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Some new integral inequalities for higher-order strongly exponentially convex functions

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Abstract

Integral inequalities with generalized convexity play an important role in both applied and theoretical mathematics. The theory of integral inequalities is currently one of the most rapidly developing areas of mathematics due to its wide range of applications. In this paper, we study the concept of higher-order strongly exponentially convex functions and establish a new Hermite–Hadamard inequality for the class of strongly exponentially convex functions of higher order. Further, we derive some new integral inequalities for Riemann–Liouville fractional integrals via higher-order strongly exponentially convex functions. These findings include several well-known results and newly obtained results as special cases. We believe that the results presented in this paper are novel and will be beneficial in encouraging future research in this field.

MSC: 26D15; 26A51; 26A33

Keywords: Convex functions; Exponentially convex functions; Hermite–Hadamard inequalities; Riemann–Liouville fractional integrals

1 Introduction

Nowadays, the generalization of convex functions is considered as an original icon in the theoretical study of mathematical inequalities [4, 10, 22, 25, 39]. Integral inequalities on different types of convex functions are applicable in many branches of mathematics such as mathematical analysis, fractional calculus, and discrete fractional calculus, see references [1, 7, 8, 12, 18, 33] and the references therein.

Karamardian [15] proposed strongly convex functions and discussed the unique existence of a solution of the nonlinear complementarity problems using the concept of strongly convex functions. However, some researchers cited that Polyak [32] had introduced the concept of strongly convex functions. Lin and Fukushima [20] introduced the concept of higher-order strongly convex functions and showed that every continuously differentiable function is a strongly convex function of higher order if and only if its gradient is strongly monotone of higher order. In 2011, Srivastava et al. [45] presented several refinements and extensions of the Hermite–Hadamard and Jensen inequalities in n variables. Mishra and Sharma [24] introduced the notion of strongly generalized convex functions of higher order and derived new Hermite–Hadamard-type integral inequalities for the class of strongly generalized convex functions of higher order. For more basic properties and applications of strongly convex functions, see references [5, 23, 26, 38, 40] and the references therein.

Antczak [3] introduced the class of exponentially convex functions that can be considered as a significant extension of the convex functions. Exponentially convex functions have applications in various fields such as mathematical programming, information geometry, big-data analysis, machine learning, statistics, sequential prediction, and stochastic optimization, see [2, 27, 28, 30]. Awan et al. [4] investigated some other kinds of exponentially convex functions and established several new Hermite–Hadamard-type integral inequalities via exponentially convex functions. Noor and Noor [28] defined and introduced some new concepts of the strongly exponentially convex functions with respect to an auxiliary nonnegative bifunction and investigated the optimality conditions for the strongly exponentially convex functions.

Kilbas et al. [17] studied some useful properties of several different families of fractional integrals and fractional derivatives and investigated integral transform methods for explicit solutions to fractional differential equations. Motivated by the importance of the fractional integral in multiple fields of pure and applied science, researchers generalized the notion of the fractional integral in various directions and discovered new integral inequalities for the generalized fractional integrals. Srivastava et al. [41] derived the fractional Steffensen–Hayashi inequality and some interesting applications to various inequalities involving ν-fractional operators. Further, Khan et al. [16] established various discrete Jensen and Schur, and Hermite–Hadamard integral inequalities for log convex fuzzy interval-valued functions. Srivastava et al. [42] obtained new Hermite–Hadamard-type inequalities for interval-valued preinvex functions via Riemann–Liouville fractional integrals.

Recently, Rashid et al. [34] established some trapezoid-type inequalities for generalized fractional integrals and related inequalities via exponentially convex functions. Rashid et al. [35] derived a new integral identity involving Riemann–Liouville fractional integrals and obtained new fractional bounds for the functions having the exponential convexity property. Further, Yu et al. [46] established certain improvements of the midpoint-type integral inequalities for mappings whose first-order derivatives in absolute value belong to the generalized (s,P)-convex mappings. For more recent results on fractional integral inequalities, see [43, 44].

The above developments have inspired us to derive new integral inequalities for generalized exponentially convex functions. Many experts have presented generalizations and extensions of the Hermite–Hadamard inequality for generalized convex functions, but their results have not included higher-order strongly exponential convexity. Hence, we present the Hermite–Hadamard integral inequality for higher-order strongly exponentially convex functions. The use of fractional calculus for finding various integral inequalities via generalized convex functions has increased in recent years. However, fractional inequalities for higher-order strongly exponentially convex functions have not been studied. We derive new integral inequalities via strongly exponentially convex functions of higher order using Riemann–Liouville fractional integrals. Moreover, some particular cases of the main results are briefly discussed.

2 Preliminaries

Let K be a nonempty, closed, and convex set in a real Hilbert space H. For $x \in K$, $\|\cdot\|$ denote the norm defined by $\|x\| = (\sum_{i=1}^{n} x_i^2)^{\frac{1}{2}}$. Let $\psi : K \longrightarrow \mathbb{R}$ be a continuous function.

Definition 2.1 ([21]) A function $\psi : K \longrightarrow \mathbb{R}$ is said to be a convex function, if

$$\psi\big(tx + (1-t)y\big) \leq t\psi(x) + (1-t)\psi(y), \quad \forall x, y \in K, t \in [0,1].$$

An interesting inequality for a convex function discovered by Hermite [13] and Hadamard [11] is known as a Hermite–Hadamard inequality in the literature, which provides a lower and an upper estimate for the integral average of any convex function defined on a compact interval. It states that if $\psi : I = [a,b] \longrightarrow \mathbb{R}$ is a convex function with $a < b$, then the following double inequality holds:

$$\psi\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b \psi(x)\,dx \leq \frac{\psi(a)+\psi(b)}{2}.$$

In recent years, there have been many extensions and generalizations of Hermite–Hadamard inequalities studied in [6, 14, 19, 29, 36, 37].

Definition 2.2 ([3]) A positive function ψ on the convex set K is said to be an exponentially convex function, if

$$e^{\psi(tx+(1-t)y)} \leq te^{\psi(x)} + (1-t)e^{\psi(y)}, \quad \forall x, y \in K, t \in [0,1].$$

Definition 2.3 ([28]) A positive function ψ on the convex set K is said to be a higher-order strongly exponentially convex function of order $\sigma > 1$ if there exists a constant $c > 0$, such that

$$e^{\psi(tx+(1-t)y)} \leq te^{\psi(x)} + (1-t)e^{\psi(y)} - ct(1-t)\|y-x\|^{\sigma}, \quad \forall x, y \in K, t \in [0,1].$$

Definition 2.4 ([31]) Let $\psi \in L_1[a,b]$. Then, the left-sided and right-sided Riemann–Liouville fractional integrals of order $\alpha > 0$ are defined by

$$J_{a^+}^{\alpha}\psi(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}\psi(t)\,dt, \quad x > a$$

and

$$J_{b^-}^{\alpha}\psi(x) = \frac{1}{\Gamma(\alpha)}\int_x^b (t-x)^{\alpha-1}\psi(t)\,dt, \quad x < b,$$

respectively, where $\Gamma(\cdot)$ is the Euler Gamma function defined by

$$\Gamma(\alpha) = \int_0^{\infty} e^{-t}t^{(\alpha-1)}\,dt.$$

Corollary 2.1 ([34]) *Let $\psi : I = [a,b] \to \mathbb{R}$ be an absolutely continuous mapping on (a,b) such that $(e^{\psi})' \in L_1[a,b]$. Then, the following equality holds*:

$$\begin{aligned}
&\frac{(x-a)^{\alpha}e^{\psi(a)} + (b-x)^{\alpha}e^{\psi(b)}}{b-a} - \frac{\Gamma(\alpha+1)[J_{x^-}^{\alpha}e^{\psi(a)} + J_{x^+}^{\alpha}e^{\psi(b)}]}{b-a} \\
&\quad = \frac{(x-a)^{\alpha+1}}{b-a}\int_0^1 \big(t^{\alpha}-1\big)e^{\psi(tx+(1-t)a)}\psi'\big(tx+(1-t)a\big)\,dt \\
&\qquad + \frac{(b-x)^{\alpha+1}}{b-a}\int_0^1 \big(1-t^{\alpha}\big)e^{\psi(tx+(1-t)b)}\psi'\big(tx+(1-t)b\big)\,dt.
\end{aligned}$$

