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SURVEY ARTICLE

Mathematical Reflections on Locality

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Abstract
Starting from the principle of locality in quantum field theory, which states that an object is influenced directly only by its immediate surroundings, we review some features of the notion of locality arising in physics and mathematics. We encode these in locality relations, given by symmetric binary relations, and locality morphisms, namely maps that factorise on products of pairs in the graph of such locality relations. This factorisation is a key property in the context of renormalisation, as illustrated on the factorisation of an exponential sum on convex cones, discussed at the end of the paper. The subject of locality is so vast and the issues it raises are so subtle, that this brief and modest presentation can only offer a small glimpse into this fascinating topic.

Keywords Locality · Causality · Renormalisation · Meromorphic germ · Convex cone

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1 Introduction

In physics, the principle of locality states that an object is influenced directly only by its immediate surroundings. Thus, one can separate events located in different

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regions of space-time and should be able to measure them independently according to the locality principle encoded in various formulations of axiomatic quantum field theory (QFT). The aim of this overview paper is to transpose the main features of the locality principle in QFT to a mathematical setting which encompasses known notions of locality. Locality relations serve here as a looking glass to get some insight on the vast and subtle topic of locality on which this presentation only provides a very partial view. Two disclaimers are in order. To avoid technical details, we have omitted most of the proofs and refer the reader to the relevant references. We also chose to focus on specific features of the principle of locality, thereby leaving out important topics which involve locality in an essential manner, such as vertex operator algebras and index theory. See e.g. [46, 47] and [29] respectively for a review on the subjects.

We distinguish two types of properties emanating from the locality principle, namely locality properties which concern nearby objects on the one hand, and separation properties required of separated objects, which are not expected to influence each other on the other hand. The latter is captured by what we call a locality relation, namely a binary symmetric relation which, loosely speaking, singles out specific pairs of elements which are "mutually separated/ independent" in a sense that depends on the specific locality.

Such a simple separation device turns out to be useful in order to deal with divergences in the context of quantum field theory, where measuring an event often requires a renormalisation step to ensure that one gets finite quantities in a consistent manner. It can come to rescue to separate divergences that need to be dealt with consistently in order to derive a reasonable finite quantity and we dedicate part of the paper to show how such a simple separation device can be used in the context of renormalisation.

Since our motivation to study locality stems from quantum field theory, we first review some features of the concept of locality and the related concept of causality in the axiomatic framework of some of the ancestors of algebraic quantum field theory (AQFT). We then propose an abstract notion of locality defined as a binary symmetric relation and discuss how it relates to the concept of causality in quantum field theory. We give prototype examples such as disjointness on sets and orthogonality on vector spaces. These play a role in the context of renormalisation where locality relation can serve as a separation device, in order to separate singularities from each other. Locality maps between two sets with locality preserve locality. Locality morphisms between two algebras with locality are locality maps which are multiplicative on the graph of the locality relation. One of the challenges of renormalisation is to build a locality renormalised morphism from a locality morphism. Our main aim is to show how this can be carried out in combining a multivariable regularisation with a suitable locality. Here is a more detailed account of the contents of the article.

The paper starts with a discussion of the locality properties of local functionals such as Lagrangians defined in terms of jets. In Sect. 2, we further list a few formulations of the locality principle among the ancestors of algebraic quantum field theory (AQFT), reviewing briefly Wightman's locality axiom in §2.3, Haag-Kastler's locality axiom in §2.4, Osterwalder-Schrader's locality axiom in §2.5. In the perturbative algebraic quantum field theory (pAQFT) setting, this can be captured by a factorisation property of scattering matrices for spacetime separated regions (§2.6), which we view as a separation property.

We then turn in Sect. 3, to an abstract notion of locality, namely a binary symmetric relation on a set X (Definition 3.1), called a locality relation. A locality relation can arise in many different ways. For instance, as discussed in §3.1.1, a locality relation can be derived from a non-causality relation, and on a poset two elements can be declared independent if they are not comparable for the partial order (16), just as spacetime separation is derived from the absence of a causal relation (6). We then view locality relation as a derived notion of the causality relation. This is complemented by a brief review of the concept of causality in the framework of causal sets given in Section 5.2. Other prototypes of locality relations are disjointness of subsets of an ambient set (Example 3.11) and orthogonality $\perp^Q$ of subspaces of a vector space equipped with an euclidean inner product Q (Example 3.13). These two examples fall in a class of locality lattices discussed in §3.1.3, whose locality relation is induced by an orthocomplementation (25), much in the same way as two sets are disjoint when one lies in the complement of the other. Disjointness and orthogonality also serve as essential building blocks for various other locality relations.

Among them are the locality relations $\perp^Q$ (both built from an inner product Q on the ambient space), on the space $\mathcal{C}$ of convex lattice cones discussed in §3.3.2 and on the space $\mathcal{M}_{\mathbb{Q}}$ of meromorphic germs at zero in several variables with linear poles discussed in §3.3.3. These two locality sets can be equipped with a locality semigroup structure discussed in §3.2.1, induced by a partial product defined on pairs in the graph of the locality relation.

Locality morphisms (in the sense of Definition 3.28) factorise on products involving pairs in the graph of the locality relation, a useful property when one cannot expect to have an ordinary multiplicative map. Examples are the exponential sums and integrals on lattice cones which yield locality morphisms from the locality semigroup of lattice cones to that of meromorphic germs, see §3.3.1.

