0$, there is $\delta = \delta(\eta) > 0$ such that

$$\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x \geq \left(K_{d,p}^{1/p}\|\nabla u\|_{L^p(\mathbb{R}^d)}-\eta\right)^p \int\limits_{B_\delta(0)} |h|^p \nu_\varepsilon(h)\,\mathrm{d}h,$$

$$\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\mathrm{d}y\,\mathrm{d}x \leq K_{d,p}\|\nabla u\|^p_{L^p(\mathbb{R}^d)} + 2^p\|u\|^p_{L^p(\mathbb{R}^d)} \int\limits_{|h|>\delta} \nu_\varepsilon(h)\,\mathrm{d}h.$$

Letting $\varepsilon \to 0$ and $\eta \to 0$ successively, using the formulas (2.3) and (2.2), we get

$$\lim_{\varepsilon\to 0} \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x = K_{d,p}\|\nabla u\|^p_{L^p(\mathbb{R}^d)}. \tag{3.1}$$

The case $p = 1$ and $u \in BV(\mathbb{R}^d)$ can be proved analogously. In fact, it can be shown that (3.1) holds if and only if up to a multiple factor $(\nu_\varepsilon)_\varepsilon$ satisfies (1.3). In other words, the class $(\nu_\varepsilon)_\varepsilon$ is the largest (the sharpest) class for which the BBM formula (3.1) holds. From now on, we assume $\Omega \neq \mathbb{R}^d$. We start with the following lemma which is somewhat a revisited version of [5, Lemma 1].

Lemma 3.1 *Assume $\nu \in L^1(\mathbb{R}^d, 1\wedge |h|^p)$ is symmetric, $p \geq 1$. Given $u \in L^p(\mathbb{R}^d)$, $\varphi \in C_c^\infty(\mathbb{R}^d)$ and a unit vector $e \in \mathbb{S}^{d-1}$ we have*

$$\begin{aligned}
&\Big|\iint\limits_{(y-x)\cdot e\geq 0} u(x)\frac{\varphi(y)-\varphi(x)}{|x-y|}(1\wedge|x-y|^p)\nu(x-y)\mathrm{d}y\mathrm{d}x\Big| \\
&\qquad + \Big|\iint\limits_{(y-x)\cdot e\leq 0} u(x)\frac{\varphi(y)-\varphi(x)}{|x-y|}(1\wedge|x-y|^p)\nu(x-y)\mathrm{d}y\mathrm{d}x\Big| \\
&\leq \iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} \frac{|u(x)-u(y)|}{|x-y|}|\varphi(x)|(1\wedge|x-y|^p)\nu(x-y)\mathrm{d}y\mathrm{d}x.
\end{aligned}$$

Proof Let us introduce the truncated measure $\widetilde{\nu}_\delta(h) = |h|^{-1}(1\wedge|h|^p)\nu(h)\,\mathbb{1}_{\mathbb{R}^d\setminus B_\delta}(h)$ for $\delta > 0$ which enables us to rule out an eventual singularity of ν at the origin. Moreover, note that $\widetilde{\nu}_\delta \in L^1(\mathbb{R}^d)$. It turns out that the mappings $(x, y) \mapsto u(x)\varphi(y)\widetilde{\nu}_\delta(x-y)$ and $(x, y) \mapsto u(x)\varphi(x)\widetilde{\nu}_\delta(x-y)$ are integrable. Indeed, using Hölder inequality combined with Fubini's theorem yield

$$\begin{aligned}
&\iint\limits_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)\varphi(x)|\widetilde{\nu}_\delta(x-y)\,\mathrm{d}y\,\mathrm{d}x \\
&= \iint\limits_{|x-y|\geq\delta} |u(x)\varphi(x)||x-y|^{-1}(1\wedge|x-y|^p)\nu(x-y)\,\mathrm{d}x\,\mathrm{d}y \\
&\leq \delta^{-1}\Big(\iint\limits_{|x-y|\geq\delta}|u(x)|^p(1\wedge|x-y|^p)\nu(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p} \\
&\quad\times\Big(\iint\limits_{|x-y|\geq\delta}|\varphi(x)|^{p'}(1\wedge|x-y|^p)\nu(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big)^{1/p'} \\
&\leq \delta^{-1}\|\varphi\|_{L^{p'}(\mathbb{R}^d)}\|u\|_{L^p(\mathbb{R}^d)}\int\limits_{\mathbb{R}^d}(1\wedge|h|^p)\nu(h)\,\mathrm{d}h < \infty.
\end{aligned}$$

Analogously, we also get

$$\iint_{\mathbb{R}^d\,\mathbb{R}^d} |u(x)\varphi(y)|\widetilde{\nu}_\delta(x-y)\,\mathrm{d}y\,\mathrm{d}x \le \delta^{-1}\|\varphi\|_{L^{p'}(\mathbb{R}^d)}\|u\|_{L^p(\mathbb{R}^d)}\int_{\mathbb{R}^d}(1\wedge|h|^p)\nu(h)\,\mathrm{d}h < \infty.$$

Consequently, by interchanging x and y, using Fubini's theorem and the symmetry of ν we obtain

$$\begin{aligned}
&\iint_{(y-x)\cdot e\ge 0} u(x)\varphi(x)\widetilde{\nu}_\delta(x-y)\,\mathrm{d}y\,\mathrm{d}x\\
&\quad= \iint_{(x-y)\cdot e\ge 0} u(y)\varphi(y)\widetilde{\nu}_\delta(x-y)\,\mathrm{d}x\,\mathrm{d}y\\
&\quad= \iint_{(y-x')\cdot e\ge 0} u(y)\varphi(y)\widetilde{\nu}_\delta(y-x')\,\mathrm{d}y\,\mathrm{d}x' \quad (x'=2y-x,\ \mathrm{d}x=\mathrm{d}x').
\end{aligned}$$

Therefore we have

$$\begin{aligned}
&\Big|\int_{\mathbb{R}^d} u(x)\,\mathrm{d}x \int_{(y-x)\cdot e\ge 0}(\varphi(y)-\varphi(x))\widetilde{\nu}_\delta(x-y)\,\mathrm{d}y\Big|\\
&\quad=\Big|\iint_{(y-x)\cdot e\ge 0} u(x)\varphi(y)\widetilde{\nu}_\delta(x-y)\,\mathrm{d}y\,\mathrm{d}x - \iint_{(y-x)\cdot e\ge 0} u(x)\varphi(x)\widetilde{\nu}_\delta(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big|\\
&=\Big|\int_{\mathbb{R}^d}\varphi(y)\,\mathrm{d}y\int_{(y-x)\cdot e\ge 0}(u(x)-u(y))\widetilde{\nu}_\delta(x-y)\,\mathrm{d}x\Big|\\
&\le \int_{\mathbb{R}^d}|\varphi(y)|\,\mathrm{d}y\int_{(y-x)\cdot e\ge 0}\frac{|u(x)-u(y)|}{|x-y|}(1\wedge|x-y|^p)\nu(x-y)\,\mathrm{d}x\\
&= \int_{\mathbb{R}^d}|\varphi(x)|\,\mathrm{d}x\int_{(y-x)\cdot e\le 0}\frac{|u(y)-u(x)|}{|x-y|}(1\wedge|x-y|^p)\nu(x-y)\,\mathrm{d}y.
\end{aligned}$$

Thus letting $\delta\to 0$ implies

$$\begin{aligned}
&\Big|\iint_{(y-x)\cdot e\ge 0} u(x)(\varphi(y)-\varphi(x))|x-y|^{-1}(1\wedge|x-y|^p)\nu(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big|\\
&\qquad\le \iint_{(y-x)\cdot e\le 0}|\varphi(x)|\frac{|u(y)-u(x)|}{|x-y|}(1\wedge|x-y|^p)\nu(x-y)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}\tag{3.2}$$

Likewise one has

$$\begin{aligned}
&\Big|\iint_{(y-x)\cdot e\le 0} u(x)(\varphi(y)-\varphi(x))|x-y|^{-1}(1\wedge|x-y|^p)\nu(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big|\\
&\qquad\qquad\le \iint_{(y-x)\cdot e\ge 0}|\varphi(x)|\frac{|u(y)-u(x)|}{|x-y|}(1\wedge|x-y|^p)\nu(x-y)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}\tag{3.3}$$

Summing the estimates (3.2) and (3.3) gives the desired inequality. □

Theorem 3.2 *Let $\Omega \subset \mathbb{R}^d$ be an open, $u \in L^p(\Omega)$, $p \geq 1$ and A_p be defined as in(1.6). Then given a unit vector $e \in \mathbb{S}^{d-1}$ and $\varphi \in C_c^\infty(\mathbb{R}^d)$ with support in Ω the following estimate holds true*

$$\left| \int_\Omega u(x) \nabla\varphi(x) \cdot e \, \mathrm{d}x \right| \leq \frac{A_p^{1/p}}{K_{d,1}} \|\varphi\|_{L^{p'}(\Omega)}. \tag{3.4}$$

Proof Throughout, to alleviate the notation we denote $\pi_\varepsilon(x-y) = (1 \wedge |x-y|^p)\nu_\varepsilon(x-y)$. Let $\overline{u} \in L^p(\mathbb{R}^d)$ be the zero extension of u off Ω. Since $\operatorname{supp}\varphi \subset \Omega$, we have the identity

$$\begin{aligned} &\int_{\mathbb{R}^d} |\varphi(x)| \, \mathrm{d}x \int_{\mathbb{R}^d} \frac{|\overline{u}(y) - \overline{u}(x)|}{|x-y|} \pi_\varepsilon(x-y) \, \mathrm{d}y \\ &= \iint_{\Omega\Omega} \frac{|u(y) - u(x)|}{|x-y|} |\varphi(x)| \pi_\varepsilon(x-y) \, \mathrm{d}y \, \mathrm{d}x \\ &+ \int_{\operatorname{supp}(\varphi)} |\varphi(x)| \, \mathrm{d}x \int_{\mathbb{R}^d \setminus \Omega} \frac{|u(x)|}{|x-y|} \pi_\varepsilon(x-y) \, \mathrm{d}y. \end{aligned}$$