Lemma 2.2 ([35]) *Let $\alpha > 0$ be a number and let $\psi : I = [a,b] \longrightarrow \mathbb{R}$ be a differentiable function on (a,b), then*

$$\begin{aligned}
\Gamma_{\psi}(a,b,\alpha) = \frac{b-a}{16}\Bigg[&\int_0^1 t^{\alpha}e^{\psi(t\frac{3a+b}{4}+(1-t)a)}\psi'\bigg(t\frac{3a+b}{4}+(1-t)a\bigg)\,dt \\
&+ \int_0^1 \big(t^{\alpha}-1\big)e^{\psi(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4})}\psi'\bigg(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\bigg)\,dt \\
&+ \int_0^1 t^{\alpha}e^{\psi(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2})}\psi'\bigg(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\bigg)\,dt \\
&+ \int_0^1 \big(t^{\alpha}-1\big)e^{\psi(tb+(1-t)\frac{a+3b}{4})}\psi'\bigg(tb+(1-t)\frac{a+3b}{4}\bigg)\,dt\Bigg],
\end{aligned}$$

where

$$\begin{aligned}
\Gamma_{\psi}(a,b,\alpha) = \frac{1}{2}\big[e^{\psi(\frac{3a+b}{4})} + e^{\psi(\frac{a+3b}{4})}\big] - \frac{4^{(\alpha-1)}\Gamma(\alpha+1)}{(b-a)^{\alpha}} \\
\times \big[J_{(\frac{3a+b}{4})^-}^{\alpha}e^{\psi(a)} + J_{(\frac{a+b}{2})^-}^{\alpha}e^{\psi(\frac{3a+b}{4})} + J_{(\frac{a+3b}{4})^-}^{\alpha}e^{\psi(\frac{a+b}{2})} + J_{b^-}^{\alpha}e^{\psi(\frac{a+3b}{4})}\big].
\end{aligned}$$

3 Main results

In this section, first, we prove the Hermite–Hadamard inequality for higher-order strongly exponentially convex functions.

Theorem 3.1 *Let $\psi : I = [a,b] \longrightarrow \mathbb{R}$ be a strongly exponentially convex function of order $\sigma > 1$ with modulus $c > 0$, then the function satisfies the following*:

$$e^{\psi(\frac{a+b}{2})} + \frac{c}{4}\|b-a\|^{\sigma} \le \frac{1}{b-a}\int_a^b e^{\psi(x)}\,dx \le \frac{e^{\psi(a)}+e^{\psi(b)}}{2} - \frac{c}{6}\|b-a\|^{\sigma}. \tag{3.1}$$

Proof Since ψ is a strongly exponentially convex function of order $\sigma > 1$ on I, we have

$$e^{\psi(tx+(1-t)y)} \le te^{\psi(x)} + (1-t)e^{\psi(y)} - ct(1-t)\|y-x\|^{\sigma}, \quad \forall x,y \in I, t \in [0,1]. \tag{3.2}$$

For $t = \frac{1}{2}$, we obtain

$$e^{\psi(\frac{x+y}{2})} \le \frac{e^{\psi(x)}+e^{\psi(y)}}{2} - \frac{c}{4}\|y-x\|^{\sigma}.$$

Letting $x = (1-t)a + tb$ and $y = ta + (1-t)b$, we have

$$e^{\psi(\frac{a+b}{2})} \le \frac{e^{\psi[(1-t)a+tb]} + e^{\psi[(ta+(1-t)b)]}}{2} - \frac{c}{4}\|b-a\|^{\sigma}.$$

Integrating the above with respect to t over $[0,1]$ and using the change of variable technique, we have

$$e^{\psi(\frac{a+b}{2})} \le \frac{1}{b-a}\int_a^b e^{\psi(x)}\,dx - \frac{c}{4}\|b-a\|^{\sigma}. \tag{3.3}$$

Integrating (3.2) with respect to t over $[0,1]$, we have

$$\frac{1}{b-a}\int_a^b e^{\psi(x)}\,dx \le \frac{e^{\psi(a)} + e^{\psi(b)}}{2} - \frac{c}{6}\|b-a\|^{\sigma}. \tag{3.4}$$

From (3.3) and (3.4), we obtain

$$e^{\psi(\frac{a+b}{2})} + \frac{c}{4}\|b-a\|^{\sigma} \le \frac{1}{b-a}\int_a^b e^{\psi(x)}\,dx \le \frac{e^{\psi(a)} + e^{\psi(b)}}{2} - \frac{c}{6}\|b-a\|^{\sigma}.$$

This completes the proof. □

Example 1 Let $I = [\frac{1}{2}, 1]$ and $c = \frac{1}{10}$. Let $\psi : I \longrightarrow \mathbb{R}$ be defined by $\psi(x) = x$ for all $x \in I$. Obviously, ψ is a higher-order strongly exponentially convex function for $c = \frac{1}{10}$. Then, the function ψ satisfies the above theorem.

Remark 3.1 When $c = 0$, Theorem 3.1 reduces to the following:

$$e^{\psi(\frac{a+b}{2})} \le \frac{1}{b-a}\int_a^b e^{\psi(x)}\,dx \le \frac{e^{\psi(a)} + e^{\psi(b)}}{2},$$

which is the Hermite–Hadamard inequality for exponentially convex functions given by Dragomir and Gomm [9].

Now, we obtain some new fractional integral inequalities using the Riemann–Liouville fractional integral via strongly exponentially convex functions of higher order.

Theorem 3.2 *Let $\psi : I = [a,b] \to \mathbb{R}$ be an absolutely continuous mapping on (a,b) such that $(e^{\psi})' \in L_1[a,b]$. If the function $|\psi|$ is a strongly exponentially convex function of order $\sigma_1 > 1$ with modulus $c_1 > 0$ and $|\psi'|$ is a strongly convex function of order $\sigma_2 > 0$ with modulus $c_2 > 0$, then*

$$\begin{aligned}
&\left|\frac{(x-a)^{\alpha}e^{\psi(a)} + (b-x)^{\alpha}e^{\psi(b)}}{b-a} - \frac{\Gamma(\alpha+1)[J^{\alpha}_{x^-}e^{\psi(a)} + J^{\alpha}_{x^+}e^{\psi(b)}]}{b-a}\right| \\
&\quad \le \frac{\alpha}{3(\alpha+3)}\phi(x)\left(\frac{(x-a)^{\alpha+1} + (b-x)^{\alpha+1}}{b-a}\right) + \frac{\alpha^3 + 6\alpha^2 + 11\alpha}{3(\alpha+1)(\alpha+2)(\alpha+3)} \\
&\qquad \times \left(\frac{(x-a)^{\alpha+1}\phi(a) + (b-x)^{\alpha+1}\phi(b)}{b-a}\right) + \frac{\alpha^2 + 5\alpha}{6(\alpha+2)(\alpha+3)}
\end{aligned}$$

$$\times\left(\frac{(x-a)^{\alpha+1}\Delta_1(x,a)+(b-x)^{\alpha+1}\Delta_1(x,b)}{b-a}\right)-\frac{\alpha^2+7\alpha}{12(\alpha+3)(\alpha+4)}$$
$$\times\left(c_1\left|\psi'(x)\right|\|b-a\|^{\sigma_1}+c_2\left|e^{\psi(x)}\right|\|b-a\|^{\sigma_2}\right)\left(\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right)$$
$$-\frac{\alpha^3+9\alpha^2+26\alpha}{12(\alpha+2)(\alpha+3)(\alpha+4)}$$
$$\times\left\{c_1\|b-a\|^{\sigma_1}\left(\frac{|\psi'(a)|(x-a)^{\alpha+1}+|\psi'(b)|(b-x)^{\alpha+1}}{b-a}\right)\right.$$
$$\left.+c_2\|b-a\|^{\sigma_2}\left(\frac{|e^{\psi(a)}|(x-a)^{\alpha+1}+|e^{\psi(b)}|(b-x)^{\alpha+1}}{b-a}\right)\right\}$$
$$+\frac{\alpha^3+12\alpha^2+47\alpha}{30(\alpha+3)(\alpha+4)(\alpha+5)}c_1c_2\|b-a\|^{\sigma_1+\sigma_2}\left(\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right),$$

where $\Delta_1(x,a)=|e^{\psi(x)}\psi'(a)|+|e^{\psi(a)}\psi'(x)|$, $\Delta_1(x,b)=|e^{\psi(x)}\psi'(b)|+|e^{\psi(b)}\psi'(x)|$ *and* $\phi(x)=|e^{\psi(x)}\psi'(x)|$, $\phi(a)=|e^{\psi(a)}\psi'(a)|$, $\phi(b)=|e^{\psi(b)}\psi'(b)|$.

