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A half-discrete Hilbert-type inequality in the whole plane with the constant factor related to a cotangent function

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Abstract

In this work, by the introduction of some parameters, a new half-discrete kernel function in the whole plane is defined, which involves both the homogeneous and the nonhomogeneous cases. By employing some techniques of real analysis, especially the method of a weight function, a new half-discrete Hilbert-type inequality with the new kernel function, as well as its equivalent Hardy-type inequalities are established. Moreover, it is proved that the constant factors of the newly obtained inequalities are the best possible. Finally, assigning special values to the parameters, some new half-discrete Hilbert-type inequalities with special kernels are presented at the end of the paper.

MSC: Primary 26D15; 26D10; secondary 47B38

Keywords: Hilbert-type inequality; Hardy-type inequality; Half-discrete; Homogeneous kernel; Nonhomogeneous kernel

1 Introduction

Suppose that $p > 1$, Θ is a measurable set, and $f(x)$, $\mu(x)$ are two nonnegative measurable functions defined on Θ. Define

$$L_{p,\mu}(\Theta) := \left\{ f : \|f\|_{p,\mu} := \left(\int_{\Theta} f^p(x)\mu(x)\,\mathrm{d}x \right)^{1/p} < \infty \right\}.$$

Specifically, if $\mu(x) \equiv 1$, then we have the abbreviations: $\|f\|_p := \|f\|_{p,\mu}$, and $L_p(\Theta) := L_{p,\mu}(\Theta)$.

In addition, let $p > 1$, $a_n, \nu_n > 0$, $n \in \Pi \subseteq \mathbb{Z}$, $\boldsymbol{a} = \{a_n\}_{n\in\Pi}$. Define

$$l_{p,\nu} := \left\{ \boldsymbol{a} : \|\boldsymbol{a}\|_{p,\nu} := \left(\sum_{n\in\Pi} a_n^p \nu_n \right)^{1/p} < \infty \right\}.$$

Specifically, if $\nu_n \equiv 1$, then we have the abbreviations: $\|a\|_p := \|a\|_{p,\nu}$, and $l_p := l_{p,\nu}$.

Consider two real-valued sequences: $\boldsymbol{a} = \{a_m\}_{m=1}^{\infty} \in l_2$, and $\boldsymbol{b} = \{b_n\}_{n=1}^{\infty} \in l_2$, then

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{m+n} < \pi \|\boldsymbol{a}\|_2 \|\boldsymbol{b}\|_2, \tag{1.1}$$

where the constant factor π is the best possible. Inequality (1.1) was proposed by Hilbert in his lectures on integral equations in 1908, and Schur established the integral analogy of (1.1) in 1911, that is,

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x+y}\,\mathrm{d}x\,\mathrm{d}y < \pi \|f\|_2 \|g\|_2, \tag{1.2}$$

where $f, g \in L_2(\mathbb{R}^+)$, and the constant factor π is also the best possible.

Inequalities (1.1) and (1.2) are commonly named as Hilbert inequalities [1]. In recent decades, especially after the 1990s, a great many extended forms of (1.1) and (1.2) were established, such as the following one provided by Krnić and Pečarić [2]:

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{a_m b_n}{(m+n)^{\lambda}} < B\left(\frac{\lambda}{2}, \frac{\lambda}{2}\right)\|\boldsymbol{a}\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu}, \tag{1.3}$$

where $0 < \lambda \leq 4$, $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mu_m = m^{p(1-\lambda/2)-1}$, $\nu_n = n^{q(1-\lambda/2)-1}$, and $B(u, v)$ is the Beta function [3].

Moreover, Yang [4] established the following extension of (1.2), that is,

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{x^{\lambda}+y^{\lambda}}\,\mathrm{d}x\,\mathrm{d}y < \frac{\pi}{\lambda\sin\beta\pi}\|f\|_{p,\mu}\|g\|_{q,\nu}, \tag{1.4}$$

where $\beta, \gamma, \lambda > 0$, $\beta + \gamma = 1$, $\mu(x) = x^{p(1-\lambda\beta)-1}$, and $\nu(x) = x^{q(1-\lambda\gamma)-1}$.

With regard to some other extended forms of inequalities (1.1) and (1.2), we refer to [5–11]. Such inequalities as (1.3) and (1.4) are commonly known as Hilbert-type inequalities. It should be pointed out that, by introducing new kernel functions, and considering the coefficient refinement, reverse form, multidimensional extension, a large number of Hilbert-type inequalities were established in the past 20 years (see [12–23]).

It should also be pointed out that the kernel function in inequalities (1.1) and (1.2) are homogeneous [11, 12], and there exists another form of (1.1) with a nonhomogeneous kernel function [12], that is,

$$\int_0^{\infty}\int_0^{\infty}\frac{f(x)g(y)}{1+xy}\,\mathrm{d}x\,\mathrm{d}y < \pi \|f\|_2 \|g\|_2. \tag{1.5}$$

The discrete form of (1.5) can also be established, but its constant factor cannot be proved to be the best possible (see [12], p. 315). In 2005, Yang provided a half-discrete form of (1.5) and proved that the constant factor is the best possible, that is [24],

$$\int_0^{\infty} f(x)\sum_{n=1}^{\infty}\frac{a_n}{1+nx}\,\mathrm{d}x < \pi \|f\|_2 \|\boldsymbol{a}\|_2. \tag{1.6}$$

With regard to some other half-discrete inequalities with homogeneous and nonhomogeneous kernels, we refer to [23, 25–32].

The main objective of this work is to establish a new class of half-discrete Hilbert-type inequalities defined in the whole plane with the kernel functions involving both the homogeneous and nonhomogeneous cases, such as the following two:

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{1+(xn)^{\beta}+(xn)^{2\beta}}\,\mathrm{d}x < \frac{2\sqrt{3}\pi}{3\beta}\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}, \tag{1.7}$$

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{x^{2\beta}-(xn)^{\beta}+n^{2\beta}}\,\mathrm{d}x < \frac{(4+\sqrt{3})\pi}{3\beta}\|f\|_{p,\tilde{\mu}}\|\boldsymbol{a}\|_{q,\tilde{\nu}}, \tag{1.8}$$

where $\mu(x)=|x|^{p(1-\beta)-1}$, $\nu_n=|n|^{q(1-\beta)-1}$, $\tilde{\mu}(x)=|x|^{p(1-3\beta/2)-1}$, and $\tilde{\nu}_n=|n|^{q(1-\beta/2)-1}$.

More generally, a new kernel function with multiple parameters, which unifies some homogeneous and nonhomogeneous cases is constructed, and then a half-discrete Hilbert-type inequality and its equivalent forms defined in the whole plane are established. The paper is organized as follows: detailed lemmas will be presented in Sect. 2, and the main results and some corollaries will be presented in Sect. 3 and Sect. 4, respectively.

2 Some lemmas

Lemma 2.1 *Let* $\delta\in\{1,-1\}$, *and*

$$\Omega := \left\{t : t=\frac{2i+1}{2j+1}, i,j\in\mathbb{Z}\right\}.$$

Suppose that $\alpha\in(0,1)$, $\beta,\gamma\in\mathbb{R}^+\cap\Omega$, *and* α, β, γ *satisfy* $\beta<\gamma$ *and* $\alpha+\beta<1$. *Define*

$$K(z) := \frac{1-\delta z^{\beta}}{1-\delta z^{\gamma}}, \tag{2.1}$$

where $z\neq 1$ *for* $\delta=1$, *and* $z\neq -1$ *for* $\delta=-1$. *Let* $K(1):=\frac{\beta}{\gamma}$ *for* $\delta=1$, *and* $K(-1):=\frac{\beta}{\gamma}$ *for* $\delta=-1$. *Then,*

$$G(z) := K(z)|z|^{\alpha-1} \tag{2.2}$$

decreases monotonically on $\mathbb{R}^+$, *and increases monotonically on* $\mathbb{R}^-$.

