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Operators Induced by Radial Measures Acting on the Dirichlet Space

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Abstract. Let $\mathbb{D}$ be the unit disc in the complex plane. Given a positive finite Borel measure μ on the radius $[0,1)$, we let μ_n denote the n-th moment of μ and we deal with the action on spaces of analytic functions in $\mathbb{D}$ of the operator of Hibert-type $\mathcal{H}_\mu$ and the operator of Cesàro-type $\mathcal{C}_\mu$ which are defined as follows: If f is holomorphic in $\mathbb{D}$, $f(z)=\sum_{n=0}^{\infty} a_n z^n$ $(z\in\mathbb{D})$, then $\mathcal{H}_\mu(f)$ is formally defined by $\mathcal{H}_\mu(f)(z)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n+k}a_k\right)z^n$ $(z\in\mathbb{D})$ and $\mathcal{C}_\mu(f)$ is defined by $\mathcal{C}_\mu(f)(z)=\sum_{n=0}^{\infty}\mu_n\left(\sum_{k=0}^{n}a_k\right)z^n$ $(z\in\mathbb{D})$. These are natural generalizations of the classical Hilbert and Cesàro operators. A good amount of work has been devoted recently to study the action of these operators on distinct spaces of analytic functions in $\mathbb{D}$. In this paper we study the action of the operators $\mathcal{H}_\mu$ and $\mathcal{C}_\mu$ on the Dirichlet space $\mathcal{D}$ and, more generally, on the analytic Besov spaces B^p $(1\le p<\infty)$.

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1. Introduction

The open unit disc in the complex plane $\mathbb{C}$ will be denoted by $\mathbb{D}$ and $\mathrm{Hol}(\mathbb{D})$ will stand for the space of all analytic functions in $\mathbb{D}$. Also, dA will denote the area measure on $\mathbb{D}$, normalized so that the area of $\mathbb{D}$ is 1. Thus $dA(z)=\frac{1}{\pi}dx\,dy=\frac{1}{\pi}r\,dr\,d\theta$.

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For $0 \le r < 1$, $0 < p \le \infty$, and f analytic in $\mathbb{D}$, the integral means $M_p(r,f)$ of f are defined by

$$M_p(r,f) = \left(\frac{1}{2\pi}\int_0^{2\pi} \left|f(re^{i\theta})\right|^p \, d\theta\right)^{1/p}, \quad 0 < p < \infty,$$
$$M_\infty(r,f) = \max_{|z|=r} |f(z)|.$$

For $0 < p \le \infty$ the Hardy space H^p consists of those functions f, analytic in $\mathbb{D}$, for which

$$||f||_{H^p} \stackrel{\text{def}}{=} \sup_{0<r<1} M_p(r,f) < \infty.$$

We refer to [20] for the theory of Hardy spaces.

For $0 < p < \infty$ and $\alpha > -1$ the weighted Bergman space A^p_α consists of those $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\|f\|_{A^p_\alpha} \stackrel{\text{def}}{=} \left((\alpha+1)\int_{\mathbb{D}} (1-|z|^2)^\alpha |f(z)|^p \, dA(z)\right)^{1/p} < \infty.$$

The unweighted Bergman space A^p_0 is simply denoted by A^p. We refer to [21,31,48] for the notation and results about Bergman spaces.

The space of Dirichlet type $\mathcal{D}^p_\alpha$ ($0 < p < \infty$, $\alpha > -1$) is the space of those $f \in \mathrm{Hol}(\mathbb{D})$ such that $f' \in A^p_\alpha$. Thus, a function $f \in \mathrm{Hol}(\mathbb{D})$ belongs to $\mathcal{D}^p_\alpha$ if and only if

$$\|f\|_{\mathcal{D}^p_\alpha} \stackrel{\text{def}}{=} |f(0)| + \left((\alpha+1)\int_{\mathbb{D}} (1-|z|^2)^\alpha |f'(z)|^p \, dA(z)\right)^{1/p} < \infty.$$

In this paper we shall be mainly concerned with the Dirichlet space $\mathcal{D} = \mathcal{D}^2_0$ which consists of those $f \in \mathrm{Hol}(\mathbb{D})$ whose image Riemann surface has a finite area. We recall that if $f \in \mathcal{D}$, $f(z) = \sum_{n=0}^\infty a_n z^n$ ($z \in \mathcal{D}$), then

$$\|f\|_{\mathcal{D}} \stackrel{\text{def}}{=} \|f\|_{\mathcal{D}^2_0} = |f(0)| + \left(\int_{\mathbb{D}} |f'(z)|^2 \, dA(z)\right)^{1/2} = |a_0| + \left(\sum_{k=1}^\infty k|a_k|^2\right)^{1/2}. \tag{1.1}$$

Throughout the paper μ will be a positive finite Borel measure on the radius $[0,1)$ and, for $n = 0,1,2,\dots$, we shall let μ_n denote the moment of order n of μ, that is, $\mu_n = \int_{[0,1)} t^n \, d\mu(t)$. The matrices $\mathcal{H}_\mu$ and $\mathcal{C}_\mu$ are defined as follows

$$\mathcal{H}_\mu = \begin{pmatrix} \mu_0 & \mu_1 & \mu_2 & \cdot & \cdot \\ \mu_1 & \mu_2 & \mu_3 & \cdot & \cdot \\ \mu_2 & \mu_3 & \mu_4 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}; \quad \mathcal{C}_\mu = \begin{pmatrix} \mu_0 & 0 & 0 & 0 & \cdot & \cdot \\ \mu_1 & \mu_1 & 0 & 0 & \cdot & \cdot \\ \mu_2 & \mu_2 & \mu_2 & 0 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{pmatrix}.$$

As we shall see in Sects. 2 and 3, these matrices induce operators acting on spaces of analytic functions which are natural generalizations of the classical Hilbert and Cesàro operators. Recently a good amount of work has been devoted to study the action of these operators of Hilbert type and of Cesàro type on distinct subspaces of $\mathrm{Hol}(\mathbb{D})$. Carleson-type measures play a basic role in this work.

Let us recall that if μ is a positive finite Borel measure on $[0,1)$ then:

- If $s>0$, then μ is said to be an s-Carleson measure if there exists a positive constant C such that
$$\mu\left([t,1)\right) \le C(1-t)^s, \quad 0\le t<1.$$
- If $0\le\alpha<\infty$, and $0<s<\infty$ we say that μ is an α-logarithmic s-Carleson measure if there exists a positive constant C such that
$$\mu\left([t,1)\right) \le C(1-t)^s\left(\log\frac{2}{1-t}\right)^{-\alpha}, \quad 0\le t<1.$$

Let us close this section by saying that, as usual, we shall be using the convention that $C=C(p,\alpha,q,\beta,\dots)$ will denote a positive constant which depends only upon the displayed parameters $p,\alpha,q,\beta\dots$ (which sometimes will be omitted) but not necessarily the same at different occurrences. Furthermore, for two real-valued functions K_1,K_2 we write $K_1\lesssim K_2$, or $K_1\gtrsim K_2$, if there exists a positive constant C independent of the arguments such that $K_1\le CK_2$, respectively $K_1\ge CK_2$. If we have $K_1\lesssim K_2$ and $K_1\gtrsim K_2$ simultaneously, then we say that K_1 and K_2 are equivalent and we write $K_1\asymp K_2$.

2. Hilbert-Type Operators

The matrix $\mathcal{H}_\mu$ induces formally an operator, which will be also called $\mathcal{H}_\mu$, on spaces of analytic functions by its action on the Taylor coefficients:
$$a_n \mapsto \sum_{k=0}^{\infty}\mu_{n+k}a_k, \quad n=0,1,2,\dots.$$
To be precise, if $f(z)=\sum_{k=0}^{\infty}a_kz^k\in\mathrm{Hol}(\mathbb{D})$ we define
$$\mathcal{H}_\mu(f)(z)=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{\infty}\mu_{n+k}a_k\right)z^n, \tag{2.1}$$
whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$.