The space $\mathcal{M}_{\mathbb{Q}}$ of meromorphic germs in several variables actually carries more structure, under the locality relation $\perp^Q$, it can be split (Proposition 3.42) as a direct sum $\mathcal{M}_{\mathbb{Q}} = \mathcal{M}_{\mathbb{Q}+} \oplus \mathcal{M}_{\mathbb{Q}-}^{Q}$ of the locality subalgebras $\mathcal{M}_{\mathbb{Q}+}$, resp. $\mathcal{M}_{\mathbb{Q}-}^{Q}$ of holomorphic, resp. polar germs at zero. More so, the projection onto $\mathcal{M}_{\mathbb{Q}+}$ along $\mathcal{M}_{\mathbb{Q}-}^{Q}$ gives rise to a locality morphism which plays a central role in the context of renormalisation (Proposition 3.44).

As mentioned previously, locality can serve as a separation device, with the disjointness relation in Example 3.11 separating sets and supports of functions and the zero intersection in Example 3.12 separating vector spaces. Its propensity to separate divergences is a key feature of a locality relation, as illustrated in §3.2.3, which reviews sufficient "separation" conditions on distributions in order to define their product, culminating with Hörmander's separation of wavefront sets (29).

Perturbative quantum field theory often involves Feynman integrals which diverge, and sophisticated methods were developed by physicists to 'cure' divergences occurring in renormalisable theories, while preserving the locality property (in the sense of (1)) of the Lagrangian defining the theory. The Bogoliubov-Parasiuk preparation, or BPHZ method (for Bogoliubov-Parasiuk-Hepp-Zimmermann) [6, 42, 73] (also known as the "forest formula") implemented in QFT is an inductive procedure on the number of loops of the graphs, which circumvents this difficulty in adding to the original Lagrangian an infinite series of counterterms relative to the singular parts, labelled by the Feynman graphs. It takes care of subdivergences in the renormalisation process while avoiding the occurrence of non-local terms in the counterterms which would not match and hence spoil the locality of the original Lagrangian in the sense of (1).

The Connes-Kreimer renormalisation scheme [20–22], known as algebraic Birkhoff factorisation (recalled in Theorem 4.1), offers an elegant coalgebraic formulation of the BPHZ method. It factorises a linear map $\phi : H \to M$ on a Hopf algebra H of Feynman graphs with values in an algebra M of meromorphic germs in one complex variable, as a convolution product of two linear maps, one of which is regular, while the other one carries the singularities. In accordance with the locality principle, the map ϕ is multiplicative on the concatenation of graphs (40) and the Birkhoff factorisation ensures that the regular part is also multiplicative. Consequently, the resulting renormalised map also enjoys this multiplicative property.

In our locality approach, $\phi : H \to \mathcal{M}_{\mathbb{Q}}$ maps the Hopf algebra to the algebra $\mathcal{M}_{\mathbb{Q}}$ of germs in several variables, and the multiplicativity of ϕ together with that of its regular part is encoded in the fact that they define locality morphisms. The fact that the canonical projection $\pi_+^Q : \mathcal{M}_{\mathbb{Q}} \to \mathcal{M}_{\mathbb{Q}+}$ along $\mathcal{M}_{\mathbb{Q}-}^Q$ defines a locality morphism of locality algebras is key to the efficiency of the locality relation $\perp^Q$ in the context of renormalisation. Indeed, it enables us to circumvent the aforementioned algebraic Birkhoff factorisation procedure in order to build a renormalised map from a locality morphism $\phi : (H, \top_H) \to \left(\mathcal{M}_{\mathbb{Q}}, \perp^Q\right)$ of locality algebras– with $\top_H$, resp. $\perp^Q$ the locality relation on H, resp. on $\mathcal{M}_{\mathbb{Q}}$– in so far as the mere composition $\pi^{\mathbb{Q}_+} \circ \phi$ gives the renormalised map directly (Theorem 4.16).

This approach using locality morphisms is applied in §4.5 to a generalisation to convex lattice cones of the classical Euler-Maclaurin formula, which relates an exponential sum to an integral. It is interpreted as a factorisation (52) of the exponential sum which gives rise to a meromorphic function in several variables as in (32), in terms of its polar part given by the integral and the regular interpolating part.

To sum up, as simple as the concept of locality relation– a mere binary symmetric relation– might seem at first sight, it can nevertheless serve as a surprisingly useful guide through the maze of divergences and subdivergences in that it separates them and provides a way to evaluate divergent expressions consistently.

2 Locality in Field Theory in Various Disguises

In physics, the **principle of locality** is a key feature of field theory which states that an object is influenced directly only by its immediate surroundings. This principle appears in various versions, and plays an important role in the construction of quantum field theory. To apply this principle, we first need to specify how objects relate, might they be close enough for them to interact, or might they lie far apart enough for no interaction to occur. The latter is interpreted in terms of a locality relation.

We distinguish two types of properties which we hold for fundamental, and which we shall refer to as

(a) **locality properties**, by which an observable should depend on the immediate neighborhood of the event only;

(b) **separation properties**, by which observables should "behave nicely" for events taking place in separated spacetime regions.

We first give a short and, considering the breadth of the subject, a very partial presentation of locality arising in various formulations for field theory.

We shall start with the locality property of Lagrangians in classical field theory, after which we shall turn to locality in quantum field theory in the sense of compatibility with separation. Our main focus is on algebraic quantum field theory which offers a rigorous mathematical approach to QFT in the tradition of axiomatic QFT, see e.g. [7] and [39] for a review of the historical developments.