Proof We first consider the case where $\delta=1$, and $z\in(0,1)\cup(1,\infty)$, then we have

$$\frac{\mathrm{d}K}{\mathrm{d}z} = \left(1-z^{\gamma}\right)^{-2} z^{\gamma-1} H(z), \tag{2.3}$$

where

$$H(z) = (\beta-\gamma)z^{\beta} - \beta z^{\beta-\gamma} + \gamma. \tag{2.4}$$

We can easily obtain that

$$\frac{\mathrm{d}H}{\mathrm{d}z} = \beta(\beta-\gamma)z^{\beta-1} - \beta(\beta-\gamma)z^{\beta-\gamma-1} = \beta(\beta-\gamma)z^{\beta-\gamma-1}\left(z^{\gamma}-1\right). \tag{2.5}$$

Therefore, we have $\frac{\mathrm{d}H}{\mathrm{d}z} > 0$ for $z \in (0,1)$, and $\frac{\mathrm{d}H}{\mathrm{d}z} < 0$ for $z \in (1,\infty)$. It follows that $H(z) \le H(1) = 0$. By (2.3), we obtain $\frac{\mathrm{d}K}{\mathrm{d}z} < 0$ for $z \in (0,1) \cup (1,\infty)$, and therefore $K(z)$ decreases monotonically on $\mathbb{R}^+$ for $\delta = 1$. Since $0 < \alpha < 1$, it can also be obtained that $G(z) = K(z)z^{\alpha-1}$ decreases monotonically on $\mathbb{R}^+$ for $\delta = 1$.

Secondly, consider the case of $\delta = 1$, and $z \in (-\infty, 0)$. Setting $z = -u$, $u \in (0,\infty)$, and observing that $\beta, \gamma \in \mathbb{R}^+ \cap \Omega$, we obtain

$$G(z) = \frac{1 - z^\beta}{1 - z^\gamma}|z|^{\alpha-1} = \frac{1 + u^\beta}{1 + u^\gamma}u^{\alpha-1} := L(u). \tag{2.6}$$

In view of $0 < \alpha < 1$, and $\alpha + \beta < 1$, we obtain

$$\begin{aligned}\frac{\mathrm{d}L}{\mathrm{d}u} &= -u^{\alpha-2}\bigl(1 + u^\gamma\bigr)^{-2}\bigl[(1 - \alpha - \beta)u^\beta \\ &\quad + (1 - \alpha - \beta + \gamma)u^{\beta+\gamma} + (1 - \alpha + \gamma)u^\gamma + 1 - \alpha\bigr] < 0.\end{aligned} \tag{2.7}$$

This implies that $L(u)$ decreases monotonically with u ($u \in \mathbb{R}^+$), and therefore $G(z)$ increases monotonically with z ($z \in \mathbb{R}^-$).

Lemma 2.1 is proved for $\delta = 1$. Additionally, in view of $\beta, \gamma \in \mathbb{R}^+ \cap \Omega$, it follows from the above discussions that Lemma 2.1 holds obviously for the case where $\delta = -1$. □

Lemma 2.2 *Let* $\delta \in \{1, -1\}$, *and*

$$\Omega := \left\{t : t = \frac{2i + 1}{2j + 1}, i, j \in \mathbb{Z}\right\}.$$

Suppose that $\alpha \in (0,1)$, $\beta, \gamma \in \mathbb{R}^+ \cap \Omega$, *and* α, β, γ *satisfy* $\alpha + \beta < \gamma$. *Let* $\Phi(z) = \cot z$, *and* $K(z)$ *be defined by* (2.1). *Then*,

$$\int_{-\infty}^{\infty} K(z)|z|^{\alpha-1}\,\mathrm{d}z = \frac{\pi}{\gamma}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha + \beta + \gamma)\pi}{2\gamma}\right)\right]. \tag{2.8}$$

Proof Consider the case where $\delta = 1$. Observing that $\beta, \gamma \in \mathbb{R}^+ \cap \Omega$, we obtain

$$\begin{aligned}\int_{-\infty}^{\infty} K(z)|z|^{\alpha-1}\,\mathrm{d}z &= \int_0^\infty \frac{1 - z^\beta}{1 - z^\gamma}z^{\alpha-1}\,\mathrm{d}z + \int_0^\infty \frac{1 + z^\beta}{1 + z^\gamma}z^{\alpha-1}\,\mathrm{d}z \\ &= \int_0^1 \frac{1 - z^\beta}{1 - z^\gamma}z^{\alpha-1}\,\mathrm{d}z + \int_1^\infty \frac{1 - z^\beta}{1 - z^\gamma}z^{\alpha-1}\,\mathrm{d}z \\ &\quad + \int_0^1 \frac{1 + z^\beta}{1 + z^\gamma}z^{\alpha-1}\,\mathrm{d}z + \int_1^\infty \frac{1 + z^\beta}{1 + z^\gamma}z^{\alpha-1}\,\mathrm{d}z \\ &= \int_0^1 \frac{z^{\alpha-1} - z^{\alpha+\beta-1}}{1 - z^\gamma}\,\mathrm{d}z + \int_0^1 \frac{z^{\gamma-\alpha-\beta-1} - z^{\gamma-\alpha-1}}{1 - z^\gamma}\,\mathrm{d}z \\ &\quad + \int_0^1 \frac{z^{\alpha-1} + z^{\alpha+\beta-1}}{1 + z^\gamma}\,\mathrm{d}z + \int_0^1 \frac{z^{\gamma-\alpha-\beta-1} + z^{\gamma-\alpha-1}}{1 + z^\gamma}\,\mathrm{d}z \\ &= 2\left[\int_0^1 \frac{z^{\alpha-1} - z^{2\gamma-\alpha-1}}{1 - z^{2\gamma}}\,\mathrm{d}z + \int_0^1 \frac{z^{\gamma-\alpha-\beta-1} - z^{\alpha+\beta+\gamma-1}}{1 - z^{2\gamma}}\,\mathrm{d}z\right] \\ &:= 2(J_1 + J_2).\end{aligned} \tag{2.9}$$

Expanding $\frac{1}{1-z^{2\gamma}}$ ($z \in (0,1)$) into a power series, and using the Lebesgue term-by-term integration theorem, we obtain

$$\begin{aligned} J_1 &= \int_0^1 \sum_{j=0}^{\infty} \left(z^{2\gamma j+\alpha-1} - z^{2\gamma j+2\gamma-\alpha-1}\right) \mathrm{d}z \\ &= \sum_{j=0}^{\infty} \int_0^1 \left(z^{2\gamma j+\alpha-1} - z^{2\gamma j+2\gamma-\alpha-1}\right) \mathrm{d}z \\ &= \sum_{j=0}^{\infty} \left(\frac{1}{2\gamma j+\alpha} - \frac{1}{2\gamma j+2\gamma-\alpha}\right). \end{aligned} \tag{2.10}$$

Observing that $\Phi(z) = \cot z$ ($0 < z < \pi$) can be written as the following rational fraction expansion [3]:

$$\Phi(z) = \frac{1}{z} + \sum_{j=1}^{\infty} \left(\frac{1}{z+j\pi} + \frac{1}{z-j\pi}\right),$$

we obtain

$$\begin{aligned} \Phi\left(\frac{\alpha\pi}{2\gamma}\right) &= \frac{2\gamma}{\pi}\left[\frac{1}{\alpha} + \sum_{j=1}^{\infty}\left(\frac{1}{2\gamma j+\alpha} + \frac{1}{\alpha-2\gamma j}\right)\right] \\ &= \frac{2\gamma}{\pi} \lim_{n\to\infty}\left(\sum_{j=0}^{n} \frac{1}{2\gamma j+\alpha} + \sum_{j=1}^{n} \frac{1}{\alpha-2\gamma j}\right) \\ &= \frac{2\gamma}{\pi} \lim_{n\to\infty}\left(\sum_{j=0}^{n} \frac{1}{2\gamma j+\alpha} - \sum_{j=0}^{n-1} \frac{1}{2\gamma j+2\gamma-\alpha}\right) \\ &= \frac{2\gamma}{\pi} \lim_{n\to\infty}\left[\frac{1}{2\gamma n+2\gamma-\alpha} + \sum_{j=0}^{n}\left(\frac{1}{2\gamma j+\alpha} - \frac{1}{2\gamma j+2\gamma-\alpha}\right)\right] \\ &= \frac{2\gamma}{\pi} \sum_{j=0}^{\infty}\left(\frac{1}{2\gamma j+\alpha} - \frac{1}{2\gamma j+2\gamma-\alpha}\right). \end{aligned} \tag{2.11}$$