If μ is the Lebesgue measure on $[0,1)$ the matrix $\mathcal{H}_\mu$ reduces to the classical Hilbert matrix $\mathcal{H}=\left((n+k+1)^{-1}\right)_{n,k\ge0}$, which induces the classical Hilbert operator $\mathcal{H}$ which has extensively studied recently (see [1,16,17,19,32–34]).

The finite positive Borel measures μ for which $\mathcal{H}_\mu$ is a bounded operator on distinct spaces of analytic functions in $\mathbb{D}$ have been characterized in a number of papers such as [9,14,25,27–29,35,37,38,45]. Obtaining an integral representation of $\mathcal{H}_\mu$ plays a basic role in these works. If μ is as above, we shall write throughout the paper

$$\mathcal{I}_\mu(f)(z) = \int_{[0,1)} \frac{f(t)}{1-tz}\, d\mu(t), \tag{2.2}$$

whenever the right hand side makes sense and defines an analytic function in $\mathbb{D}$. It turns out that the operators $\mathcal{H}_\mu$ and $\mathcal{I}_\mu$ are very closely related.

Let us mention the following results.

Theorem A. *Let μ be a positive Borel measure on $[0,1)$. Then*

- (i) *The operator $\mathcal{H}_\mu$ is bounded from H^1 into itself if and only if μ is a 1-logarithmic 1-Carleson measure. In such a case $\mathcal{H}_\mu$ and $\mathcal{I}_\mu$ coincide on H^1.*
- (ii) *If $1 < p < \infty$, then $\mathcal{H}_\mu$ is a bounded operator from H^p into itself if and only if μ is a 1-Carleson measure. In such a case $\mathcal{H}_\mu$ and $\mathcal{I}_\mu$ coincide on H^p.*
- (iii) *If $p > 1$ and $-1 < \alpha < p-2$ then the operator $\mathcal{H}_\mu$ is well defined on A^p_α and it is bounded from A^p_α into itself if and only if μ is a 1-Carleson measure. In such a case $\mathcal{H}_\mu$ and $\mathcal{I}_\mu$ coincide on A^p_α.*
- (iv) *If $p > 1$ and $p-2 < \alpha \le p-1$, then $\mathcal{H}_\mu$ is well defined on $\mathcal{D}^p_\alpha$ and it is bounded from $\mathcal{D}^p_\alpha$ into itself if and only if μ is a 1-Carleson measure. In such a case $\mathcal{H}_\mu$ and $\mathcal{I}_\mu$ coincide on $\mathcal{D}^p_\alpha$.*
- (v) *If $0 < \alpha < 2$, $\mathcal{H}_\mu$ is a bounded operator from $\mathcal{D}^2_\alpha$ into itself if and only if μ is a 1-Carleson measure. In such a case $\mathcal{H}_\mu$ and $\mathcal{I}_\mu$ coincide on $\mathcal{D}^2_\alpha$.*

The questions of characterizing those μ for which $\mathcal{H}_\mu$ is bounded on either the Dirichlet space $\mathcal{D}$ or on the Bergman space A^2 are more delicate and remain open. Regarding the Dirichlet space, the following results are proved in [28].

Theorem B. (i) *Let μ be a positive and finite Borel measure on $[0,1)$. If $\gamma > 1$ and μ is a γ-logarithmic 1-Carleson measure, then $\mathcal{H}_\mu$ is bounded from $\mathcal{D}$ into itself.*

(ii) *If $0 < \beta \le \frac{1}{2}$, then there exists a positive and finite Borel measure μ on $[0,1)$ which is a β-logarithmic 1-Carleson measure but such that $\mathcal{H}_\mu(\mathcal{D}) \not\subset \mathcal{D}$.*

We improve this result showing that being a 1-logarithmic 1-Carleson measure is enough to insure that $\mathcal{H}_\mu$ is bounded from $\mathcal{D}$ into itself and closing the gap between (i) and (ii). Indeed, we shall prove the following result.

Theorem 1. (i) *Let μ be a positive and finite Borel measure on $[0,1)$. If μ is a 1-logarithmic 1-Carleson measure, then $\mathcal{H}_\mu$ is bounded from $\mathcal{D}$ into itself.*

(ii) *If $0 < \beta < 1$, then there exists a positive and finite Borel measure μ on $[0, 1)$ which is a β-logarithmic 1-Carleson measure but such that $\mathcal{H}_\mu(\mathcal{D}) \not\subset \mathcal{D}$.*

As a corollary of part (i) we obtain the following.

Corollary 2. (a) *Let μ be a positive and finite Borel measure on $[0, 1)$ and suppose that μ is a 1-logarithmic 1-Carleson measure. Then there exists a positive constant C such that*

$$\int_{[0,1)} |tf(t)f'(t)|\, d\mu(t) \le C\|f\|_{\mathcal{D}}^2, \quad f \in \mathcal{D}. \tag{2.3}$$

(b) *There exists a positive constant C such that*

$$\int_0^1 |tf(t)f'(t)| \log \frac{2}{1-t}\, dt \le C\|f\|_{\mathcal{D}}^2, \quad f \in \mathcal{D}. \tag{2.4}$$

Regarding the Bergman space A^2, Theorem 1.5 of [25] asserts the following.

Theorem C. *Let μ be a positive and finite Borel measure on $[0, 1)$ and let h_μ be defined by $h_\mu(z) = \sum_{n=0}^\infty \mu_n z^n$ $(z \in \mathbb{D}.)$ If μ satisfies the condition*

$$\int_{[0,1)} \frac{\mu([t, 1))}{(1-t)^2}\, d\mu(t) < \infty, \tag{2.5}$$

then $\mathcal{H}_\mu$ is bounded from A^2 into itself if and only if the measure $|h'_\mu(z)|^2 dA(z)$ is a Dirichlet-Carleson measure.

We recall that a finite positive Borel measure ν on $\mathbb{D}$ is said to be a Dirichlet-Carleson messure if $\mathcal{D}$ is continuously embedded in $L^2(d\nu)$. Stegenga [43] gave a characterization of these measures involving the logarithmic capacity of a finite union of intervals of $\partial\mathbb{D}$. Shields [39] obtained a simpler characterization when dealing with measures supported on $[0, 1)$. This result of Shields will be used below.

Using Theorem 1 we shall prove the following result.

Theorem 3. (i) *Let μ be a positive and finite Borel measure on $[0, 1)$. If μ is a 1-logarithmic 1-Carleson measure, then $\mathcal{H}_\mu$ is bounded from A^2 into itself.*

(ii) *If $0 < \beta < 1$, then there exists a positive and finite Borel measure μ on $[0, 1)$ which is a β-logarithmic 1-Carleson measure but such that $\mathcal{H}_\mu(A^2) \not\subset \mathcal{A}^2$.*

In order to prove our results we start using the above mentioned result of Shields [39] to find a weak condition which insures that $\mathcal{H}_\mu$ and $\mathcal{I}_\mu$ are well defined in $\mathcal{D}$ and that $\mathcal{H}_\mu(f) = \mathcal{I}_\mu(f)$ for all $f \in \mathcal{D}$.

Proposition 4. *Let μ be a positive and finite Borel measure on $[0,1)$. If there exists a positive constant C such that*

$$\mu([t,1)) \le C\left(\log\frac{2}{1-t}\right)^{-1}, \quad 0<t<1, \tag{2.6}$$

then $\mathcal{H}_\mu$ and $\mathcal{I}_\mu$ are well defined in $\mathcal{D}$ and, furthermore, $\mathcal{H}_\mu(f)=\mathcal{I}_\mu(f)$ for all $f\in\mathcal{D}$.