First, for $\delta = \operatorname{dist}(\operatorname{supp}(\varphi), \partial\Omega) > 0$, the Hölder inequality implies

$$\begin{aligned} &\int_{\operatorname{supp}(\varphi)} |\varphi(x)| \, \mathrm{d}x \int_{\mathbb{R}^d \setminus \Omega} \frac{|u(x)|}{|x-y|} \pi_\varepsilon(x-y) \, \mathrm{d}y \\ &\leq \delta^{-1} \|u\|_{L^p(\Omega)} \|\varphi\|_{L^{p'}(\Omega)} \int_{|h| \geq \delta} (1 \wedge |h|^p) \nu_\varepsilon(h) \, \mathrm{d}h \xrightarrow{\varepsilon \to 0} 0. \end{aligned}$$

Second, using again the Hölder inequality and $|h|^{-p}(1 \wedge |h|^p) \leq 1$ we find that

$$\begin{aligned} &\iint_{\Omega\Omega} \frac{|u(y) - u(x)|}{|x-y|} |\varphi(x)| \pi_\varepsilon(x-y) \, \mathrm{d}y \, \mathrm{d}x \\ &\leq \Big(\iint_{\Omega\Omega} \frac{|u(y) - u(x)|^p}{|x-y|^p} \pi_\varepsilon(x-y) \, \mathrm{d}y \, \mathrm{d}x \Big)^{1/p} \\ &\times \Big(\iint_{\Omega\Omega} |\varphi(x)|^{p'} \pi_\varepsilon(x-y) \, \mathrm{d}y \, \mathrm{d}x \Big)^{1/p'} \\ &\leq \|\varphi\|_{L^{p'}(\Omega)} \Big(\iint_{\Omega\Omega} |u(y) - u(x)|^p \nu_\varepsilon(x-y) \, \mathrm{d}y \, \mathrm{d}x \Big)^{1/p}. \end{aligned}$$

Therefore inserting these two estimates in the previous identity and combining the resulting estimate with that of Lemma 3.1 imply

$$\liminf_{\varepsilon \to 0} \left| \int_\Omega u(x) \, \mathrm{d}x \int_{(y-x)\cdot e \geq 0} \frac{(\varphi(y) - \varphi(x))}{|x-y|} (1 \wedge |x-y|^p) \nu_\varepsilon(x-y) \, \mathrm{d}y \right| +$$

$$\liminf_{\varepsilon\to 0}\Big|\int_{\Omega} u(x)\,\mathrm{d}x \int_{(y-x)\cdot e\le 0}\frac{(\varphi(y)-\varphi(x))}{|x-y|}(1\wedge|x-y|^p)\nu_\varepsilon(x-y)\,\mathrm{d}y\Big| \le A_p^{1/p}\|\varphi\|_{L^{p'}(\Omega)}. \tag{3.5}$$

It remains to compute the limits appearing on the left hand side of (3.5). For all $x, h \in \mathbb{R}^d$ we have

$$\varphi(x+h)-\varphi(x) = \nabla\varphi(x)\cdot h + \int_0^1 \big(\nabla\varphi(x+th)-\nabla\varphi(x)\big)\cdot h\,\mathrm{d}t$$

and $|\nabla\varphi(x+h)-\nabla\varphi(x)| \le C(1\wedge|h|)$. So that Remark 2.3 implies

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{h\cdot e\ge 0}\int_0^1 \Big|\big[\nabla\varphi(x+th)-\nabla\varphi(x)\big]\cdot\frac{h}{|h|}\Big|\mathrm{d}t(1\wedge|h|^p)\nu_\varepsilon(h)\mathrm{d}h\\ &\le C\lim_{\varepsilon\to 0}\int_{\mathbb{R}^d}(1\wedge|h|^{p+1})\nu_\varepsilon(h)\mathrm{d}h = 0.\end{aligned}$$

Thus, using the above expression and the fact that $\int_{\mathbb{R}^d}(1\wedge|h|^p)\nu_\varepsilon(h)\,\mathrm{d}h = 1$ we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{(y-x)\cdot e\ge 0}\frac{(\varphi(y)-\varphi(x))}{|x-y|}(1\wedge|x-y|^p)\nu_\varepsilon(x-y)\,\mathrm{d}y\\ &\qquad = \lim_{\varepsilon\to 0}\int_{\mathbb{S}^{d-1}\cap\{w\cdot e\ge 0\}}\nabla\varphi(x)\cdot w\,\mathrm{d}\sigma_{d-1}(w)\int_0^\infty (1\wedge r^p)r^{d-1}\nu_\varepsilon(r)\,\mathrm{d}r\\ &\qquad = |\mathbb{S}^{d-1}|^{-1}\int_{\mathbb{S}^{d-1}\cap\{w\cdot e\ge 0\}}\nabla\varphi(x)\cdot w\,\mathrm{d}\sigma_{d-1}(w).\end{aligned}$$

Let $(e, v_2, \cdots v_d)$ be an orthonormal basis of $\mathbb{R}^d$ in which we write the coordinates $w = (w_1, w_2, \cdots, w_d) = (w_1, w')$ that is $w_1 = w\cdot e$ and $w_i = w\cdot v_i$. Similarly, in this basis one has $\nabla\varphi(x) = (\nabla\varphi(x)\cdot e, (\nabla\varphi(x))')$. Observe that $\nabla\varphi(x)\cdot w = \big(\nabla\varphi(x)\cdot e\big)(w\cdot e) + [\nabla\phi(x)]'\cdot w'$. We find that

$$\begin{aligned}\int_{\mathbb{S}^{d-1}\cap\{w\cdot e\ge 0\}}\nabla\varphi(x)\cdot w\,\mathrm{d}\sigma_{d-1}(w) &= \int_{\mathbb{S}^{d-1}\cap\{w\cdot e\ge 0\}}(\nabla\varphi(x)\cdot e)(w\cdot e)\,\mathrm{d}\sigma_{d-1}(w)\\ &\quad+\int_{\mathbb{S}^{d-1}\cap\{w\cdot e\ge 0\}}(\nabla\varphi(x))'\cdot w'\,\mathrm{d}\sigma_{d-1}(w).\end{aligned}$$

Consider the rotation $O(w) = (w_1, -w') = (w\cdot e, -w')$ then the rotation invariance of the Lebesgue measure entails that $\mathrm{d}\sigma_{d-1}(w) = \mathrm{d}\sigma(O(w))$ and we have

$$\int_{\mathbb{S}^{d-1}\cap\{w\cdot e\ge 0\}}(\nabla\varphi(x))'\cdot w'\,\mathrm{d}\sigma_{d-1}(w) = -\int_{\mathbb{S}^{d-1}\cap\{w\cdot e\ge 0\}}(\nabla\varphi(x))'\cdot w'\,\mathrm{d}\sigma_{d-1}(w) = 0.$$

Whereas, by symmetry we have

$$\int\limits_{\mathbb{S}^{d-1}\cap\{w\cdot e\geq 0\}} w\cdot e \;\mathrm{d}\sigma_{d-1}(w) = -\int\limits_{\mathbb{S}^{d-1}\cap\{w\cdot e\leq 0\}} w\cdot e\,\mathrm{d}\sigma_{d-1}(w) = \frac{1}{2}\int\limits_{\mathbb{S}^{d-1}} |w\cdot e|\,\mathrm{d}\sigma_{d-1}(w) = \frac{1}{2}K_{d,1}.$$

Altogether, we find that

$$\begin{aligned}|\mathbb{S}^{d-1}|^{-1}\int\limits_{\mathbb{S}^{d-1}\cap\{w\cdot e\geq 0\}} \nabla\varphi(x)\cdot w \;\mathrm{d}\sigma_{d-1}(w) &= \frac{\nabla\varphi(x)\cdot e}{2}\frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}} |w\cdot e|\;\mathrm{d}\sigma_{d-1}(w)\\ &= \frac{1}{2}K_{d,1}\nabla\varphi(x)\cdot e.\end{aligned}$$

In conclusion,

$$\lim_{\varepsilon\to 0}\int\limits_{(y-x)\cdot e\geq 0} \frac{(\varphi(y)-\varphi(x))}{|x-y|}(1\wedge|x-y|^p)\nu_\varepsilon(x-y)\;\mathrm{d}y = \frac{1}{2}K_{d,1}\nabla\varphi(x)\cdot e. \tag{3.6}$$

Analogously one is able to show that

$$\lim_{\varepsilon\to 0}\int\limits_{(y-x)\cdot e\leq 0} \frac{(\varphi(y)-\varphi(x))}{|x-y|}(1\wedge|x-y|^p)\nu_\varepsilon(x-y)\;\mathrm{d}y = \frac{1}{2}K_{d,1}\nabla\varphi(x)\cdot e. \tag{3.7}$$

By substituting the two relations (3.6) and (3.7) in (3.5) and using the dominate convergence theorem one readily ends up with the desired estimate. □

Proof of Theorem 1.1 The estimate (3.4) holds true for all $\varphi\in C_c^\infty(\Omega)$, all $1\leq p<\infty$ and $e=e_i,\;\; i=1,\cdots,d$ so that $\nabla\varphi(x)\cdot e_i=\partial_{x_i}\varphi(x)$.