Combining (2.10) and (2.11), we obtain

$$J_1 = \frac{\pi}{2\gamma}\Phi\left(\frac{\alpha\pi}{2\gamma}\right). \tag{2.12}$$

Similarly, we have

$$J_2 = -\frac{\pi}{2\gamma}\Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right). \tag{2.13}$$

Inserting (2.12) and (2.13) into (2.9), we arrive at (2.8) for $\delta = 1$. Additionally, if $\delta = -1$, then it is obvious that (2.9) is still valid owing to $\beta, \gamma \in \mathbb{R}^+ \cap \Omega$, and it follows therefore that (2.8) holds for the case where $\delta = -1$. Lemma 2.2 is proved. □

Lemma 2.3 *Let* $\delta \in \{1,-1\}$, *and*

$$\Omega := \left\{ t : t = \frac{2i+1}{2j+1}, i,j \in \mathbb{Z} \right\}.$$

Suppose that $\alpha \in (0,1)$, $\tau \in \Omega$, $\kappa \in (0,1] \cap \Omega$, $\beta, \gamma \in \mathbb{R}^+ \cap \Omega$, *and* α, β, γ *satisfy* $\beta < \gamma$ *and* $\alpha + \beta < \min\{1, \gamma\}$. *Assume that* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, $\mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$, *and* $K(z)$ *is defined by* (2.1). *For a sufficiently large positive integer* l, *set*

$$\tilde{\boldsymbol{a}} := \{\tilde{a}_n\}_{n\in\mathbb{Z}^0} := \left\{ |n|^{\alpha\kappa - 1 - \frac{2\kappa}{ql}} \right\}_{n\in\mathbb{Z}^0},$$

$$\tilde{f}(x) := \begin{cases} |x|^{\alpha\tau - 1 + \frac{2\tau}{pl}}, & x \in S, \\ 0, & x \in \mathbb{R} \setminus S, \end{cases}$$

where $S := \{x : |x|^{\operatorname{sgn}\tau} < 1\}$. *Then,*

$$\begin{aligned} \tilde{J} :&= \sum_{n\in\mathbb{Z}^0} \tilde{a}_n \int_{-\infty}^{\infty} K\bigl(x^\tau n^\kappa\bigr)\tilde{f}(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} \tilde{f}(x) \sum_{n\in\mathbb{Z}^0} \tilde{a}_n K\bigl(x^\tau n^\kappa\bigr)\,\mathrm{d}x \\ &> \frac{l}{|\tau\kappa|}\left[\int_{[-1,1]} K(z)|z|^{\alpha-1+\frac{2}{pl}}\,\mathrm{d}z + \int_{\mathbb{R}\setminus[-1,1]} K(z)|z|^{\alpha-1-\frac{2}{ql}}\,\mathrm{d}z \right]. \end{aligned} \tag{2.14}$$

Proof Write

$$\begin{aligned} \tilde{J} &= \int_{x\in S^-} \tilde{f}(x) \sum_{n\in\mathbb{Z}^+} \tilde{a}_n K\bigl(x^\tau n^\kappa\bigr)\,\mathrm{d}x + \int_{x\in S^-} \tilde{f}(x) \sum_{n\in\mathbb{Z}^-} \tilde{a}_n K\bigl(x^\tau n^\kappa\bigr)\,\mathrm{d}x \\ &\quad + \int_{x\in S^+} \tilde{f}(x) \sum_{n\in\mathbb{Z}^+} \tilde{a}_n K\bigl(x^\tau n^\kappa\bigr)\,\mathrm{d}x + \int_{x\in S^+} \tilde{f}(x) \sum_{n\in\mathbb{Z}^-} \tilde{a}_n K\bigl(x^\tau n^\kappa\bigr)\,\mathrm{d}x \\ &:= J_1 + J_2 + J_3 + J_4, \end{aligned}$$

where $S^+ := \{x : x \in S \cap \mathbb{R}^+\}$, $S^- := \{x : x \in S \cap \mathbb{R}^-\}$.

If $x \in S^-$, $n \in \mathbb{Z}^+$, then we have $x^\tau n^\kappa < 0$. By Lemma 2.1, it can be proved that $G(x^\tau n^\kappa)$ decreases with n ($n \in \mathbb{Z}^+$). Additionally, in view of $\kappa \in (0,1] \cap \Omega$, it can also be proved that $|n|^{\kappa-1-\frac{2\kappa}{ql}}$ decreases with n ($n \in \mathbb{Z}^+$). It follows therefore that

$$\tilde{a}_n K\bigl(x^\tau n^\kappa\bigr) = |x|^{\tau(1-\alpha)} G\bigl(x^\tau n^\kappa\bigr) |n|^{\kappa-1-\frac{2\kappa}{ql}}$$

decreases with n ($n \in \mathbb{Z}^+$) for a fixed x ($x \in S^-$), and it implies that

$$J_1 > \int_{x\in S^-} |x|^{\alpha\tau-1+\frac{2\tau}{pl}} \int_1^\infty K\bigl(x^\tau y^\kappa\bigr)|y|^{\alpha\kappa-1-\frac{2\kappa}{ql}}\,\mathrm{d}y\,\mathrm{d}x := P_1.$$

Similarly, it can be obtained that

$$\begin{aligned} J_2 &> \int_{x\in S^-} |x|^{\alpha\tau-1+\frac{2\tau}{pl}} \int_{-\infty}^{-1} K\bigl(x^\tau y^\kappa\bigr)|y|^{\alpha\kappa-1-\frac{2\kappa}{ql}}\,\mathrm{d}y\,\mathrm{d}x := P_2, \\ J_3 &> \int_{x\in S^+} |x|^{\alpha\tau-1+\frac{2\tau}{pl}} \int_1^\infty K\bigl(x^\tau y^\kappa\bigr)|y|^{\alpha\kappa-1-\frac{2\kappa}{ql}}\,\mathrm{d}y\,\mathrm{d}x := P_3, \end{aligned}$$

$$J_4 > \int_{x\in S^+} |x|^{\alpha\tau-1+\frac{2\tau}{pl}} \int_{-\infty}^{-1} K\bigl(x^\tau y^\kappa\bigr)|y|^{\alpha\kappa-1-\frac{2\kappa}{ql}}\,\mathrm{d}y\,\mathrm{d}x := P_4.$$

