Mathematische Annalen
https://doi.org/10.1007/s00208-023-02608-8

Mathematische Annalen

Intrinsic Diophantine approximation for overlapping iterated function systems

Simon Baker[1]

Received: 4 May 2021 / Revised: 6 March 2023 / Accepted: 11 March 2023

Abstract

In this paper we study a family of limsup sets that are defined using iterated function systems. Our main result is an analogue of Khintchine's theorem for these sets. We then apply this result to the topic of intrinsic Diophantine Approximation on self-similar sets. In particular, we define a new height function for an element of $\mathbb{Q}^d$ contained in a self-similar set in terms of its eventually periodic representations. For limsup sets defined with respect to this height function, we obtain a detailed description of their metric properties. The results of this paper hold in arbitrary dimensions and without any separation conditions on the underlying iterated function system.

Mathematics Subject Classification 11J83 · 11K55 · 28A80

1 Introduction

Diophantine Approximation is the study of approximations of vectors in $\mathbb{R}^d$ by elements of $\mathbb{Q}^d$. Given a set $X \subset \mathbb{R}^d$, it is natural to wonder how well elements of X can be approximated by elements of $\mathbb{Q}^d$ contained within X. Similarly, it is natural to wonder how well elements of X can be approximated by elements of $\mathbb{Q}^d$ lying outside of X. These two questions are the motivation behind the topics of intrinsic Diophantine Approximation and extrinsic Diophantine Approximation respectively. Often the set X is taken to be a smooth manifold or a fractal set. A tremendous amount of work has been done on these two topics when X is taken to be such a set. For further details we refer the reader to the papers [7, 9–11, 13, 14, 16–19, 26, 27, 34, 35, 37, 39, 40] and the references therein. In this paper we study intrinsic Diophantine Approximation when the set X is a self-similar set. We will provide a more thorough introduction to this topic in Sect. 2. The main result of this paper is a general theorem on the metric prop-

✉ Simon Baker
simonbaker412@gmail.com

1 Department of Mathematical Sciences, Loughborough University, Loughborough LE11 3TU, UK

Published online: 22 March 2023

erties of a family of limsup sets defined using iterated function systems. As we will see, this theorem implies a number of results in intrinsic Diophantine Approximation.

In what remains of this introductory section we will provide the relevant background from Fractal Geometry and state Theorem 1.1, which is our main result. In Sect. 2 we will show how Theorem 1.1 can be used to obtain a number of results for intrinsic Diophantine Approximation on self-similar sets. In Sect. 3 we will prove Theorem 1.1. In Sect. 4 we will apply the mass transference principle of Beresnevich and Velani together with Theorem 1.1 to deduce further results on the Hausdorff measure of certain limsup sets.

1.1 Background from Fractal Geometry

We call a map $\phi : \mathbb{R}^d \to \mathbb{R}^d$ a similarity if there exists $r \in (0, 1)$ such that $\|\phi(x) - \phi(y)\| = r\|x - y\|$ for all $x, y \in \mathbb{R}^d$. We call a finite set of similarities an iterated function system or IFS for short. An important result due to Hutchinson [25] states that for any IFS $\Phi = \{\phi_a\}_{a\in\mathcal{A}}$, there exists a unique non-empty compact set X satisfying

$$X = \bigcup_{a\in\mathcal{A}} \phi_a(X).$$

X is called the self-similar set of Φ. When the elements of Φ all have the same contraction ratio, i.e. $r_a = r_{a'}$ for all $a, a' \in \mathcal{A}$, then we say that an IFS is equicontractive. Importantly we can view X as the image of $\mathcal{A}^{\mathbb{N}}$ under an appropriate projection map: Let $\pi : \mathcal{A}^{\mathbb{N}} \to X$ be given by

$$\pi\left((a_n)_{n=1}^{\infty}\right) = \lim_{n\to\infty} \left(\phi_{a_1} \circ \cdots \circ \phi_{a_n}\right)(0).$$

Here 0 can be replaced with any other vector in $\mathbb{R}^d$. Importantly the map π is surjective and continuous (when $\mathcal{A}^{\mathbb{N}}$ is equipped with the product topology). Given an IFS $\Phi = \{\phi_a\}_{a\in\mathcal{A}}$ we define the similarity dimension of Φ to be the unique solution to the equation

$$\sum_{a\in\mathcal{A}} r_a^s = 1.$$

We denote the similarity dimension of an IFS Φ by $\dim_S(\Phi)$. Notice that if Φ is equicontractive then $\dim_S(\Phi) = \frac{\log \#\mathcal{A}}{-\log r}$ where r is the common contraction ratio. It is well known that the Hausdorff dimension of a self-similar set X always satisfies the following upper bound:

$$\dim_H(X) \leq \min\{\dim_S(\Phi), d\}. \tag{1.1}$$

For many iterated function systems this inequality is in fact an equality, see [15, 22, 23, 33]. We say that Φ satisfies the strong separation condition if $\phi_a(X) \cap \phi_{a'}(X) = \emptyset$ for all $a, a' \in \mathcal{A}$ such that $a \neq a'$. An IFS Φ is said to satisfy the open set condition if there exists a bounded open set O such that $\phi_a(O) \subset O$ for all $a \in \mathcal{A}$, and $\phi_a(O) \cap \phi_{a'}(O) = \emptyset$ whenever $a \neq a'$. It is known that the strong separation condition implies the open set condition, and that under either of these assumptions we have equality in (1.1).

To prove equality in (1.1) in the overlapping case one often uses self-similar measures. These are defined as follows: Given an IFS $\{\phi_a\}_{a\in A}$ and a probability vector $\mathbf{p} = (p_a)_{a\in\mathcal{A}}$, then there exists a unique Borel probability measure $\mu_{\mathbf{p}}$ satisfying

$$\mu_{\mathbf{p}} = \sum_{a\in\mathcal{A}} p_a \cdot \phi_a \mu_{\mathbf{p}}.$$

We call $\mu_{\mathbf{p}}$ the self-similar measure corresponding to Φ and $\mathbf{p}$. Given $\mathbf{p}$, if we let $\mathfrak{m}_{\mathbf{p}}$ denote the corresponding Bernoulli measure on $\mathcal{A}^{\mathbb{N}}$ then it is also the case that $\mu_{\mathbf{p}} = \pi \mathfrak{m}_{\mathbf{p}}$. For our purposes we will only need to focus on one particular self-similar measure, namely the one corresponding to the probability vector $\left(r_a^{\dim_S(\Phi)}\right)_{a\in\mathcal{A}}$. This self-similar measure is distinguished amongst the family of self-similar measures. Studying its properties often allows one to prove equality in (1.1). For an IFS Φ, we will denote the self-similar measure corresponding to $\left(r_a^{\dim_S(\Phi)}\right)_{a\in\mathcal{A}}$ by μ_Φ, or simply μ if the choice of Φ is implicit. Similarly, we will denote $\left(r_a^{\dim_S(\Phi)}\right)_{a\in\mathcal{A}}$ by $\mathbf{p}_\Phi$ or simply $\mathbf{p}$, and the corresponding Bernoulli measure on $\mathcal{A}^{\mathbb{N}}$ by $\mathfrak{m}_\Phi$ or $\mathfrak{m}$. For a probability vector $\mathbf{p}$ we denote the entropy of $\mathbf{p}$ by

$$h_{\mathbf{p}} := -\sum_{a\in\mathcal{A}} p_a \log p_a.$$

Suppose now that in addition to $\mathbf{p}$ we are also given an IFS Φ, we then define the Lyapnuov exponent of Φ and $\mathbf{p}$ to be

$$\chi_{\Phi,\mathbf{p}} := -\sum_{a\in\mathcal{A}} p_a \log r_a.$$

We conclude this overview of the relevant topics from Fractal Geometry by introducing some notation. In what follows, we denote an element of $\cup_{n=1}^{\infty} A^n$ or $\mathcal{A}^{\mathbb{N}}$ by $\mathbf{a}$ or $\mathbf{b}$. Given an IFS $\Phi = \{\phi_a\}_{a\in\mathcal{A}}$ and a word $\mathbf{a} = (a_1, \ldots, a_n)$, we let $\phi_{\mathbf{a}} := \phi_{a_1} \circ \cdots \circ \phi_{a_n}$ and $r_{\mathbf{a}} := \prod_{l=1}^{n} r_{a_l}$. Given a word $\mathbf{a}$ we let $X_{\mathbf{a}} = \phi_{\mathbf{a}}(X)$. Given a finite word $\mathbf{a}$ and a finite word or infinite sequence $\mathbf{b}$, we let $\mathbf{ab}$ denote the concatenation of $\mathbf{a}$ and $\mathbf{b}$. For a finite word $\mathbf{a}$ we let $\mathbf{a}^k$ denote the k-fold concatenation of $\mathbf{a}$ with itself. Similarly $\mathbf{a}^\infty$ denotes the periodic element of $\mathcal{A}^{\mathbb{N}}$ obtained by concatenating $\mathbf{a}$ with itself indefinitely. We denote the length of a finite word $\mathbf{a}$ by $|\mathbf{a}|$. Finally, given a finite word $\mathbf{a} \in \cup_{n=1}^{\infty} \mathcal{A}^n$ we let

$$[\mathbf{a}] := \left\{(b_n) \in \mathcal{A}^{\mathbb{N}} : b_1 \ldots b_{|\mathbf{a}|} = \mathbf{a}\right\}.$$

We will often refer to $[\mathbf{a}]$ as the cylinder set corresponding to $\mathbf{a}$.

1.2 Statement of Theorem 1.1

The family of limsup sets that will be the main focus of this paper are defined as follows: Given an IFS Φ and a function $\Psi : \cup_{n=1}^{\infty}\mathcal{A}^n \to [0,\infty)$, we let

$$W_{\Phi}(\Psi) := \bigcap_{N=1}^{\infty}\bigcup_{n=N}^{\infty}\bigcup_{\mathbf{a}\in\mathcal{A}^n}\bigcup_{l=0}^{n-1} B\left(\pi(a_1\cdots a_l(a_{l+1}\cdots a_n)^{\infty}), \Psi(\mathbf{a})\right).$$

Alternatively, $W_{\Phi}(\Psi)$ is the set of $x\in\mathbb{R}^d$ such that for infinitely many n, there exists $\mathbf{a}\in\mathcal{A}^n$ and $0\leq l\leq n-1$ such that

$$\|x-\pi(a_1\cdots a_l(a_{l+1}\cdots a_n)^{\infty})\| < \Psi(\mathbf{a}).$$

The connection between $W_{\Phi}(\Psi)$ and intrinsic Diophantine Approximation will be made clear in Sect. 2. Our main result demonstrates that for certain choices of Ψ, the measure of $W_{\Phi}(\Psi)$ is determined by naturally occurring volume sums. One cannot expect such a behaviour to occur for all choices of Ψ. Indeed Example 2.1 from [2] shows that for a related family of limsup sets, if we want the measure of these limsup sets to be determined by volume sums, then the underlying Ψ should reflect the different rates of scaling within the IFS. As such we will often restrict ourselves to Ψ of the form

$$\Psi(\mathbf{a}) = \mathrm{Diam}(X_{\mathbf{a}})\cdot g(|\mathbf{a}|) \tag{1.2}$$

where $g:\mathbb{N}\to[0,\infty)$. A similar restriction was also adopted in [1, 2, 5]. Functions of the form

$$\Psi(\mathbf{a}) = g(\mathrm{Diam}(X_{\mathbf{a}})) \tag{1.3}$$

for some $g:(0,\infty)\to(0,\infty)$ were considered in a similar situation in [21] and later in [8, Section 12.4]. Note that if Φ is equicontractive, then the set of Ψ of the form (1.2) coincides with the set of Ψ of the form (1.3). Moreover, both of these sets coincide with the set of Ψ such that $\Psi(\mathbf{a})$ only depends upon the length of $\mathbf{a}$.

Our main result is the following statement.

Theorem 1.1 *Let $\Phi=\{\phi_a\}_{a\in\mathcal{A}}$ be an IFS and $\Psi:\cup_{n=1}^{\infty}\mathcal{A}^n\to[0,\infty)$. Then the following statements are true:*

1. *For any $s\geq 0$, suppose that*

$$\sum_{n=1}^{\infty}\sum_{\mathbf{a}\in\mathcal{A}^n} n\cdot\Psi(\mathbf{a})^s < \infty.$$

Then $\mathcal{H}^s(W_{\Phi}(\Psi))=0$.

2. *Assume that*

$$h_{\mathbf{p}} < -2\log\sum_{a\in\mathcal{A}} p_a^2 \tag{1.4}$$

and Ψ *is of the form* $\Psi(\mathbf{a}) = \mathrm{Diam}(X_{\mathbf{a}})g(|\mathbf{a}|)$ *for some non-increasing* $g : \mathbb{N} \to [0,\infty)$. *If*

$$\sum_{n=1}^{\infty}\sum_{\mathbf{a}\in\mathcal{A}^n} n\cdot(\mathrm{Diam}(X_{\mathbf{a}})g(n))^{\dim_S(\Phi)} = \infty$$

then $\mu\,(W_{\Phi}(\Psi)) = 1$.

3. *Assume that* Φ *is equicontractive and* Ψ *is of the form* $\Psi(\mathbf{a}) = \mathrm{Diam}(X_{\mathbf{a}})g(|\mathbf{a}|)$ *for some* $g : \mathbb{N} \to [0,\infty)$. *If*

$$\sum_{n=1}^{\infty}\sum_{\mathbf{a}\in\mathcal{A}^n} n\cdot(\mathrm{Diam}(X_{\mathbf{a}})g(n))^{\dim_S(\Phi)} = \infty$$

then $\mu\,(W_{\Phi}(\Psi)) = 1$.

We conclude this section with some remarks on Theorem 1.1.

Remark 1.2 Statement 3 of Theorem 1.1 was proved for the IFS $\left\{\phi_1(x) = \frac{x}{3}, \phi_2(x) = \frac{x+2}{3}\right\}$ by Tan, Wang, and Wu in [37]. Note that this IFS has the middle third Cantor set as its self-similar set. In a recent talk Wang [38] commented that the methods used in [37] could be generalised to prove Statement 3 of Theorem 1.1 for equicontractive IFSs acting on $\mathbb{R}$ that satisfy the strong separation condition. During this talk Wang posed the question as to what happens for IFSs that are not equicontractive. This paper was in part motivated by this question and Statement 2 of Theorem 1.1 provides a partial answer. Importantly, as well as providing information in the non-equicontractive case, Theorem 1.1 also applies in arbitrary dimensions and requires no separation assumptions on the IFS. The techniques of [37] do not apply in this generality. That being said, our method of proof largely follows the same overall strategy as [37]. The major differences being that we require additional arguments to control the different rates of scaling within our potentially non-equicontractive IFS, and we also require a new argument to address the potential overlaps that may be present within the IFS. The latter argument uses ideas from [5, Section 7] where the author reinterpreted their problem in terms of a limsup set on the sequence space $\mathcal{A}^{\mathbb{N}}$. This reinterpretation addresses the problem of potential overlaps.

Remark 1.3 If Φ satisfies the open set condition then it is known that μ is equivalent to the restriction of the $\mathcal{H}^{\dim_S(\Phi)}$-dimensional Hausdorff measure on X. As such, under the open set condition, Statements 1, 2, and 3 of Theorem 1.1 provide a nearly complete description of the μ measure of $W_{\Phi}(\Psi)$ for Ψ of the form $\Psi(\mathbf{a}) = \mathrm{Diam}(X_{\mathbf{a}})g(|\mathbf{a}|)$. Moreover, if we assume that Φ is equicontractive and satisfies the open set condition, then Statements 1 and 3 do provide a complete description. In the overlapping case, i.e. when the open set condition is not satisfied, then Statements 2 and 3 can be used to deduce a number of corollaries on the Hausdorff dimension of $W_{\Phi}(\Psi)$. For if $\dim \mu = \min\{\dim_S(\Phi), d\}$ and $\mu(W_{\Phi}(\Psi)) = 1$, then we must have $\dim_H(W_{\Phi}(\Psi)) \geq \min\{\dim_S(\Phi), d\}$. Moreover because $W_{\Phi}(\Psi)$ is a subset of the self similar set X, and X satisfies (1.1), we must then have $\dim_H(W_{\Phi}(X)) = \min\{\dim_S(\Phi), d\}$. The important part in this argument is determining when we have $\dim \mu = \min\{\dim_S(\Phi), d\}$. A number of significant breakthroughs on this topic have been made in recent years, see [22, 23, 33]. These papers provide general sufficient conditions which guarantee $\dim \mu = \min\{\dim_S(\Phi), d\}$. We won't state the results of these papers in their full generality here. Instead we will focus on one particular consequence that is relevant to our purposes. Suppose that $\Phi = \{\phi_a(x) = r_a x + t_a\}$ is an IFS acting on $\mathbb{R}$ and that each r_a is algebraic, then it follows from the results of [33] that if Φ does not contain an exact overlap then $\dim \mu = \min\{\dim_S(\Phi), d\}$. We recall that an IFS is said to contain an exact overlap if there exists $\mathbf{a}, \mathbf{b} \in \cup_{n=1}^{\infty} \mathcal{A}^n$ such that $\phi_{\mathbf{a}} = \phi_{\mathbf{b}}$ and $\mathbf{a} \neq \mathbf{b}$.