Proof Using Corollary 2.1, the property of modulus, and the given hypothesis of the theorem, we obtain

$$\left|\frac{(x-a)^\alpha e^{\psi(a)}+(b-x)^\alpha e^{\psi(b)}}{b-a}-\frac{\Gamma(\alpha+1)[J^\alpha_{x^-}e^{\psi(a)}+J^\alpha_{x^+}e^{\psi(b)}]}{b-a}\right|$$
$$\le\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1\left|\left(t^\alpha-1\right)\right|\left|e^{\psi(tx+(1-t)a)}\psi'\left(tx+(1-t)a\right)\right|dt$$
$$+\frac{(b-x)^{\alpha+1}}{b-a}\int_0^1\left|\left(1-t^\alpha\right)\right|\left|e^{\psi(tx+(1-t)b)}\psi'\left(tx+(1-t)b\right)\right|dt$$
$$\le\frac{(x-a)^{\alpha+1}}{b-a}\int_0^1\left(1-t^\alpha\right)\left(t\left|e^{\psi(x)}\right|+(1-t)\left|e^{\psi(a)}\right|-c_1t(1-t)\|b-a\|^{\sigma_1}\right)$$
$$\times\left(t\left|\psi'(x)\right|+(1-t)\left|\psi'(a)\right|-c_2t(1-t)\|b-a\|^{\sigma_2}\right)dt+\frac{(b-x)^{\alpha+1}}{b-a}$$
$$\times\int_0^1\left(1-t^\alpha\right)\left(t\left|e^{\psi(x)}\right|+(1-t)\left|e^{\psi(b)}\right|-c_1t(1-t)\|b-a\|^{\sigma_1}\right)\left(t\left|\psi'(x)\right|\right.$$
$$\left.+(1-t)\left|\psi'(b)\right|-c_2t(1-t)\|b-a\|^{\sigma_2}\right)dt$$
$$=\frac{(x-a)^{\alpha+1}}{b-a}\left[\left|e^{\psi(x)}\psi'(x)\right|\int_0^1\left(1-t^\alpha\right)t^2\,dt+\left|e^{\psi(a)}\psi'(a)\right|\int_0^1\left(1-t^\alpha\right)(1-t)^2\,dt\right.$$
$$+\left(\left|e^{\psi(x)}\psi'(a)\right|+\left|e^{\psi(a)}\psi'(x)\right|\right)\int_0^1\left(1-t^\alpha\right)t(1-t)\,dt-\left(c_1\left|\psi'(x)\right|\|b-a\|^{\sigma_1}\right.$$
$$\left.+c_2\left|e^{\psi(x)}\right|\|b-a\|^{\sigma_2}\right)\int_0^1\left(1-t^\alpha\right)t^2(1-t)\,dt-\left(c_1\left|\psi'(a)\right|\|b-a\|^{\sigma_1}\right.$$
$$\left.+c_2\left|e^{\psi(a)}\right|\|b-a\|^{\sigma_2}\right)\int_0^1\left(1-t^\alpha\right)t(1-t)^2\,dt+c_1c_2\|b-a\|^{\sigma_1+\sigma_2}$$
$$\left.\times\int_0^1\left(1-t^\alpha\right)t^2(1-t)^2\,dt\right]+\frac{(b-x)^{\alpha+1}}{b-a}\left[\left|e^{\psi(x)}\psi'(x)\right|\int_0^1\left(1-t^\alpha\right)t^2\,dt\right.$$
$$+\left|e^{\psi(b)}\psi'(b)\right|\int_0^1\left(1-t^\alpha\right)(1-t)^2\,dt+\left(\left|e^{\psi(x)}\psi'(b)\right|+\left|e^{\psi(b)}\psi'(x)\right|\right)$$

$$
\begin{aligned}
&\times \int_0^1 (1-t^\alpha) t(1-t)\,dt - \left(c_1\left|\psi'(x)\right|\|b-a\|^{\sigma_1} + c_2\left|e^{\psi(x)}\right|\|b-a\|^{\sigma_2}\right) \\
&\times \int_0^1 (1-t^\alpha) t^2(1-t)\,dt - \left(c_1\left|\psi'(b)\right|\|b-a\|^{\sigma_1} + c_2\left|e^{\psi(b)}\right|\|b-a\|^{\sigma_2}\right) \\
&\times \int_0^1 (1-t^\alpha) t(1-t)^2\,dt + c_1c_2\|b-a\|^{\sigma_1+\sigma_2}\int_0^1 (1-t^\alpha) t^2(1-t)^2\,dt\Bigg] \\
= {} & \frac{\alpha}{3(\alpha+3)}\phi(x)\left(\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right) + \frac{\alpha^3+6\alpha^2+11\alpha}{3(\alpha+1)(\alpha+2)(\alpha+3)} \\
&\times \left(\frac{(x-a)^{\alpha+1}\phi(a)+(b-x)^{\alpha+1}\phi(b)}{b-a}\right) + \frac{\alpha^2+5\alpha}{6(\alpha+2)(\alpha+3)} \\
&\times \left(\frac{(x-a)^{\alpha+1}\Delta_1(x,a)+(b-x)^{\alpha+1}\Delta_1(x,b)}{b-a}\right) - \frac{\alpha^2+7\alpha}{12(\alpha+3)(\alpha+4)} \\
&\times \left(c_1\left|\psi'(x)\right|\|b-a\|^{\sigma_1} + c_2\left|e^{\psi(x)}\right|\|b-a\|^{\sigma_2}\right)\left(\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right) \\
&- \frac{\alpha^3+9\alpha^2+26\alpha}{12(\alpha+2)(\alpha+3)(\alpha+4)} \\
&\times \left\{c_1\|b-a\|^{\sigma_1}\left(\frac{|\psi'(a)|(x-a)^{\alpha+1}+|\psi'(b)|(b-x)^{\alpha+1}}{b-a}\right)\right. \\
&\left.+ c_2\|b-a\|^{\sigma_2}\left(\frac{|e^{\psi(a)}|(x-a)^{\alpha+1}+|e^{\psi(b)}|(b-x)^{\alpha+1}}{b-a}\right)\right\} \\
&+ \frac{\alpha^3+12\alpha^2+47\alpha}{30(\alpha+3)(\alpha+4)(\alpha+5)}c_1c_2\|b-a\|^{\sigma_1+\sigma_2}\left(\frac{(x-a)^{\alpha+1}+(b-x)^{\alpha+1}}{b-a}\right).
\end{aligned}
$$

This completes the proof. □

Corollary 3.3 *If we choose $\alpha = 1$, then under the assumption of Theorem 3.2, we have a new result*

$$
\begin{aligned}
&\left|\frac{(x-a)e^{\psi(a)}+(b-x)e^{\psi(b)}}{b-a} - \frac{1}{b-a}\int_a^b e^{\psi(x)}\,dx\right| \\
&\quad \le \frac{1}{12}\phi(x)\left(\frac{(x-a)^2+(b-x)^2}{b-a}\right) + \frac{1}{4}\left(\frac{(x-a)^2\phi(a)+(b-x)^2\phi(b)}{b-a}\right) \\
&\quad + \frac{1}{12}\left(\frac{(x-a)^2\Delta_1(x,a)+(b-x)^2\Delta_2(x,b)}{b-a}\right) - \frac{1}{30}\left(c_1\left|\psi'(x)\right|\|b-a\|^{\sigma_1}\right. \\
&\quad \left.+ c_2\left|e^{\psi(x)}\right|\|b-a\|^{\sigma_2}\right)\left(\frac{(x-a)^2+(b-x)^2}{b-a}\right) - \frac{1}{20}\left\{c_1\|b-a\|^{\sigma_1}\right. \\
&\quad \times \left(\frac{|\psi'(a)|(x-a)^{\alpha+1}+|\psi'(b)|(b-x)^{\alpha+1}}{b-a}\right) \\
&\quad \left.+ c_2\|b-a\|^{\sigma_2}\left(\frac{|e^{\psi(a)}|(x-a)^{\alpha+1}+|e^{\psi(b)}|(b-x)^{\alpha+1}}{b-a}\right)\right\} \\
&\quad + \frac{1}{60}c_1c_2\|b-a\|^{\sigma_1+\sigma_2}\left(\frac{(x-a)^2+(b-x)^2}{b-a}\right).
\end{aligned}
$$