Case $1<p<\infty$: In virtue of the density of $C_c^\infty(\Omega)$ in $L^{p'}(\Omega)$, it readily follows from (3.4) that for each $i=1,\cdots,d$ the mapping $\varphi\mapsto\int_\Omega u(x)\partial_{x_i}\varphi(x)\,\mathrm{d}x$ uniquely extends as a continuous linear form on $L^{p'}(\Omega)$. Since $1<p'<\infty$, the Riesz representation for Lebesgue spaces reveals that there exists a unique $g_i\in L^p(\Omega)$ and we set $\partial_{x_i}u=-g_i$, such that

$$\int\limits_\Omega u(x)\partial_{x_i}\varphi(x)\;\mathrm{d}x = \int\limits_\Omega g_i(x)\varphi(x)\;\mathrm{d}x = -\int\limits_\Omega \partial_{x_i}u(x)\varphi(x)\;\mathrm{d}x \quad\text{for all}\quad \varphi\in C_c^\infty(\Omega).$$

In order words, $u\in W^{1,p}(\Omega)$. Further, the L^p-duality and (3.4) yields the estimate (1.7) as follows

$$\begin{aligned}\|\nabla u\|_{L^p(\Omega)} &\leq \sqrt{d}\sum_{i=1}^d\|\partial_{x_i}u\|_{L^p(\Omega)} = \sqrt{d}\sum_{i=1}^d \sup_{\substack{\varphi\in C_c^\infty(\mathbb{R}^d)\\ \|\varphi\|_{L^{p'}(\Omega)}=1}}\Big|\int\limits_\Omega u(x)\,\nabla\varphi(x)\cdot e_i\,\mathrm{d}x\Big|\\ &\leq d^2\frac{A_p^{1/p}}{K_{d,1}}.\end{aligned}$$

Case p=1: Let $\chi=(\chi_1,\chi_2,\cdots,\chi_d)\in C_c^\infty(\Omega,\mathbb{R}^d)$ such that $\|\chi\|_{L^\infty(\Omega,\mathbb{R}^d)}\leq 1$ and $e=e_i,\;\; i=1,2\cdots,d$. Since $\chi_i\in C_c^\infty(\Omega)$, the estimate (3.4) implies

$$\Big|\int\limits_\Omega u(x)\;\mathrm{div}\,\chi\;\mathrm{d}x\Big| = \Big|\sum_{i=1}^d\int\limits_\Omega u(x)\,\nabla\chi_i(x)\cdot e_i\;\mathrm{d}x\Big| \leq d\frac{A_1}{K_{d,1}}.$$

Hence $u\in BV(\Omega)$ and we have $|u|_{BV(\Omega)}\leq d\frac{A_1}{K_{d,1}}$ which is the estimate (1.7). □

The next result improves the estimate (1.7).

Theorem 3.3 *Let $\Omega \subset \mathbb{R}^d$ be open. If $u \in L^p(\Omega)$ with $1 < p < \infty$ or $u \in W^{1,1}(\Omega)$ for $p = 1$ then*

$$K_{d,p}\|\nabla u\|^p_{L^p(\Omega)} \le \liminf_{\varepsilon\to 0} \iint\limits_{\Omega\Omega} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y = A_p.$$

Moreover if $p = 1$ and $u \in L^1(\Omega)$ then we have

$$K_{d,1}|u|_{BV(\Omega)} \le \liminf_{\varepsilon\to 0} \iint\limits_{\Omega\Omega} |u(x)-u(y)|\nu_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y = A_1.$$

Proof **First proof.** For $\delta > 0$ small, set $\Omega_\delta = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) > \delta\}$. Define the mollifier $\phi_\delta(x) = \frac{1}{\delta^d}\phi\left(\frac{x}{\delta}\right)$ with support in $B_\delta(0)$ where $\phi \in C_c^\infty(\mathbb{R}^d)$ is supported in $B_1(0)$, $\phi \ge 0$ and $\int \phi = 1$. We assume that u is extended by zero off Ω and let $u^\delta = u * \phi_\delta$ is the convolution product of u and ϕ_δ. If $z \in \Omega_\delta$ and $|h| \le \delta$ then $z - h \in \Omega_\delta - h \subset \Omega$. A change of variables implies

$$\begin{aligned}\iint\limits_{\Omega\Omega} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\ \mathrm{d}x &\ge \iint\limits_{\Omega_\delta - h\,\Omega_\delta - h} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\ \mathrm{d}x \\ &= \iint\limits_{\Omega_\delta\Omega_\delta} |u(x-h)-u(y-h)|^p \nu_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

Thus given that $\int \phi_\delta\,\mathrm{d}h = 1$, integrating with respect to $\phi_\delta(h)dh$, Jensen's inequality yields

$$\begin{aligned}&\iint\limits_{\Omega\Omega} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\ \mathrm{d}x \\ &\ge \int\limits_{\mathbb{R}^d} \phi_\delta(h)\,\mathrm{d}h \iint\limits_{\Omega_\delta\Omega_\delta} |u(x-h)-u(y-h)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\ \mathrm{d}x \\ &\ge \iint\limits_{\Omega_\delta\Omega_\delta} \Big|\int\limits_{\mathbb{R}^d} \big(u(x-h)-u(y-h)\big)\phi_\delta(h)\,\mathrm{d}h\Big|^p \nu_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \iint\limits_{\Omega_\delta\Omega_\delta} |u*\phi_\delta(x) - u*\phi_\delta(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y.\end{aligned}$$

In other words, we have

$$\iint\limits_{\Omega_\delta\Omega_\delta} |u^\delta(x)-u^\delta(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y \le \iint\limits_{\Omega\Omega} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y. \tag{3.8}$$

Note that $u^\delta \in C^\infty(\mathbb{R}^d)$ and $\Omega_{\delta,j} = \Omega_\delta \cap B_j(0)$ has a compact closure for each $j \ge 1$. Then for each $j \ge 1$ the Lemma 3.6 implies

$$K_{d,p}\int\limits_{\Omega_{\delta,j}} |\nabla u^\delta(x)|^p\,\mathrm{d}x = \lim_{\varepsilon\to 0} \iint\limits_{\Omega_{\delta,j}\Omega_{\delta,j}} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y$$

$$\le \liminf_{\varepsilon\to 0} \iint_{\Omega\Omega} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}x\,\mathrm{d}y = A_p.$$

Tending $j \to \infty$ in the latter we get

$$K_{d,p} \int_{\Omega_\delta} |\nabla u^\delta(x)|^p\,\mathrm{d}x \le A_p. \tag{3.9}$$

Case $1 < p < \infty$: The only interesting scenario occurs if $A_p < \infty$. In this case, Theorem 1.1 ensures that $u \in W^{1,p}(\Omega)$. Clearly we have $\nabla u^\delta = \nabla(u * \phi_\delta) = \nabla u * \phi_\delta$ and $\|\phi_\delta * \nabla u - \nabla u\|_{L^p(\Omega)} \to 0$ as $\delta \to 0$. The desired inequality follows by letting $\delta \to 0$ in (3.9) since

$$\begin{aligned}&\Big|\|\nabla u\|_{L^p(\Omega)} - \|\nabla u * \phi_\delta\|_{L^p(\Omega_\delta)}\Big|\\ &\le \|\nabla u * \phi_\delta - \nabla u\|_{L^p(\Omega)} + \Big(\int_{\Omega\setminus\Omega_\delta} |\nabla u(x)|^p\,\mathrm{d}x\Big)^{1/p} \xrightarrow{\delta\to 0} 0.\end{aligned}$$

Case $p = 1$: Again we only need to assume that $A_1 < \infty$ so that by Theorem 1.1, $u \in BV(\Omega)$. The relation (3.9) implies that

$$K_{d,1} \liminf_{\delta\to 0} \int_{\Omega_\delta} |\nabla u^\delta(x)|\,\mathrm{d}x \le A_1.$$

Let $\chi \in C_c^\infty(\Omega, \mathbb{R}^d)$ such that $\|\chi\|_{L^\infty(\Omega,\mathbb{R}^d)} \le 1$ and $\operatorname{supp} \chi \subset \Omega_\delta$ for $\delta > 0$ small. We find that

$$\begin{aligned}&\Big|\int_\Omega u(x)\,\mathrm{div}\,\chi(x)\,\mathrm{d}x - \int_{\Omega_\delta} u^\delta(x)\,\mathrm{div}\,\chi(x)\,\mathrm{d}x\Big|\\ &= \Big|\int_{\Omega_\delta} (u(x) - u * \phi_\delta(x))\,\mathrm{div}\,\chi(x)\,\mathrm{d}x\Big|\\ &\le \|\,\mathrm{div}\,\chi\|_{L^\infty(\Omega,\mathbb{R}^d)} \|u * \phi_\delta - u\|_{L^1(\Omega)} \xrightarrow{\delta\to 0} 0.\end{aligned}$$

Thus, since u is a distribution on Ω we get

$$\begin{aligned}\int_\Omega u(x)\,\mathrm{div}\,\chi(x)\,\mathrm{d}x &= \lim_{\delta\to 0} \int_{\Omega_\delta} u^\delta(x)\,\mathrm{div}\,\chi(x)\,\mathrm{d}x\\ &= -\lim_{\delta\to 0} \int_{\Omega_\delta} \nabla u^\delta(x)\cdot\chi(x)\,\mathrm{d}x \le \liminf_{\delta\to 0} \int_{\Omega_\delta} |\nabla u^\delta(x)|\,\mathrm{d}x.\end{aligned}$$

This completes the proof since the above holds for arbitrarily chosen $\chi \in C_c^\infty(\Omega, \mathbb{R}^d)$ such that $\|\chi\|_{L^\infty(\Omega,\mathbb{R}^d)} \le 1$, by definition of $|\cdot|_{BV(\Omega)}$ and the previous estimate we get

$$K_{d,1}|u|_{BV(\Omega)} \le \liminf_{\delta\to 0} \int_{\Omega_\delta} |\nabla u^\delta(x)|\,\mathrm{d}x \le A_1.$$

Second proof. Here is an alternative. Since for all $\delta > 0$, $\int_{B_\delta(0)} |h|^p \nu_\varepsilon(h)\,\mathrm{d}h \to 1$ as $\varepsilon \to 0$ (see the formula (2.3)), for each compact set $K \subset \Omega$ and $\eta > 0$ inequality (2.8) implies

$$\begin{aligned}
&\liminf_{\varepsilon\to 0} \iint_{\Omega\Omega} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\,\,\mathrm{d}x \\
&\geq \liminf_{\varepsilon\to 0} \Big(\int_{B_\delta(0)} |h|^p \nu_\varepsilon(h)\,\mathrm{d}h\Big)\big(K_{d,p}^{1/p}\|\nabla u\|_{L^p(K)} - \eta\big)^p \\
&= \big(K_{d,p}^{1/p}\|\nabla u\|_{L^p(K)} - \eta\big)^p.
\end{aligned}$$

Let $K_j = \overline{\Omega}_j \subset \Omega_{j+1}$ and $(\Omega_j)_j$ be an exhaustion of Ω. Since the above inequality is true for every compact set $K = K_j \subset \Omega$ and every $\eta > 0$ we conclude that

$$\liminf_{\varepsilon\to 0} \iint_{\Omega\Omega} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\,\,\mathrm{d}x \geq K_{d,p}\|\nabla u\|^p_{L^p(\Omega)}.$$

The case $p = 1$ and $u \in BV(\Omega)$ follows from the approximation Theorem 2.11. □

It is worth to mention that the convolution technique used in the first proof above was first used in [7] when $\Omega = \mathbb{R}^d$ and also appears in [28]. The next theorem is a the counterpart of Theorem 3.3 and is a refinement version of Theorem 2.14.