2.1 Locality Property of Functionals

In the Lagrangian formalism of field theory, the principle of locality requires the Lagrangian to be built from integrating sections of a jet bundle over the spacetime M. Let us restrict ourselves to scalar field theory on $M = \mathbb{R}^d$ and explore the locality for functionals, in which case we deal with jets of smooth real functions on M. We view the space $\mathcal{E}(M)$ of smooth real valued functions on M as the space of fields and consider a smooth functional

$$F : U \subseteq \mathcal{E}(M) \to \mathbb{C}$$

defined on an open subset U of $\mathcal{E}(M)$. According to an intuitive notion of locality widely used in physics, a functional $F : \mathcal{E}(M) \longrightarrow \mathbb{R}$ is **local** if it is of the form

$$F(\varphi) = \int_M f\big(\varphi(x), \partial_\mu \varphi(x), \ldots, \partial_{\mu_1} \ldots \partial_{\mu_k} \varphi(x)\big)\, dx, \tag{1}$$

for some finite number of indices $\mu, \mu_i \in \{1, \ldots, d\}$ and where f is a smooth function with a finite number of arguments. In other words, it is of the form $\int_M f(j_x^k \varphi)\, dx$, where $j_x^k \varphi$ is the k-th jet of φ at point x. Let us recall that the k-th jet of a function at a point x is the equivalence class of the germs of smooth functions at that point for the relation $f \sim_x^k g$ if the derivatives of f and g coincide at the point x up to order k. Alternatively, $j_x^k \varphi$ can be viewed as an element of the fibre $J_x^k(M \times \mathbb{R})$ over x of the jet bundle $J^k(M \times \mathbb{R}) \to M$ of the trivial vector bundle $M \times \mathbb{R} \to M$.

This definition of local functionals is sometimes too restrictive. For example in general relativity, where one needs a more general concept of locality, which corresponds to locality in a neighborhood of any function φ in $\mathcal{E}(M)$. More precisely,

Definition 2.1 [12, Definition I.1] A functional $F : U \to \mathbb{C}$ defined on an open subset U of $\mathcal{E}(M)$ is said to comply with the **locality property** if for every $\varphi \in U$, there is

- a neighborhood V of φ,
- an integer k depending on the choice of V, an open subset $\mathcal{V}$ of the k-th jet $J^k(M \times \mathbb{R})$, and a smooth function f in $C^\infty(\mathcal{V})$ with the map $M \ni x \mapsto f(j_x^k \psi)$ supported in a compact subset K of M,

such that

$$F(\varphi+\psi)=F(\varphi)+\int_M f(j_x^k\psi)\,dx \tag{2}$$

for any ψ in $C^\infty(M)$ with $j^k\psi$ in $\mathcal{V}$ and $\varphi+\psi$ in V.

Remark 2.2 For $\varphi=0$, (2) reads $F(\psi)=\int_M f(j_x^k\psi)\,dx+\text{cst}$ for any ψ in V, which corresponds to what is commonly known as a local functional.

A locality functional can also be detected from its behaviour for functions with separated supports. For this purpose, we introduce a symmetric binary relation $\top$ on $\mathcal{E}(M)$

$$\varphi_1\top\varphi_2 \Leftrightarrow \text{Supp}(\varphi_1)\cap\text{Supp}(\varphi_2)=\emptyset, \tag{3}$$

(here Supp(f) stands for the support of f) and the question on how to use this separation property to formulate the locality of functionals described above can be formulated using this separation property.

Definition 2.3 A functional $F: U\to\mathbb{C}$ defined on the open subset U of $\mathcal{E}(M)$ has the **Hammerstein property** if for any φ in U

$$\varphi_1\top\varphi_2 \Longrightarrow F(\varphi_1+\varphi+\varphi_2)=F(\varphi_1+\varphi)-F(\varphi)+F(\varphi+\varphi_2), \tag{4}$$

with φ_1,φ_2 in U such that $\varphi_1+\varphi$, $\varphi+\varphi_2$ and $\varphi_1+\varphi+\varphi_2$ lie in U.

Remark 2.4 For $\varphi=0$, (4) reads $F(\varphi_1+\varphi_2)=F(\varphi_1)+F(\varphi_2)+\text{cst}$, for any φ_1,φ_2 in U whose sum also lies in U.

Theorem 2.5 [12, *Theorem I.2*] *If the differential $D_\varphi F$ of F at point φ can be represented as a function $\nabla_\varphi F$ in $\mathcal{D}(M)$ such that the map $\varphi\mapsto\nabla_\varphi F$ is smooth, then the Hammerstein property* (4) *of F is equivalent to its locality* (2).

Remark 2.6 Here the differential $D_\varphi f$ and the smoothness are taken in the sense of Bastiani. We recall that f is **Bastiani differentiable** on U and write $f\in C^1(U)$ if f has a Gâteaux differential at every φ in U and the map $Df: U\times E\to F$ defined by $Df(\varphi,\psi)=D_\varphi f(\psi)$ is continuous on $U\times E$. As mentioned in [12], with this definition of differentiability, most of the usual properties commonly used by physicists, such as linearity, chain rule, Leibniz rule, are mathematically valid.