Proof. Suppose that μ satisfies (2.6). Shields proved in [39, Theorem 2] that this is equivalent to saying that there exists a positive constant A such that

$$\int_{[0,1)} |f(t)|^2\,d\mu(t) \,\le\, A\|f\|_{\mathcal{D}}^2, \quad f\in\mathcal{D}. \tag{2.7}$$

We can express (2.7) simply by saying that μ is a radial Carleson-Dirichlet measure. Also, it is easy to see that (2.6) implies that there exists $B>0$ such that

$$\mu_n \,\le\, \frac{B}{\log(n+2)}, \quad n=0,1,2,\dots. \tag{2.8}$$

Take $f\in\mathcal{D}$, $f(z)=\sum_{n=0}^{\infty}a_nz^n$ $(z\in\mathbb{D})$.
Let us prove that $\mathcal{I}_\mu(f)$ is well defined.
Using (2.7) and (2.8), we see that

$$\begin{aligned}\int_{[0,1)} t^n|f(t)|\,d\mu(t) &\le \left(\int_{[0,1)} t^{2n}\,d\mu(t)\right)^{1/2}\left(\int_{[0,1)} |f(t)|^2\,d\mu(t)\right)^{1/2}\\ &\le A^{1/2}\mu_{2n}^{1/2}\|f\|_{\mathcal{D}}\\ &\le \frac{A^{1/2}B^{1/2}\|f\|_{\mathcal{D}}}{(\log(2n+2))^{1/2}},\end{aligned}$$

for all n. Then we have

$$\sum_{n=0}^{\infty}\left(\int_{[0,1)} t^n|f(t)|\,d\mu(t)\right)|z|^n \lesssim \sum_{n=0}^{\infty}\frac{|z|^n}{(\log(2n+2))^{1/2}}, \quad z\in\mathbb{D}.$$

This implies that, for all $z\in\mathbb{D}$, the integral

$$\int_{[0,1)} \frac{f(t)}{1-tz}\,d\mu(t) \,=\, \int_{[0,1)} f(t)\left(\sum_{n=0}^{\infty}t^nz^n\right)d\mu(t)$$

converges and that

$$\int_{[0,1)} \frac{f(t)}{1-tz}\,d\mu(t) \,=\, \sum_{n=0}^{\infty}\left(\int_{[0,1)} t^nf(t)\,d\mu(t)\right)z^n, \quad z\in\mathbb{D}.$$

So $\mathcal{I}_\mu(f)$ is a well defined analytic function in $\mathbb{D}$ and

$$\mathcal{I}_\mu(f)(z) = \sum_{n=0}^{\infty}\left(\int_{[0,1)} t^nf(t)\,d\mu(t)\right)z^n, \quad z\in\mathbb{D}. \tag{2.9}$$

Let us see now that $\mathcal{H}_\mu(f)$ is also well defined and that $\mathcal{H}_\mu(f) = \mathcal{I}_\mu(f)$. Using (2.8), for all n, we have

$$\begin{aligned}\sum_{k=0}^{\infty} |\mu_{n+k} a_k| &\lesssim \mu_n |a_0| + \sum_{k=1}^{\infty} \frac{k^{1/2}|a_k|}{k^{1/2}\log(n+k+2)} \\ &\lesssim \mu_0|a_0| + \left(\sum_{k=1}^{\infty} k|a_k|^2\right)^{1/2} \left(\sum_{k=1}^{\infty} \frac{1}{k\left(\log(k+1)\right)^2}\right)^{1/2} \\ &\lesssim \|f\|_{\mathcal{D}}.\end{aligned}$$

Clearly, this implies that $\mathcal{H}_\mu$ is a well defined analytic function in $\mathbb{D}$. Also,

$$\int_{[0,1)} t^n f(t)\, d\mu(t) = \int_{[0,1)} t^n \left(\sum_{k=0}^{\infty} a_k t^k\right) d\mu(t) = \sum_{k=0}^{\infty} \mu_{n+k} a_k$$

for all k. Then (2.9) yields that $\mathcal{H}_\mu(f) = \mathcal{I}_\mu(f)$. □

Let us turn now to prove Theorem 1

Proof of Theorem 1 (i). Suppose that μ is a 1-logarithmic 1-Carleson measure. Take $f \in \mathcal{D}$, $f(z) = \sum_{k=0}^{\infty} a_k z^k$ ($z \in \mathbb{D}$). Proposition 4 implies that $\mathcal{H}_\mu(f)$ and $\mathcal{I}_\mu(f)$ are well defined and that $\mathcal{H}_\mu(f) = \mathcal{I}_\mu(f)$. The above mentioned result of Shields yields that

$$\begin{aligned}|\mathcal{H}_\mu(f)(0)| &= |\mathcal{I}_\mu(f)(0)| = \left|\int_{[0,1)} f(t)\, d\mu(t)\right| \\ &\lesssim \left(\int_{[0,1)} |f(t)|^2\, d\mu(t)\right)^{1/2} \lesssim \|f\|_{\mathcal{D}}. \end{aligned} \tag{2.10}$$

Since μ is a 1-logarithmic 1-Carleson measure,

$$\mu_n = \mathrm{O}\left(\frac{1}{n\log(n+1)}\right), \tag{2.11}$$

(see e. g. [28, pp. 380-381]). Using (2.10) and (2.11), we obtain

$$\begin{aligned}\|\mathcal{H}_\mu(f)\|_{\mathcal{D}}^2 &\lesssim |\mathcal{H}_\mu(f)(0)|^2 + \sum_{n=1}^{\infty} n \left(\sum_{k=0}^{\infty} \mu_{n+k}|a_k|\right)^2 \\ &\lesssim \|f\|_{\mathcal{D}}^2 + \sum_{n=1}^{\infty} n \left(\sum_{k=0}^{\infty} \frac{|a_k|}{(n+k)\log(n+k+1)}\right)^2 \\ &\lesssim \|f\|_{\mathcal{D}}^2 + I + II,\end{aligned}$$

where

$$I = \sum_{n=1}^{\infty} n \left(\sum_{k=0}^{n} \frac{|a_k|}{(n+k)\log(n+k+1)} \right)^2,$$

$$II = \sum_{n=1}^{\infty} n \left(\sum_{k=n+1}^{\infty} \frac{|a_k|}{(n+k)\log(n+k)} \right)^2.$$

Now, using a result of Holland and Walsh [30, Theorem 7] and simple estimates we deduce that

$$\begin{aligned} I &= \sum_{n=1}^{\infty} n \left(\sum_{k=0}^{n} \frac{|a_k|}{(n+k)\log(n+k+1)} \right)^2 \\ &\le \sum_{n=1}^{\infty} \frac{1}{n\left(\log(n+1)\right)^2} \left(\sum_{k=0}^{n} |a_k| \right)^2 \lesssim \|f\|_{\mathcal{D}}^2. \end{aligned}$$

Also, since, for every n,

$$\begin{aligned} \sum_{k=n+1}^{\infty} \frac{|a_k|}{(n+k)\log(n+k)} &\le \frac{1}{\log(n+1)} \sum_{k=n+1}^{\infty} \frac{k^{1/2}|a_k|}{k^{1/2}(n+k)} \\ &\le \frac{1}{\log(n+1)} \left(\sum_{k=n+1}^{\infty} k|a_k|^2 \right)^{1/2} \left(\sum_{k=n+1}^{\infty} \frac{1}{k(n+k)^2} \right)^{1/2} \\ &\le \frac{\|f\|_{\mathcal{D}}}{\log(n+1)} \left(\sum_{k=n+1}^{\infty} \frac{1}{k(n+k)^2} \right)^{1/2} \\ &\le \frac{\|f\|_{\mathcal{D}}}{n^{1/2}\log(n+1)} \left(\sum_{k=n+1}^{\infty} \frac{1}{(n+k)^2} \right)^{1/2} \\ &\lesssim \frac{\|f\|_{\mathcal{D}}}{n\log(n+1)}, \end{aligned}$$

it follows that

$$\begin{aligned} II &= \sum_{n=1}^{\infty} n \left(\sum_{k=n+1}^{\infty} \frac{|a_k|}{(n+k)\log(n+k)} \right)^2 \\ &\lesssim \|f\|_{\mathcal{D}}^2 \sum_{n=1}^{\infty} \frac{1}{n\left(\log(n+1)\right)^2} \\ &\lesssim \|f\|_{\mathcal{D}}^2. \end{aligned}$$