Theorem 3.4 *Let $\Omega \subset \mathbb{R}^d$ be a $W^{1,p}$-extension domain and $u \in L^p(\Omega)$, $p > 1$ then we have*

$$\limsup_{\varepsilon\to 0} \iint_{\Omega\Omega} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\mathrm{d}y\,\mathrm{d}x \leq K_{d,p}\|\nabla u\|^p_{L^p(\Omega)}.$$

Moreover, for $p = 1$, if Ω is a BV-extension domain and $u \in L^1(\Omega)$ then we have

$$\limsup_{\varepsilon\to 0} \iint_{\Omega\Omega} |u(x)-u(y)| \nu_\varepsilon(x-y)\mathrm{d}y\,\mathrm{d}x \leq K_{d,1}|u|_{BV(\Omega)}.$$

Proof The cases $\|\nabla u\|_{L^p(\Omega)} = \infty$ and $|u|_{BV(\Omega)} = \infty$ are trivial. Assume $u \in W^{1,p}(\Omega)$ and let $\overline{u} \in W^{1,p}(\mathbb{R}^d)$ be its extension to $\mathbb{R}^d$. Consider $\Omega(\delta) = \Omega + B_\delta(0) = \{x \in \mathbb{R}^d : \mathrm{dist}(x,\Omega) < \delta\}$ be a neighborhood of Ω where $0 < \delta < 1$. We claim that for each $\varepsilon > 0$, the following estimate holds

$$\iint_{\Omega\Omega} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\mathrm{d}y\,\mathrm{d}x \leq K_{d,p} \int_{\Omega(\delta)} |\nabla \overline{u}(x)|^p\,\mathrm{d}x + 2^p\|u\|^p_{L^p(\Omega)} \int_{|h|>\delta} \nu_\varepsilon(h)\,\mathrm{d}h. \tag{3.10}$$

Indeed, let $(u_n)_n$ be a sequence in $C_c^\infty(\mathbb{R}^d)$ converging to $\overline{u}$ in $W^{1,p}(\mathbb{R}^d)$. For each $n \geq 1$, passing through the polar coordinates and using the identity (1.9) we find that

$$\begin{aligned}
&\iint_{\Omega\times\Omega\cap\{|x-y|\leq\delta\}} |u_n(x)-u_n(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\,\,\mathrm{d}x \\
&\leq \int_0^1 \int_\Omega \int_{|h|\leq\delta} |\nabla u_n(x+th)\cdot h|^p\,\nu_\varepsilon(h)\,\mathrm{d}h\,\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

$$
\begin{aligned}
&\leq \int\limits_{|h|\leq\delta}\int\limits_{\Omega(\delta)} |\nabla u_n(z)\cdot h|^p \,\mathrm{d}z\, \nu_\varepsilon(h)\,\mathrm{d}h \\
&= \Big(\int\limits_{\Omega(\delta)}\int_{\mathbb{S}^{d-1}} |\nabla u_n(z)\cdot w|^p \,\mathrm{d}\sigma_{d-1}(w)\Big)\Big(\int\limits_0^\delta r^{p+d-1}\nu_\varepsilon(r)\,\mathrm{d}r\Big) \\
&= K_{d,p}\Big(\int\limits_{\Omega(\delta)} |\nabla u_n(z)|^p \,\mathrm{d}z\Big)\Big(\int\limits_{|h|\leq\delta} (1\wedge|h|^p)\nu_\varepsilon(h)\,\mathrm{d}h\Big) \\
&\leq K_{d,p}\int\limits_{\Omega(\delta)} |\nabla u_n(z)|^p \,\mathrm{d}z.
\end{aligned}
$$

Fatou's lemma implies

$$
\begin{aligned}
&\iint\limits_{\Omega\times\Omega\cap\{|x-y|\leq\delta\}} |u(x)-u(y)|^p\nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x \\
&\leq \liminf_{n\to\infty}\int\limits_{\Omega}\int\limits_{|x-y|\leq\delta} |u_n(x)-u_n(y)|^p\nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x \\
&\leq K_{d,p}\int\limits_{\Omega(\delta)} |\nabla\overline{u}(x)|^p\,\mathrm{d}x.
\end{aligned}
$$

The estimate (3.10) clearly follows since we have

$$
\int\limits_{\Omega}\int\limits_{\Omega\cap\{|x-y|>\delta\}} |u(x)-u(y)|^p\nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x \leq 2^p\|u\|^p_{L^p(\Omega)}\int\limits_{|h|>\delta}\nu_\varepsilon(h)\,\mathrm{d}h.
$$

Letting $\varepsilon\to 0$ the relation (3.10) yields

$$
\limsup_{\varepsilon\to 0}\iint\limits_{\Omega\Omega} |u(x)-u(y)|^p\nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x \leq K_{d,p}\int\limits_{\Omega(\delta)} |\nabla u(x)|^p\,\mathrm{d}x
$$

Recalling that $\overline{u}\in W^{1,p}(\mathbb{R}^d)$, $u=\overline{u}\,|_\Omega$ and using (1.11) the desired estimate follows

$$
\int\limits_{\Omega(\delta)} |\nabla\overline{u}(x)|^p\,\mathrm{d}x \xrightarrow{\delta\to 0} \int\limits_{\overline{\Omega}} |\nabla\overline{u}(x)|^p\,\mathrm{d}x = \int\limits_{\Omega} |\nabla u(x)|^p\,\mathrm{d}x.
$$

If $p=1$ and $u\in BV(\Omega)$, let $\overline{u}\in BV(\mathbb{R}^d)$ be its extension to $\mathbb{R}^d$. By Theorem 2.11 there is $(u_n)_n$ a sequence in $C^\infty(\mathbb{R}^d)\cap W^{1,1}(\mathbb{R}^d)$ which converges to $\overline{u}$ in $L^1(\mathbb{R}^d)$ and $\|\nabla u_n\|_{L^1(\mathbb{R}^d)} \xrightarrow{n\to\infty} |\overline{u}|_{BV(\mathbb{R}^d)}$.

The estimate (3.10) applied to u_n and the Fatou's lemma yield

$$
\begin{aligned}
\iint\limits_{\Omega\Omega} |u(x)-u(y)|\nu_\varepsilon(x-y)\mathrm{d}y\,\mathrm{d}x &\leq \liminf_{n\to\infty}\iint\limits_{\Omega\Omega} |u_n(x)-u_n(y)|\nu_\varepsilon(x-y)\mathrm{d}y\,\mathrm{d}x \\
&\leq \lim_{n\to\infty} K_{d,1}\int\limits_{\Omega(\delta)} |\nabla u_n(x)|\,\mathrm{d}x + 2\|u_n\|_{L^1(\Omega)}\int\limits_{|h|>\delta}\nu_\varepsilon(h)\,\mathrm{d}h.
\end{aligned}
$$

$$= K_{d,1}|\overline{u}|_{BV(\Omega(\delta))} + 2\|u\|_{L^1(\Omega)} \int\limits_{|h|>\delta} \nu_\varepsilon(h)\, \mathrm{d}h.$$

Correspondingly, we also get the estimate

$$\iint\limits_{\Omega\Omega} |u(x) - u(y)|\nu_\varepsilon(x-y)\mathrm{d}y\, \mathrm{d}x \le K_{d,1}|\overline{u}|_{BV(\Omega(\delta))} + 2\|u\|_{L^1(\Omega)} \int\limits_{|h|>\delta} \nu_\varepsilon(h)\, \mathrm{d}h. \quad (3.11)$$

Therefore, letting $\varepsilon \to 0$ implies that

$$\limsup_{\varepsilon\to 0} \iint\limits_{\Omega\Omega} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\, \mathrm{d}y\, \mathrm{d}x \le K_{d,1}|\overline{u}|_{BV(\Omega(\delta))}.$$

Recalling that $\overline{u} \in BV(\mathbb{R}^d)$, $u = \overline{u}\,|_\Omega$ and $\partial\Omega$ satisfies (1.12), i.e., $|\nabla\overline{u}|(\partial\Omega) = 0$ we have

$$|\overline{u}|_{BV(\Omega(\delta))} \xrightarrow{\delta\to 0} |\overline{u}|_{BV(\overline{\Omega})} = |u|_{BV(\Omega)}.$$

□

The following result involves the collapse across the boundary $\partial\Omega$.