2.2 Causal Separation in Spacetime

The separation in space (3), arising in the Hammerstein property (4) that detects the locality property for functionals, has a natural counterpart in special relativity, namely spacetime separation. We refer the reader to §3.1.1 and §5.2 for a discussion of the causality set approach in quantum gravity [8].

Spacetime separation is useful to express locality in algebraic quantum field theory, which offers a rigorous mathematical approach to quantum field theory. So we start with this locality relation.

To simplify the presentation, we only consider flat Minkowski space, yet the notion of causal separation generalises to Lorentzian manifolds by means of causal curves, see e.g [1, 28]. In the following $M = \mathbb{R}^{1,d-1}$ denotes the Minkowski space $(\mathbb{R}^d, g)$, where g is the Lorentzian scalar product $g(x, y) = -x_0 y_0 + \sum_{j=1}^{d-1} x_j y_j$ for $x = (x_0, x_1, \ldots, x_{d-1})$ and $y = (y_0, y_1, \ldots, y_{d-1})$ in $\mathbb{R}^d$. We set $\gamma(u) := -g(u, u)$.

A vector u in $M \setminus \{0\}$ is

- **timelike** if $\gamma(u) > 0$, a condition which defines the inner light cone,
- **lightlike** if $\gamma(u) = 0$,
- **causal** if timelike or lightlike i.e., if $\gamma(u) \geq 0$,
- **spacelike** if $\gamma(u) < 0$.

Remark 2.7 Note that the definitions fluctuate from one reference to another as to what type of vector the zero vector is.

For $d \geq 2$, the set of timelike vectors consists of two connected components. We choose a time orientation on M by picking a timelike vector $u := (1, 0, 0, \ldots, 0)$ and the component

$$I^+(0) := \{v \in \mathbb{R}^d \mid \gamma(v) > 0 \,\text{and}\, v_0 > 0\} \subseteq M$$

containing it. Timelike vectors, namely those in $I^+(0)$ (resp. $I^-(0) = -I^+(0)$), are called **future-directed** (resp. **past-directed**). Causal vectors, namely those in $J^+(0) := \overline{I^+(0)}$ (resp. $J^-(0) := -J^+(0)$) are called **future-directed** (resp. **past-directed**).

The **future** (resp. **past**) of a point m in M, are the subsets

$$I^+(m) := m + I^+(0), \quad (\text{resp.} \quad I^-(m) := m + I^-(0)).$$

The **causal future**, resp. **causal past** of a point m in M, are the subsets

$$J^+(m) := m + J^+(0), \quad (\text{resp.} \quad J^-(m) := m + J^-(0)).$$

Accordingly, we define the causal future, resp. past of a subset S of M by

$$J^\pm(S) := \bigcup_{m \in S} J^\pm(m).$$

The binary relation on M

$$p \preceq q :\Longleftrightarrow q \in J^+(p) \tag{5}$$

is reflexive, antisymmetric and transitive, thus defining a partial order on M, called a **causal partial order**. Then p and q are called **causally connected** if $p \preceq q$ or $q \preceq p$. Turning to the complement of this relation, we first note

$$p \npreceq q \Longleftrightarrow q \notin J^+(p) \Longleftrightarrow \{q\} \cap J^+(p) = \emptyset.$$

We call p and q of M **causally (or spacetime) separated** if they are not causally connected:

$$p \times q :\Longleftrightarrow p \not\leq q \text{ and } q \not\leq p \big(\Longleftrightarrow \{p\} \cap J^+(q) = \emptyset \text{ and } \{q\} \cap J^+(p) = \emptyset . \big)$$

Now define two subsets S_1 and S_2 of M to be **causally (or spacetime) separated** as follows.

$$\begin{aligned} S_1 \times S_2 &\Longleftrightarrow p \times q \quad \forall (p,q) \in S_1 \times S_2 \\ &\Longleftrightarrow p \not\leq q \text{ and } q \not\leq p \quad \forall (p,q) \in S_1 \times S_2 \\ &\Longleftrightarrow \{p\} \cap J^+(q) = \emptyset \text{ and } \{q\} \cap J^+(p) = \emptyset \quad \forall (p,q) \in S_1 \times S_2 \\ &\Longleftrightarrow S_1 \cap J^+(S_2) = \emptyset \quad \text{and} \quad S_2 \cap J^+(S_1) = \emptyset, \end{aligned} \tag{6}$$

which means that S_i lies completely outside of the future of S_j for $i \neq j$ in $\{1, 2\}$.

2.3 Wightman's Locality Axiom

In the early 1950s, Arthur Wightman presented a system of axioms [66] (see also [58, 67]) for a quantum field theory, now known as Wightman quantum field theory.

The Wightman axioms are described in spacetime $M = \mathbb{R}^{1,d-1}$ in terms of **field operators**

$$\varphi : \mathcal{S}(\mathbb{R}^d) \longrightarrow \mathcal{O}(\mathcal{H}),$$

where $\mathcal{S}(\mathbb{R}^d)$ is the space of Schwartz functions on $\mathbb{R}^d$ and $\mathcal{O}(\mathcal{H})$ is the set of all densely defined essentially self-adjoint operators (they have a unique continuation to a self-adjoint operator) in a separable complex Hilbert space $\mathcal{H}$. The Poincaré group acts on $\mathcal{H}$, sending an element p of the proper orthochronous Lorentz group $\hat{P}(1, d-1)$ (see §5.1 to a unitary transformation $U(p)$ of $\mathcal{H}$.