Theorem 3.4 *Let $\psi : I \to \mathbb{R}$ be an absolutely continuous mapping on (a,b) such that $(e^{\psi})' \in L_1[a,b]$, where $a,b \in I$ with $a < b$. If the function $|\psi|^q$ is a strongly exponentially convex function of order $\sigma_1 > 1$ with modulus $c_1 > 0$ and $|\psi'|^q$ is a strongly convex function of order $\sigma_2 > 0$ with modulus $c_2 > 0$, where $\frac{1}{p} + \frac{1}{q} = 1$ with $q > 1$, then, we have*

$$\begin{aligned}
&\left|\frac{(x-a)^{\alpha}e^{\psi(a)} + (b-x)^{\alpha}e^{\psi(b)}}{b-a} - \frac{\Gamma(\alpha+1)[J_{x^-}^{\alpha}e^{\psi(a)} + J_{x^+}^{\alpha}e^{\psi(b)}]}{b-a}\right| \\
&\leq \left(\frac{1}{\alpha}\beta\left(p+1,\frac{1}{\alpha}\right)\right)^{\frac{1}{p}}\left[\frac{(x-a)^{\alpha+1}}{b-a}\left(\frac{\Delta_2(x,a)}{3} + \frac{\Delta_3(x,a)}{6}\right.\right. \\
&\quad - \frac{c_1\|b-a\|^{\sigma_1}(|\psi'(x)|^q + |\psi'(a)|^q)}{12} - \frac{c_2\|b-a\|^{\sigma_2}(|e^{\psi(x)}|^q + |e^{\psi(a)}|^q)}{12} \\
&\quad \left. + \frac{c_1c_2\|b-a\|^{\sigma_1+\sigma_2}}{30}\right)^{\frac{1}{q}} + \frac{(b-x)^{\alpha+1}}{b-a}\left(\frac{\Delta_2(x,b)}{3} + \frac{\Delta_3(x,b)}{6}\right. \\
&\quad - \frac{c_1\|b-a\|^{\sigma_1}(|\psi'(x)|^q + |\psi'(b)|^q)}{12} - \frac{c_2\|b-a\|^{\sigma_2}(|e^{\psi(x)}|^q + |e^{\psi(b)}|^q)}{12} \\
&\quad \left.\left. + \frac{c_1c_2\|b-a\|^{\sigma_1+\sigma_2}}{30}\right)^{\frac{1}{q}}\right],
\end{aligned}$$

where $\Delta_2(x,a) = |e^{\psi(x)}\psi'(x)|^q + |e^{\psi(a)}\psi'(a)|^q$, $\Delta_2(x,b) = |e^{\psi(x)}\psi'(x)|^q + |e^{\psi(b)}\psi'(b)|^q$ *and* $\Delta_3(x,a) = |e^{\psi(x)}\psi'(a)|^q + |e^{\psi(a)}\psi'(x)|^q$, $\Delta_3(x,b) = |e^{\psi(x)}\psi'(b)|^q + |e^{\psi(b)}\psi'(x)|^q$.

Proof Using Corollary 2.1, Hölder's inequality, and the given hypothesis of the theorem, we obtain

$$\begin{aligned}
&\left|\frac{(x-a)^{\alpha}e^{\psi(a)} + (b-x)^{\alpha}e^{\psi(b)}}{b-a} - \frac{\Gamma(\alpha+1)[J_{x^-}^{\alpha}e^{\psi(a)} + J_{x^+}^{\alpha}e^{\psi(b)}]}{b-a}\right| \\
&\leq \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 \left|\left(t^{\alpha}-1\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|e^{\psi(tx+(1-t)a)}\psi'\left(tx+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 \left|\left(1-t^{\alpha}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left|e^{\psi(tx+(1-t)b)}\psi'\left(tx+(1-t)b\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 \left|\left(1-t^{\alpha}\right)\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1 \left(1-t^{\alpha}\right)\left(t\left|e^{\psi(x)}\right|^q + (1-t)\left|e^{\psi(a)}\right|^q\right.\right. \\
&\quad \left. - c_1t(1-t)\|b-a\|^{\sigma_1}\right)\left(t\left|\psi'(x)\right|^q + (1-t)\left|\psi'(a)\right|^q - c_2t(1-t)\|b-a\|^{\sigma_2}\right) dt\Big)^{\frac{1}{q}} \\
&\quad + \frac{(b-x)^{\alpha+1}}{b-a}\left(\int_0^1 \left|\left(1-t^{\alpha}\right)\right|^p\right)^{\frac{1}{p}}\left(\int_0^1 \left(t\left|e^{\psi(x)}\right|^q + (1-t)\left|e^{\psi(b)}\right|^q\right.\right. \\
&\quad \left. - c_1t(1-t)\|b-a\|^{\sigma_1}\right)\left(t\left|\psi'(x)\right|^q + (1-t)\left|\psi'(b)\right|^q - c_2t(1-t)\|b-a\|^{\sigma_2}\right) dt\Big)^{\frac{1}{q}} \\
&= \frac{(x-a)^{\alpha+1}}{b-a}\left(\int_0^1 \left(1-t^{\alpha}\right)^p dt\right)^{\frac{1}{p}}\left[\left|e^{\psi(x)}\psi'(x)\right|^q \int_0^1 t^2\, dt + \left|e^{\psi(a)}\psi'(a)\right|^q\right. \\
&\quad \times \int_0^1 (1-t)^2\, dt + \left(\left|e^{\psi(x)}\psi'(a)\right|^q + \left|e^{\psi(a)}\psi'(x)\right|^q\right)\int_0^1 t(1-t)\, dt
\end{aligned}$$

$$
\begin{aligned}
&- \left(c_1 \left|\psi'(x)\right|^q \|b-a\|^{\sigma_1} + c_2 \left|e^{\psi(x)}\right|^q \|b-a\|^{\sigma_2}\right) \int_0^1 t^2(1-t)\,dt \\
&- \left(c_1 \left|\psi'(a)\right|^q \|b-a\|^{\sigma_1} + c_2 \left|e^{\psi(a)}\right|^q \|b-a\|^{\sigma_2}\right) \int_0^1 t(1-t)^2\,dt \\
&+ c_1 c_2 \|b-a\|^{\sigma_1+\sigma_2} \int_0^1 t^2(1-t)^2\,dt \Bigg]^{\frac{1}{q}} \\
&+ \frac{(b-x)^{\alpha+1}}{b-a} \left(\int_0^1 \left(1-t^\alpha\right)^p dt \right)^{\frac{1}{p}} \Bigg[\left|e^{\psi(x)}\psi'(x)\right|^q \int_0^1 t^2\,dt \\
&+ \left|e^{\psi(b)}\psi'(b)\right|^q \int_0^1 (1-t)^2\,dt + \left(\left|e^{\psi(x)}\psi'(b)\right|^q + \left|e^{\psi(b)}\psi'(x)\right|^q \right) \int_0^1 t(1-t)\,dt \\
&- \left(c_1 \left|\psi'(x)\right|^q \|b-a\|^{\sigma_1} + c_2 \left|e^{\psi(x)}\right|^q \|b-a\|^{\sigma_2}\right) \int_0^1 t^2(1-t)\,dt \\
&- \left(c_1 \left|\psi'(b)\right|^q \|b-a\|^{\sigma_1} + c_2 \left|e^{\psi(b)}\right|^q \|b-a\|^{\sigma_2}\right) \int_0^1 t(1-t)^2\,dt \\
&+ c_1 c_2 \|b-a\|^{\sigma_1+\sigma_2} \int_0^1 t^2(1-t)^2\,dt \Bigg]^{\frac{1}{q}} \\
= {}& \left(\int_0^1 \left(1-t^\alpha\right)^p dt \right)^{\frac{1}{p}} \Bigg[\frac{(x-a)^{\alpha+1}}{b-a} \Bigg(\frac{\Delta_2(x,a)}{3} + \frac{\Delta_3(x,a)}{6} \\
&- \frac{(c_1 |\psi'(x)|^q \|b-a\|^{\sigma_1} + c_2 |e^{\psi(x)}|^q \|b-a\|^{\sigma_2})}{12} \\
&- \frac{(c_1 |\psi'(a)|^q \|b-a\|^{\sigma_1} + c_2 |e^{\psi(a)}|^q \|b-a\|^{\sigma_2})}{12} + \frac{c_1 c_2 \|b-a\|^{\sigma_1+\sigma_2}}{30} \Bigg)^{\frac{1}{q}} \\
&+ \frac{(b-x)^{\alpha+1}}{b-a} \Bigg(\frac{\Delta_2(x,b)}{3} + \frac{\Delta_3(x,b)}{6} \\
&- \frac{(c_1 |\psi'(x)|^q \|b-a\|^{\sigma_1} + c_2 |e^{\psi(x)}|^q \|b-a\|^{\sigma_2})}{12} \\
&- \frac{(c_1 |\psi'(b)|^q \|b-a\|^{\sigma_1} + c_2 |e^{\psi(b)}|^q \|b-a\|^{\sigma_2})}{12} + \frac{c_1 c_2 \|b-a\|^{\sigma_1+\sigma_2}}{30} \Bigg)^{\frac{1}{q}} \Bigg] \\
= {}& \left(\frac{1}{\alpha} \beta\left(p+1, \frac{1}{\alpha}\right) \right)^{\frac{1}{p}} \Bigg[\frac{(x-a)^{\alpha+1}}{b-a} \Bigg(\frac{\Delta_2(x,a)}{3} + \frac{\Delta_3(x,a)}{6} \\
&- \frac{c_1 \|b-a\|^{\sigma_1} (|\psi'(x)|^q + |\psi'(a)|^q)}{12} - \frac{c_2 \|b-a\|^{\sigma_2} (|e^{\psi(x)}|^q + |e^{\psi(a)}|^q)}{12} \\
&+ \frac{c_1 c_2 \|b-a\|^{\sigma_1+\sigma_2}}{30} \Bigg)^{\frac{1}{q}} + \frac{(b-x)^{\alpha+1}}{b-a} \Bigg(\frac{\Delta_2(x,b)}{3} + \frac{\Delta_3(x,b)}{6} \\
&- \frac{c_1 \|b-a\|^{\sigma_1} (|\psi'(x)|^q + |\psi'(b)|^q)}{12} - \frac{c_2 \|b-a\|^{\sigma_2} (|e^{\psi(x)}|^q + |e^{\psi(b)}|^q)}{12} \\
&+ \frac{c_1 c_2 \|b-a\|^{\sigma_1+\sigma_2}}{30} \Bigg)^{\frac{1}{q}} \Bigg].
\end{aligned}
$$