Remark 1.4 The inequality (1.4) and the non-increasing assumption on g in Statement 2 are both technical assumptions and are believed to be non-optimal. Note that in Statement 3 there are no monotonicity conditions imposed on g. We expect that both of these assumptions can be removed. For the purposes of our exposition, we highlight that in the case of an IFS consisting of two similarities $\{\phi_1(x) = r_1 x + t_1, \phi_2(x) = r_2 x + t_2\}$, then (1.4) is satisfied if $r_1^{\dim_S(\Phi)}$ satisfies

$$0.048\ldots < r_1^{\dim_S(\Phi)} < 0.951\ldots.$$

As such we see that (1.4) is satisfied by a significant proportion of those IFSs consisting of two similarities. The inequalities above can be derived by considering the functions $f : (0,1) \to \mathbb{R}$ and $g : (0,1) \to \mathbb{R}$ given by $f(x) = -x \log x - (1-x)\log(1-x)$ and $g(x) = -2\log(x^2 + (1-x)^2)$, and using a computer to determine those x for which $f(x) < g(x)$. We remark that when Φ is an equicontractive IFS then we have $h_{\mathbf{p}} = -\log \sum_{a\in\mathcal{A}} p_a^2$, and so (1.4) is automatically satisfied in this case. It follows from this observation and the fact that the quantities on each side of (1.4) depend continuously on $\mathbf{p}$, that if we fix the number of maps within our IFS to be N for some $N \in \mathbb{N}$ and identify the space of contraction ratios with $(0,1)^N$, then for a non-empty open set of contraction ratios the inequality (1.4) is satisfied. We will see another explicit example where the inequality (1.4) is satisfied in Fig. 1.

Remark 1.5 It is a simple exercise to show that a function $g : \mathbb{N} \to [0,\infty)$ satisfies

$$\sum_{n=1}^{\infty} \sum_{\mathbf{a}\in\mathcal{A}^n} n \cdot (\mathrm{Diam}(X_{\mathbf{a}}) g(n))^{\dim_S(\Phi)} = \infty$$

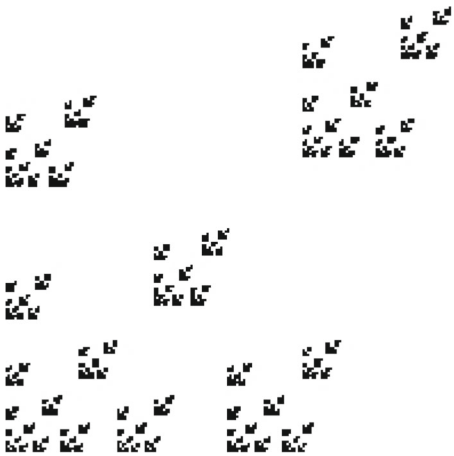


Fig. 1 $\Phi=\{\phi_1(x,y)=(\frac{x}{2},\frac{y}{2}),\phi_2(x,y)=(\frac{x+2}{4},\frac{y}{4}),\phi_3(x,y)=(\frac{x}{5},\frac{y+3}{5}),\phi_4(x,y)=(\frac{x+2}{3},\frac{y+2}{3})\}$. The self-similar set for this IFS is displayed above. One can check that this IFS satisfies the strong separation condition and (1.4). Therefore Theorem 1.1 and Theorem 2.7 apply to this IFS

if and only if

$$\sum_{n=1}^{\infty} n \cdot g(n)^{\dim_S(\Phi)} = \infty.$$

It will on occasion be more convenient to use this latter divergence condition.

Notation. In this paper we will adopt the following notational convention. Given a set S and two functions $f, g : S \to \mathbb{R}$ we write $f \ll g$ if there exists $C > 0$ such that $|f(x)| \leq C|g(x)|$ for all $x \in S$. We write $f \asymp g$ if $f \ll g$ and $g \ll f$.

2 Applications to intrinsic Diophantine approximation

2.1 Background

The study of intrinsic Diophantine Approximation for self-similar sets has its origins in a question of Mahler [30]. He asked how well can elements of the middle third Cantor set C be approximated by rational numbers lying within C. To the best of the author's knowledge, the first significant progress in this direction was the work of Levesley, Salp, and Velani [29]. They considered rational approximations of the from $p/3^n$. Or equivalently, rational approximations provided by the end points of the sets of the form $\phi_{\mathbf{a}}(C)$. They proved a general Khintchine type result for approximations of this type, see [29, Theorem 1]. Using this theorem, they were able to prove that there exist well-approximable numbers in the middle third Cantor set that are not Liouville. This was an unproved assertion attributed to Mahler. Bugeaud also proved this assertion using a different method in [13]. He in fact provided explicit examples of elements of the middle third Cantor set with any irrationality exponent. In [14] Bugeaud and Durand posed a conjecture on the value of the Hausdorff dimension of the set of points in the middle third Cantor set whose irrationality exponent exceeds a given parameter. Interestingly this conjecture suggests that a phase transition should occur for the value of the Hausdorff dimension of this set. The main result of [14] shows that a version of this conjecture holds almost surely for a particular random model of C. The following intrinsic analogue of Dirichlet's theorem for C was proved by Broderick, Fishman, and Reich [11].

Theorem 2.1 *For any $x \in C$ and $Q > 1$, there exists $p/q \in C$ with $1 \leq q \leq Q$ such that*

$$|x - p/q| < \frac{1}{q(\log_3 Q)^{\log 3/\log 2}}.$$

Theorem 2.1 was shown to be optimal by Fishman and Simmons [19], and Fishman, Merrill, and Simmons [17].

The study of intrinsic Diophantine approximation for self-similar sets naturally leads one to study limsup sets that are in a sense built using the underlying iterated function system. A number of papers have appeared which study such sets, see [1–6, 31, 32]. Despite being a problem that was originally motivated by number theoretic considerations, the study of these limsup sets is connected to topics from Ergodic Theory and Fractal Geometry. Interestingly the metric properties of these limsup sets can be related to the absolute continuity of self-similar measures, see [5].

2.2 Applications

For the rest of this section we restrict our attention to iterated function systems of the form

$$\Phi = \left\{\phi_a(x) = \frac{x + \mathfrak{p}_a}{q_a}\right\}_{a\in\mathcal{A}}.$$

Where for all $a \in \mathcal{A}$ we have $\mathfrak{p}_a \in \mathbb{Z}^d$, $q_a \in \mathbb{Z}$, and q_a also satisfies $|q_a| \geq 2$. For this IFS, the projection map π takes the following simplified form:

$$\pi(\mathbf{a}) = \sum_{n=1}^{\infty} \frac{\mathfrak{p}_{a_n}}{\prod_{l=1}^{n} q_{a_l}}. \tag{2.1}$$

For such an IFS, the following lemma connects those x contained in $X \cap \mathbb{Q}^d$ with eventually periodic sequences. It is a particular case of a more general result due to Schleischitz [34]. For this reason we omit its proof.

Lemma 2.2 *Let Φ be an IFS of the form $\Phi = \left\{\phi_a(x) = \frac{x+\mathfrak{p}_a}{q_a}\right\}$. Then for any $x \in X$ we have that $x \in \mathbb{Q}^d$ if and only if there exists $\mathbf{a} \in \cup_{n=1}^{\infty}\mathcal{A}^n$ and $0 \leq l \leq |\mathbf{a}| - 1$ such that*

$$x = \pi\left(a_1 \ldots a_l \left(a_{l+1} \ldots a_{|\mathbf{a}|}\right)^{\infty}\right).$$

If $x \in \mathbb{Q}^d$ and therefore by Lemma 2.2 is equal to $\pi\left(a_1 \ldots a_l \left(a_{l+1} \ldots a_{|\mathbf{a}|}\right)^{\infty}\right)$ for some $\mathbf{a} \in \cup_{n=1}^{\infty} \mathcal{A}^n$ and $0 \leq l \leq |\mathbf{a}| - 1$, we can use properties of geometric series and (2.1) to conclude that there exists $\mathfrak{p}_x \in \mathbb{Z}^d$ such that

$$x = \frac{\mathfrak{p}_x}{\prod_{j=1}^{l} q_{a_j} \cdot \left(\prod_{j=l+1}^{n} q_{a_j} - 1\right)}.$$

One of the major difficulties in understanding the properties of the rational numbers within a self-similar set is not knowing if any cancellation occurs between the entries in the vector $\mathfrak{p}_x$ and the $\prod_{j=1}^{l} q_{a_j} \cdot \left(\prod_{j=l+1}^{n} q_{a_j} - 1\right)$ term. It is possible that these two terms contain many common factors and as such x could be written in a significantly reduced form. This makes studying intrinsic Diophantine Approximation for self-similar sets challenging. This difficulty is avoided in this paper by considering the quantity q_{int} defined below. For ease of exposition we split what remains of this section into two cases, when Φ is equicontractive and the general case. Our applications are much simpler to state in the equicontractive case.

2.2.1 The equicontractive case

In this section we assume that Φ is equicontractive, i.e. there exists $q_{\Phi} \in \mathbb{Z}$ such that $q_a = q_{\Phi}$ for all $a \in A$. By the above we know that if $\mathfrak{p}/q \in X$ then there exists $\mathfrak{p}' \in \mathbb{Z}^d$, $n \in \mathbb{N}$, and $0 \leq l \leq n-1$ such that

$$\mathfrak{p}/q = \frac{\mathfrak{p}'}{q_{\Phi}^{l}\left(q_{\Phi}^{n-l} - 1\right)}.$$

We define the intrinsic denominator of $\mathfrak{p}/q \in X$ to be

$$q_{\text{int}}(\mathfrak{p}/q) := \inf \left\{ q_{\Phi}^{l}\left(q_{\Phi}^{|\mathbf{a}|-l} - 1\right) : \mathbf{a} \in \cup_{n=1}^{\infty} \mathcal{A}^n, 0 \leq l \leq |\mathbf{a}| - 1 \text{ satisfying } \mathfrak{p}/q = \pi\left(a_1 \ldots a_l \left(a_{l+1} \ldots a_{|\mathbf{a}|}\right)^{\infty}\right) \right\}$$

This is a generalisation of the notion of intrinsic denominator defined in [19] for Φ satisfying the strong separation condition. Given a function $\Psi : \mathbb{N} \to [0, \infty)$ we may then define a limsup set as follows:

$$W_{\Phi}^{*}(\Psi) := \{x \in X : \|x - \mathfrak{p}/q\| < \Psi(q_{\text{int}}(\mathfrak{p}/q)) \text{ for i.m. } \mathfrak{p}/q \in X\}.$$

Fishman and Simmons in [19] proved a version of Khintchine's theorem for limsup sets of the form $W_{\Phi}^{*}(\Psi)$ when the underlying Φ is equicontractive, acting on $\mathbb{R}$, and satisfies the strong separation condition. Importantly this result did not provide a complete metric description for the sets $W_{\Phi}^{*}(\Psi)$, even in the restricted case of equicontractive IFSs acting on $\mathbb{R}$ which satisfy the strong separation condition. The divergence condition they needed for a full measure statement was not optimal. This issue was addressed in a recent paper by Tan, Wang, and Wu [37] who established a complete analogue of Khintchine's theorem for the set $W_\Phi^*(\Psi)$ for the IFS $\left\{\phi_1(x)=\frac{x}{3},\phi_2(x)=\frac{x+2}{3}\right\}$. Note that this IFS has the middle third Cantor set as its self-similar set.

Theorem 2.3 [37, Theorem 1.4] *Let* $\Phi=\left\{\phi_1(x)=\frac{x}{3},\phi_2(x)=\frac{x+2}{3}\right\}$ *and* $\Psi:\mathbb{N}\to[0,\infty)$ *be a non-increasing function. Then*

$$\mu(W_\Phi^*(\Psi))=\begin{cases}0 \text{ if } \sum_{n=1}^\infty n\cdot 2^n\cdot\Psi(3^n)^{\frac{\log 2}{\log 3}}<\infty;\\ 1 \text{ if } \sum_{n=1}^\infty n\cdot 2^n\cdot\Psi(3^n)^{\frac{\log 2}{\log 3}}=\infty.\end{cases}$$

The proof given in [37] can be generalised to prove an analogue of Theorem 2.3 for any equicontractive IFS acting on $\mathbb{R}$ satisfying the strong separation condition. Our main result in this direction is the following statement. It generalises Theorem 2.3 to arbitrary dimensions and requires no separation conditions for Φ.

Theorem 2.4 *Let* Φ *be an equicontractive IFS of the form* $\Phi=\left\{\phi_a(x)=\frac{x+\mathfrak{p}_a}{q_\Phi}\right\}$. *Let* $\Psi:\mathbb{N}\to[0,\infty)$ *be a non-increasing function. Then the following statements are true:*

1. *For any* $s\geq 0$, *suppose that* $\sum_{n=1}^\infty n\cdot\#\mathcal{A}^n\cdot\Psi(q_\Phi^n)^s<\infty$. *Then* $\mathcal{H}^s(W_\Phi^*(\Psi))=0$.
2. *If* $\sum_{n=1}^\infty n\cdot\#\mathcal{A}^n\cdot\Psi(q_\Phi^n)^{\frac{\log\#\mathcal{A}}{\log q_\Phi}}=\infty$ *then* $\mu(W_\Phi^*(\Psi))=1$.

Proof We begin by remarking that for any word $\mathbf{a}$ and $0\leq l\leq|\mathbf{a}|-1$ we have

$$q_\Phi^{|\mathbf{a}|-1}\leq q_\Phi^l\left(q_\Phi^{|\mathbf{a}|-l}-1\right)\leq q_\Phi^{|\mathbf{a}|}. \tag{2.2}$$

Proof of the convergence case. To prove the convergence case let $\Psi':\cup_{n=1}^\infty\mathcal{A}^n\to[0,\infty)$ be given by

$$\Psi'(\mathbf{a})=\Psi\left(q_\Phi^{|\mathbf{a}|-1}\right).$$

Now notice that for any $\mathfrak{p}/q\in X$, there must exist $\mathbf{a}$ and $0\leq l\leq|\mathbf{a}|-1$ such that

$$B\left(\mathfrak{p}/q,\Psi(q_{\text{int}}(\mathfrak{p}/q))\right)=B\left(\pi\left(a_1\ldots a_l\left(a_{l+1}\ldots a_{|\mathbf{a}|}\right)^\infty\right),\Psi\left(q_\Phi^l(q_\Phi^{|\mathbf{a}|}-1\right)\right).$$

Therefore, using the fact Ψ is non-increasing together with (2.2) we have

$$\begin{aligned}B(\mathfrak{p}/q,\Psi(q_{\text{int}}(\mathfrak{p}/q)))&\subseteq B\left(\pi\left(a_1\ldots a_l\left(a_{l+1}\ldots a_{|\mathbf{a}|}\right)^\infty\right),\Psi\left(q_\Phi^{|\mathbf{a}|-1}\right)\right)\\&=B\left(\pi\left(a_1\ldots a_l\left(a_{l+1}\ldots a_{|\mathbf{a}|}\right)^\infty\right),\Psi'(\mathbf{a})\right).\end{aligned}$$

Therefore $W_\Phi^*(\Psi)\subseteq W_\Phi(\Psi')$. By Theorem 1.1 it will follow that $\mathcal{H}^s(W_\Phi^*(\Psi))=0$ if $\sum_{n=1}^\infty\sum_{\mathbf{a}\in\mathcal{A}^n}n\cdot\Psi'(\mathbf{a})^s<\infty$. This latter inequality is equivalent to $\sum_{n=1}^\infty n\cdot\#\mathcal{A}^n\cdot\Psi(q_\Phi^{n-1})^s<\infty$. However, this inequality is implied by our assumption

$\sum_{n=1}^{\infty} n \cdot \#\mathcal{A}^n \cdot \Psi(q_\Phi^n)^s < \infty$.