Theorem 3.5 *Assume $\Omega \subset \mathbb{R}^d$ is open then for any $u \in W^{1,p}(\mathbb{R}^d)$ we have*

$$\limsup_{\varepsilon\to 0} 2 \iint\limits_{\Omega\Omega^c} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\, \mathrm{d}y\ \mathrm{d}x \le K_{d,p} \int\limits_{\partial\Omega} |\nabla u(x)|^p\, \mathrm{d}x,$$

$$\liminf_{\varepsilon\to 0} 2 \iint\limits_{\Omega\Omega^c} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\, \mathrm{d}y\ \mathrm{d}x \ge K_{d,p} \int\limits_{\partial\overline{\Omega}} |\nabla u(x)|^p\, \mathrm{d}x.$$

The same holds for $p = 1$by replacing $W^{1,1}(\mathbb{R}^d)$ with $BV(\mathbb{R}^d)$.

Proof We only prove for $u \in W^{1,p}(\mathbb{R}^d)$, the case $u \in BV(\mathbb{R}^d)$ is analogous. The sets Ω and $U_\delta = \{x \in \mathbb{R}^d : \mathrm{dist}(x, \Omega) > \delta\}$, $\delta > 0$ are open. By Theorem 3.3, we get

$$K_{d,p} \int\limits_{\Omega} |\nabla u(x)|^p\, \mathrm{d}x \le \liminf_{\varepsilon\to 0} \iint\limits_{\Omega\Omega} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\, \mathrm{d}y\ \mathrm{d}x,$$

$$K_{d,p} \int\limits_{U_\delta} |\nabla u(x)|^p\, \mathrm{d}x \le \liminf_{\varepsilon\to 0} \iint\limits_{U_\delta U_\delta} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\, \mathrm{d}y\ \mathrm{d}x.$$

Since $U_\delta \subset \Omega^c$ and $\mathbb{1}_{U_\delta}(x) \to \mathbb{1}_{\mathbb{R}^d\setminus\overline{\Omega}}(x)$, for all $x \in \mathbb{R}^d$ as $\delta \to 0$, it follows that

$$K_{d,p} \int\limits_{\mathbb{R}^d\setminus\overline{\Omega}} |\nabla u(x)|^p\, \mathrm{d}x \le \liminf_{\varepsilon\to 0} \iint\limits_{\Omega^c\Omega^c} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\, \mathrm{d}y\ \mathrm{d}x.$$

Accordingly, together with (3.1), we deduce the desired result as follows

$$\limsup_{\varepsilon\to 0} 2 \iint\limits_{\Omega\Omega^c} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\, \mathrm{d}y\ \mathrm{d}x$$

$$= \limsup_{\varepsilon\to 0} \Big(\iint\limits_{\mathbb{R}^d\mathbb{R}^d} - \iint\limits_{\Omega\Omega} - \iint\limits_{\Omega^c\Omega^c} \Big) |u(x) - u(y)|^p \nu_\varepsilon(x-y)\, \mathrm{d}y\ \mathrm{d}x$$

$$
\begin{aligned}
&\leq K_{d,p}\Big(\|\nabla u\|^p_{L^p(\mathbb{R}^d)} - \|\nabla u\|^p_{L^p(\Omega)} - \|\nabla u\|^p_{L^p(\mathbb{R}^d\setminus\overline{\Omega})}\Big)\\
&= K_{d,p}\int_{\partial\Omega} |\nabla u(x)|^p\,\mathrm{d}x.
\end{aligned}
$$

The reverse inequality follows analogously, since by exploiting (3.10) (or (3.11)) one easily gets

$$
\begin{aligned}
K_{d,p}\int_{\overline{\Omega}} |\nabla u(x)|^p\,\mathrm{d}x &\geq \limsup_{\varepsilon\to 0}\iint_{\Omega\Omega} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x,\\
K_{d,p}\int_{\Omega^c} |\nabla u(x)|^p\,\mathrm{d}x &\geq \limsup_{\varepsilon\to 0}\iint_{\Omega^c\Omega^c} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x.
\end{aligned}
$$

□

Next we establish a pointwise and $L^1(\Omega)$ convergence when u is a sufficiently smooth function.

Lemma 3.6 *Let $\Omega\subset\mathbb{R}^d$ be open and $u\in C^1_c(\mathbb{R}^d)$. The following convergence occurs in both pointwise and $L^1(\Omega)$ sense:*

$$
\lim_{\varepsilon\to 0}\int_\Omega |u(x)-u(y)|^p\nu_\varepsilon(x-y)\mathrm{d}y = K_{d,p}|\nabla u(x)|^p.
$$

Proof **First proof of Lemma 3.6**. Let $\sigma>0$ be sufficiently small. By assumption ∇u is uniformly continuous and hence one can find $0<\eta=\eta(\sigma)<1$ such that if $|x-y|<\eta$ then

$$
|\nabla u(y)-\nabla u(x)|\leq\sigma. \tag{3.12}
$$

Let $\eta_x=\min(\eta,\delta_x)$ with $\delta_x=\operatorname{dist}(x,\partial\Omega)$ so that $B(x,\eta_x)\subset\Omega$ for all $x\in\Omega$. Consider the mapping $F:\Omega\times(0,1)\to\mathbb{R}$ with

$$
F(x,\varepsilon):=\int_{\Omega\cap\{|x-y|\leq\eta_x\}} |u(x)-u(y)|^p\nu_\varepsilon(x-y)\mathrm{d}y = \int_{|h|\leq\eta_x} |u(x)-u(x+h)|^p\nu_\varepsilon(h)\,\mathrm{d}h.
$$

In virtue of the fundamental theorem of calculus, we have

$$
\begin{aligned}
F(x,\varepsilon) &= \int_{|h|\leq\eta_x}\Big|\int_0^1 \nabla u(x+th)\cdot h\mathrm{d}t\Big|^p \nu_\varepsilon(h)\,\mathrm{d}h\\
&= \int_{|h|\leq\eta_x} |\nabla u(x)\cdot h|^p\,\nu_\varepsilon(h)\,\mathrm{d}h + R(x,\varepsilon), \text{ with theremainder}
\end{aligned}
$$

$$
R(x,\varepsilon)=\int_{|h|\leq\eta_x}\left(\Big|\int_0^1 \nabla u(x+th)\cdot h\mathrm{d}t\Big|^p - \Big|\int_0^1 \nabla u(x)\cdot h\mathrm{d}t\Big|^p\right)\nu_\varepsilon(h)\,\mathrm{d}h.
$$

The mapping $s \mapsto G_p(s) = |s|^p$ belongs to $C^1(\mathbb{R}^d\setminus\{0\})$ and $G_p'(s) = pG_p(s)s^{-1}$. Thus, we have

$$G_p(b) - G_p(a) = (b-a)\int_0^1 G_p'(a + s(b-a))\, \mathrm{d}s.$$

Set $a = \nabla u(x)\cdot h$ and $b = \int_0^1 \nabla u(x+th)\cdot h\, \mathrm{d}t$ so that the relation (3.12) yields

$$\begin{aligned}|G_p(b) - G_p(a)| &\le p|b-a|\int_0^1 |(1-s)a + sb|^{p-1}\, \mathrm{d}s\\ &\le p\|\nabla u\|^{p-1}_{L^\infty(\mathbb{R}^d)}|h|^{p-1}\int_0^1 |\nabla u(x+th) - \nabla u(x)||h|\, \mathrm{d}t\\ &\le p\sigma\|\nabla u\|^{p-1}_{L^\infty(\mathbb{R}^d)}|h|^p.\end{aligned}$$

Integrating both sides with respect to $\nu_\varepsilon(h)\,\mathrm{d}h$, implies that

$$|R(x,\varepsilon)| \le p\sigma\|\nabla u\|^{p-1}_{L^\infty(\mathbb{R}^d)}\int_{|h|\le\eta_x} |h|^p\nu_\varepsilon(h)\, \mathrm{d}h.$$

Since $\int_{|h|\le\eta_x}|h|^p\nu_\varepsilon(h)\,\mathrm{d}h \to 1$ as $\varepsilon\to 0$, by the formula (2.3), letting $\varepsilon\to 0$ and $\sigma\to 0$ successively yields $R(x,\varepsilon)\to 0$. Whereas, using polar coordinates, the relation (1.9) and the Remark 2.3 gives

$$\begin{aligned}\int_{|h|\le\eta_x} |\nabla u(x)\cdot h|^p\,\nu_\varepsilon(h)\,\mathrm{d}h &= \int_0^{\eta_x} r^{d+p-1}\mathrm{d}r\int_{\mathbb{S}^{d-1}}|\nabla u(x)\cdot w|^p\mathrm{d}\sigma_{d-1}(w)\\ &= K_{d,p}|\nabla u(x)|^p\int_{|h|\le\eta_x}|h|^p\nu_\varepsilon(h)\,\mathrm{d}h \xrightarrow{\varepsilon\to0} K_{d,p}|\nabla u(x)|^p.\end{aligned}$$

Therefore, we have $F(x,\varepsilon)\xrightarrow{\varepsilon\to0} K_{d,p}|\nabla u(x)|^p$. Furthermore, a close look to our reasoning reveals that we have subsequently shown that

$$\lim_{\varepsilon\to0}\int_{\Omega\cap\{|x-y|\le\delta\}}|u(x)-u(y)|^p\nu_\varepsilon(x-y)\,\mathrm{d}y = K_{d,p}|\nabla u(x)|^p, \quad \text{for all } \delta>0. \tag{3.13}$$

This is due to the fact that, for all $\delta>0$ we have

$$\int_{\Omega\cap\{|x-y|>\delta\}}|u(x)-u(y)|^p\nu_\varepsilon(x-y)\,\mathrm{d}y \le 2^p\|u\|^p_{L^\infty(\mathbb{R}^d)}\int_{|h|>\delta}\nu_\varepsilon(h)\mathrm{d}h \xrightarrow{\varepsilon\to0} 0. \tag{3.14}$$