□

Corollary 3.5 *If we choose* $\alpha = 1$, *then under the assumption of Theorem* 3.4, *we have a new result*

$$\begin{aligned}&\left|\frac{(x-a)e^{\psi(a)}+(b-x)e^{\psi(b)}}{b-a}-\frac{1}{b-a}\int_a^b e^{\psi(x)}\,dx\right|\\&\quad\le\left(\frac{1}{p+1}\right)^{\frac{1}{p}}\left[\frac{(x-a)^2}{b-a}\left(\frac{\Delta_2(x,a)}{3}+\frac{\Delta_3(x,a)}{6}\right.\right.\\&\qquad-\frac{c_1\|b-a\|^{\sigma_1}(|\psi'(x)|^q+|\psi'(a)|^q)}{12}-\frac{c_2\|b-a\|^{\sigma_2}(|e^{\psi(x)}|^q+|e^{\psi(a)}|^q)}{12}\\&\qquad\left.+\frac{c_1c_2\|b-a\|^{\sigma_1+\sigma_2}}{30}\right)^{\frac{1}{q}}+\frac{(b-x)^2}{b-a}\left(\frac{\Delta_2(x,b)}{3}+\frac{\Delta_3(x,b)}{6}\right.\\&\qquad-\frac{c_1\|b-a\|^{\sigma_1}(|\psi'(x)|^q+|\psi'(b)|^q)}{12}-\frac{c_2\|b-a\|^{\sigma_2}(|e^{\psi(x)}|^q+|e^{\psi(b)}|^q)}{12}\\&\qquad\left.\left.+\frac{c_1c_2\|b-a\|^{\sigma_1+\sigma_2}}{30}\right)^{\frac{1}{q}}\right].\end{aligned}$$

Theorem 3.6 *Let* $\alpha > 0$ *be a number and let* $\psi : I = [a,b] \longrightarrow \mathbb{R}$ *be a differentiable function on* (a,b). *If the function* $|\psi|$ *is a strongly exponentially convex function of order* $\sigma_1 > 1$ *with modulus* $c_1 > 0$ *and* $|\psi'|$ *is a strongly convex function of order* $\sigma_2 > 0$ *with modulus* $c_2 > 0$, *then, we have*

$$\begin{aligned}&\left|\Gamma_\psi(a,b,\alpha)\right|\\&\quad\le\frac{b-a}{16}\left[\frac{\alpha^3+9\alpha^2+20\alpha+6}{3(\alpha+1)(\alpha+2)(\alpha+3)}\left(\left|e^{\psi(\frac{3a+b}{4})}\psi'\left(\frac{3a+b}{4}\right)\right|\right.\right.\\&\qquad\left.+\left|e^{\psi(\frac{a+3b}{4})}\psi'\left(\frac{a+3b}{4}\right)\right|\right)+\frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}\left|e^{\psi(a)}\psi'(a)\right|\\&\qquad+\frac{\alpha^3+3\alpha^2+2\alpha+6}{3(\alpha+1)(\alpha+2)(\alpha+3)}\left|e^{\psi(\frac{a+b}{2})}\psi'\left(\frac{a+b}{2}\right)\right|+\frac{\alpha}{3(\alpha+3)}\left|e^{\psi(b)}\psi'(b)\right|\\&\qquad+\frac{(A_1(a,b)+A_6(a,b))}{(\alpha+2)(\alpha+3)}-\frac{\alpha^3+9\alpha^2+38\alpha+24}{12(\alpha+2)(\alpha+3)(\alpha+4)}\big(A_2(a,b)\\&\qquad+A_7(a,b)\big)-\frac{2A_3(a,b)}{(\alpha+2)(\alpha+3)(\alpha+4)}-\frac{\alpha^3+9\alpha^2+14\alpha+24}{12(\alpha+2)(\alpha+3)(\alpha+4)}A_5(a,b)\\&\qquad+\frac{\alpha(\alpha+5)}{6(\alpha+2)(\alpha+3)}\big(A_4(a,b)+A_8(a,b)\big)-\frac{\alpha(\alpha+7)}{12(\alpha+3)(\alpha+4)}A_9(a,b)\\&\qquad\left.+\frac{4(\alpha^3+12\alpha^2+47\alpha+30)}{30(\alpha+3)(\alpha+4)(\alpha+5)}c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2}\right],\end{aligned}\tag{3.5}$$

where

$$A_1(a,b)=\left|e^{\psi(a)}\psi'\left(\frac{3a+b}{4}\right)\right|+\left|e^{\psi(\frac{3a+b}{4})}\psi'(a)\right|,$$

$$A_2(a,b)=c_1\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\left|\psi'\left(\frac{3a+b}{4}\right)\right|+c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\left|e^{\psi(\frac{3a+b}{4})}\right|,$$

$$A_3(a,b)=c_1\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\left|\psi'(a)\right|+c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\left|e^{\psi(a)}\right|,$$

$$A_4(a,b) = \left|e^{\psi(\frac{3a+b}{4})}\psi'\left(\frac{a+b}{2}\right)\right| + \left|e^{\psi(\frac{a+b}{2})}\psi'\left(\frac{3a+b}{4}\right)\right|,$$
$$A_5(a,b) = c_1\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\left|\psi'\left(\frac{a+b}{2}\right)\right| + c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\left|e^{\psi(\frac{a+b}{2})}\right|,$$
$$A_6(a,b) = \left|e^{\psi(\frac{a+b}{2})}\psi'\left(\frac{a+3b}{4}\right)\right| + \left|e^{\psi(\frac{a+3b}{4})}\psi'\left(\frac{a+b}{2}\right)\right|,$$
$$A_7(a,b) = c_1\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\left|\psi'\left(\frac{a+3b}{4}\right)\right| + c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\left|e^{\psi(\frac{a+3b}{4})}\right|,$$
$$A_8(a,b) = \left|e^{\psi(\frac{a+3b}{4})}\psi'(b)\right| + \left|e^{\psi(b)}\psi'\left(\frac{a+3b}{4}\right)\right|$$

and

$$A_9(a,b) = c_1\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\left|\psi'(b)\right| + c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\left|e^{\psi(b)}\right|.$$

Proof Using Lemma 2.2 and the property of modulus, we have

$$\begin{aligned}
&\left|\Gamma_\psi(a,b,\alpha)\right| \\
&\leq \frac{b-a}{16}\left[\int_0^1 t^\alpha\left|e^{\psi(t\frac{3a+b}{4}+(1-t)a)}\psi'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|dt\right. \\
&\quad+\int_0^1(1-t^\alpha)\left|e^{\psi(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4})}\psi'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|dt \\
&\quad+\int_0^1 t^\alpha\left|e^{\psi(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2})}\psi'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|dt \\
&\quad\left.+\int_0^1(1-t^\alpha)\left|e^{\psi(tb+(1-t)\frac{a+3b}{4})}\psi'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|dt\right].
\end{aligned}$$

This implies

$$\left|\Gamma_\psi(a,b,\alpha)\right| \leq \frac{b-a}{16}\sum_{i=1}^4 I_i, \tag{3.6}$$

where

$$I_1 = \int_0^1 t^\alpha\left|e^{\psi(t\frac{3a+b}{4}+(1-t)a)}\psi'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|dt.$$