Proof of the divergence case. Let $g : \mathbb{N} \to [0, \infty)$ be given by

$$g(n) = \frac{\Psi(q_\Phi^n)}{\mathrm{Diam}(X) \cdot q_\Phi^{-n}}.$$

Now define $\Psi' : \cup_{n=1}^{\infty} \mathcal{A}^n \to [0, \infty)$ by $\Psi'(\mathbf{a}) = \mathrm{Diam}(X_\mathbf{a}) \cdot g(|\mathbf{a}|)$. For any $\mathbf{a} \in \mathcal{A}^n$ and $0 \leq l \leq |\mathbf{a}| - 1$ there exists $\mathfrak{p}/q \in X$ such that

$$B\left(\pi\left(a_1 \ldots a_l\,(a_{l+1} \ldots a_n)^\infty\right), \Psi'(\mathbf{a})\right) = B\left(\mathfrak{p}/q, \Psi\left(q_\Phi^n\right)\right).$$

By the non-increasing assumption on Ψ, the definition of intrinsic denominator, and (2.2), it follows that

$$B(\pi(a_1 \ldots a_l(a_{l+1} \ldots a_n)^\infty), \Psi'(\mathbf{a})) \subseteq B(\mathfrak{p}/q, \Psi(q_{\mathrm{int}}(\mathfrak{p}/q))).$$

Therefore $W_\Phi(\Psi') \subseteq W_\Phi^*(\Psi)$. By Theorem 1.1, it will follow that $\mu(W_\Phi^*(\Psi)) = 1$ if we can show that

$$\sum_{n=1}^{\infty} \sum_{\mathbf{a} \in \mathcal{A}^n} n \cdot (\mathrm{Diam}(X_\mathbf{a}) g(|\mathbf{a}|))^{\frac{\log \#\mathcal{A}}{\log q_\Phi}} = \infty.$$

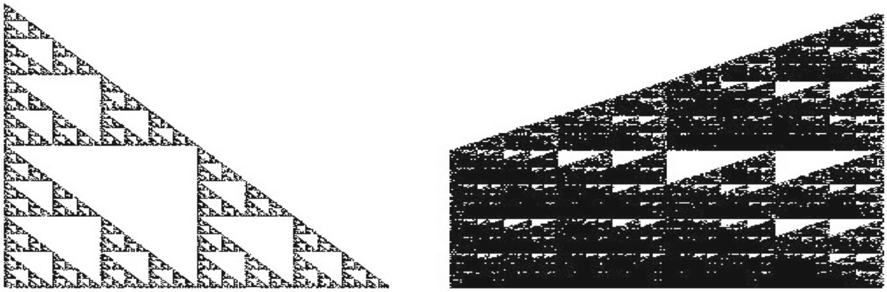
By the definition of g this is equivalent to our assumption $\sum_{n=1}^{\infty} n \cdot \#\mathcal{A}^n \cdot \Psi(q_\Phi^n)^{\frac{\log \#\mathcal{A}}{\log q_\Phi}} = \infty$. Therefore our result follows. □

If a rational vector $\mathfrak{p}/q \in X$ is in its reduced form then we must have $q \leq q_{\mathrm{int}}(\mathfrak{p}/q)$. This observation together with Theorem 2.4 implies the following statement for traditional rational approximations, where the neighbourhood is defined in terms of the denominator of $\mathfrak{p}/q$ rather than $q_{\mathrm{int}}(\mathfrak{p}/q)$.

Corollary 2.5 *Let Φ be an equicontractive IFS of the form $\Phi = \{\phi_\mathbf{a}(x) = \frac{x + \mathfrak{p}_a}{q_\Phi}\}$. Let $\Psi : \mathbb{N} \to [0, \infty)$ be a non-increasing function. If $\sum_{n=1}^{\infty} n \cdot \#\mathcal{A}^n \cdot \Psi(q_\Phi^n)^{\frac{\log \#\mathcal{A}}{\log q_\Phi}} = \infty$ then*

$$\mu\left(\{x \in X : \|x - \mathfrak{p}/q\| < \Psi(q) \textit{ for i.m. } \mathfrak{p}/q \in X\}\right) = 1.$$

See Fig. 2 for an example of two IFSs to which Theorem 2.4 can be applied.



(a) The self-similar set for Φ_1. (b) The self-similar set for Φ_2.

Fig. 2 We let $\Phi_1 = \left\{\phi(x,y) = (\frac{x+i}{2}, \frac{y+j}{2}) : (i,j) \in \{(0,0),(1,0),(0,1)\}\right\}$ and $\Phi_2 = \left\{\phi(x,y) = (\frac{x+i}{2}, \frac{y+j}{2}) : (i,j) \in \{(0,0),(1,0),(0,1),(2,2),(2,0)\}\right\}$. Φ_1 satisfies the open set condition and Φ_2 does not. Theorem 1.1 and Theorem 2.4 apply to both of these iterated function systems

2.2.2 The general case

In this section we no longer assume that Φ is equicontractive. We formulate two statements, one when Φ is potentially overlapping, and one when Φ satisfies the strong separation condition.

Given $g : \mathbb{N} \to [0,\infty)$ and $\mathfrak{p}/q \in X$ we define

$$\Psi_g(\mathfrak{p}/q) := \sup\Big\{ \mathrm{Diam}(X_{\mathbf{a}}) \cdot g(|\mathbf{a}|) : \mathbf{a} \in \cup_{n=1}^{\infty} \mathcal{A}^n,\ 0 \le l \le |\mathbf{a}| - 1 \text{ satisfying } \mathfrak{p}/q = \pi\left(a_1 \ldots a_l \left(a_{l+1} \ldots a_{|\mathbf{a}|}\right)^{\infty}\right)\Big\}.$$

To each $g : \mathbb{N} \to [0,\infty)$ we associate the set

$$W_{\Phi}^{**}(g) = \left\{x \in X : \|x - \mathfrak{p}/q\| < \Psi_g(\mathfrak{p}/q) \text{ for i.m. } \mathfrak{p}/q \in X\right\}.$$

The following statement is essentially Theorem 1.1 rephrased in terms of rational approximations.

Theorem 2.6 *Let Φ be an IFS of the form $\Phi = \left\{\phi_{\mathbf{a}}(x) = \frac{x+\mathfrak{p}_a}{q_a}\right\}$ and $g : \mathbb{N} \to [0,\infty)$. Then the following statements are true*

1. *For any $s \ge 0$, suppose that $\sum_{n=1}^{\infty} \sum_{\mathbf{a} \in \mathcal{A}^n} n \cdot (\mathrm{Diam}(X_{\mathbf{a}}) g(n))^s < \infty$. Then $\mathcal{H}^s(W_{\Phi}^{**}(g)) = 0$.*
2. *If g is non-increasing, $h_{\mathbf{p}} < -2\log\sum_{a\in\mathcal{A}} p_a^2$, and $\sum_{n=1}^{\infty}\sum_{\mathbf{a}\in\mathcal{A}^n} n \cdot (\mathrm{Diam}(X_{\mathbf{a}}) g(n))^{\dim_S(\Phi)} = \infty$ then $\mu(W_{\Phi}^{**}(g)) = 1$.*

Proof This result follows from Theorem 1.1 together with the observation that if $x \notin \mathbb{Q}^d$ then $x \in W_{\Phi}^{**}(\Psi_g)$ if and only if $x \in W_{\Phi}(\Psi)$ for $\Psi(\mathbf{a}) = \mathrm{Diam}(X_{\mathbf{a}}) g(|\mathbf{a}|)$. □

For what remains of this section we will always assume that Φ satisfies the strong separation condition. Because of the strong separation condition, for any $\mathfrak{p}/q \in X$ there exists a unique sequence $(a_n) \in \mathcal{A}^{\mathbb{N}}$ satisfying $\pi((a_n)) = \mathfrak{p}/q$. We emphasise that (a_n) must be eventually periodic. Given $\mathfrak{p}/q \in X$ we define the intrinsic denominator of $\mathfrak{p}/q$ to be

$$\inf\left\{\prod_{j=1}^{l} q_{a_j} \cdot \left(\prod_{j=l+1}^{n} q_{a_j} - 1\right)\right.$$
$$\left. : n \in \mathbb{N},\ 0 \le l \le n-1, \text{ and } \mathfrak{p}/q = \pi\left(a_1 \ldots a_l \left(a_{l+1} \ldots a_n\right)^{\infty}\right)\right\}.$$

We denote this quantity by $q_{\mathrm{int}}(\mathfrak{p}/q)$. For any $\mathfrak{p}/q \in X$ we define

$$n(\mathfrak{p}/q) := \inf\left\{n \in \mathbb{N} : \mathfrak{p}/q = \pi\left(a_1 \ldots a_l \left(a_{l+1} \ldots a_n\right)^{\infty}\right) \text{ for some } 0 \le l \le n-1\right\}$$

Similarly, we let

$$l(\mathfrak{p}/q) := \inf\left\{0 \le l \le n(\mathfrak{p}/q) - 1 : \mathfrak{p}/q = \pi\left(a_1 \ldots a_l \left(a_{l+1} \ldots a_{n_{p/q}}\right)^{\infty}\right)\right\}.$$

It can be shown that $n(\mathfrak{p}/q)$ and $l(\mathfrak{p}/q)$ are the unique parameters satisfying

$$q_{\mathrm{int}}(\mathfrak{p}/q) = \prod_{j=1}^{l(\mathfrak{p}/q)} q_{a_j} \cdot \left(\prod_{j=l(\mathfrak{p}/q)+1}^{n(\mathfrak{p}/q)} q_{a_j} - 1\right)$$

and

$$\mathfrak{p}/q = \pi\left(a_1 \ldots a_{l(\mathfrak{p}/q)} \left(a_{l(\mathfrak{p}/q)+1} \ldots a_{n(\mathfrak{p}/q)}\right)^{\infty}\right).$$

Given $g : \mathbb{N} \to [0, \infty)$ we define a limsup set as follows, let

$$W_{\Phi}^{***}(g) := \left\{x \in X : \|x - \mathfrak{p}/q\| < \frac{g(n(\mathfrak{p}/q))}{q_{\mathrm{int}}(\mathfrak{p}/q)} \text{ for i.m. } \mathfrak{p}/q \in X\right\}.$$

Theorem 2.7 *Let Φ be an IFS of the form $\Phi = \left\{\phi_a(x) = \frac{x+\mathfrak{p}_a}{q_a}\right\}$ satisfying the strong separation condition and let $g : \mathbb{N} \to [0, \infty)$. Then the following statements are true:*

1. *For any $s \ge 0$, suppose that $\sum_{n=1}^{\infty} \sum_{\mathbf{a} \in \mathcal{A}^n} n \cdot (\mathrm{Diam}(X_{\mathbf{a}}) g(n))^s < \infty$. Then $\mathcal{H}^s(W_{\Phi}^{***}(g)) = 0$.*
2. *If g is non-increasing, $h_{\mathbf{p}} < -2 \log \sum_{a \in \mathcal{A}} p_a^2$, and $\sum_{n=1}^{\infty} \sum_{\mathbf{a} \in \mathcal{A}^n} n \cdot (\mathrm{Diam}(X_{\mathbf{a}}) g(n))^{\dim_S(\Phi)} = \infty$, then $\mu(W_{\Phi}^{***}(g)) = 1$.*

Proof We begin our proof by remarking that there exist constants $C_1, C_2 > 0$ such that for any word $\mathbf{a} \in \cup_{n=1}^{\infty} \mathcal{A}^n$ and $0 \le l \le |\mathbf{a}| - 1$ we have

$$\frac{C_1}{\prod_{j=1}^{l} q_{a_j} \cdot \left(\prod_{j=l+1}^{|\mathbf{a}|} q_{a_j} - 1\right)} < \mathrm{Diam}(X_{\mathbf{a}}) < \frac{C_2}{\prod_{j=1}^{l} q_{a_j} \cdot \left(\prod_{j=l+1}^{|\mathbf{a}|} q_{a_j} - 1\right)}. \tag{2.3}$$

The convergence case. It follows from (2.3) that for any $\mathfrak{p}/q \in X$ we have

$$B\left(\frac{\mathfrak{p}}{q}, \frac{g(n(\mathfrak{p}/q))}{q_{\mathrm{int}}(\mathfrak{p}/q)}\right) \subseteq B\left(\pi\left(a_1 \ldots a_{l(\mathfrak{p}/q)}\left(a_{l(\mathfrak{p}/q)+1} \ldots a_{n(\mathfrak{p}/q)}\right)^{\infty}\right),\right.$$
$$\left.\frac{g(n(\mathfrak{p}/q))\,\mathrm{Diam}\left(X_{a_1\ldots a_{n(\mathfrak{p}/q)}}\right)}{C_1}\right).$$

Therefore $W_{\Phi}^{***}(g) \subseteq W_{\Phi}(\Psi)$ for $\Psi(\mathbf{a}) = \frac{g(|\mathbf{a}|)\,\mathrm{Diam}(X_{\mathbf{a}})}{C_1}$. In which case $\mathcal{H}^s(W_{\Phi}^{***}(g)) = 0$ if $\mathcal{H}^s(W_{\Phi}(\Psi)) = 0$. However this last inequality follows from Theorem 1.1 and our assumption $\sum_{n=1}^{\infty}\sum_{\mathbf{a}\in\mathcal{A}^n} n \cdot (\mathrm{Diam}(X_{\mathbf{a}})g(n))^s < \infty$.

The divergence case. For any $\mathbf{a} \in \cup_{n=1}^{\infty}\mathcal{A}^n$ and $0 \le l \le |\mathbf{a}| - 1$ there exists $\mathfrak{p}/q \in X$ such that $\mathfrak{p}/q = \pi\left(a_1 \ldots a_l\left(a_{l+1} \ldots a_{|\mathbf{a}|}\right)^{\infty}\right)$. Therefore

$$B\left(\pi\left(a_1 \ldots a_l\left(a_{l+1} \ldots a_{|\mathbf{a}|}\right)^{\infty}\right), \frac{g(|\mathbf{a}|)\,\mathrm{Diam}(X_{\mathbf{a}})}{C_2}\right) = B\left(\mathfrak{p}/q, \frac{g(|\mathbf{a}|)\,\mathrm{Diam}(X_{\mathbf{a}})}{C_2}\right).$$

It now follows from the fact g is non-increasing together with (2.3) that

$$\begin{aligned} &B\left(\pi\left(a_1 \ldots a_l\left(a_{l+1} \ldots a_{|\mathbf{a}|}\right)^{\infty}\right), \frac{g(|\mathbf{a}|)\,\mathrm{Diam}(X_{\mathbf{a}})}{C_2}\right)\\ &\subseteq B\left(\mathfrak{p}/q, \frac{g(n(\mathfrak{p}/q))\,\mathrm{Diam}\left(X_{a_1\ldots a_{n(\mathfrak{p}/q)}}\right)}{C_2}\right)\\ &\subseteq B\left(\mathfrak{p}/q, \frac{g(n(\mathfrak{p}/q))}{q_{\mathrm{int}}(\mathfrak{p}/q)}\right) \end{aligned}$$

Therefore $W_{\Phi}(\Psi) \subset W_{\Phi}^{***}(g)$ for Ψ given by $\Psi(\mathbf{a}) = \frac{g(|\mathbf{a}|)\,\mathrm{Diam}(X_{\mathbf{a}})}{C_2}$, and our result follows if $\mu(W_{\Phi}(\Psi)) = 1$. However $\mu(W_{\Phi}(\Psi)) = 1$ follows from Theorem 1.1 together with our assumption $\sum_{n=1}^{\infty}\sum_{\mathbf{a}\in\mathcal{A}^n} n \cdot (\mathrm{Diam}(X_{\mathbf{a}})g(n))^{\dim_S(\Phi)} = \infty$. □

As remarked upon in the equicontractive case, if $\mathfrak{p}/q \in X$ is in its reduced from then we must have $q \le q_{\mathrm{int}}(\mathfrak{p}/q)$. This observation together with Theorem 2.7 implies the following corollary.

Corollary 2.8 *Let Φ be an IFS of the form $\Phi = \left\{\phi_a(x) = \frac{x+\mathfrak{p}_a}{q_a}\right\}$ satisfying the strong separation condition and $h_{\mathbf{p}} < -2\log\sum_{a\in\mathcal{A}} p_a^2$. Let $g : \mathbb{N} \to [0,\infty)$ be a non-increasing function. If $\sum_{n=1}^{\infty}\sum_{\mathbf{a}\in\mathcal{A}^n} n \cdot (\mathrm{Diam}(X_{\mathbf{a}})g(n))^{\dim_S(\Phi)} = \infty$ then*

$$\mu\left(\left\{x \in X : \|x - \mathfrak{p}/q\| < \frac{g(n(\mathfrak{p}/q))}{q} \text{ for i.m. } \mathfrak{p}/q \in X\right\}\right) = 1.$$

3 Proof of Theorem 1.1

In this section we prove Theorem 1.1. Statement 1 of this theorem is proved by a standard covering argument and as such is omitted. We will only prove Statement 2 of Theorem 1.1 in full. Statement 3 is proved via an almost identical method. Where appropriate we will indicate in the footnotes where the proofs differ and why Statement 3 does not require the assumption g is non-increasing.

3.1 Technical preliminaries

Given a word $\mathbf{a} \in \cup_{n=1}^{\infty} \mathcal{A}^n$ we let $C_t(\mathbf{a})$ denote the number of distinct words of length t appearing within $\mathbf{a}$. Given a probability vector $\mathbf{p}$ and $n \in \mathbb{N}$ we let

$$k_{n,\mathbf{p}} := \left\lfloor \frac{-\log n}{\log \sum_{a \in \mathcal{A}} p_a^2} \right\rfloor + 1.$$

We also let

$$\mathcal{F}_{n,\mathbf{p}} := \left\{ \mathbf{a} \in \mathcal{A}^n : C_{k_{n,\mathbf{p}}}(\mathbf{a}) \geq \left\lfloor \frac{n}{10} \right\rfloor \right\}.$$

When the choice of $\mathbf{p}$ is implicit, we simply denote $k_{n,\mathbf{p}}$ by k_n and $\mathcal{F}_{n,\mathbf{p}}$ by $\mathcal{F}_n$.