If $\tau < 0$, that is, $\tau \in \Omega \cap \mathbb{R}^-$, then $S^- = S \cap \mathbb{R}^- = (-\infty, -1)$. Let $x^\tau y^\kappa = z$, and observe that $x^{-\frac{\tau}{\kappa}} = -|x|^{-\frac{\tau}{\kappa}}$ $(x < 0)$ and $z^{\frac{1}{\kappa}-1} = |z|^{\frac{1}{\kappa}-1}$ $(z < 0)$, then we have

$$\begin{aligned} P_1 &= \int_{-\infty}^{-1} |x|^{\alpha\tau-1+\frac{2\tau}{pl}} \int_{1}^{\infty} K\bigl(x^\tau y^\kappa\bigr)|y|^{\alpha\kappa-1-\frac{2\kappa}{ql}}\,\mathrm{d}y\,\mathrm{d}x \\ &= \frac{1}{\kappa}\int_{-\infty}^{-1} |x|^{-1+\frac{2\tau}{l}} \int_{-\infty}^{x^\tau} K(z)|z|^{\alpha-1-\frac{2}{ql}}\,\mathrm{d}z\,\mathrm{d}x \\ &= \frac{1}{\kappa}\int_{-\infty}^{-1} |x|^{-1+\frac{2\tau}{l}} \int_{-\infty}^{-1} K(z)|z|^{\alpha-1-\frac{2}{ql}}\,\mathrm{d}z\,\mathrm{d}x \\ &\quad + \frac{1}{\kappa}\int_{-\infty}^{-1} |x|^{-1+\frac{2\tau}{l}} \int_{-1}^{x^\tau} K(z)|z|^{\alpha-1-\frac{2}{ql}}\,\mathrm{d}z\,\mathrm{d}x \\ &= \frac{l}{2|\tau\kappa|}\int_{-\infty}^{-1} K(z)|z|^{\alpha-1-\frac{2}{ql}}\,\mathrm{d}z \\ &\quad + \frac{1}{\kappa}\int_{-\infty}^{-1} |x|^{-1+\frac{2\tau}{l}} \int_{-1}^{x^\tau} K(z)|z|^{\alpha-1-\frac{2}{ql}}\,\mathrm{d}z\,\mathrm{d}x. \end{aligned} \tag{2.15}$$

It follows from Fubini's theorem that

$$\begin{aligned} &\int_{-\infty}^{-1} |x|^{-1+\frac{2\tau}{l}} \int_{-1}^{x^\tau} K(z)|z|^{\alpha-1-\frac{2}{ql}}\,\mathrm{d}z\,\mathrm{d}x \\ &\quad= \int_{-1}^{0} K(z)|z|^{\alpha-1-\frac{2}{ql}} \int_{-\infty}^{z^{1/\tau}} |x|^{-1+\frac{2\tau}{l}}\,\mathrm{d}x\,\mathrm{d}z \\ &\quad= \frac{l}{2|\tau|}\int_{-1}^{0} K(z)|z|^{\alpha-1+\frac{2}{pl}}\,\mathrm{d}z. \end{aligned} \tag{2.16}$$

Inserting (2.16) back into (2.15), we obtain

$$P_1 = \frac{l}{2|\tau\kappa|}\biggl[\int_{-\infty}^{-1} K(z)|z|^{\alpha-1-\frac{2}{ql}}\,\mathrm{d}z + \int_{-1}^{0} K(z)|z|^{\alpha-1+\frac{2}{pl}}\,\mathrm{d}z\biggr].$$

Similarly, it can be obtained that $P_4 = P_1$, and

$$P_2 = P_3 = \frac{l}{2|\tau\kappa|}\biggl[\int_{1}^{\infty} K(z)|z|^{\alpha-1-\frac{2}{ql}}\,\mathrm{d}z + \int_{0}^{1} K(z)|z|^{\alpha-1+\frac{2}{pl}}\,\mathrm{d}z\biggr].$$

This implies that

$$\begin{aligned} \tilde{J} &> P_1 + P_2 + P_3 + P_4 \\ &= \frac{l}{|\tau\kappa|}\biggl[\int_{[-1,1]} K(z)|z|^{\alpha-1+\frac{2}{pl}}\,\mathrm{d}z + \int_{\mathbb{R}\setminus[-1,1]} K(z)|z|^{\alpha-1-\frac{2}{ql}}\,\mathrm{d}z\biggr]. \end{aligned}$$

Hence, Lemma 2.3 is proved when $\tau < 0$. If $\tau > 0$. It can also be proved that (2.14) holds true. The proof of Lemma 2.3 is completed. □

3 Main results

Theorem 3.1 *Let* $\delta \in \{1,-1\}$, *and*

$$\Omega := \left\{ t : t = \frac{2i+1}{2j+1}, i, j \in \mathbb{Z} \right\}.$$

Suppose that $\alpha \in (0,1)$, $\tau \in \Omega$, $\kappa \in (0,1] \cap \Omega$, $\beta, \gamma \in \mathbb{R}^+ \cap \Omega$, *and* α, β, γ *satisfy* $\beta < \gamma$ *and* $\alpha + \beta < \min\{1, \gamma\}$. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Assume that* $\mu(x) = |x|^{p(1-\alpha\tau)-1}$, $\nu_n = |n|^{q(1-\alpha\kappa)-1}$, *where* $n \in \mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$. *Let* $f(x), a_n \geq 0$ *be such that* $f(x) \in L_{p,\mu}(\mathbb{R})$, *and* $\boldsymbol{a} = \{a_n\}_{n \in \mathbb{Z}^0} \in l_{q,\nu}$. *Let* $\Phi(z) = \cot z$, *and* $K(z)$ *be defined by* (2.1). *Then, the following inequalities hold and are equivalent*:

$$\begin{aligned} J &:= \sum_{n \in \mathbb{Z}^0} a_n \int_{-\infty}^{\infty} K\left(x^\tau n^\kappa\right) f(x)\, \mathrm{d}x = \int_{-\infty}^{\infty} f(x) \sum_{n \in \mathbb{Z}^0} K\left(x^\tau n^\kappa\right) a_n \, \mathrm{d}x \\ &< \frac{\pi}{\gamma} |\tau|^{-\frac{1}{q}} \kappa^{-\frac{1}{p}} \left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right) \right] \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \end{aligned} \tag{3.1}$$

$$\begin{aligned} L_1 &:= \sum_{n \in \mathbb{Z}^0} |n|^{p\alpha\kappa - 1} \left[\int_{-\infty}^{\infty} K\left(x^\tau n^\kappa\right) f(x) \, \mathrm{d}x \right]^p \\ &< \left\{ \frac{\pi}{\gamma} |\tau|^{-\frac{1}{q}} \kappa^{-\frac{1}{p}} \left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right) \right] \right\}^p \|f\|_{p,\mu}^p, \end{aligned} \tag{3.2}$$

$$\begin{aligned} L_2 &:= \int_{-\infty}^{\infty} |x|^{q\alpha\tau - 1} \left[\sum_{n \in \mathbb{Z}^0} K\left(x^\tau n^\kappa\right) a_n \right]^q \mathrm{d}x \\ &< \left\{ \frac{\pi}{\gamma} |\tau|^{-\frac{1}{q}} \kappa^{-\frac{1}{p}} \left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right) \right] \right\}^q \|\boldsymbol{a}\|_{q,\nu}^q, \end{aligned} \tag{3.3}$$

where the constant $\frac{\pi}{\gamma} |\tau|^{-\frac{1}{q}} \kappa^{-\frac{1}{p}} [\Phi(\frac{\alpha\pi}{2\gamma}) - \Phi(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma})]$ *in* (3.1), (3.2), *and* (3.3) *is the best possible.*