Applying the given hypothesis of the theorem, we obtain

$$\begin{aligned}
I_1 &\leq \int_0^1 t^\alpha\left(t\left|e^{\psi(\frac{3a+b}{4})}\right| + (1-t)\left|e^{\psi(a)}\right| - c_1t(1-t)\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\right) \\
&\quad\times\left(t\left|\psi'\left(\frac{3a+b}{4}\right)\right| + (1-t)\left|\psi'(a)\right| - c_2t(1-t)\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\right)dt \\
&= \left|e^{\psi(\frac{3a+b}{4})}\psi'\left(\frac{3a+b}{4}\right)\right|\int_0^1 t^{\alpha+2}\,dt + \left|e^{\psi(a)}\psi'(a)\right|\int_0^1 t^\alpha(1-t)^2\,dt
\end{aligned}$$

$$
\begin{aligned}
&+\left(\left|e^{\psi(a)}\psi'\left(\frac{3a+b}{4}\right)\right|+\left|e^{\psi(\frac{3a+b}{4})}\psi'(a)\right|\right)\int_0^1 t^{\alpha+1}(1-t)\,dt\\
&-\left(c_1\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\left|\psi'\left(\frac{3a+b}{4}\right)\right|+c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\left|e^{\psi(\frac{3a+b}{4})}\right|\right)\\
&\times\int_0^1 t^{\alpha+2}(1-t)\,dt-\left(c_1\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\left|\psi'(a)\right|+c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\left|e^{\psi(a)}\right|\right)\\
&\times\int_0^1 t^{\alpha+1}(1-t)^2\,dt+c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2}\int_0^1 t^{\alpha+2}(1-t)^2\,dt\\
&=\frac{1}{\alpha+3}\left|e^{\psi(\frac{3a+b}{4})}\psi'\left(\frac{3a+b}{4}\right)\right|+\frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}\left|e^{\psi(a)}\psi'(a)\right|\\
&+\frac{1}{(\alpha+2)(\alpha+3)}A_1(a,b)-\frac{1}{(\alpha+3)(\alpha+4)}A_2(a,b)\\
&-\frac{2}{(\alpha+2)(\alpha+3)(\alpha+4)}A_3(a,b)\\
&+\frac{2}{(\alpha+3)(\alpha+4)(\alpha+5)}c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2},
\end{aligned}
\tag{3.7}
$$

$$
\begin{aligned}
I_2&=\int_0^1\left(1-t^\alpha\right)\left|e^{\psi(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4})}\psi'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|dt\\
&\le\int_0^1\left(1-t^\alpha\right)\left(t\left|e^{\psi(\frac{a+b}{2})}\right|+(1-t)\left|e^{\psi(\frac{3a+b}{4})}\right|-c_1t(1-t)\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\right)\\
&\quad\times\left(t\left|\psi'\left(\frac{a+b}{2}\right)\right|+(1-t)\left|\psi'\left(\frac{3a+b}{4}\right)\right|-c_2t(1-t)\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\right)dt\\
&=\frac{\alpha}{3(\alpha+3)}\left|e^{\psi(\frac{a+b}{2})}\psi'\left(\frac{a+b}{2}\right)\right|\\
&\quad+\frac{\alpha(\alpha^2+6\alpha+11)}{3(\alpha+1)(\alpha+2)(\alpha+3)}\left|e^{\psi(\frac{3a+b}{4})}\psi'\left(\frac{3a+b}{4}\right)\right|\\
&\quad+\frac{\alpha(\alpha+5)}{6(\alpha+2)(\alpha+3)}A_4(a,b)-\frac{\alpha(\alpha+7)}{12(\alpha+3)(\alpha+4)}A_5(a,b)\\
&\quad-\frac{\alpha(\alpha^2+9\alpha+26)}{12(\alpha+2)(\alpha+3)(\alpha+4)}A_2(a,b)\\
&\quad+\frac{2\alpha(\alpha^2+12\alpha+47)}{30(\alpha+3)(\alpha+4)(\alpha+5)}c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2},
\end{aligned}
\tag{3.8}
$$

$$
\begin{aligned}
I_3&=\int_0^1 t^\alpha\left|e^{\psi(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2})}\psi'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|dt\\
&\le\int_0^1 t^\alpha\left(t\left|e^{\psi(\frac{a+3b}{4})}\right|+(1-t)\left|e^{\psi(\frac{a+b}{2})}\right|-c_1t(1-t)\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\right)\\
&\quad\times\left(t\left|\psi'\left(\frac{a+3b}{4}\right)\right|+(1-t)\left|\psi'\left(\frac{a+b}{2}\right)\right|-c_2t(1-t)\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\right)dt\\
&=\frac{1}{\alpha+3}\left|e^{\psi(\frac{a+3b}{4})}\psi'\left(\frac{a+3b}{4}\right)\right|\\
&\quad+\frac{2}{(\alpha+1)(\alpha+2)(\alpha+3)}\left|e^{(\frac{a+b}{2})}\psi'\left(\frac{a+b}{2}\right)\right|
\end{aligned}
$$

$$+ \frac{A_6(a,b)}{(\alpha+2)(\alpha+3)} - \frac{A_7(a,b)}{(\alpha+3)(\alpha+4)} - \frac{2A_5(a,b)}{(\alpha+2)(\alpha+3)(\alpha+4)} + \frac{2}{(\alpha+3)(\alpha+4)(\alpha+5)} c_1 c_2 \left\| \frac{(b-a)}{4} \right\|^{\sigma_1+\sigma_2} \tag{3.9}$$

and

$$\begin{aligned} I_4 &= \int_0^1 (1-t^\alpha) \left| e^{\psi(tb+(1-t)\frac{a+3b}{4})} \psi'\left(tb+(1-t)\frac{a+3b}{4}\right) \right| dt \\ &\le \int_0^1 (1-t^\alpha) \left(t\left|e^{\psi(b)}\right| + (1-t)\left|e^{\psi(\frac{a+3b}{4})}\right| - c_1 t(1-t) \left\| \frac{(b-a)}{4} \right\|^{\sigma_1} \right) \\ &\quad \times \left(t\left|\psi'(b)\right| + (1-t)\left|\psi'\left(\frac{a+3b}{4}\right)\right| - c_2 t(1-t) \left\| \frac{(b-a)}{4} \right\|^{\sigma_2} \right) dt \\ &= \frac{\alpha}{3(\alpha+3)} \left|e^{\psi(b)}\psi'(b)\right| + \frac{\alpha(\alpha^2+6\alpha+11)}{3(\alpha+1)(\alpha+2)(\alpha+3)} \left| e^{\psi(\frac{a+3b}{4})} \psi'\left(\frac{a+3b}{4}\right) \right| \\ &\quad + \frac{\alpha(\alpha+5)}{6(\alpha+2)(\alpha+3)} A_8(a,b) - \frac{\alpha(\alpha+7)}{12(\alpha+3)(\alpha+4)} A_9(a,b) \\ &\quad - \frac{\alpha(\alpha^2+9\alpha+26)}{12(\alpha+2)(\alpha+3)(\alpha+4)} A_7(a,b) \\ &\quad + \frac{2\alpha(\alpha^2+12\alpha+47)}{30(\alpha+3)(\alpha+4)(\alpha+5)} c_1 c_2 \left\| \frac{(b-a)}{4} \right\|^{\sigma_1+\sigma_2}. \end{aligned} \tag{3.10}$$

Substituting (3.7), (3.8), (3.9), and (3.10) into (3.6), we obtain the desired inequality (3.5). Hence, the proof is completed. □

Corollary 3.7 *If we choose* $\alpha = 1$, *then under the assumption of Theorem* 3.6, *we have a new result*

$$\begin{aligned} &\left| \frac{1}{2}\left[e^{\psi(\frac{3a+b}{4})} + e^{\psi(\frac{a+3b}{4})} \right] - \frac{1}{b-a} \int_a^b e^{\psi(x)}\,dx \right| \\ &\quad \le \frac{b-a}{16} \left[\frac{1}{2} \left(\left| e^{\psi(\frac{3a+b}{4})} \psi'\left(\frac{3a+b}{4}\right) \right| + \left| e^{\psi(\frac{a+3b}{4})} \psi'\left(\frac{a+3b}{4}\right) \right| \right) \right. \\ &\quad + \frac{1}{12}\left|e^{\psi(a)}\psi'(a)\right| + \frac{1}{6}\left| e^{\psi(\frac{a+b}{2})} \psi'\left(\frac{a+b}{2}\right) \right| + \frac{1}{12}\left|e^{\psi(b)}\psi'(b)\right| \\ &\quad + \frac{1}{12}\big(A_1(a,b)+A_6(a,b)\big) - \frac{1}{10}\big(A_2(a,b)+A_7(a,b)\big) - \frac{1}{30}A_3(a,b) \\ &\quad - \frac{1}{15}A_5(a,b) + \frac{1}{12}\big(A_4(a,b)+A_8(a,b)\big) - \frac{1}{30}A_9(a,b) \\ &\quad \left. + \frac{1}{10} c_1 c_2 \left\| \frac{(b-a)}{4} \right\|^{\sigma_1+\sigma_2} \right]. \end{aligned}$$