The following lemma is a suitable adaptation of Lemma 4.1. from [37].

Lemma 3.1 *Let* $\mathbf{p}$ *be a probability vector and* $\mathfrak{m}$ *be the corresponding Bernoulli measure on* $\mathcal{A}^{\mathbb{N}}$. *Then for* n *sufficiently large, we have*

$$\mathfrak{m}\left(\bigcup_{\mathbf{a} \in \mathcal{F}_n} [\mathbf{a}] \right) \geq \frac{7}{32}.$$

Proof Given $\mathbf{a} \in \mathcal{A}^n$ and $\mathbf{b} \in \mathcal{A}^{k_n}$, let

$$|\mathbf{a}|_{\mathbf{b}} := \# \left\{ 0 \leq l \leq n - k_n : a_{l+1} \ldots a_{l+k_n} = \mathbf{b} \right\}.$$

It is convenient to express our proof using the language of probability theory. As such let $(Z_l)_{l=1}^n$ be a sequence of independent and identically distributed random variables taking values in $\mathcal{A}$ such that $\mathbb{P}(Z_l = a) = p_a$ for all $a \in \mathcal{A}$. Let $\Sigma_{k_n}^n$ be the real valued random variable given by

$$\Sigma_{k_n}^n := \sum_{\mathbf{b} \in \mathcal{A}^{k_n}} |Z_1 \ldots Z_n|_{\mathbf{b}}^2.$$

We start by bounding the expectation of $\Sigma_{k_n}^n$:

$$\begin{aligned}
\mathbb{E}(\Sigma_{k_n}^n) &= \sum_{\mathbf{b}\in\mathcal{A}^{k_n}} \mathbb{E}\left(|Z_1\ldots Z_n|_{\mathbf{b}}^2\right)\\
&= \sum_{\mathbf{b}\in\mathcal{A}^{k_n}} \mathbb{E}\left(\left(\sum_{l=0}^{n-k_n} \mathbb{1}_{\mathbf{b}}(Z_{l+1}\ldots Z_{l+k_n})\right)^2\right)\\
&= \sum_{\mathbf{b}\in\mathcal{A}^{k_n}} \mathbb{E}\left(\sum_{l,j=0}^{n-k_n} \mathbb{1}_{\mathbf{b}}(Z_{l+1}\ldots Z_{l+k_n})\cdot \mathbb{1}_{\mathbf{b}}(Z_{j+1}\ldots Z_{j+k_n})\right)\\
&= \sum_{l,j=0}^{n-k_n} \mathbb{E}\left(\sum_{\mathbf{b}\in\mathcal{A}^{k_n}} \mathbb{1}_{\mathbf{b}}(Z_{l+1}\ldots Z_{l+k_n})\cdot \mathbb{1}_{\mathbf{b}}(Z_{j+1}\ldots Z_{j+k_n})\right)\\
&= \sum_{l,j=0}^{n-k_n} \mathbb{E}\left(\mathbb{1}_{Z_{l+1}\ldots Z_{l+k_n}=Z_{j+1}\ldots Z_{j+k_n}}\right)\\
&= \sum_{l,j=0}^{n-k_n} \mathbb{P}\left(Z_{l+1}\ldots Z_{l+k_n}=Z_{j+1}\ldots Z_{j+k_n}\right). \qquad (3.1)
\end{aligned}$$

With (3.1) in mind, we now bound $\mathbb{P}(Z_{l+1}\ldots Z_{l+k_n}=Z_{j+1}\ldots Z_{j+k_n})$ from above. If $l=j$ then clearly $\mathbb{P}(Z_{l+1}\ldots Z_{l+k_n}=Z_{j+1}\ldots Z_{j+k_n})=1$. We remark that for any parameters $l, j, k\in\mathbb{N}$, if $l+k<j+1$ then by independence we have

$$\begin{aligned}
&\mathbb{P}(Z_{l+1}\ldots Z_{l+k}=Z_{j+1}\ldots Z_{j+k})\\
&\quad=\prod_{m=1}^{k}\mathbb{P}(Z_{l+m}=Z_{j+m})=\prod_{m=1}^{k}\left(\sum_{a\in\mathcal{A}}p_a^2\right)=\left(\sum_{a\in\mathcal{A}}p_a^2\right)^k.
\end{aligned}$$

We will use the fact that if $l+k<j+1$ then

$$\mathbb{P}(Z_{l+1}\ldots Z_{l+k}=Z_{j+1}\ldots Z_{j+k})=\left(\sum_{a\in\mathcal{A}}p_a^2\right)^k \qquad (3.2)$$

throughout our proof.

We now proceed via a case analysis. If $l+k_n<j+1$ then (3.2) immediately implies

$$\mathbb{P}(Z_{l+1}\ldots Z_{l+k_n}=Z_{j+1}\ldots Z_{j+k_n})=\left(\sum_{a\in\mathcal{A}}p_a^2\right)^{k_n}. \qquad (3.3)$$

Now suppose that $l + \frac{k_n}{4} \leq j + 1 \leq l + k_n$. Observe that $Z_{l+1} \dots Z_{l+k_n} = Z_{j+1} \dots Z_{j+k_n}$ implies that $Z_{2l-j+k_n+1} \dots Z_{l+k_n} = Z_{l+k_n+1} \dots Z_{j+k_n}$. Therefore by (3.2) we have

$$\begin{aligned}\mathbb{P}(Z_{l+1} \dots Z_{l+k_n} = Z_{j+1} \dots Z_{j+k_n}) &\leq \mathbb{P}(Z_{2l-j+k_n+1} \dots Z_{l+k_n} = Z_{l+k_n+1} \dots Z_{j+k_n}) \\ &= \left(\sum_{a \in \mathcal{A}} p_a^2\right)^{j-l+1} \\ &\leq \left(\sum_{a \in \mathcal{A}} p_a^2\right)^{\frac{k_n}{4}}. \end{aligned} \tag{3.4}$$

Let us now suppose that $l+1 < j+1 < l+\frac{k_n}{4}$ and $Z_{l+1} \dots Z_{l+k_n} = Z_{j+1} \dots Z_{j+k_n}$. Notice that $Z_{l+1} \dots Z_{l+k_n} = Z_{j+1} \dots Z_{j+k_n}$ implies that $Z_{l+i+1} \dots Z_{l+i+\lfloor \frac{k_n}{4} \rfloor} = Z_{j+i+1} \dots Z_{j+i+\lfloor \frac{k_n}{4} \rfloor}$ for any $0 \leq i < k_n - \lfloor \frac{k_n}{4} \rfloor$. Repeatedly applying this identity, it follows that

$$\begin{aligned} Z_{l+1} \dots Z_{l+\lfloor \frac{k_n}{4} \rfloor} &= Z_{j+1} \dots Z_{j+\lfloor \frac{k_n}{4} \rfloor} = Z_{l+2(j-l)+1} \dots Z_{l+2(j-l)+\lfloor \frac{k_n}{4} \rfloor} \\ &= \cdots \\ &= Z_{l+d(j-l)+1} \dots Z_{l+d(j-l)+\lfloor \frac{k_n}{4} \rfloor} \end{aligned}$$

for any d such that $l + d(j-l) + 1 < k_n - \lfloor \frac{k_n}{4} \rfloor$. Since $j - l < \frac{k_n}{4}$, it follows that we can pick d such that $l + \lfloor \frac{k_n}{4} \rfloor < l + d(j-l) + 1 \leq k_n - \lfloor \frac{k_n}{4} \rfloor$. Taking such a d, it then follows from (3.2) that

$$\begin{aligned} \mathbb{P}(Z_{l+1} \dots Z_{l+k_n} &= Z_{j+1} \dots Z_{j+k_n}) \\ &\leq \mathbb{P}\left(Z_{l+1} \dots Z_{l+\lfloor \frac{k_n}{4} \rfloor} = Z_{l+d(j-l)+1} \dots Z_{l+d(j-l)+\lfloor \frac{k_n}{4} \rfloor}\right) \\ &= \left(\sum_{a \in \mathcal{A}} p_a^2\right)^{\lfloor \frac{k_n}{4} \rfloor}. \end{aligned} \tag{3.5}$$

Recalling (3.1), we have

$$\begin{aligned} \mathbb{E}(\Sigma_{k_n}^n) &= \sum_{l,j=0}^{n-k_n} \mathbb{P}(Z_{l+1} \dots Z_{l+k_n} = Z_{j+1} \dots Z_{j+k_n}) \\ &= \sum_{l=0}^{n-k_n} 1 + 2 \sum_{l=0}^{n-k_n-1} \sum_{j=l+1}^{n-k_n} \mathbb{P}(Z_{l+1} \dots Z_{l+k_n} = Z_{j+1} \dots Z_{j+k_n}) \\ &= \sum_{l=0}^{n-k_n} 1 + 2 \sum_{l=0}^{n-k_n-1} \left(\sum_{l+k_n < j+1 \leq n-k_n} \mathbb{P}(\cdot) + \sum_{l+\frac{k_n}{4} \leq j+1 \leq l+k_n} \mathbb{P}(\cdot) + \sum_{l+1 < j+1 < l+\frac{k_n}{4}} \mathbb{P}(\cdot) \right). \end{aligned}$$

Applying the bounds provided by (3.3), (3.4), (3.5) we have

$$\begin{aligned}\mathbb{E}(\Sigma_{k_n}^n) &\leq n + 2\sum_{l=0}^{n-k_n-1}\left(\sum_{l+k_n<j+1\leq n-k_n}\left(\sum_{a\in\mathcal{A}}p_a^2\right)^{k_n} + \sum_{l+\frac{k_n}{4}\leq j+1\leq l+k_n}\left(\sum_{a\in\mathcal{A}}p_a^2\right)^{\frac{k_n}{4}}\right.\\ &\qquad\left.+\sum_{l+1<j+1<l+\frac{k_n}{4}}\left(\sum_{a\in\mathcal{A}}p_a^2\right)^{\lfloor\frac{k_n}{4}\rfloor}\right)\\ &\leq n+2\sum_{l=0}^{n-k_n-1}\left(n\left(\sum_{a\in\mathcal{A}}p_a^2\right)^{k_n}+k_n\left(\sum_{a\in\mathcal{A}}p_a^2\right)^{\lfloor\frac{k_n}{4}\rfloor}\right).\end{aligned}$$

By the definition of k_n we know that

$$n\left(\sum_{a\in\mathcal{A}}p_a^2\right)^{k_n}\leq 1.$$

Moreover, as k_n grows logarithmically in n, and $\left(\sum_{a\in\mathcal{A}}p_a^2\right)^{\lfloor\frac{k_n}{4}\rfloor}$ decays to zero polynomially fast, we know that

$$k_n\left(\sum_{a\in\mathcal{A}}p_a^2\right)^{\lfloor\frac{k_n}{4}\rfloor}\leq 1$$

for all n sufficiently large. Therefore for n sufficiently large we have

$$\mathbb{E}(\Sigma_{k_n}^n)\leq n+2\sum_{l=0}^{n-k_n-1}2\leq 5n. \tag{3.6}$$

Let B be the event that

$$\#\left\{\mathbf{b}\in\mathcal{A}^{k_n}:|Z_1\ldots Z_n|_{\mathbf{b}}\geq 1\right\}<\left\lfloor\frac{n}{10}\right\rfloor.$$

We now bound the probability of B from above. It follows from (3.6) that for n sufficiently large we have

$$\begin{aligned}5n &\geq \mathbb{E}(\Sigma_{k_n}^n)\\ &\geq \mathbb{E}\left(\sum_{\mathbf{b}\in\mathcal{A}^{k_n}}|Z_1\ldots Z_n|_{\mathbf{b}}^2\middle| B\right)\mathbb{P}(B)\end{aligned}$$

$$\begin{aligned}
&\geq \mathbb{P}(B) \cdot \min \left\{ m_1^2 + \cdots + m_{\lfloor \frac{n}{10} \rfloor}^2 : m_1 + \cdots + m_{\lfloor \frac{n}{10} \rfloor} = n - k_n + 1 \right\} \\
&= \mathbb{P}(B) \cdot \left(\frac{n - k_n + 1}{\lfloor \frac{n}{10} \rfloor} \right)^2 \cdot \left\lfloor \frac{n}{10} \right\rfloor \\
&= \mathbb{P}(B) \cdot (n - k_n + 1)^2 \cdot \left\lfloor \frac{n}{10} \right\rfloor^{-1} \\
&\geq \mathbb{P}(B) \cdot 10 \cdot \frac{(4n/5)^2}{n} \\
&= \mathbb{P}(B) \cdot \frac{32n}{5}.
\end{aligned}$$

In the penultimate line we used that for n sufficiently large we have $n - k_n + 1 \geq \frac{4n}{5}$. This is because k_n grows logarithmically in n. Therefore

$$\mathbb{P}(B) \leq \frac{25}{32}.$$

This means that $\mathbb{P}(B^c) \geq 7/32$. Since

$$\mathfrak{m}\left(\bigcup_{\mathbf{a} \in \mathcal{F}_n} [\mathbf{a}] \right) = \mathbb{P}(B^c)$$

this completes our proof. □

Let us now suppose that we are given a probability vector $\mathbf{p}$ and an IFS Φ. Recall that the entropy of $\mathbf{p}$ and the Lyapunov exponent of $\mathbf{p}$ are defined to be

$$h_{\mathbf{p}} = -\sum_{a \in \mathcal{A}} p_a \log p_a \quad \text{and} \quad \chi_{\Phi, p} = -\sum_{a \in \mathcal{A}} p_a \log r_a$$

respectively. Notice that when $\mathbf{p} = \left(r_a^{\dim_S(\Phi)} \right)_{a \in \mathcal{A}}$ then $\dim_S(\Phi) = \frac{h_{\mathbf{p}}}{\chi_{\Phi,\mathbf{p}}}$. When the choice of $\mathbf{p}$ and Φ is implicit, we simply denote $h_{\mathbf{p}}$ by h and $\chi_{\Phi,\mathbf{p}}$ by χ. Given a word $\mathbf{a} \in \mathcal{A}^n$ and $\epsilon > 0$ we let

$$\begin{aligned}
Bad(\mathbf{a}, \epsilon) := &\left\{ 0 \leq l \leq n - k_n : \prod_{i=1}^{k_n} r_{a_{l+i}} \notin \left[e^{k_n(-\chi-\epsilon)}, e^{k_n(-\chi+\epsilon)} \right] \right\} \\
&\cup \left\{ 0 \leq l \leq n - k_n : \prod_{i=1}^{k_n} p_{a_{l+i}} \notin \left[e^{k_n(-h-\epsilon)}, e^{k_n(-h+\epsilon)} \right] \right\}.
\end{aligned}$$

We then define

$$Bad(n, \epsilon) := \left\{ \mathbf{a} \in \mathcal{A}^n : \# Bad(\mathbf{a}, \epsilon) \geq \left\lfloor \frac{n}{20} \right\rfloor \right\}.$$

Lemma 3.2 *Let Φ be an IFS,* $\mathbf{p}$ *be a probability vector, and* $\mathfrak{m}$ *be the Bernoulli measure corresponding to* $\mathbf{p}$. *For any* $\epsilon > 0$, *there exists* $\gamma \in (0, 1)$ *such that*

$$\mathfrak{m}\left(\bigcup_{\mathbf{a}\in Bad(n,\epsilon)} [\mathbf{a}]\right) \ll \gamma^{k_n}.$$

Proof As in the proof of the previous lemma, it is useful to express this proof in terms of random variables. Let $(Z_l)_{l=1}^n$ be a sequence of independent and identically distributed random variables taking values in $\mathcal{A}$ such that $\mathbb{P}(Z_l = a) = p_a$, and let $\epsilon > 0$ be arbitrary. We start our proof by bounding from above the expectation

$$\mathbb{E}\left(\sum_{l=0}^{n-k_n} \mathbb{1}_{[e^{k_n(-\chi-\epsilon)},e^{k_n(-\chi+\epsilon)}]^c}\left(\prod_{i=1}^{k_n} r_{Z_{l+i}}\right)\right).$$

By the linearity of expectation we have

$$\begin{aligned}
&\mathbb{E}\left(\sum_{l=0}^{n-k_n} \mathbb{1}_{[e^{k_n(-\chi-\epsilon)}e^{k_n(-\chi+\epsilon)}]^c}\left(\prod_{i=1}^{k_n} r_{Z_{l+i}}\right)\right)\\
&\quad=\sum_{l=0}^{n-k_n} \mathbb{E}\left(\mathbb{1}_{[e^{k_n(-\chi-\epsilon)},e^{k_n(-\chi+\epsilon)}]^c}\left(\prod_{i=1}^{k_n} r_{Z_{l+i}}\right)\right)\\
&\quad=\sum_{l=0}^{n-k_n} \mathbb{P}\left(\prod_{i=1}^{k_n} r_{Z_{l+i}} \notin \left[e^{k_n(-\chi-\epsilon)}, e^{k_n(-\chi+\epsilon)}\right]\right).
\end{aligned}$$