Putting everything together, we obtain $\|\mathcal{H}_\mu(f)\|_{\mathcal{D}}^2 \lesssim \|f\|_{\mathcal{D}}^2$. □

Proof of Theorem 1 (ii). Suppose that $0 < \beta < 1$. Take $\alpha \in \mathbb{R}$ with

$$\frac{1}{2} < \alpha < \min\left(1, \frac{3-2\beta}{2}\right).$$

Let μ be the Borel measure on $[0,1)$ defined by $d\mu(t) = \left(\log \frac{2}{1-t}\right)^{-\beta} dt$. Then (see [28, p. 392]) μ is a β-logarithmic 1-Carleson measure and

$$\mu_n \asymp \frac{1}{n\left[\log(n+1)\right]^{\beta}}.$$

Set $a_n = \frac{1}{(n+1)[\log(n+1)]^{\alpha}}$ $(n = 1, 2, \dots)$ and $g(z) = \sum_{n=1}^{\infty} a_n z^n$ $(z \in \mathbb{D})$.

The condition $\alpha > \frac{1}{2}$ implies that $g \in \mathcal{D}$. We are going to see that $\mathcal{H}_\mu(g) \notin \mathcal{D}$, this will finish the proof.

We have

$$\begin{aligned}
\|\mathcal{H}_\mu(g)\|_{\mathcal{D}}^2 &\gtrsim \sum_{n=2}^{\infty} n \left(\sum_{k=2}^{n} \mu_{n+k} a_k\right)^2 \\
&\asymp \sum_{n=2}^{\infty} n \left(\sum_{k=2}^{n} \frac{1}{(n+k)\left[\log(n+k)\right]^{\beta} k \left[\log k\right]^{\alpha}}\right)^2 \\
&\gtrsim \sum_{n=2}^{\infty} \frac{n}{n^2 \left[\log n\right]^{2\beta}} \left(\sum_{k=2}^{n} \frac{1}{k\left[\log k\right]^{\alpha}}\right)^2 \\
&= \sum_{n=2}^{\infty} \frac{1}{n \left[\log n\right]^{2\beta}} \left(\sum_{k=2}^{n} \frac{1}{k\left[\log k\right]^{\alpha}}\right)^2 \\
&\gtrsim \sum_{n=2}^{\infty} \frac{1}{n \left[\log n\right]^{2\beta+2\alpha-2}}.
\end{aligned}$$

Since $2\alpha + 2\beta - 2 < 1$, $\sum_{n=2}^{\infty} \frac{1}{n[\log n]^{2\beta+2\alpha-2}} = \infty$ and, hence, $\mathcal{H}_\mu(g) \notin \mathcal{D}$ as desired. $\square$

Proof of Corollary 2. The Dirichlet space is a Hilbert space with the inner product

$$< f, g >= f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{g'(z)}\, dA(z), \quad f, g \in \mathcal{D}.$$

Hence, $\mathcal{D}$ is identifiable with its dual with this pairing.

Assume that μ is a finite Borel measure on $[0,1)$ which is a 1-logarithmic 1-Carleson measure. If $f \in \mathcal{D}$, using Theorem 1, we see that $\mathcal{H}_\mu(f) \in \mathcal{D}$ and $\|\mathcal{H}_\mu(f)\|_{\mathcal{D}} \lesssim \|f\|_{\mathcal{D}}$. Then $\mathcal{H}_\mu(f)$ induces a bounded linear functional on $\mathcal{D}$ with norm controlled by $\|f\|_{\mathcal{D}}$. Thus

$$\left|\int_{\mathbb{D}} \mathcal{H}_\mu(f)'(z)\overline{g'(z)}\, dA(z)\right| \lesssim \|f\|_{\mathcal{D}}\|g\|_{\mathcal{D}}, \quad f, g \in \mathcal{D}. \tag{2.12}$$

Now, using the definitions, Fubini's theorem, and the reproducing formula for the Bergman space A^2, we have

$$\begin{aligned}\int_{\mathbb{D}} \mathcal{H}_\mu(f)'(z)\overline{g'(z)}\,dA(z) &= \int_{\mathbb{D}} \left(\int_{[0,1)} \frac{tf(t)}{(1-tz)^2}\,d\mu(t)\right)\overline{g'(z)}\,dA(z)\\ &= \int_{[0,1)} tf(t)\left(\int_{\mathbb{D}} \frac{\overline{g'(z)}}{(1-tz)^2}\,dA(z)\right)d\mu(t)\\ &= \int_{[0,1)} tf(t)\overline{g'(t)}\,d\mu(t).\end{aligned}$$

Using (2.12), we obtain

$$\left|\int_{[0,1)} tf(t)\overline{g'(t)}\,d\mu(t)\right| \lesssim \|f\|_{\mathcal{D}}\|g\|_{\mathcal{D}}, \quad f,g\in\mathcal{D}. \tag{2.13}$$

Take $f,g\in\mathcal{D}$, $f(z)=\sum_{n=0}^\infty a_nz^n$, $g(z)=\sum_{n=0}^\infty b_nz^n$ $(z\in\mathbb{D})$. Set

$$f_1(z)=\sum_{n=0}^\infty |a_n|z^n,\quad g_1(z)=\sum_{n=0}^\infty |b_n|z^n\quad (z\in\mathbb{D}).$$

Then $f_1,g_1\in\mathcal{D}$, $\|f_1\|_{\mathcal{D}}=\|f\|_{\mathcal{D}}$, and $\|g_1\|_{\mathcal{D}}=\|g\|_{\mathcal{D}}$. Using (2.13) with f_1 and g_1 in the places of f and g, we obtain

$$\begin{aligned}\int_{[0,1)} \left|tf(t)\overline{g'(t)}\right|\,d\mu(t) &\le \int_{[0,1)} \left|tf_1(t)\overline{g_1'(t)}\right|\,d\mu(t)\\ &\lesssim \|f_1\|_{\mathcal{D}}\|g_1\|_{\mathcal{D}}\\ &= \|f\|_{\mathcal{D}}\|g\|_{\mathcal{D}}.\end{aligned}$$

Taking $f=g$, (2.3) follows.

Part (b) follows taking $d\mu(t)=\log\frac{2}{1-t}\,dt$ in part (a). $\square$

Proof of Theorem 3. Our proof of Theorem 3 is based on the fact that the pairing

$$< f,g >= f(0)\overline{g(0)} + \int_{\mathbb{D}} f'(z)\overline{\left(\frac{g(z)-g(0)}{z}\right)}\,dA(z),\quad f\in\mathcal{D},\ g\in A^2$$

is a "duality paring" between the Dirichlet space $\mathcal{D}$ and the Bergman space A^2. Notice that if $f(z)=\sum_{n=0}^\infty a_nz^n$ and $g(z)=\sum_{n=0}^\infty b_nz^n$ $(z\in\mathbb{D})$, then

$$< f,g >= \sum_{n=0}^\infty a_n\overline{b_n}.$$

It is a simple exercise to show that $< \mathcal{H}_\mu(P),Q >=< P,\mathcal{H}_\mu(Q) >$ if P and Q are polynomials. Then it follows that if $\mathcal{H}_\mu$ is a bounded operator from $\mathcal{D}$ into itself then its adjoint (via this pairing) is $\mathcal{H}_\mu$, and then we see that $\mathcal{H}_\mu$ is a bounded operator from A^2 into itself. Using this and Theorem 1 (i) we obtain part (a) of Theorem 3.