Theorem 3.8 *Let* $\alpha > 0$ *be a number and let* $\psi : I = [a,b] \longrightarrow \mathbb{R}$ *be a differentiable function on* (a,b). *If the function* $|\psi|^q$ *is a strongly exponentially convex function of order* $\sigma_1 > 1$ *with modulus* $c_1 > 0$ *and* $|\psi'|^q$ *is a strongly convex function of order* $\sigma_2 > 0$ *with modulus* $c_2 > 0$, *where* $\frac{1}{p}+\frac{1}{q}=1$, $q>1$, *then*

$$\begin{aligned}
&\left|\Gamma_{\psi}(a,b,\alpha)\right| \\
&\leq \frac{b-a}{16\times 60^{\frac{1}{q}}}\left(\frac{1}{\alpha}\right)^{\frac{1}{p}}\Bigg[\left(\frac{\alpha}{1+p\alpha}\right)^{\frac{1}{p}}\Bigg\{\Bigg(20\Bigg(\left|e^{\psi(\frac{3a+b}{4})}\psi'\left(\frac{3a+b}{4}\right)\right|^{q} \\
&\quad+\left|e^{\psi(a)}\psi'(a)\right|^{q}\Bigg)+10B_1(a,b)-5B_2(a,b)+2c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2}\Bigg)^{\frac{1}{q}} \\
&\quad+\Bigg(20\Bigg(\left|e^{\psi(\frac{a+3b}{4})}\psi'\left(\frac{a+3b}{4}\right)\right|^{q}+\left|e^{\psi(\frac{a+b}{2})}\psi'\left(\frac{a+b}{2}\right)\right|^{q}\Bigg)+10B_3(a,b) \\
&\quad-5B_4(a,b)+2c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2}\Bigg)^{\frac{1}{q}}\Bigg\}+\left(\beta\left(p+1,\frac{1}{\alpha}\right)\right)^{\frac{1}{p}} \\
&\quad\times\Bigg\{\Bigg(20\Bigg(\left|e^{\psi(\frac{a+b}{2})}\psi'\left(\frac{a+b}{2}\right)\right|^{q}+\left|e^{\psi(\frac{3a+b}{4})}\psi'\left(\frac{3a+b}{4}\right)\right|^{q}\Bigg)+10B_5(a,b) \\
&\quad-5B_6(a,b)+2c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2}\Bigg)^{\frac{1}{q}}+\Bigg(20\Bigg(\left|e^{\psi(b)}\psi'(b)\right|^{q} \\
&\quad+\left|e^{\psi(\frac{a+3b}{4})}\psi'\left(\frac{a+3b}{4}\right)\right|^{q}\Bigg)+10B_7(a,b)-5B_8(a,b) \\
&\quad+2c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2}\Bigg)^{\frac{1}{q}}\Bigg\}\Bigg],
\end{aligned} \tag{3.11}$$

where

$$\begin{aligned}
B_1(a,b)&=\left|e^{\psi(a)}\psi'\left(\frac{3a+b}{4}\right)\right|^{q}+\left|e^{\psi(\frac{3a+b}{4})}\psi'(a)\right|^{q}, \\
B_2(a,b)&=c_1\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\left(\left|\psi'\left(\frac{3a+b}{4}\right)\right|^{q}+\left|\psi'(a)\right|^{q}\right) \\
&\quad+c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\left(\left|e^{\psi(\frac{3a+b}{4})}\right|^{q}+\left|e^{\psi(a)}\right|^{q}\right), \\
B_3(a,b)&=\left|e^{\psi(\frac{a+b}{2})}\psi'\left(\frac{a+3b}{4}\right)\right|^{q}+\left|e^{\psi(\frac{a+3b}{4})}\psi'\left(\frac{a+b}{2}\right)\right|^{q}, \\
B_4(a,b)&=c_1\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\left(\left|\psi'\left(\frac{a+3b}{4}\right)\right|^{q}+\left|\psi'\left(\frac{a+b}{2}\right)\right|^{q}\right) \\
&\quad+c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\left(\left|e^{\psi(\frac{a+3b}{4})}\right|^{q}+\left|e^{\psi(\frac{a+b}{2})}\right|^{q}\right), \\
B_5(a,b)&=\left|e^{\psi(\frac{3a+b}{4})}\psi'\left(\frac{a+b}{2}\right)\right|^{q}+\left|e^{\psi(\frac{a+b}{2})}\psi'\left(\frac{3a+b}{4}\right)\right|^{q}, \\
B_6(a,b)&=c_1\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\left(\left|\psi'\left(\frac{a+b}{2}\right)\right|^{q}+\left|\psi'\left(\frac{3a+b}{4}\right)\right|^{q}\right) \\
&\quad+c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\left(\left|e^{\psi(\frac{a+b}{2})}\right|^{q}+\left|e^{\psi(\frac{3a+b}{4})}\right|^{q}\right), \\
B_7(a,b)&=\left|e^{\psi(\frac{a+3b}{4})}\psi'(b)\right|^{q}+\left|e^{\psi(b)}\psi'\left(\frac{a+3b}{4}\right)\right|^{q}
\end{aligned}$$

and

$$B_8(a,b) = c_1 \left\| \frac{(b-a)}{4} \right\|^{\sigma_1} \left(\left|\psi'(b)\right|^q + \left|\psi'\left(\frac{a+3b}{4}\right)\right|^q \right) + c_2 \left\| \frac{(b-a)}{4} \right\|^{\sigma_2} \left(\left|e^{\psi(b)}\right|^q + \left|e^{\psi(\frac{a+3b}{4})}\right|^q \right).$$

Proof Using Lemma 2.2 and Hölder's inequality, we have

$$\begin{aligned}
&\left|\Gamma_\psi(a,b,\alpha)\right| \\
&\leq \frac{b-a}{16}\left[\left(\int_0^1 (t^\alpha)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|e^{\psi(t\frac{3a+b}{4}+(1-t)a)}\psi'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt\right)^{\frac{1}{q}}\right. \\
&\quad + \left(\int_0^1 (1-t^\alpha)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|e^{\psi(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4})}\psi'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + \left(\int_0^1 (t^\alpha)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|e^{\psi(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2})}\psi'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad \left. + \left(\int_0^1 (1-t^\alpha)^p\,dt\right)^{\frac{1}{p}}\left(\int_0^1\left|e^{\psi(tb+(1-t)\frac{a+3b}{4})}\psi'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q dt\right)^{\frac{1}{q}}\right].
\end{aligned}$$

This implies

$$\begin{aligned}
&\left|\Gamma_\psi(a,b,\alpha)\right| \\
&\leq \frac{b-a}{16}\left[\left(\int_0^1 (t^\alpha)^p\,dt\right)^{\frac{1}{p}}\left(\sum_{r=1}^{2} J_r^{\frac{1}{q}}\right) + \left(\int_0^1 (1-t^\alpha)^p\,dt\right)^{\frac{1}{p}}\left(\sum_{r=3}^{4} J_r^{\frac{1}{q}}\right)\right],
\end{aligned} \tag{3.12}$$

where

$$J_1 = \int_0^1 \left|e^{\psi(t\frac{3a+b}{4}+(1-t)a)}\psi'\left(t\frac{3a+b}{4}+(1-t)a\right)\right|^q dt.$$