By Hoeffding's inequality for large deviations [24], there exists $\gamma_1 := \gamma_1(\epsilon, \mathbf{p}, \Phi) \in (0, 1)$ such that

$$\mathbb{P}\left(\prod_{i=1}^{k_n} r_{Z_{l+i}} \notin \left[e^{k_n(-\chi-\epsilon)}, e^{k_n(-\chi+\epsilon)}\right]\right) \ll \gamma_1^{k_n}.$$

Therefore

$$\mathbb{E}\left(\sum_{l=0}^{n-k_n} \mathbb{1}_{[e^{k_n(-\chi-\epsilon)},e^{k_n(-\chi+\epsilon)}]^c}\left(\prod_{i=1}^{k_n} r_{Z_{l+i}}\right)\right) \ll n\cdot\gamma_1^{k_n}.$$

Now by Markov's inequality, we have

$$\mathbb{P}\left(\sum_{l=0}^{n-k_n} \mathbb{1}_{[e^{k_n(-\chi-\epsilon)},e^{k_n(-\chi+\epsilon)}]^c}\left(\prod_{i=1}^{k_n} r_{Z_{l+i}}\right) \geq \frac{1}{2}\left\lfloor\frac{n}{20}\right\rfloor\right) \ll \gamma_1^{k_n}.$$

By an analogous argument, it can be shown that there exists $\gamma_2 \in (0, 1)$ such that

$$\mathbb{P}\left(\sum_{l=0}^{n-k_n} \mathbb{1}_{[e^{k_n(-h-\epsilon)}, e^{k_n(-h+\epsilon)}]^c}\left(\prod_{i=1}^{k_n} p_{Z_{l+i}}\right) \geq \frac{1}{2}\left\lfloor \frac{n}{20} \right\rfloor\right) \ll \gamma_2^{k_n}.$$

Now let $\gamma = \max\{\gamma_1, \gamma_2\}$. Clearly if $(Z_l)_{l=1}^n$ is such that either

$$\prod_{i=1}^{k_n} r_{Z_{l+i}} \notin \left[e^{k_n(-\chi-\epsilon)}, e^{k_n(-\chi+\epsilon)}\right] \quad \text{or} \quad \prod_{i=1}^{k_n} p_{Z_{l+i}} \notin \left[e^{k_n(-h-\epsilon)}, e^{k_n(-h+\epsilon)}\right]$$

for $\lfloor \frac{n}{20} \rfloor$ values of l, it must satisfy

$$\prod_{i=1}^{k_n} r_{a_{l+i}} \notin \left[e^{k_n(-\chi-\epsilon)}, e^{k_n(-\chi+\epsilon)}\right]$$

for at least $\frac{1}{2}\lfloor \frac{n}{20} \rfloor$ values of l or

$$\prod_{i=1}^{k_n} p_{a_{l+i}} \notin \left[e^{k_n(-h-\epsilon)}, e^{k_n(-h+\epsilon)}\right]$$

for at least $\frac{1}{2}\lfloor \frac{n}{20} \rfloor$ values of l. As such we may conclude that

$$\begin{aligned}
&\mathfrak{m}\left(\bigcup_{\mathbf{a} \in Bad(n,\epsilon)} [\mathbf{a}]\right) \\
&= \mathbb{P}\left(\#\left\{l : \prod_{i=1}^{k_n} r_{Z_{l+i}} \notin \left[e^{k_n(-\chi-\epsilon)}, e^{k_n(-\chi+\epsilon)}\right] \text{ or } \prod_{i=1}^{k_n} p_{Z_{l+i}} \notin \left[e^{k_n(-h-\epsilon)}, e^{k_n(-h+\epsilon)}\right]\right\}\right. \\
&\geq \left.\left\lfloor \frac{n}{20} \right\rfloor\right) \\
&\leq \mathbb{P}\left(\sum_{l=0}^{n-k_n} \mathbb{1}_{[e^{k_n(-\chi-\epsilon)}, e^{k_n(-\chi+\epsilon)}]^c}\left(\prod_{i=1}^{k_n} r_{a_{l+i}}\right) \geq \frac{1}{2}\left\lfloor \frac{n}{20} \right\rfloor\right) \\
&\quad + \mathbb{P}\left(\sum_{l=0}^{n-k_n} \mathbb{1}_{[e^{k_n(-h-\epsilon)}, e^{k_n(-h+\epsilon)}]^c}\left(\prod_{i=1}^{k_n} p_{a_{l+i}}\right) \geq \frac{1}{2}\left\lfloor \frac{n}{20} \right\rfloor\right) \\
&\ll \gamma^{k_n}.
\end{aligned}$$

This completes our proof. □

Combining Lemmas 3.1 and 3.2, we see that the following statement holds.

Lemma 3.3 *Let Φ be an IFS, $\mathbf{p}$ be a probability vector, and $\mathfrak{m}$ be the Bernoulli measure corresponding to $\mathbf{p}$. For any $\epsilon > 0$, it is the case that for all n sufficiently large there exists a set $Good(n, \epsilon) \subset \mathcal{A}^n$ satisfying:*

1. $\mathfrak{m}\left(\bigcup_{\mathbf{a}\in Good(n,\epsilon)}[\mathbf{a}]\right)\geq 7/64.$
2. *For each* $\mathbf{a}\in Good(n,\epsilon)$ *there exists a set* $W_{\mathbf{a}}\subset\{0,\ldots,n-k_n\}$ *satisfying:*
   (a) $\#W_{\mathbf{a}}\geq\left\lfloor\frac{n}{20}\right\rfloor.$
   (b) *If* $l,l'\in W_{\mathbf{a}}$ *and* $l\neq l'$ *then* $a_{l+1}\ldots a_{l+k_n}\neq a_{l'+1}\ldots a_{l'+k_n}.$
   (c) *For each* $l\in W_{\mathbf{a}}$ *we have*

$$\prod_{i=1}^{k_n} r_{a_{l+i}}\in\left[e^{k_n(-\chi-\epsilon)},e^{k_n(-\chi+\epsilon)}\right]$$

and

$$\prod_{i=1}^{k_n} p_{a_{l+i}}\in\left[e^{k_n(-h-\epsilon)},e^{k_n(-h+\epsilon)}\right].$$

Lemma 3.3 is the main technical result in this section and will play an important part in our proof of Theorem 1.1.[1]

3.2 Proof of Statement 2 from Theorem 1.1

Before moving on to our proof of Statement 2, it is useful to record for later reference a number of technical results. We start by recalling a well known lemma.

Lemma 3.4 *Let* $(X,A,\mathfrak{m})$ *be a finite measure space and* $E_n\in A$ *be a sequence of sets such that* $\sum_{n=1}^{\infty}\mathfrak{m}(E_n)=\infty$. *Then*

$$\mathfrak{m}\left(\limsup_{n\to\infty}E_n\right)\geq\limsup_{Q\to\infty}\frac{\left(\sum_{n=1}^{Q}\mathfrak{m}(E_n)\right)^2}{\sum_{n,m=1}^{Q}\mathfrak{m}(E_n\cap E_m)}.$$

Lemma 3.4 is due to Kochen and Stone [28]. For a proof of this lemma see either [20, Lemma 2.3] or [36, Lemma 5]. The following density lemma follows from [8, Lemma 6]. It has been phrased for our purposes.

Lemma 3.5 *Let* $\mathfrak{m}$ *be a Bernoulli measure on* $\mathcal{A}^{\mathbb{N}}$ *and* $E\subset\mathcal{A}^{\mathbb{N}}$. *Suppose that there exists* $c>0$ *such that for each finite word* $\mathbf{c}\in\cup_{n=1}^{\infty}\mathcal{A}^n$ *we have*

$$\mathfrak{m}([\mathbf{c}]\cap E)\geq c\cdot\mathfrak{m}([\mathbf{c}]).$$

Then $\mathfrak{m}(E)=1.$

[1] For the proof of Statement 3 from Theorem 1.1 we do not require item (2*c*) from Lemma 3.3. This is because our IFS is equicontractive and so the probability vector $\mathbf{p}$ is the uniform vector $(\mathcal{A}^{-1})_{a\in\mathcal{A}}$. Therefore we know exactly how the products in item (2*c*) of Lemma 3.3 will behave. It is instructive to think that the proof of Statement 3 follows the proof of Statement 2 without the introduction of the parameter ϵ.

For the rest of this section we fix an IFS Φ satisfying

$$h_{\mathbf{p}} < -2\log \sum_{a\in\mathcal{A}} p_a^2.$$

We emphasise that throughout this section $\mathbf{p}$ will always be the probability vector corresponding to $(r_a^{\dim_S(\Phi)})$. It follows from this inequality that we can pick $\epsilon > 0$ sufficiently small such that

$$\frac{-2\chi}{h} < \frac{-\chi-\epsilon}{-\log\sum_{a\in\mathcal{A}} p_a^2} \tag{3.7}$$

and

$$h+\epsilon < -2\log \sum_{a\in A} p_a^2. \tag{3.8}$$

For the rest of this section we fix an $\epsilon > 0$ such that (3.7) and (3.8) are both satisfied. These are the only conditions ϵ will need to satisfy. Let $\Psi : \cup_{n=1}^{\infty}\mathcal{A}^n \to [0,\infty)$ be an arbitrary function of the form $\Psi(\mathbf{a}) = \mathrm{Diam}(X_{\mathbf{a}})g(|\mathbf{a}|)$ for some non-increasing $g : \mathbb{N} \to [0,\infty)$. We also assume that g is such that

$$\sum_{n=1}^{\infty}\sum_{\mathbf{a}\in\mathcal{A}^n} n\cdot(\mathrm{Diam}(X_{\mathbf{a}})g(n))^{\dim_S(\Phi)} = \infty.$$

As previously remarked, this divergence condition is equivalent to

$$\sum_{n=1}^{\infty} n\cdot g(n)^{\dim_S(\Phi)} = \infty. \tag{3.9}$$

The following two lemmas allow us to replace g with a function whose decaying behaviour we know more about.

Lemma 3.6 *Assume that g is a non-increasing function satisfying* (3.9)*. Let $g_1 : \mathbb{N} \to [0,\infty)$ be given by*

$$g_1(n) = \min\left\{g(n), \frac{1}{n^{2/\dim_S(\Phi)}}\right\}.$$

Then

$$\sum_{n=1}^{\infty} n\cdot g_1(n)^{\dim_S(\Phi)} = \infty.$$

Proof Our argument is an adaptation of the proof of Lemma 6.3 from [12]. Since g is non-increasing, and so is the sequence $\left(n^{-2/\dim_S(\Phi)}\right)$, we see that the function g_1 is also non-increasing. Let us now suppose that

$$\sum_{n=1}^{\infty} n \cdot g_1(n)^{\dim_S(\Phi)} < \infty. \tag{3.10}$$

Since g_1 is non-increasing, for any m sufficiently large we have

$$m^2 g_1(m)^{\dim_S(\Phi)} \leq 10 \sum_{n=\lfloor m/2 \rfloor}^{m} n \cdot g_1(n)^{\dim_S(\Phi)}.$$

Equation (3.10) then implies that

$$\lim_{m\to\infty} \sum_{n=\lfloor m/2 \rfloor}^{m} n \cdot g_1(n)^{\dim_S(\Phi)} = 0.$$

Combining the two equations above, we may conclude that

$$g_1(m) < \frac{1}{m^{2/\dim_S(\Phi)}}$$

for all m sufficiently large. This means that $g_1(n) = g(n)$ for n sufficiently large. Therefore by (3.10) we must have

$$\sum_{n=1}^{\infty} n \cdot g(n)^{\dim_S(\Phi)} < \infty.$$

This contradicts our initial assumption that g satisfies (3.9). Therefore we must have.[2]

$$\sum_{n=1}^{\infty} n \cdot g_1(n)^{\dim_S(\Phi)} = \infty.$$

□

Lemma 3.7 *Assume that g is a non-increasing function satisfying* (3.9) *and let g_1 be as in Lemma* 3.6. *Let $g_2 : \mathbb{N} \to [0, \infty)$ be given by*

$$g_2(n) = \begin{cases} g_1(n) & \text{if } g_1(n) \geq \frac{1}{n^{4/\dim_S(\Phi)}}; \\ 0 & \text{if } g_1(n) < \frac{1}{n^{4/\dim_S(\Phi)}}. \end{cases}$$

[2] This is the only part in our proof of Statement 2 where the assumption g is non-increasing is used. The proof of Statement 3 differs here in that we define $g_1 : \mathbb{N} \to [0, \infty)$ by $g_1(n) = \min\{g(n), \frac{1}{n^{1/\dim_S(\Phi)}}\}$. Then the appropriate analogue of Lemma 3.6 holds for any g satisfying (3.9). This is why Statement 3 holds for arbitrary g, not just those g that are non-increasing.

Then

$$\sum_{n=1}^{\infty} n \cdot g_2(n)^{\dim_S(\Phi)} = \infty.$$

Proof This follows from Lemma 3.6 and the fact $\sum_{n=1}^{\infty} n \cdot \left(\frac{1}{n^{4/\dim_S(\Phi)}}\right)^{\dim_S(\Phi)} < \infty$. □

Let g_2 be as in Lemma 3.7. We now define $\Psi_2 : \cup_{n=1}^{\infty} \mathcal{A}^n \to [0, \infty)$ by $\Psi_2(\mathbf{a}) = \mathrm{Diam}(X_{\mathbf{a}}) g_2(|\mathbf{a}|)$. Since $g_2(n) \leq g(n)$ for all n it follows that $W_\Phi(\Psi_2) \subset W_\Phi(\Psi)$. Therefore to prove Statement 2 of Theorem 1.1 it is sufficient to show that $\mu(W_\Phi(\Psi_2)) = 1$.

Let $\mathbf{c} \in \cup_{n=1}^{\infty} \mathcal{A}^n$ be arbitrary and fixed. We will show that

$$\mathfrak{m}\left([\mathbf{c}] \cap \pi^{-1}\left(W_\Phi(\Psi_2)\right)\right) \geq c \cdot \mathfrak{m}([\mathbf{c}]) \tag{3.11}$$

for some c independent of $\mathbf{c}$. Lemma 3.5 then implies that $\mathfrak{m}\left(\pi^{-1}(W_\Phi(\Psi_2)\right) = 1$. Since $\mu = \pi\mathfrak{m}$ this implies that $\mu\left(W_\Phi(\Psi_2)\right) = 1$. Therefore to complete our proof it suffices to show that (3.11) holds.

Let us fix an N sufficiently large so that Lemma 3.3 applies for all $n \geq N$ for our choice of ϵ. We may also assume that N is sufficiently large so that

$$\frac{1}{n^{2/\dim_S(\Phi)}} < e^{k_n(-\chi-\epsilon)} \tag{3.12}$$

for $n \geq N$, and so that there exists $\gamma \in (1, 2)$ for which

$$e^{(h+\epsilon)k_n} \leq n^{\gamma} \tag{3.13}$$

for $n \geq N$. The existence of γ, and the fact that (3.12) and (3.13) are satisfied for n sufficiently large, follows from (3.7), (3.8) and the fact $\dim_S(\Phi) = \frac{h}{\chi}$.

Let $n \geq N$, for each $\mathbf{a} \in Good(n, \epsilon)$ and $l \in W_{\mathbf{a}}$ we consider the ball

$$B\left(\pi(\mathbf{c}a_1 \ldots a_l(a_{l+1} \ldots a_n)^{\infty}), \mathrm{Diam}(X_{\mathbf{ca}}) g_2(|\mathbf{c}| + n)\right).$$

If $g_2(|\mathbf{c}| + n) \neq 0$ then there exists $h_{\mathbf{a},l} \in \mathbb{N}$ and $l + 1 \leq j_{\mathbf{a},l} \leq n$ such that

$$X_{\mathbf{c}a_1 \ldots a_l(a_{l+1} \ldots a_n)^{h_{\mathbf{a},l}} a_{l+1} \ldots a_{j_{\mathbf{a},l}}} \subseteq B\left(\pi(\mathbf{c}a_1 \ldots a_l(a_{l+1} \ldots a_n)^{\infty}), \mathrm{Diam}(X_{\mathbf{ca}}) g_2(|\mathbf{c}| + n)\right) \tag{3.14}$$

and

$$\begin{aligned}\min_{a \in \mathcal{A}} r_a \cdot \mathrm{Diam}(X_{\mathbf{ca}}) g_2(|\mathbf{c}| + n) &\leq \mathrm{Diam}\left(X_{\mathbf{c}a_1 \ldots a_l(a_{l+1} \ldots a_n)^{h_{\mathbf{a},l}} a_{l+1} \ldots a_{j_{\mathbf{a},l}}}\right) \\ &< \mathrm{Diam}(X_{\mathbf{ca}}) g_2(|\mathbf{c}| + n).\end{aligned} \tag{3.15}$$

It follows from the fact $\mathbf{p} = (r_{\mathbf{a}}^{\dim_S(\Phi)})$ and $\mathfrak{m}$ is the Bernoulli measure on $\mathcal{A}^{\mathbb{N}}$ corresponding to $\mathbf{p}$, that for any word $\mathbf{a} \in \cup_{n=1}^{\infty}\mathcal{A}^n$ we have

$$\mathfrak{m}([\mathbf{a}]) \asymp \mathrm{Diam}(X_{\mathbf{a}})^{\dim_S(\Phi)}. \tag{3.16}$$

Combining (3.15) together with (3.16) we can deduce that

$$\mathfrak{m}\left(\left[\mathbf{c}a_1 \dots a_l (a_{l+1} \dots a_n)^{h_{\mathbf{a},l}} a_{l+1} \dots a_{j_{\mathbf{a},l}}\right]\right) \asymp \mathfrak{m}([\mathbf{ca}]) g_2(|\mathbf{c}| + n)^{\dim_S(\Phi)}. \tag{3.17}$$

We will use the cylinder sets $\left[\mathbf{c}a_1 \dots a_l (a_{l+1} \dots a_n)^{h_{\mathbf{a},l}} a_{l+1} \dots a_{j_{\mathbf{a},l}}\right]$ to prove that (3.11) holds. Before doing that it is useful to prove some properties of the parameters $h_{\mathbf{a},l}$ and $j_{\mathbf{a},l}$.