The operators $\varphi(f)$, $f \in \mathcal{S}(\mathbb{R}^d)$ are defined on a common dense subspace $D \subseteq \mathcal{H}$ that contains a vector Ω (the vacuum state) of norm 1. This common domain is required to be invariant under $\varphi(f)$, $f \in \mathcal{S}(\mathbb{R}^d)$, and for any v in D and any w in $\mathcal{H}$, the field $f \longmapsto \langle w, \varphi(f)\, v\rangle$ should define a tempered distribution. The vacuum state is required to be cyclic, meaning that the linear span of

$$\left\{ \varphi(f_1) \ldots \varphi(f_n)\Omega \,\middle|\, f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^d), n \in \mathbb{Z}_{\geq 0} \right\}$$

is a dense subspace of $(\mathcal{H}, \langle \cdot, \cdot \rangle)$.

As well as i) the **covariance** axiom, which encompasses an invariance condition on Ω and an equivariance condition on the fields $f \mapsto \varphi(f)$ under the action of $\hat{P}(1, d-1)$, and ii) a **spectrum condition** by which the energy-momentum tensor built from the generators of $\hat{P}(1, d-1)$ is required to lie in the forward cone. A third Wightman axiom which we want to focus on here, requires:

(Locality of the Wightman fields) For f_1 and f_2 in $\mathcal{S}(\mathbb{R}^d)$:

$$\operatorname{Supp}(f_1) \times \operatorname{Supp}(f_2) \Longrightarrow [\varphi(f_1), \varphi(f_2)] = 0, \tag{7}$$

where as before, $\mathrm{Supp}(f)$ stands for the support of f, namely the closure of the set of points where f does not vanish, and $[\cdot, \cdot]$ for the operator bracket.

The covariance, locality and the spectrum condition for the Wightman fields induce corresponding covariance, locality and the spectrum condition axioms for the **Wightman distributions** (also called vacuum expectations or correlation functions) in $\mathcal{S}'\left(\mathbb{R}^d\right)^n$, defined as

$$\mathcal{W}_n(f_1, \ldots, f_n) = \langle \varphi(f_1) \ldots \varphi(f_n)\Omega, \, \Omega \rangle \quad \forall f_1, \ldots, f_n \in \mathcal{S}(\mathbb{R}^d). \tag{8}$$

Locality of the Wightman fields translates to the following symmetry condition for the Wightman distributions:

(Locality of the Wightman distributions) For $f_1, \ldots, f_n$ in $\mathcal{S}(\mathbb{R}^d)$:

$$\begin{aligned} &\mathrm{Supp}(f_i) \times \mathrm{Supp}(f_j) \Longrightarrow \\ &\quad \mathcal{W}_n(f_1, \ldots, f_i, \ldots, f_j, \ldots, f_n) = \mathcal{W}_n(f_1, \ldots, f_j, \ldots, f_i, \ldots, f_n) \end{aligned} \tag{9}$$

for any $i \neq j$ in $\{1, \ldots, n\}$.

For distributions $\mathcal{W} \in \mathcal{S}'(\mathbb{R}^d)^n$, which as well as covariance, locality and a spectrum condition, obey a positivity condition, the Wightman reconstruction theorem [66] determines up to unitary equivalence, a separable Hilbert space $\mathcal{H}$, a vector Ω in $\mathcal{H}$, a dense domain $D \subseteq \mathcal{H}$ containing Ω, and a field $\varphi(f)$, $f \in \mathcal{S}(\mathbb{R}^n)$ with a domain D, such that the Wightman distributions $\mathcal{W}_n$ are of the form (8).

2.4 Haag-Kastler Locality Axiom

The Wightman axioms were later abstracted by Haag and Kastler in 1964 [40], see also [28], in terms of algebras of quantum observables associated to spacetime regions, namely as local nets of observables.

From the Wightman axioms, one can construct for each open subset U of M a subalgebra $\mathcal{A}(U)$ of $\mathcal{O}(\mathcal{H})$ generated by the operators $\varphi(f)$ where f is any smooth function which has support included in U. In the framework of the Wightman axioms, the operators $\varphi(f)$ are typically unbounded. In contrast, in the formalism of Haag and Kastler, these operators are assumed to be bounded so that the algebra they generate is endowed with the operator norm. Accordingly, $\mathcal{A}(U)$ is equipped with a norm $\|\cdot\|$. Due to the fact that the Wightman fields act on a Hilbert space, a notion of adjointness is required and $\mathcal{A}(U)$ is endowed with an involution $x \mapsto x^*$, such that $\|x^* x\| = \|x\|^2$ as in the case of the operator norm. This leads to the requirement that $\mathcal{A}(U)$ be a C^*-algebra.

Since a function which has support in $U_2 \subsetneq U_1$ also has support in U_1, one requires monomorphisms $i_{U_2,U_1} : \mathcal{A}(U_2) \hookrightarrow \mathcal{A}(U_1)$ between the algebras.

Correspondingly, we consider

(a) the category $\mathrm{Caus}(M)$ whose objects are relatively compact and causally convex open subsets U in M and whose morphisms are given by subset inclusions $U_1 \hookrightarrow U_2$. By U causally convex we mean that the segment $c(t) := (1-t)p + tq$, $t \in [0, 1]$ linking two causally related points $p \leq q$ in U, should lie in U.

In the modern framework of AQFT, M is a Lorentzian manifold and the two points should be related by a causal curve;

(b) a category **A** of algebras of a certain type, e.g. C^*-algebras or von Neumann algebras, whose morphisms are algebra monomorphisms $\iota_{A_1,A_2} : A_1 \hookrightarrow A_2$.