Applying the given hypothesis of the theorem, we obtain

$$\begin{aligned}
J_1 &\leq \int_0^1 \left(t\left|e^{\psi(\frac{3a+b}{4})}\right|^q + (1-t)\left|e^{\psi(a)}\right|^q - c_1 t(1-t)\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\right) \\
&\quad \times \left(t\left|\psi'\left(\frac{3a+b}{4}\right)\right|^q + (1-t)\left|\psi'(a)\right|^q - c_2 t(1-t)\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\right) dt \\
&= \left|e^{\psi(\frac{3a+b}{4})}\psi'\left(\frac{3a+b}{4}\right)\right|^q \int_0^1 t^2\,dt + \left|e^{\psi(a)}\psi'(a)\right|^q \int_0^1 (1-t)^2\,dt \\
&\quad + \left(\left|e^{\psi(a)}\psi'\left(\frac{3a+b}{4}\right)\right|^q + \left|e^{\psi(\frac{3a+b}{4})}\psi'(a)\right|^q\right)\int_0^1 t(1-t)\,dt \\
&\quad - \left(c_1\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\left|\psi'\left(\frac{3a+b}{4}\right)\right|^q + c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\left|e^{\psi(\frac{3a+b}{4})}\right|^q\right) \\
&\quad \times \int_0^1 t^2(1-t)\,dt - \left(c_1\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\left|\psi'(a)\right|^q + c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\left|e^{\psi(a)}\right|^q\right)
\end{aligned}$$

$$
\begin{aligned}
&\times \int_0^1 t(1-t)^2\,dt + c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2}\int_0^1 t^2(1-t)^2\,dt \\
&= \frac{1}{3}\left|e^{\psi(\frac{3a+b}{4})}\psi'\left(\frac{3a+b}{4}\right)\right|^q + \frac{1}{3}\left|e^{\psi(a)}\psi'(a)\right|^q + \frac{1}{6}\left(\left|e^{\psi(a)}\psi'\left(\frac{3a+b}{4}\right)\right|^q \right.\\
&\left.\quad + \left|e^{\psi(\frac{3a+b}{4})}\psi'(a)\right|^q\right) - \frac{1}{12}\left(c_1\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\left|\psi'\left(\frac{3a+b}{4}\right)\right|^q\right.\\
&\left.\quad + c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\left|e^{\psi(\frac{3a+b}{4})}\right|^q\right) - \frac{1}{12}\left(c_1\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\left|\psi'(a)\right|^q\right.\\
&\left.\quad + c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\left|e^{\psi(a)}\right|^q\right) + \frac{1}{30}c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2} \\
&= \frac{1}{60}\left[20\left(\left|e^{\psi(\frac{3a+b}{4})}\psi'\left(\frac{3a+b}{4}\right)\right|^q + \left|e^{\psi(a)}\psi'(a)\right|^q\right) + 10B_1(a,b)\right.\\
&\left.\quad - 5B_2(a,b) + 2c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2}\right],
\end{aligned} \tag{3.13}
$$

$$
\begin{aligned}
J_2 &= \int_0^1 \left|e^{\psi(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2})}\psi'\left(t\frac{a+3b}{4}+(1-t)\frac{a+b}{2}\right)\right|^q dt \\
&\le \int_0^1 \left(t\left|e^{\psi(\frac{a+3b}{4})}\right|^q + (1-t)\left|e^{\psi(\frac{a+b}{2})}\right|^q - c_1t(1-t)\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\right)\\
&\quad \times \left(t\left|\psi'\left(\frac{a+3b}{4}\right)\right|^q + (1-t)\left|\psi'\left(\frac{a+b}{2}\right)\right|^q - c_2t(1-t)\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\right)dt \\
&= \frac{1}{60}\left[20\left(\left|e^{\psi(\frac{a+3b}{4})}\psi'\left(\frac{a+3b}{4}\right)\right|^q + \left|e^{\psi(\frac{a+b}{2})}\psi'\left(\frac{a+b}{2}\right)\right|^q\right)\right.\\
&\left.\quad + 10B_3(a,b) - 5B_4(a,b) + 2c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2}\right],
\end{aligned} \tag{3.14}
$$

$$
\begin{aligned}
J_3 &= \int_0^1 \left|e^{\psi(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4})}\psi'\left(t\frac{a+b}{2}+(1-t)\frac{3a+b}{4}\right)\right|^q dt \\
&\le \int_0^1 \left(t\left|e^{\psi(\frac{a+b}{2})}\right|^q + (1-t)\left|e^{\psi(\frac{3a+b}{4})}\right|^q - c_1t(1-t)\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\right)\\
&\quad \times \left(t\left|\psi'\left(\frac{a+b}{2}\right)\right|^q + (1-t)\left|\psi'\left(\frac{3a+b}{4}\right)\right|^q - c_2t(1-t)\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\right)dt \\
&= \frac{1}{60}\left[20\left(\left|e^{\psi(\frac{a+b}{2})}\psi'\left(\frac{a+b}{2}\right)\right|^q + \left|e^{\psi(\frac{3a+b}{4})}\psi'\left(\frac{3a+b}{4}\right)\right|^q\right)\right.\\
&\left.\quad + 10B_5(a,b) - 5B_6(a,b) + 2c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2}\right]
\end{aligned} \tag{3.15}
$$

and

$$
\begin{aligned}
J_4 &= \int_0^1 \left|e^{\psi(tb+(1-t)\frac{a+3b}{4})}\psi'\left(tb+(1-t)\frac{a+3b}{4}\right)\right|^q dt \\
&\le \int_0^1 \left(t\left|e^{\psi(b)}\right|^q + (1-t)\left|e^{\psi(\frac{a+3b}{4})}\right|^q - c_1t(1-t)\left\|\frac{(b-a)}{4}\right\|^{\sigma_1}\right)\\
&\quad \times \left(t\left|\psi'(b)\right|^q + (1-t)\left|\psi'\left(\frac{a+3b}{4}\right)\right|^q - c_2t(1-t)\left\|\frac{(b-a)}{4}\right\|^{\sigma_2}\right)dt
\end{aligned}
$$

$$= \frac{1}{60}\Bigg[20\bigg(\left|e^{\psi(b)}\psi'(b)\right|^q + \left|e^{\psi(\frac{a+3b}{4})}\psi'\left(\frac{a+3b}{4}\right)\right|^q\bigg) + 10B_7(a,b) - 5B_8(a,b) + 2c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2}\Bigg]. \tag{3.16}$$

Substituting (3.13), (3.14), (3.15), and (3.16) into (3.12), we obtain the desired inequality (3.11). This completes the proof. □

Remark 3.2 When $c_1, c_2 = 0$, the above theorem reduces to Theorem 2.2 of [35]. If $c_1, c_2 = 0$ and $\alpha = 1$, the above theorem reduces to Corollary 2.2 of [35].

Corollary 3.9 *If we choose $\alpha = 1$, then under the assumption of Theorem 3.8, we have a new result*

$$\begin{aligned}
&\left|\frac{1}{2}\left[e^{\psi(\frac{3a+b}{4})} + e^{\psi(\frac{a+3b}{4})}\right] - \frac{1}{b-a}\int_a^b e^{\psi(x)}\,dx\right| \\
&\le \frac{b-a}{16\times 60^{\frac{1}{q}}}\left(\frac{1}{1+p}\right)^{\frac{1}{p}}\Bigg[\bigg\{20\bigg(\left|e^{\psi(\frac{3a+b}{4})}\psi'\left(\frac{3a+b}{4}\right)\right|^q + \left|e^{\psi(a)}\psi'(a)\right|^q\bigg) \\
&\quad + 10B_1(a,b) - 5B_2(a,b) + 2c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2}\bigg\}^{\frac{1}{q}} \\
&\quad + \bigg\{20\bigg(\left|e^{\psi(\frac{a+3b}{4})}\psi'\left(\frac{a+3b}{4}\right)\right|^q + \left|e^{\psi(\frac{a+b}{2})}\psi'\left(\frac{a+b}{2}\right)\right|^q\bigg) \\
&\quad + 10B_3(a,b) - 5B_4(a,b) + 2c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2}\bigg\}^{\frac{1}{q}} \\
&\quad + \bigg\{20\bigg(\left|e^{\psi(\frac{a+b}{2})}\psi'\left(\frac{a+b}{2}\right)\right|^q + \left|e^{\psi(\frac{3a+b}{4})}\psi'\left(\frac{3a+b}{4}\right)\right|^q\bigg) \\
&\quad + 10B_5(a,b) - 5B_6(a,b) + 2c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2}\bigg\}^{\frac{1}{q}} \\
&\quad + \bigg\{20\bigg(\left|e^{\psi(b)}\psi'(b)\right|^q + \left|e^{\psi(\frac{a+3b}{4})}\psi'\left(\frac{a+3b}{4}\right)\right|^q\bigg) + 10B_7(a,b) \\
&\quad - 5B_8(a,b) + 2c_1c_2\left\|\frac{(b-a)}{4}\right\|^{\sigma_1+\sigma_2}\bigg\}^{\frac{1}{q}}\Bigg].
\end{aligned}$$

4 Conclusion

In this paper, we have studied the concept of higher-order strongly exponentially convex functions that is the generalization of the concept of strongly exponentially convex functions. We have proved the Hermite–Hadamard inequality for higher-order strongly exponentially convex functions. Further, we have combined the concept of inequalities with fractional integral operators. By using Riemann–Liouville fractional integrals, we have established some integral inequalities for strongly exponentially convex functions of higher order. The results obtained in this paper are the generalization and extension of previously known results. The method followed to derive fractional inequalities for these generalized strongly exponentially convex functions is innovative and simple. It could be followed to generalize and extend further consequences for other kinds of convexities using generalized fractional integral operators.

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