Lemma 3.8 *Let $n \geq N$ be such that $g_2(|\mathbf{c}| + n) \neq 0$, and let $\mathbf{a} \in Good(n, \epsilon)$ and $l \in W_{\mathbf{a}}$. For $h_{\mathbf{a},l}$ and $j_{\mathbf{a},l}$ as defined above, if $h_{\mathbf{a},l} = 1$ then $j_{\mathbf{a},l} > l + k_n$.*

Proof If $h_{\mathbf{a},l} = 1$ then

$$\mathrm{Diam}\left(X_{\mathbf{c}a_1 \dots a_l (a_{l+1} \dots a_n)^{h_{\mathbf{a},l}} a_{l+1} \dots a_{j_{\mathbf{a},l}}}\right) \leq \mathrm{Diam}(X_{\mathbf{ca}}) g_2(|\mathbf{c}| + n)$$

implies

$$\prod_{i=1}^{j_{\mathbf{a},l} - l} r_{a_{l+i}} \leq g_2(|\mathbf{c}| + n).$$

By Lemma 3.6 and Lemma 3.7 we know that $g_2(|\mathbf{c}| + n) \leq n^{-2/\dim_S(\Phi)}$. Therefore

$$\prod_{i=1}^{j_{\mathbf{a},l} - l} r_{a_{l+i}} \leq \frac{1}{n^{2/\dim_S(\Phi)}}. \tag{3.18}$$

Importantly, by $(2c)$ from Lemma 3.3 we know that

$$\prod_{i=1}^{k_n} r_{a_{l+i}} \geq e^{k_n(-\chi - \epsilon)}. \tag{3.19}$$

Equation (3.12) states that

$$\frac{1}{n^{2/\dim_S(\Phi)}} < e^{k_n(-\chi - \epsilon)}$$

for $n \geq N$. It follows therefore from (3.18) and (3.19) that

$$\prod_{i=1}^{j_{\mathbf{a},l} - l} r_{a_{l+i}} < \prod_{i=1}^{k_n} r_{a_{l+i}}.$$

Therefore we must have $j_{\mathbf{a},l} > l + k_n$. □

If we combine (2b). from Lemma 3.3 together with Lemma 3.8, we may conclude the following lemma.

Lemma 3.9 *Assume that $n \geq N$ is such that $g_2(|\mathbf{c}|+n) \neq 0$ and let $\mathbf{a} \in Good(n,\epsilon)$. If $l, l' \in W_{\mathbf{a}}$ and $l \neq l'$ then*

$$\mathbf{c}a_1 \ldots a_l (a_{l+1} \ldots a_n)^{h_{\mathbf{a},l}} a_{l+1} \ldots a_{j_{\mathbf{a},l}} \neq \mathbf{c}a_1 \ldots a_{l'} (a_{l'+1} \ldots a_n)^{h_{\mathbf{a},l'}} a_{l'+1} \ldots a_{j_{\mathbf{a},l'}}.$$

Lemma 3.10 *Let $n \geq N$ be such that $g_2(|\mathbf{c}|+n) \neq 0$, and let $\mathbf{a} \in Good(n,\epsilon)$ and $l \in W_{\mathbf{a}}$. There exists $C = C(\mathbf{c})$ such that for $h_{\mathbf{a},l}$ and $j_{\mathbf{a},l}$ as defined above, we have*

$$(n-l)(h_{\mathbf{a},l}-1) + j_{\mathbf{a},l} - l < C \log n.$$

Proof If $g_2(|\mathbf{c}|+n) \neq 0$ then by Lemma 3.7 we know that it must satisfy $g_2(|\mathbf{c}|+n) \geq \frac{1}{(|\mathbf{c}|+n)^{4/\dim_S(\Phi)}}$. Equation (3.15) then implies that if $g_2(|\mathbf{c}|+n) \neq 0$ then

$$\frac{\min_{\mathbf{a}\in A} r_a}{(|\mathbf{c}|+n)^{4/\dim_S(\Phi)}} \leq \left(\prod_{i=1}^{n-l} r_{a_{l+i}}\right)^{h_{\mathbf{a},l}-1} \cdot \prod_{i=1}^{j_{\mathbf{a},l}-l} r_{a_{l+i}}.$$

This in turn implies that

$$\frac{\min_{\mathbf{a}\in A} r_a}{(|\mathbf{c}|+n)^{4/\dim_S(\Phi)}} \leq \left(\max_{a\in A} r_a\right)^{(n-l)(h_{\mathbf{a},l}-1)+j_{\mathbf{a},l}-l}.$$

Taking logarithms and then manipulating the resulting expression, one can show that the above implies that there exists $C = C(\mathbf{c})$ such that

$$(n-l)(h_{\mathbf{a},l}-1) + j_{\mathbf{a},l} - l < C \log n.$$

□

For each $n \geq N$ such that $g_2(|\mathbf{c}|+n) \neq 0$ we let

$$E_n := \bigcup_{a\in Good(n,\epsilon)} \bigcup_{l\in W_{\mathbf{a}}} \left[\mathbf{c}a_1 \ldots a_l \, (a_{l+1} \ldots a_n)^{h_{\mathbf{a},l}} \, a_{l+1} \ldots a_{j_{\mathbf{a},l}}\right].$$

Lemma 3.9 tells us that any pair of cylinder sets in this union are disjoint. If $n \geq N$ is such that $g_2(|\mathbf{c}|+n) = 0$ then set $E_n = \emptyset$. Importantly (3.14) implies that

$$\limsup_{n\to\infty} E_n \subset [\mathbf{c}] \cap \pi^{-1}(W_\Phi(\Psi_2)).$$

Therefore to prove (3.11) it is sufficient to show that

$$\mathfrak{m}\left(\limsup_{n\to\infty} E_n\right) \gg \mathfrak{m}([\mathbf{c}]). \tag{3.20}$$

We will prove that (3.20) holds using Lemma 3.4. Before that it is necessary to check that the hypothesis of this lemma are satisfied.

Lemma 3.11 *For $n \geq N$ we have $\mathfrak{m}(E_n) \asymp m([\mathbf{c}]) \cdot n \cdot g_2(|\mathbf{c}| + n)^{\dim_S(\Phi)}$.*

Proof This lemma is obviously true if n is such that $g_2(|\mathbf{c}| + n) = 0$. As such we restrict our attention to those $n \geq N$ for which $g_2(|\mathbf{c}| + n) \neq 0$. Recall that by Lemma 3.9, for distinct $l, l' \in W_{\mathbf{a}}$ we have $\mathbf{c}a_1 \ldots a_l(a_{l+1} \ldots a_n)^{h_{\mathbf{a},l}} a_{l+1} \ldots a_{j_{\mathbf{a},l}} \neq \mathbf{c}a_1 \ldots a_{l'}(a_{l'+1} \ldots a_n)^{h_{\mathbf{a},l'}} a_{l'+1} \ldots a_{j_{\mathbf{a},l'}}$. Therefore we have

$$\begin{aligned}
\mathfrak{m}(E_n) &= \mathfrak{m}\left(\bigcup_{\mathbf{a}\in Good(n,\epsilon)} \bigcup_{l\in W_{\mathbf{a}}} \left[\mathbf{c}a_1 \ldots a_l(a_{l+1} \ldots a_n)^{h_{\mathbf{a},l}} a_{l+1} \ldots a_{j_{\mathbf{a},l}}\right]\right)\\
&= \sum_{\mathbf{a}\in Good(n,\epsilon)} \sum_{l\in W_{\mathbf{a}}} \mathfrak{m}\left(\left[\mathbf{c}a_1 \ldots a_l(a_{l+1} \ldots a_n)^{h_{\mathbf{a},l}} a_l \ldots a_{j_{\mathbf{a},l}}\right]\right)\\
&\overset{(3.16)}{\asymp} \sum_{\mathbf{a}\in Good(n,\epsilon)} \sum_{l\in W_{\mathbf{a}}} \mathrm{Diam}\left(X_{\mathbf{c}a_1 \ldots a_l(a_{l+1} \ldots a_n)^{h_{\mathbf{a},l}} a_{l+1} \ldots a_{j_{\mathbf{a},l}}}\right)^{\dim_S(\Phi)}\\
&\overset{(3.15)}{\asymp} \sum_{\mathbf{a}\in Good(n,\epsilon)} \sum_{l\in W_{\mathbf{a}}} (\mathrm{Diam}(X_{\mathbf{ca}})\, g_2(|\mathbf{c}| + n))^{\dim_S(\Phi)}\\
&\overset{Lemma\ 3.3}{\asymp} n \cdot g_2(|\mathbf{c}| + n)^{\dim_S(\Phi)} \sum_{\mathbf{a}\in Good(n,\epsilon)} \mathrm{Diam}(X_{\mathbf{ca}})^{\dim_S(\Phi)}\\
&\overset{(3.16)}{\asymp} n \cdot g_2(|\mathbf{c}| + n)^{\dim_S(\Phi)} \sum_{\mathbf{a}\in Good(n,\epsilon)} \mathfrak{m}([\mathbf{ca}])\\
&= \mathfrak{m}([\mathbf{c}]) \cdot n \cdot g_2(|\mathbf{c}| + n)^{\dim_S(\Phi)} \sum_{\mathbf{a}\in Good(n,\epsilon)} \mathfrak{m}([\mathbf{a}])\\
&\overset{Lemma\ 3.3}{\asymp} \mathfrak{m}([\mathbf{c}]) \cdot n \cdot g_2(|\mathbf{c}| + n)^{\dim_S(\Phi)}.
\end{aligned}$$

□

It follows from Lemma 3.7 and Lemma 3.11 that $\sum_{n=1}^{\infty} \mathfrak{m}(E_n) = \infty$. So our sequence of sets (E_n) satisfies the hypothesis of Lemma 3.4. To complete our proof we need to get good upper bounds for $\mathfrak{m}(E_n \cap E_m)$. We restrict our attention to those n and m satisfying $n < m$, $g_2(|\mathbf{c}| + n) \neq 0$, and $g_2(|\mathbf{c}| + m) \neq 0$. For these n and m we see that

$$\mathfrak{m}(E_n \cap E_m) = \sum_{\mathbf{a}\in Good(n,\epsilon)} \sum_{l\in W_{\mathbf{a}}} \mathfrak{m}\left(\left[\mathbf{c}a_1 \ldots a_l(a_{l+1} \ldots a_n)^{h_{\mathbf{a},l}} a_{l+1} \ldots a_{j_{\mathbf{a},l}}\right] \cap E_m\right).$$

The following proposition gives good upper bounds for the terms in this sum. The parameter C in the statement of this proposition is the same C as in Lemma 3.10.

Proposition 3.12 *Let $n, m \geq N$ be such that $n < m$, $g_2(|\mathbf{c}|+n) \neq 0$, and $g_2(|\mathbf{c}|+m) \neq 0$. Then for $\mathbf{a} \in Good(n, \epsilon)$ and $l \in W_{\mathbf{a}}$ the following holds:*

I. *If* $n < m \leq n + C\log n$ *then*

$$\begin{aligned}
&\mathfrak{m}\left(\left[\mathbf{c}a_1\ldots a_l(a_{l+1}\ldots a_n)^{h_{\mathbf{a},l}}a_{l+1}\ldots a_{j_{\mathbf{a},l}}\right]\cap E_m\right)\\
&\quad\ll \mathfrak{m}([\mathbf{ca}])e^{(h+\epsilon)k_m}g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}\\
&\quad\quad+\mathfrak{m}([\mathbf{ca}])\left(\max_{a\in\mathcal{A}}p_a\right)^{m-n}g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}\\
&\quad\quad+m\cdot\mathfrak{m}([\mathbf{ca}])g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}.
\end{aligned}$$

II. *If* $m > n + C\log n$ *then*

$$\begin{aligned}
&\mathfrak{m}\left(\left[\mathbf{c}a_1\ldots a_l(a_{l+1}\ldots a_n)^{h_{\mathbf{a},l}}a_{l+1}\ldots a_{j_{\mathbf{a},l}}\right]\cap E_m\right)\\
&\quad\ll m\cdot\mathfrak{m}([\mathbf{ca}])g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}.
\end{aligned}$$

Proof We prove each statement separately.
Proof of Statement I. Assume that $n < m \leq n + C\log n$. Let $\mathbf{a} \in Good(n,\epsilon)$ and $l \in W_{\mathbf{a}}$. If $m \leq l + h_{\mathbf{a},l}(n-l) + (j_{\mathbf{a},l} - l)$ then at most one $\mathbf{b} \in Good(m,\epsilon)$ is such that $[\mathbf{cb}]$ has a non-empty intersection with $\left[\mathbf{c}a_1\ldots a_l(a_{l+1}\ldots a_n)^{h_{\mathbf{a},l}}a_{l+1}\ldots a_{j_{\mathbf{a},l}}\right]$. Let us assume that such a $\mathbf{b}$ exists. Otherwise $\mathfrak{m}([\mathbf{c}a_1\ldots a_l(a_{l+1}\ldots a_n)^{h_{\mathbf{a},l}}a_{l+1}\ldots a_{j_{\mathbf{a},l}}]\cap E_m) = 0$ and our upper bound holds trivially. In this case we see that

$$\begin{aligned}
&\mathfrak{m}\left(\left[\mathbf{c}a_1\ldots a_l(a_{l+1}\ldots a_n)^{h_{\mathbf{a},l}}a_{l+1}\ldots a_{j_{\mathbf{a},l}}\right]\cap E_m\right)\\
&\quad=\sum_{l'\in W_{\mathbf{b}}}\mathfrak{m}\left(\left[\mathbf{c}a_1\ldots a_l(a_{l+1}\ldots a_n)^{h_{\mathbf{a},l}}a_{l+1}\ldots a_{j_{\mathbf{a},l}}\right]\right.\\
&\quad\quad\left.\cap\left[\mathbf{c}b_1\ldots b_{l'}(b_{l'+1}\ldots b_m)^{h_{\mathbf{b},l'}}b_{l'+1}\ldots b_{j_{\mathbf{b},l'}}\right]\right).
\end{aligned}$$

Lemma 3.8 implies that if

$$\begin{aligned}
&\mathfrak{m}\left(\left[\mathbf{c}a_1\ldots a_l(a_{l+1}\ldots a_n)^{h_{\mathbf{a},l}}a_{l+1}\ldots a_{j_{\mathbf{a},l}}\right]\right.\\
&\quad\left.\cap\left[\mathbf{c}b_1\ldots b_{l'}(b_{l'+1}\ldots b_m)^{h_{\mathbf{b},l'}}b_{l'+1}\ldots b_{j_{\mathbf{b},l'}}\right]\right)\neq 0
\end{aligned}$$

then we must have

$$\left[\mathbf{c}a_1\ldots a_l(a_{l+1}\ldots a_n)^{h_{\mathbf{a},l}}a_{l+1}\ldots a_{j_{\mathbf{a},l}}\right]\cap\left[\mathbf{c}b_1\ldots b_m b_{l'+1}\ldots b_{l'+k_m}\right]\neq\emptyset.$$

By Lemma 3.3 we know that for each $l' \in W_{\mathbf{b}}$ the cylinder set $\left[\mathbf{c}b_1\ldots b_m b_{l'+1}\ldots b_{l'+k_m}\right]$ satisfies

$$\mathfrak{m}\left(\left[\mathbf{c}b_1\ldots b_m b_{l'+1}\ldots b_{l'+k_m}\right]\right)\geq\mathfrak{m}([\mathbf{cb}])\cdot e^{k_m(-h-\epsilon)}.$$

Therefore, by (3.17) and a measure argument we have

$$\#\left\{l' \in W_b : \left[\mathbf{c}a_1 \dots a_l (a_{l+1} \dots a_n)^{h_{\mathbf{a},l}} a_{l+1} \dots a_{j_{\mathbf{a},l}}\right] \cap \left[\mathbf{c}b_1 \dots b_m b_{l'+1} \dots b_{l'+k_m}\right] \neq \emptyset\right\}$$
$$\ll \frac{\mathfrak{m}([\mathbf{ca}]) g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}}{\mathfrak{m}([\mathbf{cb}]) e^{k_m(-h-\epsilon)}} + 1. \tag{3.21}$$