In the modern framework of pAQFT, for a classical (resp. a quantised theory), **A** is the category of nuclear, topological locally convex unital $\star$-algebras, (resp. deformed) Poisson algebras and the morphisms are continuous algebra monomorphisms.

In the Haag-Kastler axiomatic, a field theory can be expressed as a functor with a locality condition.

Definition 2.8 A field theory model on Minkowskian spacetime M is a functor (see e.g. [38])

$$\mathcal{A} : \mathrm{Caus}(M) \to \mathbf{A}, \tag{10}$$

(with **A** defined in above item (b)) which obeys **Einstein locality**:

$$\forall U_1, U_2 \in \mathrm{Caus}(M), \quad U_1 \times U_2 \Longrightarrow [\mathcal{A}(U_1), \mathcal{A}(U_2)]_U = \{0\}, \tag{11}$$

where $[\cdot,\cdot]_U$ stands for the algebra bracket in $\mathcal{A}(U)$ for a subset U of $\mathrm{Caus}(M)$ (see above item (a)) which contains them both.

Whether it is a classical or a quantum field theory is determined by the category of algebras **A** one chooses in (10).

2.5 Osterwalder-Schrader Locality Axiom

The **Wick rotation** map

$$W : \mathbb{R}^d \ni (x_0, x_1, \ldots, x_n) \longrightarrow (-i\, x_0, x_1, \ldots, x_n) \in i\mathbb{R} \times \mathbb{R}^{d-1}$$

is an isomorphism of vector spaces which transforms the Minkowskian scalar product g into the Euclidean scalar product

$$\langle x, y\rangle := g(W(x), W(y)) = \sum_{i=0}^{d} x_i^2.$$

In [49], Osterwalder and Schrader proved that the Wightman axioms induce via the Wick rotation, a Euclidean field theory on $\mathbb{R}^d$ equipped with the canonical inner product $\langle x, y\rangle = \sum_{i=0}^{d} x_i^2$ through analytic continuation of the Wightman distributions. In other words, they show that Wick rotation is a well defined isomorphism of quantum field theories on Minkowski and on Euclidean spacetime. Via Wick rotation, Wightman distributions give rise to Euclidean Green's functions also called **Schwinger functions** $\mathcal{S}(\mathbf{f}), \mathbf{f} \in \mathcal{S}((\mathbb{R}^d)^n)$ that satisfy the so called Osterwalder-Schrader axioms and locality is expressed in terms of a symmetry condition of the Schwinger distributions [49, Axiom E3] which can be seen as the Euclidean counterpart of (9).

For $\mathbf{f} = f_1 \otimes \cdots \otimes f_n$, with $f_i \in \mathcal{S}(\mathbb{R}^d), \quad i = 1, \ldots, n$, it reads

(Locality of the Schwinger distributions)

$$\begin{aligned} & \mathrm{Supp}(f_i) \cap \mathrm{Supp}(f_j) = \emptyset && (12)\\ \Longrightarrow \quad & \mathcal{S}_n(f_1, \dots, f_i, \dots, f_j \dots, f_n) = \mathcal{S}_n(f_1, \dots, f_j, \dots, f_i \dots, f_n) \quad \forall i \neq j. \end{aligned}$$

Symanzik [62] later advocated a purely Euclidean approach to quantum field theory, in which Euclidean Green's functions are built directly from a formal Lagrangian density and in their book [30], Glimm and Jaffe built a measure on the space of distributions $D'(\mathbb{R}^d)$ whose moments satisfy the Osterwalder-Schrader axioms. In all these approaches, locality for Euclidean Green's functions also corresponds to a symmetry condition.

2.6 Factorisation Property in Perturbative AQFT

Perturbative AQFT offers a framework to handle interacting field theories via deformation quantisation. Roughly speaking, algebras $\mathcal{A}(O)$ associated with open subsets O arising in the Haag-Kastler approach briefly mentioned above, are considered formal power series with coefficients in topological $*$-algebras, and the resulting interacting formal deformation quantisation may be expressed in terms of scattering amplitudes. These are the probability amplitudes for plane waves of free fields to come in from the far past, then interact in a compact region of spacetime via the given interaction and to emerge again as free fields into the far future.

The perturbative scattering matrix of the interacting field theory, or S-matrix, encodes the collection of all these scattering amplitudes, as the types and wave vectors of the incoming and outgoing free fields varies. The S- matrix associated with a Lagrangian perturbative quantum field theory, is usually thought of as a (formal) perturbation series over Feynman diagrams extracted from the Lagrangian density.

Building on the operator-valued distribution approach in Epstein and Glaser's **causal perturbation theory** [27], in the framework of AQFT, scattering matrices are viewed as maps $S : G(M) \longrightarrow \mathcal{U}(\mathcal{A})$ from the group $G(M) := (\mathcal{D}(M), +, 0)$ to the (non necessarily abelian) group $(\mathcal{U}(\mathcal{A}), \cdot, 1)$ of unitary elements of a unital topological $*$-algebra $\mathcal{A}$.