Similarly, if $\mathcal{H}_\mu$ is a bounded operator from A^2 into itself, then $\mathcal{H}_\mu$ is also a bounded operator from $\mathcal{D}$ into itself and then part (b) of Theorem 3 follows using Theorem 1 (ii). □

3. Cesàro-Type Operators

For μ a finite positive Borel measure on $[0, 1)$ as above, the matrix $\mathcal{C}_\mu$ induces a linear operator, also called $\mathcal{C}_\mu$, from Hol($\mathbb{D}$) into itself as follows: If $f \in \mathrm{Hol}(\mathbb{D})$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$,

$$\mathcal{C}_\mu(f)(z) = \sum_{n=0}^{\infty} \left(\mu_n \sum_{k=0}^{n} a_k \right) z^n, \quad z \in \mathbb{D}.$$

Let us remark that the operator $\mathcal{C}_\mu$ has the following integral representation: If $f \in \mathrm{Hol}(\mathbb{D})$ then

$$\mathcal{C}_\mu(f)(z) = \int_{[0,1)} \frac{f(tz)}{1 - tz}\, d\mu(t), \quad z \in \mathbb{D}. \tag{3.1}$$

When μ is the Lebesgue measure on $[0, 1)$, the operator $\mathcal{C}_\mu$ reduces to the classical Cesàro operator $\mathcal{C}$.

The Cesàro operator $\mathcal{C}$ acting on distinct subspaces of Hol($\mathbb{D}$) has been extensively studied in a good number of articles such as [2,10,12,15,23,36,40–42,44]. Let us recall that it is bounded on H^p $(0 < p < \infty)$ and on A^p_α $(0 < p < \infty,\ \alpha > -1)$.

The operators $\mathcal{C}_\mu$ were introduced in [23] where, among other results, it was proved that the following conditions are equivalent:

(i) μ is a Carleson measure, that is, $\mu([t, 1)) \le C(1 - t)$ $(0 < t < 1)$.
(ii) $\mu_n = \mathrm{O}\left(\frac{1}{n}\right)$.
(iii) $1 \le p < \infty$ and $\mathcal{C}_\mu$ is bounded from H^p into itself.
(iv) $1 < p < \infty$, $\alpha > -1$, and $\mathcal{C}_\mu$ is bounded from A^p_α into itself.

Blasco [12] has generalized the definition of the operators $\mathcal{C}_\mu$ by dealing with complex Borel measures on $[0, 1)$ and he has extended results of [23] to this more general setting.

A further generalization has been given in [24] by working with the operators $\mathcal{C}_\mu$ associated to arbitrary complex Borel measures on $\mathbb{D}$, not necessarily supported on a radius. The complex Borel measures on $\mathbb{D}$ for which the operator $\mathcal{C}_\mu$ is bounded or Hilbert-Schmidt on H^2 or on A^2_α $(\alpha > -1)$ are characterized in the mentioned paper [24].

We devote this section to study the operators $\mathcal{C}_\mu$ on the Dirichlet space, a question which has not been considered in the just mentioned papers. Our main results are contained in the following two theorems.

Theorem 5. *Let μ be a finite positive Borel measure on $[0,1)$.*

(i) *If μ is a 1-logarithmic 1-Carleson measure, then $\mathcal{C}_\mu$ is a bounded operator from the Dirichlet space $\mathcal{D}$ into itself.*
(ii) *If $\mathcal{C}_\mu$ is a bounded operator from $\mathcal{D}$ into itself then μ is a 1/2-logarithmic 1-Carleson measure.*

Theorem 6. *Suppose that $\frac{1}{2} < \beta < 1$. Then there exists a finite positive Borel measure μ on $[0,1)$ which is β-logarithmic 1-Carleson measure for which $\mathcal{C}_\mu(\mathcal{D}) \not\subset \mathcal{D}$.*

Proof of Theorem 5 (i). Since μ is a 1-logarithmic 1-Carleson measure, we have that

$$\mu_n = \mathrm{O}\left(\frac{1}{(n+1)\log(n+2)}\right). \tag{3.2}$$

Take $f \in \mathcal{D}$, $f(z) = \sum_{n=0}^{\infty} a_n z^n$ $(z \in \mathbb{D})$. Using (3.2) and Theorem 7 of [30], we obtain

$$\begin{aligned}\|\mathcal{C}_\mu(f)\|_{\mathcal{D}}^2 &\le \sum_{n=0}^{\infty}(n+1)\mu_n^2\left(\sum_{k=0}^{n}|a_k|\right)^2\\ &\lesssim \sum_{n=0}^{\infty}\frac{\left(\sum_{k=0}^{n}|a_k|\right)^2}{(n+1)[\log(n+2)]^2}\\ &\lesssim \|f\|_{\mathcal{D}}^2.\end{aligned}$$

□

Proof of Theorem 5 (ii). Suppose that $\mathcal{C}_\mu$ is a bounded operator from $\mathcal{D}$ into itself. For $N \in \mathbb{N}$, set

$$f_N(z) = \sum_{n=1}^{N}\frac{z^n}{n}, \quad z \in \mathbb{D}.$$

Then,

$$\|f_N\|_{\mathcal{D}}^2 = \sum_{n=1}^{N}\frac{1}{n} \asymp \log(N+1).$$

Since $\mathcal{C}_\mu$ is bounded on $\mathcal{D}$, bearing in mind that the sequence of moments $\{\mu_n\}$ is decreasing, we have

$$\begin{aligned}\log(N+1) \asymp \|f_N\|_{\mathcal{D}}^2 &\gtrsim \sum_{n=1}^{\infty} n\mu_n^2\left(\sum_{k=1}^{n}\frac{1}{k}\right)^2\\ &\gtrsim \mu_N^2\sum_{n=1}^{N} n[\log(n+1)]^2 \asymp \mu_N^2 N^2[\log(N+1)]^2.\end{aligned}$$

Then it follows that $\mu_N = \mathrm{O}\left(\frac{1}{N[\log(N+1)]^{1/2}}\right)$. This implies that μ is a 1/2-logarithmic 1-Carleson measure. $\square$

Proof of Theorem 6. Assume that $1/2 < \beta < 1$. Let μ be the Borel measure on $[0,1)$ defined by $d\mu(t) = \left(\log \frac{2}{1-t}\right)^{-\beta} dt$. Then, as mentioned before, μ is a β-logarithmic 1-Carleson measure and $\mu_n \asymp \frac{1}{n[\log(n+1)]^{\beta}}$.

Set $\alpha = \beta - \frac{1}{2}$. Then $0 < \alpha < \frac{1}{2}$. Define

$$g(z) = \left(\log \frac{2}{1-z}\right)^{\alpha} = \sum_{n=0}^{\infty} A_n z^n, \quad z \in \mathbb{D}.$$

We have that

$$A_n \asymp \frac{1}{(n+1)[\log(n+2)]^{1-\alpha}}.$$

Since $\alpha < \frac{1}{2}$, we have that $g \in \mathcal{D}$. Also

$$\begin{aligned}\|\mathcal{C}_\mu(g)\|_{\mathcal{D}}^2 &\geq \sum_{n=2}^{\infty} n\mu_n^2 \left(\sum_{k=2}^{n} A_k\right)^2 \gtrsim \sum_{n=2}^{\infty} \frac{n}{n^2[\log n]^{2\beta}[\log n]^{-2\alpha}} \\ &= \sum_{n=2}^{\infty} \frac{1}{n[\log n]^{2(\beta-\alpha)}} = \sum_{n=2}^{\infty} \frac{1}{n[\log n]} = \infty.\end{aligned}$$

$\square$

Danikas and Siskakis [15] proved that $\mathcal{C}(H^\infty) \not\subset H^\infty$ and that $\mathcal{C}(H^\infty) \subset BMOA$. This was improved by Essén and Xiao who proved in [22] that $\mathcal{C}(H^\infty) \subset Q^p$ for $0 < p < \infty$. This result has been sharpened in [10].