Applying the above observations we see that

$$\sum_{l' \in W_{\mathbf{b}}} \mathfrak{m}\left(\left[\mathbf{c}a_1 \dots a_l (a_{l+1} \dots a_n)^{h_{\mathbf{a},l}} a_{l+1} \dots a_{j_{\mathbf{a},l}}\right] \cap \left[\mathbf{c}b_1 \dots b_{l'} (b_{l'+1} \dots b_m)^{h_{\mathbf{b},l'}} b_{l'+1} \dots b_{j_{\mathbf{b},l'}}\right]\right)$$
$$\leq \sum_{\substack{l' \in W_{\mathbf{b}} \\ \left[\mathbf{c}a_1 \dots a_l (a_{l+1} \dots a_n)^{h_{\mathbf{a},l}} a_{l+1} \dots a_{j_{\mathbf{a},l}}\right] \cap \left[\mathbf{c}b_1 \dots b_m b_{l'+1} \dots b_{l'+k_m}\right] \neq \emptyset}} \mathfrak{m}\left(\left[\mathbf{c}b_1 \dots b_{l'} (b_{l'+1} \dots b_m)^{h_{\mathbf{b},l'}} b_{l'+1} \dots b_{j_{\mathbf{b},l'}}\right]\right)$$
$$\overset{(3.17)}{\ll} \sum_{\substack{l' \in W_{\mathbf{b}} \\ \left[\mathbf{c}a_1 \dots a_l (a_{l+1} \dots a_n)^{h_{\mathbf{a},l}} a_{l+1} \dots a_{j_{\mathbf{a},l}}\right] \cap \left[\mathbf{c}b_1 \dots b_m b_{l'+1} \dots b_{l'+k_m}\right] \neq \emptyset}} \mathfrak{m}([\mathbf{cb}]) g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}$$
$$\overset{(3.21)}{\ll} \mathfrak{m}([\mathbf{ca}]) e^{(h+\epsilon)k_m} g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)} g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)} + \mathfrak{m}([\mathbf{cb}]) g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}.$$

Because $\mathbf{b}$ must have $\mathbf{a}$ as a prefix we see that

$$\mathfrak{m}([\mathbf{cb}]) \leq \mathfrak{m}([\mathbf{ca}]) \left(\max_{a \in \mathcal{A}} p_a\right)^{m-n}. \tag{3.22}$$

Substituting (3.22) into the last line in the above, we have shown that if $m \leq l + h_{\mathbf{a},l}(n-l) + (j_{\mathbf{a},l} - l)$ then[3]

$$m([\mathbf{c}a_1 \dots a_l (a_{l+1} \dots a_n)^{h_{\mathbf{a},l}} a_{l+1} \dots a_{j_{\mathbf{a},l}}] \cap E_m)$$
$$\ll \mathfrak{m}([\mathbf{ca}]) e^{(h+\epsilon)k_m} g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)} g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}$$
$$+ \mathfrak{m}([\mathbf{ca}]) \left(\max_{a \in \mathcal{A}} p_a\right)^{m-n} g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}. \tag{3.23}$$

[3] For the proof of Statement 3 from Theorem 1.1 we know that $\mathbf{p} = \left(\mathcal{A}^{-1}\right)_{a \in \mathcal{A}}$, and as such we can make more precise statements about the measure of cylinders. Indeed in the above we do not need to introduce the parameter ϵ and (3.21) holds with $\epsilon = 0$. This means that we can strengthen (3.23) to $\mathfrak{m}\left(\left[\mathbf{c}a_1 \dots a_l (a_{l+1} \dots a_n)^{h_{\mathbf{a},l}} a_{l+1} \dots a_{j_{\mathbf{a},l}}\right] \cap E_m\right) \ll \mathfrak{m}([\mathbf{ca}]) e^{hk_m} g_2(|\mathbf{c}| + n)^{\dim_S(\Phi)} g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)} + \mathfrak{m}([\mathbf{ca}])(\max_{a \in \mathcal{A}} p_a)^{m-n} g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}$. Which by the definition of k_m implies $\mathfrak{m}\left(\left[\mathbf{c}a_1 \dots a_l (a_{l+1} \dots a_n)^{h_{\mathbf{a},l}} a_{l+1} \dots a_{j_{\mathbf{a},l}}\right] \cap E_m\right) \ll \mathfrak{m}([\mathbf{ca}]) m g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)} g_2(|\mathbf{c}| + m)^{\dim_S(\Phi)} + \mathfrak{m}([\mathbf{ca}])(\max_{a \in \mathcal{A}} p_a)^{m-n} g_2(|\mathbf{c}| + m)^{\dim_S(\Phi)}$. The rest of the proof follows identically.

Now suppose that $m > l + h_{\mathbf{a},l}(n-l) + (j_{\mathbf{a},l} - l)$. In this case

$$\begin{aligned}
&\mathfrak{m}\left(\left[\mathbf{c}a_1 \dots a_l (a_{l+1} \dots a_n)^{h_{\mathbf{a},l}} a_{l+1} \dots a_{j_{\mathbf{a},l}}\right] \cap E_m\right) \\
&= \sum_{\substack{\mathbf{b} \in Good(m,\epsilon) \\ \mathbf{b} \text{ begins with } a_1 \dots a_l (a_{l+1} \dots a_n)^{h_{\mathbf{a},l}} a_{l+1} \dots a_{j_{\mathbf{a},l}}}} \sum_{l' \in W_{\mathbf{b}}} \mathfrak{m}\left(\left[\mathbf{c}b_1 \dots b_{l'} (b_{l'+1} \dots b_m)^{h_{\mathbf{b},l'}} b_{l'+1} \dots b_{j_{\mathbf{b},l'}}\right]\right) \\
&\overset{(3.17)}{\ll} \sum_{\substack{\mathbf{b} \in Good(m,\epsilon) \\ \mathbf{b} \text{ begins with } a_1 \dots a_l (a_{l+1} \dots a_n)^{h_{\mathbf{a},l}} a_{l+1} \dots a_{j_{\mathbf{a},l}}}} \sum_{l' \in W_{\mathbf{b}}} \mathfrak{m}([\mathbf{cb}]) g_2(|\mathbf{c}| + m)^{\dim_S(\Phi)} \\
&\leq m \cdot g_2(|\mathbf{c}| + m)^{\dim_S(\Phi)} \sum_{\substack{\mathbf{b} \in Good(m,\epsilon) \\ \mathbf{b} \text{ begins with } a_1 \dots a_l (a_{l+1} \dots a_n)^{h_{\mathbf{a},l}} a_{l+1} \dots a_{j_{\mathbf{a},l}}}} \mathfrak{m}([\mathbf{cb}]) \\
&\leq m \cdot g_2(|\mathbf{c}| + m)^{\dim_S(\Phi)} \sum_{\substack{\mathbf{b} \in \mathcal{A}^m \\ \mathbf{b} \text{ begins with } a_1 \dots a_l (a_{l+1} \dots a_n)^{h_{\mathbf{a},l}} a_{l+1} \dots a_{j_{\mathbf{a},l}}}} \mathfrak{m}([\mathbf{cb}]) \\
&= m \cdot g_2(|\mathbf{c}| + m)^{\dim_S(\Phi)} \cdot \mathfrak{m}\left(\left[\mathbf{c}a_1 \dots a_l (a_{l+1} \dots a_n)^{h_{\mathbf{a},l}} a_{l+1} \dots a_{j_{\mathbf{a},l}}\right]\right) \\
&\overset{(3.17)}{\ll} m \cdot \mathfrak{m}([\mathbf{ca}]) g_2(|\mathbf{c}| + m)^{\dim_S(\Phi)} g_2(|\mathbf{c}| + n)^{\dim_S(\Phi)}.
\end{aligned} \tag{3.24}$$

Adding together the upper bounds obtained in (3.23) and (3.24) we obtain the desired upper bound which holds for all m satisfying $n < m \leq n + C \log n$.

Proof of Statement II. Assume $m > n + C \log n$. Let $\mathbf{a} \in Good(n, \epsilon)$ and $l \in W_{\mathbf{a}}$. If $m > n + C \log n$ then by Lemma 3.10 we must have $m > l + h_{\mathbf{a},l}(n-l) + (j_{\mathbf{a},l} - l)$. In which case the same argument as is used in the proof of the second part of Statement I applies and we have the desired bound

$$\begin{aligned}
&\mathfrak{m}\left(\left[\mathbf{c}a_1 \dots a_l (a_{l+1} \dots a_n)^{h_{\mathbf{a},l}} a_{l+1} \dots a_{j_{\mathbf{a},l}}\right] \cap E_m\right) \\
&\quad \leq m \cdot \mathfrak{m}([\mathbf{ca}]) g_2(|\mathbf{c}| + m)^{\dim_S(\Phi)} g_2(|\mathbf{c}| + n)^{\dim_S(\Phi)}.
\end{aligned}$$

□

Equipped with Proposition 3.12 we will now prove the following statement.

Proposition 3.13 *There exists a constant $C_1 = C_1(\mathbf{c})$ such that*

$$\sum_{n,m=N}^{Q} \mathfrak{m}(E_n \cap E_m) \ll \mathfrak{m}([\mathbf{c}]) \left(\sum_{n=N}^{Q} n \cdot g_2(|\mathbf{c}| + n)^{\dim_S(\Phi)} + \left(\sum_{n=N}^{Q} n \cdot g_2(|\mathbf{c}| + n)^{\dim_S(\Phi)} \right)^2 \right) + C_1.$$

Proof We start our proof by rewriting $\sum_{n,m=N}^{Q}\mathfrak{m}(E_n\cap E_m):$

$$\begin{aligned}\sum_{n,m=N}^{Q}\mathfrak{m}(E_n\cap E_m)&=\sum_{n=N}^{Q}\mathfrak{m}(E_n)+2\sum_{n=N}^{Q-1}\sum_{m=n+1}^{Q}\mathfrak{m}(E_n\cap E_m)\\&=\underbrace{\sum_{n=N}^{Q}\mathfrak{m}(E_n)}_{A}+\underbrace{2\sum_{n=N}^{Q-1}\sum_{m=n+1}^{\min\{Q,n+C\log n\}}\mathfrak{m}(E_n\cap E_m)}_{B}\\&\quad+\underbrace{2\sum_{n=N}^{Q-1}\sum_{n+C\log n<m\leq Q}\mathfrak{m}(E_n\cap E_m)}_{C}.\end{aligned}$$

We will focus on the three terms A, B, and C individually. By Lemma 3.11 we have the following bound for term A:

$$\sum_{n=N}^{Q}\mathfrak{m}(E_n)\asymp\mathfrak{m}([\mathbf{c}])\sum_{n=N}^{Q}n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}. \tag{3.25}$$

Now focusing on the term B, if we apply Statement I from Proposition 3.12 we have

$$\begin{aligned}&\sum_{n=N}^{Q-1}\sum_{m=n+1}^{\min\{Q,n+C\log n\}}\mathfrak{m}(E_n\cap E_m)\\&=\sum_{\substack{n=N\\g_2(n)\neq 0}}^{Q-1}\sum_{\substack{m=n+1\\g_2(m)\neq 0}}^{\min\{Q,n+C\log n\}}\sum_{\mathbf{a}\in Good(n,\epsilon)}\sum_{l\in W_{\mathbf{a}}}\mathfrak{m}\left(\left[\mathbf{c}a_1\ldots a_l(a_{l+1}\ldots a_n)^{h_{\mathbf{a},l}}a_{l+1}\ldots a_{j_{\mathbf{a},l}}\right]\cap E_m\right)\\&\ll\underbrace{\sum_{n=N}^{Q-1}\sum_{m=n+1}^{\min\{Q,n+C\log n\}}\sum_{\mathbf{a}\in Good(n,\epsilon)}\sum_{l\in W_{\mathbf{a}}}\mathfrak{m}([\mathbf{ca}])e^{(h+\epsilon)k_m}g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}}_{B1}\\&\quad+\underbrace{\sum_{n=N}^{Q-1}\sum_{m=n+1}^{\min\{Q,n+C\log n\}}\sum_{\mathbf{a}\in Good(n,\epsilon)}\sum_{l\in W_{\mathbf{a}}}\mathfrak{m}([\mathbf{ca}])\left(\max_{a\in\mathcal{A}}p_a\right)^{m-n}g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}}_{B2}\\&\quad+\underbrace{\sum_{n=N}^{Q-1}\sum_{m=n+1}^{\min\{Q,n+C\log n\}}\sum_{\mathbf{a}\in Good(n,\epsilon)}\sum_{l\in W_{\mathbf{a}}}m\cdot\mathfrak{m}([\mathbf{ca}])g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}}_{B3}.\end{aligned}$$

Focusing on the term B1 in the above, we know by (3.13) that

$$e^{(h+\epsilon)k_m}\leq m^{\gamma}$$

for some $\gamma \in (1, 2)$. Using this inequality we have

$$\begin{aligned}
&\sum_{n=N}^{Q-1} \sum_{m=n+1}^{\min\{Q,n+C\log n\}} \sum_{\mathbf{a}\in Good(n,\epsilon)} \sum_{l\in W_{\mathbf{a}}} \mathfrak{m}([\mathbf{ca}])e^{(h+\epsilon)k_m} g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)} g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)} \\
&\leq \sum_{n=N}^{Q-1} \sum_{m=n+1}^{\min\{Q,n+C\log n\}} \sum_{\mathbf{a}\in Good(n,\epsilon)} \sum_{l\in W_{\mathbf{a}}} m^{\gamma} \mathfrak{m}([\mathbf{ca}]) g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)} g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)} \\
&\leq \sum_{n=N}^{Q-1} \sum_{m=n+1}^{\min\{Q,n+C\log n\}} \sum_{\mathbf{a}\in Good(n,\epsilon)} n\cdot m^{\gamma} \mathfrak{m}([\mathbf{ca}]) g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)} g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)} \\
&\leq \mathfrak{m}([\mathbf{c}]) \sum_{n=N}^{Q-1} \sum_{m=n+1}^{\min\{Q,n+C\log n\}} n\cdot m^{\gamma} g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)} g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)} \\
&\leq \mathfrak{m}([\mathbf{c}]) \sum_{n=N}^{Q-1} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)} \sum_{m=n+1}^{\min\{Q,n+C\log n\}} m^{\gamma}\cdot g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}
\end{aligned}$$

It follows from Lemma 3.6 and Lemma 3.7 that $g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)} \leq m^{-2}$ for all $m \in \mathbb{N}$. Therefore we have

$$\begin{aligned}
&\mathfrak{m}([\mathbf{c}]) \sum_{n=N}^{Q-1} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)} \sum_{m=n+1}^{\min\{Q,n+C\log n\}} m^{\gamma}\cdot g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)} \\
&\leq \mathfrak{m}([\mathbf{c}]) \sum_{n=N}^{Q-1} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)} \sum_{m=n+1}^{\min\{Q,n+C\log n\}} m^{\gamma-2} \\
&\ll \mathfrak{m}([\mathbf{c}]) \sum_{n=N}^{Q-1} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)} \int_{n+1}^{n+C\log n} x^{\gamma-2}\,dx \\
&\ll \mathfrak{m}([\mathbf{c}]) \sum_{n=N}^{Q-1} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)} \left((n+C\log n)^{\gamma-1} - (n+1)^{\gamma-1}\right) \\
&\overset{M.V.T.}{\ll} \mathfrak{m}([\mathbf{c}]) \sum_{n=N}^{Q-1} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)} \cdot \left(C\log n \cdot \frac{1}{n^{2-\gamma}}\right) \\
&\ll \mathfrak{m}([\mathbf{c}]) \sum_{n=N}^{Q-1} \frac{C\log n}{n^{3-\gamma}} \\
&\ll \mathfrak{m}([\mathbf{c}]) \sum_{n=1}^{\infty} \frac{C\log n}{n^{3-\gamma}}.
\end{aligned}$$

In the line marked $M.V.T.$ we use the mean value theorem. In the penultimate line in the above we used that $g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)} \leq n^{-2}$. Because $\gamma \in (1, 2)$ we know that $\sum_{n=1}^{\infty} \frac{C\log n}{n^{3-\gamma}} < \infty$. Therefore we can assert that there exists a constant $C_1 = C_1(\mathbf{c})$

so that the term $B1$ satisfies

$$\sum_{n=N}^{Q-1}\sum_{m=n+1}^{\min\{Q,n+C\log n\}}\sum_{\mathbf{a}\in Good(n,\epsilon)}\sum_{l\in W_{\mathbf{a}}}\mathfrak{m}([\mathbf{ca}])e^{(h+\epsilon)k_m}$$
$$g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}\leq C_1. \tag{3.26}$$