Following Bogoliubov and Shirkov [5], who used work by Stückelberg [60], [59] and [50], Epstein and Glaser express the **causality condition** by means of the following **factorisation property** which compares with (4). With the notations of (6) it reads:

$$\mathrm{Supp}(\varphi_1) \times \mathrm{Supp}(\varphi_2) \Longrightarrow S(\varphi_1 + \varphi + \varphi_2) = S(\varphi_2 + \varphi)\, S(\varphi)^{-1}\, S(\varphi + \varphi_1), \quad (13)$$

for $\varphi_1, \varphi_2, \varphi$ in $G(M)$, which amounts to

$$\mathrm{Supp}(\varphi_1) \times \mathrm{Supp}(\varphi_2) \Longrightarrow S_\varphi(\varphi_1 + \varphi_2) = S_\varphi(\varphi_2)\, S_\varphi(\varphi_1), \quad (14)$$

where we have set $S_\varphi(\psi) := S(\varphi)^{-1}\, S(\varphi + \psi)$, called the relative S-matrix, see [27, Causality cdt. (C.A.)], also [13, Formula (15)], [26, Formula (2)] and [53, Formula (6.21)].

Relative scattering matrices satisfy the locality condition required for local observables (cfr. (7))

$$\begin{aligned} &\mathrm{Supp}(\varphi_1) \times \mathrm{Supp}(\varphi_2) \\ \Longrightarrow\ & S_\varphi(\varphi_1+\varphi_2) = S_\varphi(\varphi_1)\, S_\varphi(\varphi_2) = S_\varphi(\varphi_2)\, S_\varphi(\varphi_1) \\ \Longrightarrow\ & [S_\varphi(\varphi_1), S_\varphi(\varphi_2)] = 0, \end{aligned} \tag{15}$$

and serve as generating functionals for the interacting fields, giving rise to a functor as in Definition 2.8, which defines a perturbative quantum field theory built from the S-matrices. A construction of the local net of quantum observables from causal perturbation theory by means of functionals with the locality property (15) was given in [13, 26] (see also [53] and references therein).

In summary, the principle of locality is a key feature of field theory, it reveals some non-global structures in field theory, and it plays a key role in the construction of pAQFTs.

3 Locality Relations in Mathematics

The locality axioms in AQFTs suggest various formulations of the concept of locality, which we now want to study in a more general setting. Leaving the realm of quantum field theory, we single out features common to the various formulations of locality reviewed above.

3.1 Locality Relations

3.1.1 Causality and Locality

Definition 3.1 A **locality structure** in a set X is a symmetric binary relation $\top \subseteq X \times X$ (we sometimes write $X \times_\top X$ for $\top$), the pair $(X, \top)$ is called a **locality set**, and $\top$ is called **locality relation**. Two elements u and v such that $u \top v$ are called **independent**.

Note that we do not require that a locality relation be neither reflexive ($x \top x$ for any x in X) nor irreflexive ($x \not\top x$ for any x in X). In practice, the relations we consider are almost irreflexive in the following sense.

Definition 3.2 A relation $\top \subseteq X \times X$ is called **almost irreflexive** if for each $x \in X$, $x \top x$ implies $x \in X^\top$. In particular, an irreflexive locality relation is an almost irreflexive locality relation.

The complement relation $\top^c := X \times X \setminus \top$ of a locality relation $\top \subseteq X \times X$ is also a locality relation. Yet irreflexivity is never stable under complements:

Proposition 3.3 *Let a locality relation $\top$ be neither $\emptyset$ nor $X \times X$. If $\top$ is almost irreflexive, then its complement $\top^c$ is not almost irreflexive.*

Proof Let $\top$ be almost irreflexive. We consider two cases.

First let $\top$ be irreflexive. Then $\top^c$ is reflexive. If $\top^c$ is also almost irreflexive, then we have $\top^c = X \times X$. This contradicts that $\top \neq \emptyset$. So $\top^c$ is not almost irreflexive.

Next let $\top$ be not irreflexive. So there is $x_1 \in X$ such that $x_1 \top x_1$. Since $\top$ is almost irreflexive, $(x_1, y) \in \top$ for all $y \in X$. Since $\top$ is not $X \times X$, the condition that $\top$ is almost irreflexive means that it is not reflexive. So there is $x_2 \in X$ such that $x_2 \top^c x_2$. Suppose that $\top^c$ is also almost irreflexive. Then $(x_2, y) \in \top^c$ for all $y \in X$. In particular, $(x_2, x_1) \in \top^c$. By symmetry, $(x_1, x_2) \in \top^c$, implying $(x_1, x_2) \in \top \cap \top^c$. This is a contradiction, showing that $\top^c$ is not almost irreflexive. □

A locality on X can be extended to one on the power set $\mathcal{P}(X)$ by

$$A \top B \Leftrightarrow \forall (a, b) \in A \times B, \quad a \top b.$$

Causality has been studied quite extensively, for example in the general context of causal sets in quantum gravity [8, 56]. Further details on this approach to field theory are given in Sect. 5. Like locality, causality can also be studied abstractly, to specify causally related pairs-causal relation. We also refer to [54] for an interesting study in the context of AQFT.

Definition 3.4 Given a set X equipped with a relation $\preceq$, the pair $(X, \preceq)$ is called a **causal set (causet)** if $\preceq$ is a partial order (so $\preceq$ is reflexive, antisymmetric and transitive) on X that is locally finite in the sense that any interval $[a, b]$ for $a \preceq b$ is finite.

Remark 3.5 Here we adapt the notion from [52]. There are variations of the above definition of causal sets. Their compatibility is discussed in Sect. 5, in particular Proposition 5.1.