Proof Let $\widetilde{K}(x^\tau y^\kappa) := K(x^\tau n^\kappa)$, $g(y) := a_n$, and $h(y) := n$ for $y \in [n-1, n)$, $n \in \mathbb{Z}^+$. Let $\widetilde{K}(x^\tau y^\kappa) := K(x^\tau n^\kappa)$, $g(y) := a_n$, and $h(y) := |n|$ for $y \in [n, n+1)$, $n \in \mathbb{Z}^-$. By Hölder's inequality, we have

$$\begin{aligned} &\sum_{n \in \mathbb{Z}^0} a_n \int_{-\infty}^{\infty} K\left(x^\tau n^\kappa\right) f(x) \, \mathrm{d}x \\ &\quad = \int_{-\infty}^{\infty} f(x) \sum_{n \in \mathbb{Z}^0} K\left(x^\tau n^\kappa\right) a_n \, \mathrm{d}x = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{K}\left(x^\tau y^\kappa\right) f(x) g(y) \, \mathrm{d}x \, \mathrm{d}y \\ &\quad = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left[\widetilde{K}\left(x^\tau y^\kappa\right)\right]^{1/p} \left[h(y)\right]^{(\alpha\kappa-1)/p} |x|^{(1-\alpha\tau)/q} f(x) \\ &\qquad \times \left[\widetilde{K}\left(x^\tau y^\kappa\right)\right]^{1/q} |x|^{(\alpha\tau-1)/q} \left[h(y)\right]^{(1-\alpha\kappa)/p} g(y) \, \mathrm{d}x \, \mathrm{d}y \\ &\quad \leq \left\{ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \widetilde{K}\left(x^\tau y^\kappa\right) \left[h(y)\right]^{\alpha\kappa - 1} |x|^{p(1-\alpha\tau)/q} f^p(x) \, \mathrm{d}y \, \mathrm{d}x \right\}^{1/p} \end{aligned} \tag{3.4}$$

$$\times\left\{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\widetilde{K}\bigl(x^{\tau}y^{\kappa}\bigr)|x|^{\alpha\tau-1}\bigl[h(y)\bigr]^{q(1-\alpha\kappa)/p}g^{q}(y)\,\mathrm{d}x\,\mathrm{d}y\right\}^{1/q}$$

$$=\left[\int_{-\infty}^{\infty}\psi(x)|x|^{p(1-\alpha\tau)/q}f^{p}(x)\,\mathrm{d}x\right]^{1/p}\left[\sum_{n\in\mathbb{Z}^{0}}\omega(n)|n|^{q(1-\alpha\kappa)/p}a_{n}^{q}\right]^{1/q},$$

where

$$\psi(x)=\sum_{n\in\mathbb{Z}^{0}}K\bigl(x^{\tau}n^{\kappa}\bigr)|n|^{\alpha\kappa-1},\tag{3.5}$$

$$\omega(n)=\int_{-\infty}^{\infty}K\bigl(x^{\tau}n^{\kappa}\bigr)|x|^{\alpha\tau-1}\,\mathrm{d}x.\tag{3.6}$$

Since $\kappa\leq 1$, it can be shown that $|n|^{\kappa-1}$ decreases monotonically if $n\in\mathbb{Z}^{+}$, and increases monotonically if $n\in\mathbb{Z}^{-}$. Moreover, by Lemma 2.1, and observing that $\tau\in\Omega$ and $\kappa\in(0,1]\cap\Omega$, it can be proved that whether $x\in\mathbb{R}^{+}$ or $x\in\mathbb{R}^{-}$, $G(x^{\tau}n^{\kappa})$ decreases monotonically with n if $n\in\mathbb{Z}^{+}$, and increases monotonically with n if $n\in\mathbb{Z}^{-}$. Therefore, for a fixed x,

$$K\bigl(x^{\tau}n^{\kappa}\bigr)|n|^{\alpha\kappa-1}=|x|^{\tau-\alpha\tau}G\bigl(x^{\tau}n^{\kappa}\bigr)|n|^{\kappa-1}$$

decreases monotonically with n if $n\in\mathbb{Z}^{+}$, and increases monotonically with n if $n\in\mathbb{Z}^{-}$. It follows therefore that

$$\psi(x)=\sum_{n\in\mathbb{Z}^{0}}K\bigl(x^{\tau}n^{\kappa}\bigr)|n|^{\alpha\kappa-1}<\int_{-\infty}^{\infty}K\bigl(x^{\tau}y^{\kappa}\bigr)|y|^{\alpha\kappa-1}\,\mathrm{d}y.$$

Supposing that $x<0$, and observing that $\tau\in\Omega$ and $\kappa\in(0,1]\cap\Omega$, we obtain $x^{-\frac{\tau}{\kappa}}=-|x|^{-\frac{\tau}{\kappa}}$ and $z^{\frac{1}{\kappa}-1}=|z|^{\frac{1}{\kappa}-1}$. Letting $x^{\tau}y^{\kappa}=z$, it follows that

$$\psi(x)<\int_{-\infty}^{\infty}K\bigl(x^{\tau}y^{\kappa}\bigr)|y|^{\alpha\kappa-1}\,\mathrm{d}y=\frac{|x|^{-\alpha\tau}}{\kappa}\int_{-\infty}^{\infty}K(z)|z|^{\alpha-1}\,\mathrm{d}z.\tag{3.7}$$

By a similar discussion, it can also be proved that (3.7) is valid for $x>0$. Inserting (2.8) into (3.7), we obtain

$$\psi(x)<\frac{\pi|x|^{-\alpha\tau}}{\kappa\gamma}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right)-\Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right].\tag{3.8}$$

Additionally, it can be obtained that

$$\omega(n)=\frac{\pi|n|^{-\alpha\kappa}}{|\tau|\gamma}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right)-\Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right].\tag{3.9}$$

Inserting (3.8) and (3.9) back into (3.4), we obtain (3.1). In what follows, it is to be proved that (3.2) and (3.3) hold under the condition that inequality (3.1) holds. In fact, Let $\boldsymbol{b}=\{b_{n}\}_{n\in\mathbb{Z}^{0}}$, where

$$b_{n}:=|n|^{p\alpha\kappa-1}\left[\int_{-\infty}^{\infty}K\bigl(x^{\tau}n^{\kappa}\bigr)f(x)\,\mathrm{d}x\right]^{p-1},$$

then,

$$\begin{aligned} L_1 &= \sum_{n\in\mathbb{Z}^0} |n|^{p\alpha\kappa-1}\left[\int_{-\infty}^{\infty} K\left(x^\tau n^\kappa\right)f(x)\,\mathrm{d}x\right]^p \\ &= \sum_{n\in\mathbb{Z}^0} b_n \int_{-\infty}^{\infty} K\left(x^\tau n^\kappa\right)f(x)\,\mathrm{d}x \\ &< \frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right)-\Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right]\|f\|_{p,\mu}\|\boldsymbol{b}\|_{q,\nu} \\ &= \frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right)-\Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right]\|f\|_{p,\mu}L_1^{1/q}. \end{aligned} \tag{3.10}$$

It follows from (3.10) that (3.2) holds true. Similarly, inequality (3.3) can be proved. In fact, setting

$$g(x) := |x|^{q\alpha\tau-1}\left[\sum_{n\in\mathbb{Z}^0} K\left(x^\tau n^\kappa\right)a_n\right]^{q-1},$$

and using (3.1), it follows that

$$\begin{aligned} L_2 &= \int_{-\infty}^{\infty} |x|^{q\alpha\tau-1}\left[\sum_{n\in\mathbb{Z}^0} K\left(x^\tau n^\kappa\right)a_n\right]^q \mathrm{d}x \\ &= \int_{-\infty}^{\infty} g(x)\sum_{n\in\mathbb{Z}^0} K\left(x^\tau n^\kappa\right)a_n\,\mathrm{d}x \\ &< \frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right)-\Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right]\|g\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu} \\ &= \frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right)-\Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right]\|\boldsymbol{a}\|_{q,\nu}L_2^{1/p}. \end{aligned} \tag{3.11}$$

Therefore, (3.3) follows obviously. Furthermore, it can be proved that (3.1) holds true when inequality (3.2) or (3.3) is valid. In fact, assuming (3.2) holds true, it follows from Hölder's inequality that

$$\begin{aligned} J &= \sum_{n\in\mathbb{Z}^0}\left[|n|^{\alpha\tau-1/p}\int_{-\infty}^{\infty} K\left(x^\tau n^\kappa\right)f(x)\,\mathrm{d}x\right]\left(a_n|n|^{-\alpha\tau+1/p}\right) \\ &\le L_1^{1/p}\left[\sum_{n\in\mathbb{Z}^0} a_n^q|n|^{q(1-\alpha\tau)-1}\right]^{1/q} = L_1^{1/p}\|\boldsymbol{a}\|_{q,\nu}. \end{aligned} \tag{3.12}$$

Applying inequality (3.2) to (3.12), we arrive at (3.1). Similarly, if we suppose that inequality (3.3) holds true, it can also be proved that (3.1) is valid. Thus, inequalities (3.1), (3.2), and (3.3) are equivalent.