Turning our attention to the term $B2$ we see that

$$\begin{aligned}
&\sum_{n=N}^{Q-1}\sum_{m=n+1}^{\min\{Q,n+C\log n\}}\sum_{\mathbf{a}\in Good(n,\epsilon)}\sum_{l\in W_{\mathbf{a}}}\mathfrak{m}([\mathbf{ca}])\left(\max_{a\in\mathcal{A}}p_a\right)^{m-n}g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}\\
&\ll\sum_{n=N}^{Q-1}\sum_{m=n+1}^{\min\{Q,n+C\log n\}}\sum_{\mathbf{a}\in Good(n,\epsilon)}n\cdot\mathfrak{m}([\mathbf{ca}])\left(\max_{a\in\mathcal{A}}p_a\right)^{m-n}g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}\\
&\ll\mathfrak{m}([\mathbf{c}])\sum_{n=N}^{Q-1}\sum_{m=n+1}^{\min\{Q,n+C\log n\}}n\cdot\left(\max_{a\in\mathcal{A}}p_a\right)^{m-n}g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}\\
&\ll\mathfrak{m}([\mathbf{c}])\sum_{m=N+1}^{Q}\sum_{n=1}^{m-1}n\cdot\left(\max_{a\in\mathcal{A}}p_a\right)^{m-n}g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}\\
&=\mathfrak{m}([\mathbf{c}])\sum_{m=N+1}^{Q}g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}\sum_{n=1}^{m-1}n\cdot\left(\max_{a\in\mathcal{A}}p_a\right)^{m-n}.
\end{aligned}$$

Now using the fact that $\sum_{n=1}^{m-1}n\cdot(\max_{a\in\mathcal{A}}p_a)^{m-n}\ll m$ we see that

$$\mathfrak{m}([\mathbf{c}])\sum_{m=N+1}^{Q}g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}\sum_{n=1}^{m-1}n\cdot\left(\max_{a\in\mathcal{A}}p_a\right)^{m-n}$$
$$\ll\mathfrak{m}([\mathbf{c}])\sum_{m=N+1}^{Q}m\cdot g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}.$$

So our term $B2$ must satisfy

$$\sum_{n=N}^{Q-1}\sum_{m=n+1}^{\min\{Q,n+C\log n\}}\sum_{\mathbf{a}\in Good(n,\epsilon)}\sum_{l\in W_{\mathbf{a}}}\mathfrak{m}([\mathbf{ca}])\left(\max_{a\in\mathcal{A}}p_a\right)^{m-n}g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}$$
$$\ll\mathfrak{m}([\mathbf{c}])\sum_{n=N}^{Q}n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}. \tag{3.27}$$

Now focusing on the term $B3$, we have

$$\begin{aligned}
&\sum_{n=N}^{Q-1}\sum_{m=n+1}^{\min\{Q,n+C\log n\}}\sum_{\mathbf{a}\in Good(n,\epsilon)}\sum_{l\in W_{\mathbf{a}}} m\cdot\mathfrak{m}([\mathbf{ca}])g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}\\
&\leq \sum_{n=N}^{Q-1}\sum_{m=n+1}^{\min\{Q,n+C\log n\}}\sum_{\mathbf{a}\in Good(n,\epsilon)} n\cdot m\cdot\mathfrak{m}([\mathbf{ca}])g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}\\
&\leq \mathfrak{m}([\mathbf{c}])\sum_{n=N}^{Q-1}\sum_{m=n+1}^{\min\{Q,n+C\log n\}} n\cdot m\cdot g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}\\
&\leq \mathfrak{m}([\mathbf{c}])\sum_{n=N}^{Q-1} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}\sum_{m=n+1}^{\min\{Q,n+C\log n\}} m\cdot g_2(|\mathbf{c}|+m)^{\dim_S(\Phi)}\\
&\leq \mathfrak{m}([\mathbf{c}])\left(\sum_{n=N}^{Q} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}\right)^2. \qquad (3.28)
\end{aligned}$$

Combining (3.26), (3.27), and (3.28) we have the following bound for the term B

$$\begin{aligned}
&2\sum_{n=N}^{Q-1}\sum_{m=n+1}^{\min\{Q,n+C\log n\}}\mathfrak{m}(E_n\cap E_m)\\
&\quad\ll \mathfrak{m}([\mathbf{c}])\left(\sum_{n=N}^{Q} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}+\sum_{n=N}^{Q} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}\right)^2+C_1. \qquad (3.29)
\end{aligned}$$

By an analogous argument to that used to bound $B3$, by applying Statement II from Proposition 3.12 it can also be shown that the term C satisfies

$$2\sum_{n=N}^{Q-1}\sum_{n+C\log n<m\leq Q}\mathfrak{m}(E_n\cap E_m)\ll\mathfrak{m}([\mathbf{c}])\left(\sum_{n=N}^{Q} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}\right)^2 \qquad (3.30)$$

Combining (3.25), (3.29), and (3.30) we may conclude our desired bound

$$\begin{aligned}
\sum_{n,m=N}^{Q}\mathfrak{m}(E_n\cap E_m)\ll\mathfrak{m}([\mathbf{c}])&\left(\sum_{n=N}^{Q} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}\right.\\
&\left.+\left(\sum_{n=N}^{Q} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}\right)^2\right)+C_1.
\end{aligned}$$

□

Combining Proposition 3.13 together with Lemma 3.4 and Lemma 3.11 we may conclude that

$$
\begin{aligned}
&\mathfrak{m}\left(\limsup_{n\to\infty} E_n\right)\\
&\geq \limsup_{Q\to\infty} \frac{\left(\sum_{n=N}^{Q} \mathfrak{m}(E_n)\right)^2}{\sum_{n,m=N}^{Q} \mathfrak{m}(E_n\cap E_m)}\\
&\gg \limsup_{Q\to\infty} \frac{\mathfrak{m}([\mathbf{c}])^2\left(\sum_{n=N}^{Q} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}\right)^2}{\mathfrak{m}([\mathbf{c}])\left(\sum_{n=N}^{Q} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}+\left(\sum_{n=N}^{Q} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}\right)^2\right)+C_1}\\
&= \limsup_{Q\to\infty} \frac{\mathfrak{m}([\mathbf{c}])^2\left(\sum_{n=N}^{Q} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}\right)^2}{\mathfrak{m}([\mathbf{c}])\left(\sum_{n=N}^{Q} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}+\left(\sum_{n=N}^{Q} n\cdot g_2(|\mathbf{c}|+n)^{\dim_S(\Phi)}\right)^2\right)}\\
&\gg \mathfrak{m}([\mathbf{c}]).
\end{aligned}
$$

Thus (3.20) holds and our proof of Statement 2 from Theorem 1.1 is complete. We emphasise that in the penultimate line in the above we used the fact that the constant C_1 does not affect the limit. This is important because the implicit constant in (3.20) needs to be independent of $\mathbf{c}$ if we want to apply Lemma 3.5.

4 An application of the mass transference principle

The mass transference principle of Beresnevich and Velani [10] is a powerful tool that allows one to derive information on the Hausdorff measure of a limsup set. We do not state it in its full generality, but instead content ourselves with the following which is better suited for our purposes.

Let $X\subset\mathbb{R}^d$. Then X is said to be Ahlfors regular if there exists $C_1, C_2>0$ such that

$$C_1 r^{\dim_H(X)} \leq \mathcal{H}^{\dim_H(X)}(B(x,r)\cap X)\leq C_2 r^{\dim_H(X)}$$

for all $x\in X$ and r sufficiently small. Given an Ahlfors regular set X, a ball $B(x,r)$ in X, and $s>0$, we let $B^s=B\left(x, r^{s/\dim_H(X)}\right)$. The following theorem is a simplified version of Theorem 3 from [10].

Theorem 4.1 *Let X be Ahlfors regular and (B_l) be a sequence of balls in X with radii tending to zero. Let $s>0$ and suppose that for any ball B in X we have*

$$\mathcal{H}^{\dim_H(X)}\left(B\cap\limsup_{l\to\infty} B_l^s\right)=\mathcal{H}^{\dim_H(X)}(B).$$

Then, for any ball B in X

$$\mathcal{H}^s\left(B \cap \limsup_{l \to \infty} B_l\right) = \mathcal{H}^s(B).$$

It is a well known fact that if an IFS Φ satisfies the open set condition then the corresponding self-similar set is Ahlfors regular. It is also well known that if Φ satisfies the open set condition then μ is equivalent to the restriction of $\mathcal{H}^{\dim_S(\Phi)}$ on X. Combining these facts together with Theorem 1.1 and Theorem 4.1, we may deduce the following statement.

Theorem 4.2 *Let $\Phi = \{\phi_a\}_{a \in \mathcal{A}}$ be an IFS which satisfies the open set condition. Let $\Psi : \cup_{n=1}^{\infty} \mathcal{A}^n \to [0, \infty)$ be given by $\Psi(\mathbf{a}) = \mathrm{Diam}(X_{\mathbf{a}})g(|\mathbf{a}|)$ for some function $g : \mathbb{N} \to [0, \infty)$ satisfying*

$$\sum_{n=1}^{\infty} \sum_{\mathbf{a} \in \mathcal{A}^n} n \cdot (\mathrm{Diam}(X_{\mathbf{a}})g(n))^{\dim_S(\Phi)} = \infty.$$

For $t \geq 1$ let $\Psi^t : \cup_{n=1}^{\infty} \mathcal{A}^n \to [0, \infty)$ be given by $\Psi^t(\mathbf{a}) = \Psi(\mathbf{a})^t$. Then the following statements are true:

1. *Assume that*

$$h_{\mathbf{p}} < -2 \log \sum_{a \in \mathcal{A}} p_a^2$$

and that g is non-increasing. Then for any $t \geq 1$ we have $\mathcal{H}^{\dim_H(X)/t}(W_\Phi(\Psi^t)) = \mathcal{H}^{\dim_H/t}(X)$.

2. *If Φ is equicontractive then for any $t \geq 1$ we have $\mathcal{H}^{\dim_H(X)/t}(W_\Phi(\Psi^t)) = \mathcal{H}^{\dim_H/t}(X)$.*

We emphasise that if Φ is not equicontractive and Ψ is of the form $\Psi(\mathbf{a}) = \mathrm{Diam}(X_{\mathbf{a}})g(|\mathbf{a}|)$, then for $t > 1$ there exists no $g_t : \mathbb{N} \to [0, \infty)$ such that Ψ^t satisfies $\Psi^t(\mathbf{a}) = \mathrm{Diam}(X_{\mathbf{a}})g_t(|\mathbf{a}|)$ for all $\mathbf{a} \in \mathcal{A}$. As such Theorem 4.2 allows us to make positive measure statements about a new class of functions.

We conclude this section by mentioning that by following the arguments used in Sect. 2, one can use Theorem 4.2 to prove a number of statements on the Hausdorff measure of certain limsup sets arising from the study of intrinsic Diophantine Approximation on self-similar sets. We leave the details to the interested reader.

Acknowledgements The author would like to thank Baowei Wang for his feedback on an initial draft. The author would also like to thank the anonymous referee for their detailed comments. This research was supported by an EPSRC New Investigator Award (EP/W003880/1).

Data Availability Statement No data was created during this work.

Declarations

Conflict of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

References

1. Allen, D., Bárány, B.: Hausdorff measures of shrinking targets on self-conformal sets. Mathematika **67**(4), 807–839 (2021)
2. Baker, S.: An analogue of Khintchine's theorem for self-conformal sets. Math. Proc. Cambridge Philos. Soc. **167**(3), 567–597 (2019)
3. Baker, S.: Approximation properties of β-expansions. Acta Arith. **168**, 269–287 (2015)
4. Baker, S.: Approximation properties of β-expansion II. Ergodic Theory Dynam. Syst. **38**(5), 1627–1641 (2018)
5. Baker, S.: Overlapping iterated function systems from the perspective of Metric Number Theory. Mem. Amer. Math. Soc. (to appear)
6. Baker, S., Troscheit, S.: Analogues of Khintchine's theorem for random attractors. Trans. Am. Math. Soc. **375**(2), 1411–1441 (2022)
7. Beresnevich, V.: Badly approximable points on manifolds. Invent. Math. **202**(3), 1199–1240 (2015)
8. Beresnevich, V., Dickinson, D., Velani, S.: Measure theoretic laws for limsup sets. Mem. Am. Math. Soc. **179**(846), 1–98 (2006)
9. Beresnevich, V., Vaughan, R., Velani, S., Zorin, E.: Diophantine approximation on manifolds and the distribution of rational points: contributions to the convergence theory. Int. Math. Res. Not. IMRN **10**, 2885–2908 (2017)
10. Beresnevich, V., Velani, S.: A mass transference principle and the Duffin-Schaeffer conjecture for Hausdorff measures. Ann. Math. (2) **164**(3), 971–992 (2006)
11. Broderick, R., Fishman, L.: A, Reich, Intrinsic approximation on Cantor-like sets, a problem of Mahler. Mosc. J. Comb. Number Theory **1**(4), 3–12 (2011)
12. Bugeaud, Y.: Approximation by algebraic numbers, Cambridge Tracts in Mathematics, 160. Cambridge University Press, Cambridge (2004)
13. Bugeaud, Y.: Diophantine approximation and Cantor sets. Math. Ann. **341**(3), 677–684 (2008)
14. Bugeaud, Y., Durand, A.: Metric Diophantine approximation on the middle-third Cantor set. J. Eur. Math. Soc. (JEMS) **18**(6), 1233–1272 (2016)
15. Falconer, K.: Fractal geometry. Mathematical foundations and applications, 3rd edn. Wiley, Chichester (2014). xxx+368 pp. ISBN: 978-1-119-94239-9
16. Fishman, L., Kleinbock, D., Merrill, K., Simmons, D.: Intrinsic Diophantine approximation on manifolds: general theory. Trans. Am. Math. Soc. **370**(1), 577–599 (2018)
17. Fishman, L., Merrill, K., Simmons, D.: Uniformly de Bruijn sequences and symbolic Diophantine approximation on fractals. Ann. Comb. **22**(2), 271–293 (2018)
18. Fishman, L., Simmons, D.: Extrinsic Diophantine approximation on manifolds and fractals. J. Math. Pures Appl. (9) **104**(1), 83–101 (2015)
19. Fishman, L., Simmons, D.: Intrinsic approximation for fractals defined by rational iterated function systems: Mahler's research suggestion. Proc. Lond. Math. Soc. (3) **109**(1), 189–212 (2014)
20. Harman, G.: Metric Number Theory, LMS Monographs 18. Clarendon Press, Oxford (1998)

21. Hill, R., Velani, S.: The ergodic theory of shrinking targets. Invent. Math. **119**, 175–198 (1995)
22. Hochman, M.: On self-similar sets with overlaps and inverse theorems for entropy. Ann. Math. **180**, 773–822 (2014)
23. Hochman, M.: On self-similar sets with overlaps and inverse theorems for entropy in $\mathbb{R}^d$. Mem. Amer. Math. Soc. (to appear)
24. Hoeffding, W.: Probability inequalities for sums of bounded random variables. J. Am. Statist. Assoc. **58**, 13–30 (1963)
25. Hutchinson, J.: Fractals and self-similarity. Indiana Univ. Math. J. **30**(5), 713–747 (1981)
26. Khalil, O., Luethi, M.: Random walks, spectral gaps, and Khintchine's Theorem on fractals. Invent. Math. (to appear)
27. Kleinbock, D., Lindenstrauss, E., Weiss, B.: On fractal measures and Diophantine approximation. Selecta Math. (N.S.) **10**(4), 479–523 (2004)
28. Kochen, S., Stone, C.: A note on the Borel-Cantelli lemma. Illinois J. Math. **8**, 248–251 (1964)
29. Levesley, J., Salp, C., Velani, S.: On a problem of K. Mahler: Diophantine approximation and Cantor sets. Math. Ann. **338**(1), 97–118 (2007)
30. Mahler, K.: Some suggestions for further research. Bull. Austral. Math. Soc. **29**(1), 101–108 (1984)
31. Persson, T., Reeve, H.: A Frostman type lemma for sets with large intersections, and an application to Diophantine approximation. Proc. Edinburgh Math. Soc. **58**(02), 521–542 (2015)
32. Persson, T., Reeve, H.: On the Diophantine properties of λ-expansions. Mathematika **59**(1), 65–86 (2013)
33. Rapaport, A.: Proof of the exact overlaps conjecture for systems with algebraic contractions. Annales Scientifique de l'ENS **55**, 1357–1377 (2022)
34. Schleischitz, J.: On intrinsic and extrinsic rational approximation to Cantor sets. Ergod. Theory Dyn. Syst. **41**(5), 1560–1589 (2021)
35. Simmons, D., Weiss, B.: Random walks on homogeneous spaces and Diophantine approximation on fractals. Invent. Math. **216**(2), 337–394 (2019)
36. Sprindžuk, V.G.: Metric Theory of Diophantine Approximation. V. H. Winston & Sons, Huntington (1979)
37. Tan, B., Wang, B., Wu, J.: Mahler's question for intrinsic Diophantine approximation on triadic Cantor set: the divergence theory. Preprint at arXiv:2103.00544
38. Wang, B.: Bristol Ergodic Theory and Dynamical Systems Seminar (February 2021)
39. Weiss, B.: Almost no points on a Cantor set are very well approximable. R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **457**(2008), 949–952 (2001)
40. Yu, H.: Rational points near self-similar sets. Preprint at arXiv:2101.05910