Hence we have the pointwise convergence as claimed, i.e., for all $x\in\Omega$ we have

$$\lim_{\varepsilon\to0}\int_\Omega |u(x)-u(y)|^p\nu_\varepsilon(x-y)\,\mathrm{d}y\;\mathrm{d}x = K_{d,p}|\nabla u(x)|^p. \tag{3.15}$$

To proceed with the convergence in $L^1(\Omega)$, according to the Schéffé lemma [34, p. 55], it suffices to show the convergence of $L^1(\Omega)$-norm. Choosing $R \geq 1$ such that $\operatorname{supp} u \subset B_R(0)$, we write

$$\iint_{\Omega\Omega} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x = \iint_{\Omega\times\Omega\cap\{|x-y|\leq R\}} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x + \iint_{\Omega\times\Omega\cap\{|x-y|> R\}} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x.$$

Since $|u(x)-u(x+h)|^p \leq 2^p(1\wedge|h|^p)\|u\|^p_{W^{1,\infty}(\mathbb{R}^d)}$ and $\int_{\mathbb{R}^d}(1\wedge|h|^p)\nu_\varepsilon(h)\,\mathrm{d}h = 1$ one gets

$$H_\varepsilon(x) = \int_{\Omega\cap\{|x-y|\leq R\}} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y \leq \int_{|h|\leq R} |u(x)-u(x+h)|^p \nu_\varepsilon(h)\,\mathrm{d}h \leq 2^p\|u\|^p_{W^{1,\infty}(\mathbb{R}^d)}.$$

Noting that $\operatorname{supp} H_\varepsilon \subset B_{2R}(0)$, one finds that $H_\varepsilon(x) \leq 2^p\|u\|^p_{W^{1,\infty}(\mathbb{R}^d)}\mathbb{1}_{B_{2R}(0)}$ with $\mathbb{1}_{B_{2R}(0)} \in L^1(\Omega)$, the pointwise limit in (3.13) and the dominated convergence theorem imply that

$$\lim_{\varepsilon\to 0} \iint_{\Omega\times\Omega\cap\{|x-y|\leq R\}} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x = K_{d,p}\int_\Omega |\nabla u(x)|^p\,\mathrm{d}x.$$

We thus obtain the following convergence of $L^1(\Omega)$-norm as expected

$$\lim_{\varepsilon\to 0}\iint_{\Omega\Omega} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x = K_{d,p}\int_\Omega |\nabla u(x)|^p\,\mathrm{d}x,$$

since, by assumption on ν_ε, one has

$$\iint_{\Omega\times\Omega\cap\{|x-y|> R\}} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x \leq 2^p\|u\|^p_{L^p(\Omega)}\int_{|h|>R}\nu_\varepsilon(h)\mathrm{d}h \xrightarrow{\varepsilon\to 0} 0.$$

Second proof of Lemma 3.6 when $u \in C^2_c(\mathbb{R}^d)$. Note that if we put $G_p(s) = |s|^p$ then $G_p \in C^2(\mathbb{R}^d\setminus\{0\})$. The Taylor formula implies

$$\begin{aligned} u(y)-u(x) &= \nabla u(x)\cdot(y-x) + O(|x-y|^2), \quad x,y\in\mathbb{R}^d,\\ G_p(b)-G_p(a) &= G_p'(a)(b-a) + O(b-a)^2, \quad a,b\in\mathbb{R}\setminus\{0\}. \end{aligned}$$

Hence for almost all $x, y \in \mathbb{R}^d$, we have

$$\begin{aligned} |u(y)-u(x)|^p &= G_p(\nabla u(x)\cdot(y-x) + O(|y-x|^2))\\ &= |\nabla u(x)\cdot(y-x)|^p + O(|y-x|^{p+1}). \end{aligned}$$

Set $\delta_x = \operatorname{dist}(x,\partial\Omega)$. Passing through polar coordinates and using the relation (1.9) yields

$$\begin{aligned}
&\int\limits_{B(x,\delta_x)} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y \\
&= \int\limits_{|h|\le\delta_x} \Big|\nabla u(x)\cdot h\Big|^p \nu_\varepsilon(h)\,\mathrm{d}h + O\Big(\int\limits_{|h|\le\delta_x} |h|^{p+1}\nu_\varepsilon(h)\,\mathrm{d}h\Big)\\
&= \int\limits_{\mathbb{S}^{d-1}} \Big|\nabla u(x)\cdot w\Big|^p \mathrm{d}\sigma_{d-1}(w)\int_0^{\delta_x} r^{d-1}\nu_\varepsilon(r)\mathrm{d}r + O\Big(\int\limits_{|h|\le\delta_x} |h|^{p+1}\nu_\varepsilon(h)\mathrm{d}h\Big)\\
&= K_{d,p}\,|\nabla u(x)|^p \int\limits_{|h|\le\delta_x} \nu_\varepsilon(h)\mathrm{d}h + O\Big(\int\limits_{|h|\le\delta_x} |h|^{p+1}\nu_\varepsilon(h)\mathrm{d}h\Big).
\end{aligned}$$

Therefore, letting $\varepsilon \to 0$ in the latter expression and taking into account Remark 2.3 gives

$$\lim_{\varepsilon\to 0} \int\limits_{B(x,\delta_x)} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\mathrm{d}y = K_{d,p}|\nabla u(x)|^p.$$

The pointwise convergence (3.15) readily follows. Since on the other side, we have

$$\int\limits_{\Omega\setminus B(x,\delta_x)} |u(x)-u(y)|^p \nu_\varepsilon(x-y)\mathrm{d}y \le 2^p \|u\|^p_{L^\infty(\Omega)} \int\limits_{|h|\ge\delta_x} \nu_\varepsilon(h)\,\mathrm{d}h \xrightarrow{\varepsilon\to 0} 0.$$

Thus the remaining details follow by proceeding as in the previous proof. □

We are now in position to prove Theorem 1.3.

Proof of Theorem 1.3 Assume $A_p = \infty$ then by Theorem 1.1 we have $\|\nabla u\|_{L^p(\Omega)} = \infty$ for $1 < p < \infty$ and $|u|_{BV(\Omega)} = \infty$ for $p = 1$. In either case the relation (1.13) or (1.14) is verified. The interesting situation is when $A_p < \infty$, i.e., by Theorem 1.1, $u \in W^{1,p}(\Omega)$ if $1 < p < \infty$ and $u \in BV(\Omega)$ if $p = 1$. We provide two alternative proofs. As first alternative, the result immediately follows by combining Theorem 3.3 and Theorem 3.4. For the second alternative, consider $1 < p < \infty$ or $u \in W^{1,1}(\Omega)$. By Lemma 2.12 there is $C > 0$ independent of ε such that for $u, v \in W^{1,p}(\Omega)$,

$$\Big|\|U_\varepsilon\|_{L^p(\Omega\times\Omega)} - \|V_\varepsilon\|_{L^p(\Omega\times\Omega)}\Big| \le \|U_\varepsilon - V_\varepsilon\|_{L^p(\Omega\times\Omega)} \le C\|u-v\|_{W^{1,p}(\Omega)},$$

$$\text{where we deifine} U_\varepsilon(x,y) = |u(x)-u(y)|\nu_\varepsilon^{1/p}(x-y) \quad \text{and}$$
$$V_\varepsilon(x,y) = |v(x)-v(y)|\nu_\varepsilon^{1/p}(x-y)\,.$$

Therefore, it suffices to establish the result for u in a dense subset of $W^{1,p}(\Omega)$. Note that $C_c^\infty(\mathbb{R}^d)$ is dense in $W^{1,p}(\Omega)$ since Ω is a $W^{1,p}$-extension domain. We conclude by using Lemma 3.6. □

As consequence of Theorem 1.3 we have the following concrete examples.

Corollary 3.7 *Assume $\Omega \subset \mathbb{R}^d$ is an extension domain and $u \in L^p(\Omega)$. If we abuse the notation $\|\nabla u\|_{L^1(\Omega)} = |u|_{BV(\Omega)}$ for $p = 1$, then there holds*

$$\lim_{s\to 1}(1-s)\iint\limits_{\Omega\Omega}\frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\mathrm{d}y\mathrm{d}x=\frac{|\mathbb{S}^{d-1}|}{p}K_{d,p}\|\nabla u\|^p_{L^p(\Omega)},$$

$$\lim_{\varepsilon\to 0}\varepsilon^{-d-p}\iint\limits_{\Omega\times\Omega\cap\{|x-y|<\varepsilon\}}|u(x)-u(y)|^p\,\mathrm{d}y\,\mathrm{d}x=\frac{|\mathbb{S}^{d-1}|}{d+p}K_{d,p}\|\nabla u\|^p_{L^p(\Omega)},$$

$$\lim_{\varepsilon\to 0}\varepsilon^{-d}\iint\limits_{\Omega\times\Omega\cap\{|x-y|<\varepsilon\}}\frac{|u(x)-u(y)|^p}{|x-y|^p}\,\mathrm{d}y\,\mathrm{d}x=\frac{|\mathbb{S}^{d-1}|}{d}K_{d,p}\|\nabla u\|^p_{L^p(\Omega)},$$

$$\lim_{\varepsilon\to 0}\frac{1}{|\log\varepsilon|}\iint\limits_{\Omega\times\Omega\cap\{|x-y|>\varepsilon\}}\frac{|u(x)-u(y)|^p}{|x-y|^{d+p}}\mathrm{d}y\mathrm{d}x=|\mathbb{S}^{d-1}|K_{d,p}\|\nabla u\|^p_{L^p(\Omega)}.$$