We recall that $BMOA$ is the space of those functions $f \in H^1$ whose boundary values have bounded mean oscillation. Alternatively, a function $f \in \mathrm{Hol}(\mathbb{D})$ belongs to $BMOA$ if and only if

$$\sup_{T \in \mathrm{Aut}(\mathbb{D})} \|f \circ T - f(T(0))\|_{H^2} < \infty,$$

where $\mathrm{Aut}(\mathbb{D})$ denotes the set of all Möbius transformations from $\mathbb{D}$ onto itself. We refer to [26] for the theory of $BMOA$-functions.

For $0 < s < \infty$ the space Q_s consists of those $f \in \mathrm{Hol}(\mathbb{D})$ such that

$$\sup_{T \in \mathrm{Aut}(\mathbb{D})} \int_{\mathbb{D}} |f'(z)|^2 (1 - |T(z)|^2)^s \, dA(z) < \infty.$$

The spaces Q_s were introduced in [6] and [7]. We refer to [46] for the theory of Q_s spaces. Let us recall that

$$\mathcal{D} \subsetneq Q_{s_1} \subsetneq Q_{s_2} \subsetneq Q_1 = BMOA, \quad 0 < s_1 < s_2 < 1.$$

For $s > 1$ the space Q_s coincides with the Bloch space $\mathcal{B}$ of those functions $f \in \mathrm{Hol}(\mathbb{D})$ for which

$$\|f\|_{\mathcal{B}} \stackrel{\text{def}}{=} |f(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)|f'(z)| < \infty.$$

The paper [3] is an excellent reference for the theory of Bloch functions. Let us recall that $BMOA \subsetneq \mathcal{B}$.

Blasco [12] has proved that

$$\mathcal{C}(H^\infty) \subset \bigcap_{1<p<\infty} \Lambda^p_{1/p}. \tag{3.3}$$

Here, for $p \geq 1$, $\Lambda^p_{1/p}$ is the space of those functions $f \in \mathrm{Hol}(\mathbb{D})$ having a non-tangential limit at almost every point of $\partial\mathbb{D}$ and so that $\omega_p(\cdot, f)$, the integral modulus of continuity of order p of the boundary values $f(e^{i\theta})$ of f, satisfies $\omega_p(\delta, f) = \mathrm{O}(\delta^{1/p})$, as $\delta \to 0$. Classical results of Hardy and Littlewood (see [13] and [20, Chapter 5]) show that $\Lambda^p_{1/p} \subset H^p$ and that

$$\Lambda^p_{1/p} = \left\{ f \text{ analytic in } \mathbb{D} : M_p(r, f') = \mathrm{O}\left(\frac{1}{(1-r)^{1-\frac{1}{p}}}\right), \quad \text{as } r \to 1 \right\}.$$

In particular, Λ^1_1 is the space of those $f \in \mathrm{Hol}(\mathbb{D})$ such that $f' \in H^1$. The spaces $\Lambda^p_{1/p}$ increase with p and they are all contained in $BMOA$ [13]. Since $\Lambda^2_{1/2} \subset Q_s$ for all $s > 0$ (see [5, p. 427]), (3.3) improves the mentioned result in [22].

Bao, Sun and Wulan [8, Theorem 3.1] have proved that for any given $s > 0$, $\mathcal{C}_\mu(H^\infty) \subset Q_s$ if and only if μ is a Carleson measure.

It is natural to look for a result like (3.3) with $\mathcal{D}$ in the place of H^∞. It is easy to see that

$$\mathcal{C}(\mathcal{D}) \not\subset \mathcal{B}. \tag{3.4}$$

Indeed, set $a_n = \frac{1}{(n+1)\log(n+1)}$ $(n \geq 1)$ and $f(z) = \sum_{n=1}^\infty a_n z^n$ $(z \in \mathbb{D})$. Then $f \in \mathcal{D}$ and, setting $A_n = \sum_{k=1}^n a_k$, we have, for $0 < r < 1$,

$$\begin{aligned}
(1-r)\mathcal{C}(f)'(r) &= (1-r)\sum_{n=1}^\infty \frac{n}{n+1} A_n r^{n-1} \geq \frac{1}{2}(1-r)\sum_{n=1}^\infty A_n r^{n-1} \\
&= \frac{1}{2}\left[A_1 + \sum_{n=2}^\infty (A_n - A_{n-1}) r^{n-1}\right] = \frac{1}{2}\left[A_1 + \sum_{n=2}^\infty a_n r^{n-1}\right] \\
&\asymp \log\log\frac{2}{1-r}.
\end{aligned}$$

Hence, $C(f) \notin \mathcal{B}$.

The next natural step is trying to characterize the measures μ such that $\mathcal{C}_\mu(\mathcal{D}) \subset \mathcal{B}$. We have the following result.

Theorem 7. *Let X be a Banach space of analytic functions in $\mathbb{D}$ with $\Lambda^2_{1/2} \subset X \subset \mathcal{B}$ and let μ be a positive finite Borel measure on $[0,1)$.*

(i) *If μ is a $\frac{1}{2}$-logarithmic 1-Carleson measure, then $\mathcal{C}_\mu$ is a bounded operator from $\mathcal{D}$ into X.*

(ii) *If $\mathcal{C}_\mu$ is a bounded operator from $\mathcal{D}$ into X and $0<\beta<\frac{1}{2}$, then μ is a β-logarithmic 1-Carleson measure.*

Proof. Suppose that μ is a $\frac{1}{2}$-logarithmic 1-Carleson measure. Then

$$\mu_n \lesssim \frac{1}{n[\log(n+1)]^{1/2}}. \tag{3.5}$$

Take $f \in \mathcal{D}$, $f(z)=\sum_{n=0}^\infty a_n z^n$ $(z \in \mathbb{D})$. We have

$$\mathcal{C}_\mu(f)(z) = \sum_{n=0}^\infty \mu_n \left(\sum_{k=0}^n a_k\right) z^n = \sum_{n=0}^\infty A_n z^n,$$

where $A_n = \mu_n \left(\sum_{k=0}^n a_k\right)$. We have,

$$\begin{aligned}\left|\sum_{k=0}^n a_k\right| &\le |a_0| + \sum_{k=1}^n \frac{k^{1/2}|a_k|}{k^{1/2}}\\ &\le |a_0| + \left(\sum_{k=1}^n k|a_k|^2\right)^{1/2}\left(\sum_{k=1}^n \frac{1}{k}\right)^{1/2} \lesssim \|f\|_{\mathcal{D}}[\log(n+1)]^{1/2}.\end{aligned}$$

This and (3.5) imply that $|A_n| \lesssim \frac{\|f\|_{\mathcal{D}}}{n}$ a fact which easily yields that $\mathcal{C}_\mu(f) \in \Lambda^2_{1/2}$. This finishes the proof of (i).

Let us turn to prove (ii). Assume that $0<\beta<\frac{1}{2}$ and that $\mathcal{C}_\mu$ is a bounded operator from $\mathcal{D}$ into X.

Since $X \subset \mathcal{B}$, $\mathcal{C}_\mu$ is a bounded operator from $\mathcal{D}$ into $\mathcal{B}$.