In what follows, it will be proved that the constant factors in (3.1), (3.2), and (3.3) are the best possible. In fact, suppose that there exists a constant C that satisfies

$$0 < C \le \frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right)-\Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right], \tag{3.13}$$

so that

$$\begin{aligned} J &= \sum_{n\in\mathbb{Z}^0} a_n \int_{-\infty}^{\infty} K\left(x^{\tau} n^{\kappa}\right) f(x)\,\mathrm{d}x = \int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} K\left(x^{\tau} n^{\kappa}\right) a_n\,\mathrm{d}x \\ &< C\|f\|_{p,\mu}\|\boldsymbol{a}\|_{q,\nu}. \end{aligned} \tag{3.14}$$

Replace a_n and $f(x)$ in (3.14) with $\tilde{a}_n$ and $\tilde{f}(x)$ defined in Lemma 2.3, respectively, and use (2.14), then we have

$$\begin{aligned} &\int_{[-1,1]} K(z)|z|^{\alpha-1+\frac{2}{pl}}\,\mathrm{d}z + \int_{\mathbb{R}\setminus[-1,1]} K(z)|z|^{\alpha-1-\frac{2}{ql}}\,\mathrm{d}z \\ &\quad< \frac{|\tau\kappa|}{l}\tilde{J} < \frac{|\tau\kappa|C}{l}\|\tilde{f}\|_{p,\mu}\|\tilde{\boldsymbol{a}}\|_{q,\nu} \\ &\quad= \frac{|\tau\kappa|C}{l}\left(2\int_{S^+} x^{\frac{2\tau}{l}-1}\,\mathrm{d}x\right)^{\frac{1}{p}}\left(2+2\sum_{n=2}^{\infty} n^{\frac{-2\kappa}{l}-1}\right)^{\frac{1}{q}} \\ &\quad< \frac{2|\tau\kappa|C}{l}\left(\int_{S^+} x^{\frac{2\tau}{l}-1}\,\mathrm{d}x\right)^{\frac{1}{p}}\left(1+\int_{1}^{\infty} x^{-\frac{2\kappa}{l}-1}\,\mathrm{d}x\right)^{\frac{1}{q}} \\ &\quad= 2|\tau\kappa|C\left(\frac{1}{2|\tau|}\right)^{\frac{1}{p}}\left(\frac{1}{l}+\frac{1}{2\kappa}\right)^{\frac{1}{q}}. \end{aligned} \tag{3.15}$$

Apply Fatou's lemma to (3.15), and use (2.8), then it follows that

$$\begin{aligned} &\frac{\pi}{\gamma}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right] \\ &\quad= \int_{-\infty}^{\infty} K(z)|z|^{\alpha-1}\,\mathrm{d}z \\ &\quad= \int_{[-1,1]} \underline{\lim}_{l\to\infty} K(z)|z|^{\alpha-1+\frac{2}{pl}}\,\mathrm{d}z + \int_{\mathbb{R}\setminus[-1,1]} \underline{\lim}_{l\to\infty} K(z)|z|^{\alpha-1-\frac{2}{ql}}\,\mathrm{d}z \\ &\quad\le \underline{\lim}_{l\to\infty}\left[\int_{[-1,1]} K(z)|z|^{\alpha-1+\frac{2}{pl}}\,\mathrm{d}z + \int_{\mathbb{R}\setminus[-1,1]} K(z)|z|^{\alpha-1-\frac{2}{ql}}\,\mathrm{d}z\right] \\ &\quad\le \underline{\lim}_{l\to\infty}\left[2|\tau\kappa|C\left(\frac{1}{2|\tau|}\right)^{\frac{1}{p}}\left(\frac{1}{l}+\frac{1}{2\kappa}\right)^{\frac{1}{q}}\right] = C|\tau|^{\frac{1}{q}}\kappa^{\frac{1}{p}}. \end{aligned}$$

It follows that

$$C \ge \frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right]. \tag{3.16}$$

Combine (3.13) and (3.16), then we have

$$C = \frac{\pi}{\gamma}|\tau|^{-\frac{1}{q}}\kappa^{-\frac{1}{p}}\left[\Phi\left(\frac{\alpha\pi}{2\gamma}\right) - \Phi\left(\frac{(\alpha+\beta+\gamma)\pi}{2\gamma}\right)\right].$$

Hence, it is proved that the constant factor in inequality (3.1) is the best possible. Observing that inequalities (3.1), (3.2), and (3.3) are equivalent, it can also be proved that the constant factors in (3.2) and (3.3) are the best possible. Theorem 3.1 is proved. □

4 Corollaries

Let $\gamma = 3\beta, \tau = \kappa = 1$ in Theorem 3.1. Then, (3.1) is transformed into the following Hilbert-type inequality with a nonhomogeneous kernel.

Corollary 4.1 *Let* $\delta \in \{1, -1\}$, *and*

$$\Omega := \left\{ t : t = \frac{2i+1}{2j+1}, i, j \in \mathbb{Z} \right\}.$$

Suppose that $\alpha \in (0,1)$, $\beta \in \Omega$, *and* α, β *satisfy* $0 < \alpha < 2\beta$ *and* $\alpha + \beta < 1$. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Assume that* $\mu(x) = |x|^{p(1-\alpha)-1}$, $\nu_n = |n|^{q(1-\alpha)-1}$, *where* $n \in \mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$. *Let* $f(x), a_n \geq 0$ *be such that* $f(x) \in L_{p,\mu}(\mathbb{R})$, *and* $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. *Let* $\Phi(z) = \cot z$. *Then,*

$$\begin{aligned} &\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{1 + \delta(xn)^\beta + (xn)^{2\beta}} \,\mathrm{d}x \\ &\quad < \frac{\pi}{3\beta}\left[\Phi\left(\frac{\alpha\pi}{6\beta}\right) - \Phi\left(\frac{(\alpha+4\beta)\pi}{6\beta}\right)\right] \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \end{aligned} \tag{4.1}$$

where the constant factor $\frac{\pi}{3\beta}[\Phi(\frac{\alpha\pi}{6\beta}) - \Phi(\frac{(\alpha+4\beta)\pi}{6\beta})]$ *in* (4.1) *is the best possible.*

Set $\alpha = \frac{\beta}{2}$ in Corollary 4.1, then $\beta \in \Omega$, and $0 < \beta < \frac{2}{3}$. Since $\Phi(\frac{\pi}{12}) = 3 + \sqrt{3}$, we obtain

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{1 + \delta(xn)^\beta + (xn)^{2\beta}} \,\mathrm{d}x < \frac{(4+\sqrt{3})\pi}{3\beta} \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{4.2}$$

where $\mu(x) = |x|^{p(1-\beta/2)-1}$, $\nu_n = |n|^{q(1-\beta/2)-1}$. Letting $\delta = 1$, we have (1.7).

Set $\alpha = \beta$ in Corollary 4.1, then $\beta \in \Omega$, $0 < \beta < \frac{1}{2}$, and (4.1) reduces to the following inequality.

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{1 + \delta(xn)^\beta + (xn)^{2\beta}} \,\mathrm{d}x < \frac{2\sqrt{3}\pi}{3\beta} \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{4.3}$$

where $\mu(x) = |x|^{p(1-\beta)-1}$, $\nu_n = |n|^{q(1-\beta)-1}$.