Proof For the first relation, take $\nu_\varepsilon(h)=a_{\varepsilon,d,p}|h|^{-d-p(1-\varepsilon)}$ with $a_{\varepsilon,d,p}=\frac{p\varepsilon(1-\varepsilon)}{|\mathbb{S}^{d-1}|}$. For the second and third take $\nu_\varepsilon(h)=\frac{d+\beta}{|\mathbb{S}^{d-1}|}\varepsilon^{-d-\beta}\mathbb{1}_{B_\varepsilon}(h)$ $\beta\in\{0,p\}$. For the last one, fixed $\varepsilon_0\geq 1$, take $\nu_\varepsilon(h)=\frac{b_\varepsilon}{|\mathbb{S}^{d-1}||\log\varepsilon|}\mathbb{1}_{B_{\varepsilon_0}\setminus B_\varepsilon}(h)$, $b_\varepsilon=\frac{p|\log\varepsilon|}{(1-\varepsilon_0^{-p})+p|\log\varepsilon|}$, where one notes that $b_\varepsilon\to 1$ as $\varepsilon\to 0$. □

Proof of Theorem 1.6 Let $E\subset\Omega$ be compact with a nonempty interior. Consider the open set $E(\delta)=E+B_\delta(0)\subset\Omega$ where $0<\delta<1\wedge\operatorname{dist}(\partial\Omega,E)$ so that $\int_{|h|>\delta}\nu_\varepsilon(h)\,\mathrm{d}h\leq 1$. Denote $\mathrm{d}|\nabla u|^p(x)=|\nabla u(x)|^p\,\mathrm{d}x$, $u\in W^{1,p}(\Omega)$. Using (3.10) and (3.11) with Ω replaced by E imply

$$\int\limits_E\mu_\varepsilon(x)\,\mathrm{d}x\leq K_{d,p}\int\limits_{E(\delta)}\mathrm{d}|\nabla u|^p(x)+2^p\|u\|^p_{L^p(\Omega)}\int\limits_{|h|>\delta}\nu_\varepsilon(h)\,\mathrm{d}h. \tag{3.16}$$

Hence, since $\int_{|h|>\delta}\nu_\varepsilon(h)\,\mathrm{d}h\leq 1$, the family of functions $(\mu_\varepsilon)_\varepsilon$ is bounded in $L^1(E)$. In virtue of the weak compactness of $L^1(E)$ (see [8, p. 116]) we may assume that $(\mu_\varepsilon)_\varepsilon$ converges in the weak-* sense to a Radon measure μ_E, i.e., $\langle\mu_\varepsilon-\mu_E,\varphi\rangle\xrightarrow{\varepsilon\to 0}0$ for all $\varphi\in C(E)$ otherwise, one may pick a converging subsequence. For a suitable $(\Omega_j)_{j\in\mathbb{N}}$ exhaustion of Ω, i.e., $\Omega_j's$ are open, each $K_j=\overline{\Omega}_j$ is compact, $K_j=\overline{\Omega}_j\subset\Omega_{j+1}$ and $\Omega=\bigcup_{j\in\mathbb{N}}\Omega_j$, it is sufficient to let $\mu=\mu_{K_j}=K_{d,p}|\nabla u|^p$ on K_j. We aim to show that $\mu=K_{d,p}|\nabla u|^p$. Noticing μ and $K_{d,p}|\nabla u|^p$ are Radon measures it sufficient to show that both measures coincide on compact sets, i.e., we have to show that $\mu_E(E)=K_{d,p}\int_E\mathrm{d}|\nabla u|^p(x)$. On the one hand, since $\mu_\varepsilon(E)\to\mu(E)$ and $\int_{|h|>\delta}\nu_\varepsilon(h)\,\mathrm{d}h\to 0$ as $\varepsilon\to 0$, the fact that $u\in W^{1,p}(\Omega)$ or $u\in BV(\Omega)$ enables us to successively let $\varepsilon\to 0$ and $\delta\to 0$ in (3.16) which amounts to the following

$$\int\limits_E\mathrm{d}\mu_E(x)\leq K_{d,p}\int\limits_E\mathrm{d}|\nabla u|^p(x).$$

On other hand, since E has a nonempty interior, Theorem 3.3 implies

$$\begin{aligned}K_{d,p}\int\limits_E\mathrm{d}|\nabla u|^p(x)&\leq\liminf_{\varepsilon\to 0}\iint\limits_{EE}|u(x)-u(y)|^p\nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x\\&\leq\lim_{\varepsilon\to 0}\int\limits_E\mu_\varepsilon(x)\,\mathrm{d}x=\int\limits_E\mathrm{d}\mu_E(x).\end{aligned}$$

Finally $\mu(E) = \mu_E(E) = K_{d,p}\int_E \mathrm{d}|\nabla u|^p(x)$. Whence we get $\mathrm{d}\mu = K_{d,p}\,\mathrm{d}|\nabla u|^p$ as claimed. □

A consequence of Theorem 1.6 is given by the following analog result.

Corollary 3.8 *Let $\Omega \subset \mathbb{R}^d$ be open. Let $u \in W^{1,p}(\mathbb{R}^d)$ and define the Radon measures*

$$\mathrm{d}\widetilde{\mu}_\varepsilon(x) = \int_{\mathbb{R}^d} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\mathrm{d}y\,\mathrm{d}x.$$

The sequence $(\widetilde{\mu}_\varepsilon)_\varepsilon$ converges weakly on Ω to the Radon measure $\mathrm{d}\mu(x) = K_{d,p}|\nabla u(x)|^p\,\mathrm{d}x$, i.e. $\widetilde{\mu}_\varepsilon(E) \xrightarrow{\varepsilon\to 0} \mu(E)$ for every compact set $E \subset \Omega$. If $u \in BV(\Omega)$, $p = 1$, then $\mathrm{d}\mu(x) = K_{d,1}\,\mathrm{d}|\nabla u|(x)$.

Proof Let $E \subset \Omega$ be a compact set so that $\delta > 0$ with $\delta = \mathrm{dist}(E, \Omega^c) > 0$. Thus, we have

$$\widehat{\mu}_\varepsilon(E) := \iint_{E\Omega^c} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x \le 2^{p-1}\|u\|^p_{L^p(\mathbb{R}^d)} \int_{|h|>\delta} \nu_\varepsilon(h)\,\mathrm{d}h \xrightarrow{\varepsilon\to 0} 0.$$

Form this and Theorem 1.6 we get $\widetilde{\mu}_\varepsilon(E) = \mu_\varepsilon(E) + \widehat{\mu}_\varepsilon(E) \xrightarrow{\varepsilon\to 0} \mu(E)$. □

Next, we sate without proof the asymptotically compactness involving the case where the function u also varies. The full proof can be found in [18, Theorem 5.40] and [28].

Theorem 3.9 *Assume $\Omega \subset \mathbb{R}^d$ is open, bounded and Lipschitz. Let the family $(u_\varepsilon)_\varepsilon$ such that*

$$\sup_{\varepsilon>0}\Big(\|u_\varepsilon\|^p_{L^p(\Omega)} + \iint_{\Omega\Omega} |u_\varepsilon(x) - u_\varepsilon(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x\Big) < \infty.$$

There exists a subsequence $(\varepsilon_n)_n$ with $\varepsilon_n \to 0^+$ as $n \to \infty$ such that $(u_{\varepsilon_n})_n$ converges in $L^p(\Omega)$ to a function $u \in L^p(\Omega)$. Moreover, $u \in W^{1,p}(\Omega)$ if $1 < p < \infty$ or $u \in BV(\Omega)$ if $p = 1$.

Counterexample 1 We consider the fractional kernel $\nu_\varepsilon(h) = a_{\varepsilon,d,p}|h|^{-d-(1-\varepsilon)p}$, $p \ge 1$, where $a_{\varepsilon,d,p} = \frac{p\varepsilon(1-\varepsilon)}{|\mathbb{S}^{d-1}|}$. We put $s = 1-\varepsilon > 0$ and consider the nonlocal seminorm

$$|u|^p_{W^{s,p}(\Omega)} = |u|^p_{W^p_{\nu_\varepsilon}(\Omega)} = \iint_{\Omega\Omega} |u(x) - u(y)|^p \nu_\varepsilon(x-y)\,\mathrm{d}y\,\mathrm{d}x$$
$$= \frac{ps(1-s)}{|\mathbb{S}^{d-1}|}\iint_{\Omega\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{d+sp}}\,\mathrm{d}y\,\mathrm{d}x.$$

Case $d = 1$. For an illustrative purpose we start with the case $d = 1$. Consider $\Omega = (-1, 0) \cup (0, 1)$ and put $u(x) = -\frac{1}{2}$ if $x \in (-1, 0)$ and $u(x) = \frac{1}{2}$ if $x \in [0, 1)$. If we put $s = 1 - \varepsilon$ then we have

$$|u|^p_{W^{s,p}(\Omega)} = ps(1-s)\int_0^1\int_0^1 \frac{\mathrm{d}y\ \mathrm{d}x}{(x+y)^{1+sp}} = \begin{cases} \infty & \text{if } sp \ge 1, \\ \frac{(1-s)}{1-sp}(2 - 2^{1-sp}) & \text{if } sp < 1. \end{cases}$$

(i) Clearly, $u \in W^{1,p}(\Omega)$ for all $1 \le p < \infty$ with $\nabla u = 0$ on Ω. Note however that, the weak derivative of u on $(-1, 1)$ is δ_0; the Dirac mass at the origin. It follows that $u \notin W^{1,p}(-1, 1)$ for all $1 \le p < \infty$ and $u \in BV(-1, 1)$ with $|u|_{BV(-1,1)} = 1$.

(ii) Moreover, Ω is not a $W^{1,p}$-extension domain. Indeed, assume $\overline{u} \in W^{1,p}(\mathbb{R})$ is an extension of u defined. In particular, $\overline{u} \in W^{1,p}(-1,1)$ and $\overline{u} = u$ on Ω. The distributional derivative of $\overline{u}$ on $(-1,1)$ is $\nabla\overline{u} = \delta_0$, This contradicts the fact that $\overline{u} \in W^{1,p}(\mathbb{R})$.