Set $\alpha = 1-\beta$, and $f(z) = \sum_{n=0}^\infty \frac{z^n}{(n+1)[\log(n+2)]^\alpha}$ $(z \in \mathbb{D})$.

Notice that $\frac{1}{2}<\alpha<1$. This implies that $f \in \mathcal{D}$ and, hence, $\mathcal{C}_\mu(f) \in \mathcal{B}$. Then, bearing in mind that the sequence $\{\mu_n\}$ is decreasing, we see that, for $0<r<1$ and $N \in \mathbb{N}$,

$$\begin{aligned}\frac{1}{1-r} &\gtrsim \sum_{n=1}^\infty n\mu_n \left(\sum_{k=1}^n \frac{1}{(k+1)[\log(k+2)]^\alpha}\right) r^{n-1}\\ &\ge \sum_{n=1}^N n\mu_n \left(\sum_{k=1}^n \frac{1}{(k+1)[\log(k+2)]^\alpha}\right) r^n\\ &\gtrsim \mu_N \sum_{n=1}^N n[\log(n+2)]^{1-\alpha} r^n.\end{aligned}$$

Taking $r = 1-\frac{1}{N}$, we obtain

$$N \gtrsim \mu_N N^2[\log(N+2)]^{1-\alpha} = \mu_N N^2[\log(N+2)]^\beta$$

and, hence, $\mu_N \lesssim \frac{1}{N[\log(N+2)]^\beta}$. This implies that μ is a β-logarithmic 1-Carleson measure. $\square$

4. Extensions to Besov Spaces

The Dirichlet space is one among the analytic Besov spaces B^p. For $1 < p < \infty$, the analytic Besov space B^p is the space $\mathcal{D}^p_{p-2}$. Thus $B^2 = \mathcal{D}$.

The minimal Besov space B^1 requires a special definition. It is the space of all $f \in \mathrm{Hol}(\mathbb{D})$ such that $f'' \in A^1$. It is a Banach space with the norm $\|\cdot\|_{B^1}$ defined by $\|f\|_{B^1} = |f(0)| + |f'(0)| + \|f''\|_{A^1}$.

The Besov spaces B^p form a nested scale of conformally invariant spaces and they are all contained in $BMOA$:

$$B^p \subsetneq B^q \subsetneq BMOA \subsetneq \mathcal{B}, \quad 1 \leq p < q < \infty.$$

Also $B^p \subsetneq \Lambda^p_{1/p}$ for all $p \in [1, \infty)$. We mention [4,11,18,30,47,48] for information on Besov spaces. Let us remark that, letting $d\lambda$ be the Möbius invariant measure on $\mathbb{D}$ defined by $d\lambda(z) = \frac{dA(z)}{(1-|z|^2)^2}$, we have:

(a) The Bergman projection P is a continuous linear operator from $L^\infty(\mathbb{D})$ onto the Bloch space $\mathcal{B}$,
(b) For $1 < p < \infty$, the Bergman projection P is a continuous linear operator from $L^p(d\lambda)$ onto B^p

(see [48, Chapter 5]).

Our aim in this section is trying to extend to the spaces B^p some of the results obtained in the preceding ones for the Dirichlet space.

For the space B^1 we have the following result.

Theorem 8. *Let μ be positive finite Borel measure on $[0, 1)$. Then the following conditions are equivalent.*

(i) $\int_{[0,1)} \frac{d\mu(t)}{1-t} < \infty$.
(ii) $\sum_{n=0}^\infty \mu_n < \infty$.
(iii) *The operator $\mathcal{H}_\mu$ is a bounded operator from B^1 into itself.*
(iv) *The operator $\mathcal{C}_\mu$ is a bounded operator from B^1 into itself.*

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is clear.

Suppose that (iii) holds. Let f be the constant function $f(z) = 1$, for all $z \in \mathbb{D}$. Then $\mathcal{H}_\mu(f) = \mathcal{I}_\mu(f) \in B^1 \subset H^\infty$ and then

$$\int_{[0,1)} \frac{d\mu(t)}{1-t} = \lim_{r\to 1^-} \mathcal{I}_\mu(f)(r) \leq \|\mathcal{I}_\mu(f)\|_{H^\infty} < \infty.$$

Thus (i) holds.

Conversely, suppose that (i) holds. Take $f \in B^1$. We have

$$\mathcal{H}_\mu(f)''(z) = \int_{[0,1)} \frac{2t^2 f(t)}{(1-tz)^3}\, d\mu(t), \quad z \in \mathbb{D}.$$

Then using Fubini's theorem, [48, Lemma 3.10], and the fact that $B^1 \subset H^\infty$, we obtain

$$\begin{aligned}\int_{\mathbb{D}} |\mathcal{H}_\mu(f)''(z)|\, dA(z) &\lesssim \int_{\mathbb{D}} \int_{[0,1)} \frac{|f(t)|}{|1-tz|^3}\, d\mu(t)\, dA(z)\\ &= \int_{[0,1)} |f(t)| \int_{\mathbb{D}} \frac{dA(z)}{|1-tz|^3}\, d\mu(t)\\ &\lesssim \|f\|_{H^\infty} \int_{[0,1)} \frac{d\mu(t)}{1-t}\\ &\lesssim \|f\|_{B^1} \int_{[0,1)} \frac{d\mu(t)}{1-t}.\end{aligned}$$

Thus, (iii) follows.

Let us prove next the equivalence (i) $\Leftrightarrow$ (iv).

Suppose (i). Take $f \in B^1$. Bearing in mind (3.1) and using Fubini's theorem, we see that

$$\begin{aligned}&\int_{\mathbb{D}} |\mathcal{C}_\mu(f)''(z)|\, dA(z)\\ &\quad\lesssim \int_{[0,1)} \int_{\mathbb{D}} \frac{|f''(tz)| dA(z)}{|1-tz|} d\mu(t) + \int_{[0,1)} \int_{\mathbb{D}} \frac{|f'(tz)| dA(z)}{|1-tz|^2} d\mu(t)\\ &\qquad + \int_{[0,1)} \int_{\mathbb{D}} \frac{|f(tz)| dA(z)}{|1-tz|^3} d\mu(t).\end{aligned}$$

We now estimate each of the three terms in the last formula separately. For the first one we have

$$\begin{aligned}\int_{[0,1)} \int_{\mathbb{D}} \frac{|f''(tz)|}{|1-tz|}\, dA(z)\, d\mu(t) &\le \int_{[0,1)} \frac{1}{1-t} \int_{\mathbb{D}} |f''(tz)|\, dA(z)\, d\mu(t)\\ &\lesssim \|f\|_{B^1} \int_{[0,1)} \frac{d\mu(t)}{1-t}.\end{aligned}$$

For the second one, we use the fact that $B^1 \subset \Lambda^1_1$ to obtain

$$\begin{aligned}\int_{[0,1)} \int_{\mathbb{D}} \frac{|f'(tz)|}{|1-tz|^2}\, dA(z)\, d\mu(t) &\lesssim \int_{[0,1)} \int_0^1 \frac{M_1(tr, f')}{(1-tr)^2}\, dr\, d\mu(t)\\ &\le \|f\|_{\Lambda^1_1} \int_{[0,1)} \frac{d\mu(t)}{1-t} \lesssim \|f\|_{B^1} \int_{[0,1)} \frac{d\mu(t)}{1-t}.\end{aligned}$$

For the last integral, we use that $B^1 \subset H^\infty$ and Lemma 3.10 of [48] to see that

$$\begin{aligned}\int_{[0,1)} \int_{\mathbb{D}} \frac{|f(tz)|}{|1-tz|^3}\, dA(z)\, d\mu(t) &\le \|f\|_{H^\infty} \int_{[0,1)} \int_{\mathbb{D}} \frac{dA(z)}{|1-tz|^3}\, d\mu(t)\\ &\lesssim \|f\|_{B^1} \int_{[0,1)} \frac{d\mu(t)}{1-t}.\end{aligned}$$

Putting everything together we obtain (iv).