Set $\alpha = \frac{3\beta}{2}$ in Corollary 4.1, then $\beta \in \Omega$, and $0 < \beta < \frac{2}{5}$. In view of $\Phi(\frac{11\pi}{12}) = 3 + \sqrt{3}$, we arrive at

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{1 + \delta(xn)^\beta + (xn)^{2\beta}} \,\mathrm{d}x < \frac{(4+\sqrt{3})\pi}{3\beta} \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{4.4}$$

where $\mu(x) = |x|^{p(1-3\beta/2)-1}$, $\nu_n = |n|^{q(1-3\beta/2)-1}$.

Let $\alpha = \frac{\gamma-\beta}{2}$, $\tau = \kappa = 1$ in Theorem 3.1. Then, another Hilbert-type inequality with a nonhomogeneous kernel can be obtained.

Corollary 4.2 *Let* $\delta \in \{1, -1\}$, *and*

$$\Omega := \left\{ t : t = \frac{2i+1}{2j+1}, i, j \in \mathbb{Z} \right\}.$$

Suppose that $\beta, \gamma \in \Omega$, $0 < \beta < \gamma$ *and* $\beta + \gamma < 2$. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Assume that* $\mu(x) = |x|^{p(\beta-\gamma+2)/2-1}$, $\nu_n = |n|^{q(\beta-\gamma+2)/2-1}$, *where* $n \in \mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$. *Let* $f(x), a_n \geq 0$ *be such that* $f(x) \in L_{p,\mu}(\mathbb{R})$, *and* $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. *Let* $\Phi(z) = \cot z$. *Then,*

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{1-\delta(xn)^\beta}{1-\delta(xn)^\gamma} a_n \,\mathrm{d}x < \frac{2\pi}{\gamma} \Phi\left(\frac{(\gamma-\beta)\pi}{4\gamma}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{4.5}$$

where the constant factor $\frac{2\pi}{3\beta}\Phi(\frac{(\gamma-\beta)\pi}{4\gamma})$ *in* (4.5) *is the best possible.*

Letting $\gamma = (2k+1)\beta$, $k \in \mathbb{N}^+$, we have $0 < (k+1)\beta < 1$, $\beta \in \Omega$, and (4.5) is transformed into the following inequality

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{\sum_{j=0}^{2k} \delta^{2k-j}(xn)^{j\beta}} \,\mathrm{d}x < \frac{2\pi}{(2k+1)\beta} \Phi\left(\frac{k\pi}{4k+2}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{4.6}$$

where $\mu(x) = |x|^{p(1-k\beta)-1}$, $\nu_n = |n|^{q(1-k\beta)-1}$.

Setting $k = 1$ in (4.6), we can also obtain (4.3). Moreover, Setting $k = 2$ in (4.6), we have $0 < \beta < \frac{1}{3}$, $\beta \in \Omega$. It follows that

$$\int_{-\infty}^{\infty} \sum_{n\in\mathbb{Z}^0} \frac{a_n f(x)}{1 + \delta(xn)^\beta + (xn)^{2\beta} + \delta(xn)^{3\beta} + (xn)^{4\beta}} \,\mathrm{d}x < \frac{2\pi}{5\beta} \Phi\left(\frac{\pi}{5}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{4.7}$$

where $\mu(x) = |x|^{p(1-2\beta)-1}$, $\nu_n = |n|^{q(1-2\beta)-1}$.

Let $\gamma = 3\beta$, $\tau = -1$, $\kappa = 1$ in Theorem 3.1, and replace $f(x)x^{2\beta}$ with $f(x)$. Then, the following Hilbert-type inequality with a homogeneous kernel of degree 2β can be obtained.

Corollary 4.3 *Let* $\delta \in \{1, -1\}$, *and*

$$\Omega := \left\{ t : t = \frac{2i+1}{2j+1}, i, j \in \mathbb{Z} \right\}.$$

Suppose that $\alpha \in (0,1)$, $\beta \in \Omega$, *and* α, β *satisfy* $0 < \alpha < 2\beta$ *and* $\alpha + \beta < 1$. *Let* $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. *Assume that* $\mu(x) = |x|^{p(1+\alpha-2\beta)-1}$, $\nu_n = |n|^{q(1-\alpha)-1}$, *where* $n \in \mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$. *Let* $f(x), a_n \geq 0$ *be such that* $f(x) \in L_{p,\mu}(\mathbb{R})$, *and* $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. *Let* $\Phi(z) = \cot z$. *Then,*

$$\begin{aligned} &\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{x^{2\beta} + \delta(xn)^\beta + n^{2\beta}} \,\mathrm{d}x \\ &\quad< \frac{\pi}{3\beta} \left[\Phi\left(\frac{\alpha\pi}{6\beta}\right) - \Phi\left(\frac{(\alpha+4\beta)\pi}{6\beta}\right) \right] \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \end{aligned} \tag{4.8}$$

where the constant factor $\frac{\pi}{3\beta}[\Phi(\frac{\alpha\pi}{6\beta}) - \Phi(\frac{(\alpha+4\beta)\pi}{6\beta})]$ *in* (4.8) *is the best possible.*

Set $\alpha = \frac{\beta}{2}$ in Corollary 4.3, then $\beta \in \Omega$, and $0 < \beta < \frac{2}{3}$. Since $\Phi(\frac{\pi}{12}) = 3 + \sqrt{3}$, we obtain the following inequality

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{x^{2\beta} + \delta(xn)^\beta + n^{2\beta}} \,\mathrm{d}x < \frac{(4+\sqrt{3})\pi}{3\beta} \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{4.9}$$

where $\mu(x) = |x|^{p(1-3\beta/2)-1}$, $\nu_n = |n|^{q(1-\beta/2)-1}$. Letting $\delta = -1$, we obtain inequality (1.8).

Set $\alpha = \beta$ in Corollary 4.3, then $\beta \in \Omega$, and $0 < \beta < \frac{1}{2}$, and (4.8) is transformed into the following inequality

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{x^{2\beta} + \delta(xn)^{\beta} + n^{2\beta}} \,\mathrm{d}x < \frac{2\sqrt{3}\pi}{3\beta} \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{4.10}$$

where $\mu(x) = |x|^{p(1-\beta)-1}$, $\nu_n = |n|^{q(1-\beta)-1}$.

Let $\alpha = \frac{\gamma-\beta}{2}$, $\tau = -1$, $\kappa = 1$ in Theorem 3.1, and replace $f(x)x^{\gamma-\beta}$ with $f(x)$. Then, the following Hilbert-type inequality involving a homogeneous kernel with degree $\gamma - \beta$ can be obtained.

Corollary 4.4 *Let $\delta \in \{1, -1\}$, and*

$$\Omega := \left\{ t : t = \frac{2i+1}{2j+1}, i, j \in \mathbb{Z} \right\}.$$

Suppose that $\beta, \gamma \in \Omega$, $0 < \beta < \gamma$ and $\beta + \gamma < 2$. Let $p > 1$, $\frac{1}{p} + \frac{1}{q} = 1$. Assume that $\mu(x) = |x|^{p(\beta-\gamma+2)/2-1}$, $\nu_n = |n|^{q(\beta-\gamma+2)/2-1}$, where $n \in \mathbb{Z}^0 := \mathbb{Z} \setminus \{0\}$. Let $f(x), a_n \geq 0$ be such that $f(x) \in L_{p,\mu}(\mathbb{R})$, and $\boldsymbol{a} = \{a_n\}_{n\in\mathbb{Z}^0} \in l_{q,\nu}$. Let $\Phi(z) = \cot z$. Then,

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{x^{\beta} - \delta n^{\beta}}{x^{\gamma} - \delta n^{\gamma}} a_n \,\mathrm{d}x < \frac{2\pi}{\gamma} \Phi\left(\frac{(\gamma-\beta)\pi}{4\gamma}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{4.11}$$

where the constant factor $\frac{2\pi}{\gamma}\Phi(\frac{(\gamma-\beta)\pi}{4\gamma})$ in (4.11) is the best possible.