(iii) Since integrals disregard null sets, we have $\|u\|_{W^{s,p}(\Omega)} = \|u\|_{W^{s,p}(-1,1)}$ for all $1 \le p < \infty$. If $1 < p < \infty$ and $s \ge 1/p$ then $\|u\|_{W^{s,p}(\Omega)} = \|u\|_{W^{s,p}(-1,1)} = \infty$ and hence $u \notin W^{s,p}(\Omega)$. Thus the embedding $W^{1,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$ fails. However, if $p = 1$ we get $u \in W^{s,1}(-1,1)$. Since $s = 1-\varepsilon$ this also implies that

$$A_p = \liminf_{\varepsilon\to 0} |u|^p_{W^p_{\nu_\varepsilon}(\Omega)} = \begin{cases} \infty & \text{if } p > 1, \\ 1 & \text{if } p = 1. \end{cases}$$

Case $d \ge 2$. The above example persists in higher dimension. Consider Ω be the unit ball $B_1(0)$ deprived with the hyperplane $\{x_d = 0\}$ that is, $\Omega = B_1^+(0) \cup B_1^-(0)$ where $B_1^{\pm}(0) = B_1(0) \cap \{(x', x_d) \in \mathbb{R}^d : \pm x_d > 0\}$ and $u(x) = \frac{1}{2}\mathbb{1}_{B_1^+(0)}(x) - \frac{1}{2}\mathbb{1}_{B_1^-(0)}(x)$. Denoting balls in $\mathbb{R}^{d-1}$ as $B_r'(x')$, we have $\big\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x_d, y_d \in (0,1/2),\ x', y' - x' \in B'_{1/4}(0)\big\} \subset B_1^+(0) \times B_1^+(0)$. Enforcing the change of variables $y' = x' + (x_d + y_d)h'$ so that $\mathrm{d}y' = (x_d + y_d)^{d-1}\,\mathrm{d}h'$ yields

$$\begin{aligned}
|u|^p_{W^{s,p}(\Omega)} &= 2a_{\varepsilon,d,p} \int\limits_{B_1^+(0)} \int\limits_{B_1^+(0)} \big(|x'-y'|^2 + (x_d+y_d)^2\big)^{-\frac{d+sp}{2}}\,\mathrm{d}y\,\mathrm{d}x \\
&\ge 2a_{\varepsilon,d,p} \int_0^{1/2}\int_0^{1/2} \int\limits_{B'_{1/4}(0)} \int\limits_{B'_{1/4}(x')} \big(|x'-y'|^2 + (x_d+y_d)^2\big)^{-\frac{d+sp}{2}}\,\mathrm{d}y'\,\mathrm{d}x'\,\mathrm{d}y_d\,\mathrm{d}x_d \\
&= 2a_{\varepsilon,d,p}|B'_{1/4}(0)| \int_0^{1/2}\int_0^{1/2} \int\limits_{|h'|\le \frac{1}{4(x_d+y_d)}} \frac{\mathrm{d}h'}{(1+|h'|^2)^{\frac{d+sp}{2}}} \frac{\mathrm{d}x_d\,\mathrm{d}y_d}{(x_d+y_d)^{1+sp}} \\
&\overset{\frac{1}{x_d+y_d}\ge 1}{\ge} ps(1-s)\kappa^1_{d,p,s} \int_0^{1/2}\int_0^{1/2} \frac{\mathrm{d}x_d\,\mathrm{d}y_d}{(x_d+y_d)^{1+sp}}, \\
\kappa^1_{d,p,s} &= 2\frac{|B'_{1/4}(0)|}{|\mathbb{S}^{d-1}|} \int\limits_{B'_{1/4}(0)} \frac{\mathrm{d}h'}{(1+|h'|^2)^{\frac{d+sp}{2}}}.
\end{aligned}$$

Analogously, since $B_1^+(0) \times B_1^+(0) \subset \big\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : x_d, y_d \in (0,1),\ x' \in B_1'(0)\big\}$ we have

$$\begin{aligned}
|u|^p_{W^{s,p}(\Omega)} &= 2a_{\varepsilon,d,p} \int\limits_{B_1^+(0)} \int\limits_{B_1^+(0)} \big(|x'-y'|^2 + (x_d+y_d)^2\big)^{-\frac{d+sp}{2}}\,\mathrm{d}y\,\mathrm{d}x \\
&\le 2a_{\varepsilon,d,p} \int_0^1\int_0^1 \int\limits_{B_1'(0)} \int\limits_{\mathbb{R}^{d-1}} \big(|x'-y'|^2 + (x_d+y_d)^2\big)^{-\frac{d+sp}{2}}\,\mathrm{d}y'\,\mathrm{d}x'\,\mathrm{d}y_d\,\mathrm{d}x_d \\
&= ps(1-s)\kappa^2_{d,p,s} \int_0^1\int_0^1 \frac{\mathrm{d}x_d\,\mathrm{d}y_d}{(x_d+y_d)^{1+sp}},
\end{aligned}$$

$$\kappa^2_{d,p,s} = 2\frac{|B_1'(0)|}{|\mathbb{S}^{d-1}|}\int\limits_{\mathbb{R}^{d-1}} \frac{\mathrm{d}h'}{(1+|h'|^2)^{\frac{d+sp}{2}}}.$$

Note that $\kappa^i_{d,p,1} \le \kappa^i_{d,p,s} \le \kappa^i_{d,p,0}$. Using the case $d = 1$ we draw the following conclusion,

$$|u|^p_{W^p_{\nu_\varepsilon}(\Omega)} = |u|^p_{W^{s,p}(\Omega)} \asymp \begin{cases} \infty & \text{if } sp \ge 1, \\ \frac{(1-s)}{1-sp}(2-2^{1-sp}) & \text{if } sp < 1. \end{cases}$$

(i) Clearly, $u \in W^{1,p}(\Omega)$ and $u \notin W^{1,p}(B_1(0))$ for all $1 \le p < \infty$. However $u \in BV(B_1(0))$.

(ii) Moreover, (i) implies that Ω is not a $W^{1,p}$-extension domain for $1 \le p < \infty$.

(iii) As integrals disregard null sets, we have $\|u\|_{W^{s,p}(\Omega)} = \|u\|_{W^{s,p}(B_1(0))} = \infty$ for $1 < p < \infty$ and $s \ge 1/p$ and hence $u \notin W^{s,p}(\Omega)$. Thus the embedding $W^{1,p}(\Omega) \hookrightarrow W^{s,p}(\Omega)$ fails. However, if $p = 1$ we get $u \in W^{s,1}(B_1(0))$. Furthermore, we have

$$A_p = \liminf_{\varepsilon\to 0} |u|^p_{W^p_{\nu_\varepsilon}(\Omega)} \asymp \begin{cases} \infty & \text{if } p > 1, \\ 1 & \text{if } p = 1. \end{cases}$$

Proposition 3.10 *For any $e \in \mathbb{S}^{d-1}$ we have*

$$K_{d,p} = \frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}} |w\cdot e|^p \,\mathrm{d}\sigma_{d-1}(w) = \frac{\Gamma\big(\frac{d}{2}\big)\Gamma\big(\frac{p+1}{2}\big)}{\Gamma\big(\frac{d+p}{2}\big)\Gamma\big(\frac{1}{2}\big)}.$$

Proof The case $d = 1$ is obvious and we only prove for $d \ge 2$. Since $K_{d,p}$ is independent of $e \in \mathbb{S}^{d-1}$, it is sufficient to take $e = (0, \cdots, 0, 1)$. Let $w = (w', t) \in \mathbb{S}^{d-1}$ with $t \in (-1, 1)$ so that $w' \in \sqrt{1-t^2}\mathbb{S}^{d-2}$. The Jacobian for spherical coordinates gives $\mathrm{d}\sigma_{d-1}(w) = \frac{\mathrm{d}\sigma_{d-2}(w')dt}{\sqrt{1-t^2}}$ (see [20, Appendix D.2]). Therefore, noting $|\mathbb{S}^{d-1}| = \omega_{d-1}$, we have

$$\begin{aligned} K_{d,p} &= \frac{1}{|\mathbb{S}^{d-1}|}\int_{\mathbb{S}^{d-1}} |w_d|^p d\sigma_{d-1}(w) && = \frac{1}{\omega_{d-1}}\int\limits_{-1}^{1}\int\limits_{\sqrt{1-t^2}\mathbb{S}^{d-2}} |t|^p \frac{\mathrm{d}\sigma_{d-2}(w')\,\mathrm{d}t}{\sqrt{1-t^2}} \\ &= \frac{2}{\omega_{d-1}}\int\limits_0^1 t^p\Big|\sqrt{1-t^2}\mathbb{S}^{d-2}\Big|\frac{dt}{\sqrt{1-t^2}} && = \frac{2\omega_{d-2}}{\omega_{d-1}}\int\limits_0^1 (1-t^2)^{\frac{d-3}{2}} t^p dt \\ &= \frac{\omega_{d-2}}{\omega_{d-1}}\int\limits_0^1 (1-t)^{\frac{d-1}{2}-1} t^{\frac{p+1}{2}-1} dt && = \frac{\omega_{d-2}}{\omega_{d-1}} B\Big(\frac{d-1}{2}, \frac{p+1}{2}\Big) = \frac{\omega_{d-2}}{\omega_{d-1}}\frac{\Gamma\left(\frac{d-1}{2}\right)\Gamma\left(\frac{p+1}{2}\right)}{\Gamma\left(\frac{d+p}{2}\right)}. \end{aligned}$$

Here $B(x, y) := \int_0^1 (1-t)^{x-1}t^{y-1}\,\mathrm{d}t$, $x > 0$, $y > 0$ is the beta function which links to the Gamma function by the relation $B(x, y)\Gamma(x + y) = \Gamma(x)\Gamma(y)$. The claim follows by using the formula $\omega_{d-1} = \frac{2\pi^{d/2}}{\Gamma(d/2)}$ along with $\Gamma(\frac{1}{2}) = \pi^{1/2}$. □

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