Suppose now that (iv) holds. Let f be the constant function given by $f(z) = 1$, for all $z \in \mathbb{D}$. Then $\mathcal{C}_\mu(f) \in B^1 \subset H^\infty$. Using the integral representation of $\mathcal{C}_\mu$ we see that

$$\int_{[0,1)} \frac{d\mu(t)}{1-t} = \lim_{r\to 1^-} \mathcal{C}_\mu(f)(r) \le \|\mathcal{C}_\mu(f)\|_{H^\infty}.$$

Thus, $\int_{[0,1)} \frac{d\mu(t)}{1-t} < \infty$. This is (i). □

Let us turn now to deal with the possible extensions in the range $1 < p < \infty$. The following result comes from [28, Theorem 2.4] and [23, Theorem 7].

Theorem D. *Let μ be a positive finite Borel measure on $[0, 1)$. If μ is a 1-logarithmic 1-Carleson measure then the operators $\mathcal{H}_\mu$ and $\mathcal{C}_\mu$ are bounded from the Bloch space $\mathcal{B}$ into itself.*

Using this result and those obtained in Sects. 2 and 3 we will prove the following.

Theorem 9. *Suppose that $2 < p < \infty$ and let μ be a positive finite Borel measure on $[0, 1)$. If μ is a 1-logarithmic 1-Carleson measure then the operators $\mathcal{H}_\mu$ and $\mathcal{C}_\mu$ are bounded from the Besov space B^p into itself.*

Proof. We shall use complex interpolation in the proof. Let us refer to [48, Chapter 2] for the terminology and basic results concerning complex interpolation.

If X_0 and X_1 are two compatible Banach spaces then, for $0 < \theta < 1$, $[X_0, X_1]_\theta$ stands for the space obtained by the complex method of interpolation of Calderón. As a consequence of the above mentioned results characterizing the spaces B^p as the image of $L^p(d\lambda)$ under the Bergman projection and the Bloch space as the image of $L^\infty(d\lambda)$ under the Bergman projection, Zhu proves in [48, Theorem 5.25] that if $1 < p_0 < \infty$, $0 < \theta < 1$, and $1/p = (1-\theta)/p_0$, then

$$[B^{p_0}, \mathcal{B}]_\theta = B^p. \tag{4.1}$$

In particular,

$$B^p = [\mathcal{D}, \mathcal{B}]_\theta, \quad \text{if } 2 < p < \infty \text{ and } \theta = 1 - \frac{2}{p}. \tag{4.2}$$

Theorem 9 follows using (4.2), Theorem 1 (i), Theorem 5 (i), and the interpolation theorem of operators [48, Theorem 2.4]. □

Regarding the sharpness of Theorem 9, we have the following result.

Theorem 10. *Suppose that $0 < \beta < 1$.*

(i) *If $1 < p < \infty$ then there exists a positive Borel measure μ on $[0, 1)$ which is a β-logarithmic 1-Carleson measure with the property that $\mathcal{H}_\mu(B^p) \not\subset B^p$.*

(ii) *If* $1 < p \le 2$ *then there exists a positive Borel measure* μ *on* $[0,1)$ *which is a* β*-logarithmic* 1*-Carleson measure with the property that* $\mathcal{C}_\mu(B^p) \not\subset B^p$*.*

Proof. Assume that $1 < p < \infty$ and $0 < \beta < 1$. Take $\alpha \in \mathbb{R}$ with

$$\frac{1}{p} < \alpha < \min\left(1, 1 + \frac{1}{p} - \beta\right).$$

Let μ be the Borel measure on $[0,1)$ defined by $d\mu(t) = \left(\log\frac{2}{1-t}\right)^{-\beta} dt$. We know that μ is a β-logarithmic 1-Carleson measure and that $\mu_n \asymp \frac{1}{(n+1)[\log(n+2)]^\beta}$.

For $n \ge 1$, set $a_n = \frac{1}{n[\log(n+1)]^\alpha}$ and $g(z) = \sum_{n=1}^\infty a_n z^n$ $(z \in \mathbb{D})$.

Since the sequence $\{a_n\}$ is decreasing and $\sum_{n=1}^\infty n^{p-1}|a_n|^p < \infty$, using [28, Theorem 3.10] we see that $g \in B^p$.

We have that $\mathcal{H}_\mu(g)(z) = \sum_{n=0}^\infty \left(\sum_{k=0}^\infty \mu_{n+k} a_k\right) z^n$ $(z \in \mathbb{D})$. Since the a_k's are positive and the sequence of moments $\{\mu_n\}$ is decreasing, it follows that the sequence $\{\sum_{k=0}^\infty \mu_{n+k} a_k\}$ is also decreasing. Then using again [28, Theorem 3.10] we see that

$$H_\mu(g) \in B^p \quad \Leftrightarrow \quad \sum_{n=1}^\infty n^{p-1}\left(\sum_{k=0}^\infty \mu_{n+k} a_k\right)^p < \infty. \tag{4.3}$$

Now,

$$\begin{aligned}
\sum_{n=1}^\infty n^{p-1}\left(\sum_{k=0}^\infty \mu_{n+k} a_k\right)^p &\gtrsim \sum_{n=2}^\infty n^{p-1}\left(\sum_{k=2}^\infty \frac{1}{(n+k)[\log(n+k)]^\beta k(\log k)^\alpha}\right)^p \\
&\ge \sum_{n=2}^\infty n^{p-1}\left(\sum_{k=2}^n \frac{1}{(n+k)[\log(n+k)]^\beta k(\log k)^\alpha}\right)^p \\
&\gtrsim \sum_{n=2}^\infty \frac{n^{p-1}}{n^p(\log n)^{p\beta}}\left(\sum_{k=2}^n \frac{1}{k(\log k)^\alpha}\right)^p \\
&\asymp \sum_{n=2}^\infty \frac{1}{n(\log n)^{p\beta}(\log n)^{p(\alpha-1)}} \\
&= \sum_{n=2}^\infty \frac{1}{n(\log n)^{p(\beta+\alpha-1)}}.
\end{aligned}$$

Since $p(\beta+\alpha-1) < 1$, it follows that $\sum_{n=1}^\infty n^{p-1}\left(\sum_{k=0}^\infty \mu_{n+k} a_k\right)^p = \infty$ and then (4.3) gives that $H_\mu(g) \notin B^p$.

Assume now that $1 < p \le 2$. We have

$$\mathcal{C}_\mu(g)(z) = \sum_{n=0}^\infty \mu_n\left(\sum_{k=0}^n a_k\right) z^n.$$

Using the fact that $1 < p \leq 2$ and [20, Theorem 6.s2] it readily follows that

$$\mathcal{C}_\mu(g) \in B^p \quad \Rightarrow \quad \sum_{n=1}^{\infty} n^{p-1}\mu_n^p \left(\sum_{k=1}^{n} a_k\right)^p < \infty. \tag{4.4}$$

But,

$$\begin{aligned}\sum_{n=1}^{\infty} n^{p-1}\mu_n^p \left(\sum_{k=1}^{n} a_k\right)^p &\gtrsim \sum_{n=1}^{\infty} \frac{1}{n[\log(n+1)]^{\beta p}} \left(\sum_{k=2}^{n} \frac{1}{k(\log k)^\alpha}\right)^p \\ &\gtrsim \sum_{n=1}^{\infty} \frac{1}{n[\log(n+1)]^{p(\beta+\alpha-1)}} \\ &= \infty.\end{aligned}$$

Using (4.4) we obtain that $\mathcal{C}_\mu(g) \notin B^p$. □

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Declarations

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