Letting $\gamma = (2k+1)\beta$, $k \in \mathbb{N}^+$, we have $0 < (k+1)\beta < 1$, $\beta \in \Omega$, and (4.11) is transformed into the following inequality

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{\sum_{j=0}^{2k} \delta^{2k-j} x^{j\beta} n^{(2k-j)\beta}} \,\mathrm{d}x < \frac{2\pi}{(2k+1)\beta} \Phi\left(\frac{k\pi}{4k+2}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{4.12}$$

where $\mu(x) = |x|^{p(1-k\beta)-1}$, $\nu_n = |n|^{q(1-k\beta)-1}$.

Setting $k = 1$, and $\delta = 1$ in (4.12), (4.10) can also be obtained. Additionally, let $k = 2$ in (4.12), then $0 < \beta < \frac{1}{3}$, $\beta \in \Omega$, and it follows that

$$\int_{-\infty}^{\infty} f(x) \sum_{n\in\mathbb{Z}^0} \frac{a_n}{x^{4\beta} + x^{3\beta}n^{\beta} + (xn)^{2\beta} + x^{\beta}n^{3\beta} + n^{4\beta}} \,\mathrm{d}x < \frac{2\pi}{5} \Phi\left(\frac{\pi}{5}\right) \|f\|_{p,\mu} \|\boldsymbol{a}\|_{q,\nu}, \tag{4.13}$$

where $\mu(x) = |x|^{p(1-2\beta)-1}$, $\nu_n = |n|^{q(1-2\beta)-1}$.

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Declarations

Competing interests

The authors declare no competing interests.

Author contributions

Minghui You carried out the results, and read and approved the current version of the manuscript.

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References

1. Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities. Cambridge University Press, London (1952)
2. Krnić, M., Pečarić, J.: Extension of Hilbert's inequality. J. Math. Anal. Appl. **324**, 150–160 (2006)
3. Wang, Z.X., Guo, D.R.: Introduction to Special Functions. Higher Education Press, Beijing (2012)
4. Yang, B.C.: On an extension of Hilbert's integral inequality with some parameters. Aust. J. Math. Anal. Appl. **1**(1), 1–8 (2004)
5. Yang, B.C., Debnath, L.: On a new generalization of Hardy–Hilbert's inequality and its application. J. Math. Anal. Appl. **23**(2), 484–497 (1999)
6. Gao, M.Z., Yang, B.C.: On the extended Hilbert's inequality. Proc. Am. Math. Soc. **126**(3), 751–759 (1998)
7. You, M.H.: On a new discrete Hilbert-type inequality and applications. Math. Inequal. Appl. **18**(4), 1575–1578 (2015)
8. You, M.H.: On an extension of the discrete Hilbert inequality and applications. J. Wuhan Univ. Natur. Sci. Ed. **67**(2), 179–184 (2021)
9. Krnić, M., Pečarić, J., Vuković, P.: Discrete Hilbert-type inequalities with general homogeneous kernels. Rend. Circ. Mat. Palermo **60**(1), 161–171 (2011)
10. Krnić, M., Pečarić, J., Perić, I., et al.: Advances in Hilbert-Type Inequalities. Element Press, Zagreb (2012)
11. Yang, B.C.: The Norm of Operator and Hilbert-Type Inequalities. Science Press, Beijing (2009)
12. Rassias, M.T., Yang, B.C.: On a Hilbert-type integral inequality in the whole plane related to the extended Riemann zeta function. Complex Anal. Oper. Theory **13**(4), 1765–1782 (2019)
13. Rassias, M.T., Yang, B.C.: On an equivalent property of a reverse Hilbert-type integral inequality related to the extended Hurwitz-zeta function. J. Math. Inequal. **13**(2), 315–334 (2019)
14. Rassias, M.T., Yang, B.C.: A Hilbert-type integral inequality in the whole plane related to the hypergeometric function and the beta function. J. Math. Anal. Appl. **428**(2), 1286–1308 (2015)
15. Rassias, M.T., Yang, B.C., Raigorodskii, A.: On a nore accurate reverse Hilbert-type inequlity in the whole plane. J. Math. Inequal. **14**(4), 1359–1374 (2020)
16. Hong, Y., He, B., Yang, B.C.: Necessary and sufficient conditions for the validity of Hilbert-type inequalities with a class of quasi-homogeneous kernels ans its applications in operator theory. J. Math. Inequal. **12**(3), 777–788 (2018)
17. Liu, Q.: A Hilbert-type integral inequality under configuring free power and its applications. J. Inequal. Appl. **2019**, 91 (2019). https://doi.org/10.1186/s13660-019-2039-1
18. You, M.H., Sun, X.: On a Hilbert-type inequality with the kernel involving extended Hardy operator. J. Math. Inequal. **15**(3), 1239–1253 (2021)
19. You, M.H., Dong, F., He, Z.H.: A Hilbert-type inequality in the whole plane with the constant factor related to some special constants. J. Math. Inequal. **16**(1), 35–50 (2022)
20. You, M.H.: On a class of Hilbert-type inequalities in the whole plane involving some classical kernel functions. Proc. Edinb. Math. Soc. **65**(3), 833–846 (2022)
21. You, M.H.: A unified extension of some classical Hilbert-type inequalities and applications. Rocky Mt. J. Math. **51**(5), 1865–1877 (2021)
22. Mo, H.M., Yang, B.C.: On a new Hilbert-type integral inequality involving the upper limit functions. J. Inequal. Appl. **2020**, 5 (2020). https://doi.org/10.1186/s13660-019-2280-7
23. Batbold, T., Krnić, M., Pečarić, J., Vuković, P.: Further Development of Hilbert-Type Inequalities. Element Press, Zagreb (2017)
24. Yang, B.C.: A mixed Hilbert-type inequality with a best constant factor. Int. J. Pure Appl. Math. **20**(3), 319–328 (2005)
25. Yang, B.C., Wu, S.H., Wang, A.Z.: On a reverse half-discrete Hardy–Hilbert's inequality with parameters. Mathematics **7**(11), 1054 (2019). https://doi.org/10.3390/math7111054
26. Yang, B.C., Chen, Q.: A half-discrete Hilbert-type inequality with a homogeneous kernel and an extension. J. Inequal. Appl. **2011**, 124 (2011). https://doi.org/10.1186/1029-242X-2011-124
27. Rassias, M.T., Yang, B.C.: On half-discrete Hilbert's inequality. Appl. Math. Comput. **220**, 75–93 (2013)
28. Rassias, M.T., Yang, B.C., Raigorodskii, A.: On a half-discrete Hilbert-type inequality in the whole plane with the kernel of hyperbolic secant function related to the Hurwitz zeta function. In: Trigonometric Sums and Their Applications, pp. 229–259. Springer, Berlin (2020)
29. He, B., Yang, B.C., Chen, Q.: A new multiple half-discrete Hilbert-type inequality with parameters and a best possible constant factor. Mediterr. J. Math. (2014). https://doi.org/10.1007/s00009-014-0468-0
30. Krnić, M., Pečarić, J., Vuković, P.: A unified treatment of half-discrete Hilbert-type inequalities with a homogeneous kernel. Mediterr. J. Math. **10**, 1697–1716 (2013)
31. You, M.H.: More accurate and strengthened forms of half-discrete Hilbert inequality. J. Math. Anal. Appl. **512**(2), 126141 (2022). https://doi.org/10.1016/j.jmaa.2022.126141
32. You, M.H., Sun, X., Fan, X.S.: On a more accurate half-discrete Hilbert-type inequality involving hyperbolic functions. Open Math. **20**(1), 544–559 (2022). https://doi.org/10.1515/math-2022-0041