In order to take the principle of locality in consideration, we introduce a locality relation which singles out spacetime separated pairs, so not causally related pairs.

Definition 3.6 Let X be a set and let $\preceq$ be a relation on X.

(a) The relation is called a **causality relation** if it is a partial order;
(b) The relation is called a **non-causality relation** if it is irreflexive, and **dominant** in the sense that for all $x \neq y \in X$, either $x \preceq y$ or $y \preceq x$; We call such a pair (x, y) **causally separated**.
(c) The relation is called an **irreflexive locality relation** if it is irreflexive as well as being symmetric.

The three notions, of causality, non-causality and locality, relate as follows.

Lemma 3.7 (a) *Let $\preceq$ be a causality relation, in the sense of a partial order (leaving out the locally finiteness condition). Its complement $\not\preceq$ is a non-causality relation;*

(b) [54] *Let $\not\preceq$ be a non-causality relation. Its symmetric intersection $\top := \not\preceq \cap \not\succeq$:*

$$a \top b \Leftrightarrow \big(a \not\preceq b \quad \text{and} \quad b \not\preceq a\big), \tag{16}$$

defines an irreflexive locality relation.

Proof (a) The reflexivity of $\preceq$ gives the irreflexivity of $\not\preceq$. The antisymmetry of $\preceq$ gives the dominant property of $\not\preceq$.

(b) The irreflexivity of $\top$ follows from that of $\not\preceq$. As the symmetric intersection of $\not\preceq$, $\top$ is symmetric. □

Here is the prototype example motivating the definition of a causality relation. A partial order $\preceq$ on X induces a binary relation on the power set $\mathcal{P}(X)$ defined as

$$A \preceq B \Leftrightarrow \forall (a, b) \in A \times B, \quad a \preceq b,$$

which in turn gives rise to an induced locality relation on the power set $\mathcal{P}(X)$

$$A \top B \Leftrightarrow \big(\forall (a, b) \in A \times B, \quad a \not\preceq b \quad \text{and} \quad b \not\preceq a\big). \tag{17}$$

Example 3.8 With the notations of the above example, Equation (17) yields back causality separation (6):

$$S_1 \top S_2 \Longleftrightarrow \big(S_1 \cap J^+(S_2) = \emptyset \quad \text{and} \quad S_2 \cap J^+(S_1) = \emptyset\big) \Longleftrightarrow S_1 \times S_2.$$

3.1.2 First Examples

Locality relations are ubiquitous in mathematics.

Example 3.9 (**Coprimality** of natural numbers) The set $\mathbb{N}$ of positive integers equipped with the coprime relation:

$$n \top_{\text{co}} m \Longleftrightarrow m \wedge n := \gcd(m, n) = 1,$$

is a locality set. The locality $\top_{\text{co}}$ is almost irreflexive since only 1 is coprime to itself and it is coprime to any other integer.

Independence of probabilistic events is a natural locality relation.

Example 3.10 (**Independence** of events) On a probability space $\mathcal{P} := (\Omega, \Sigma, P)$ with σ-algebra Σ, define a locality relation by independence of events:

$$A \top B \Longleftrightarrow P(A \cap B) = P(A)\, P(B) \quad \forall A, B \in \Sigma.$$

Then $(\Sigma, \top)$ is a locality set. This locality relation is also almost irreflexive. This is because $P(A \cap A) = P(A)P(A)$ exactly when $P(A) = 0$ or 1. If $P(A) = 0$, then $P(A \cap B) = 0 = P(A)P(B)$; while if $P(A) = 1$, then $P(A^c) = 0$ and so $P(A \cap B) = P(B) - P(A^c \cap B) = P(B) = P(A)P(B)$.

Disjointness of sets defines a locality relation, which underlies many other locality relations.

Example 3.11 (**Disjointness** of sets) The disjointness of sets induces a locality relation on the power set $\mathcal{P}(X)$:

$$A \top_\cap B :\Longleftrightarrow A \cap B = \emptyset \quad \forall A, B \subseteq X. \tag{18}$$

The locality $\top_\cap$ is almost irreflexive since only $\emptyset$ is disjoint with itself and it is also disjoint with any other subset.

Example 3.12 **Zero intersection** on the set $G(V)$ of subspaces of a finite dimensional vector space V defines a locality relation

$$W_1 \top_{\{0\}} W_2 \Longleftrightarrow W_1 \cap W_2 = \{0\} \quad \forall W_1, W_2 \in G(V). \tag{19}$$

The locality $\top_{\{0\}}$ is almost irreflexive for the same reason as the previous example.

Orthogonality on an euclidean vector space defines a locality relation, which underlies many others.

Example 3.13 (**Orthogonality**) On an euclidean vector space (V, Q), where Q stands for the inner product, the orthogonality relation

$$u \perp^Q w \Longleftrightarrow Q(u, w) = 0 \quad \forall u, w \in V, \tag{20}$$

defines a locality relation on V and, for linear subspaces U and W of V,

$$U \perp^Q W \Longleftrightarrow Q(u, w) = 0 \quad \forall (u, w) \in U \times W, \tag{21}$$

defines a locality relation on the set $G(V)$ of linear subspaces of a finite dimensional vector space V.

The locality $\perp^Q$ is almost irreflexive since only $\{0\}$ is independent of itself and it is independent of any other subspace. Note that this would not be the case were the bilinear form Q is not positive definite.

3.1.3 Locality Lattices

Examples 3.9, 3.11 and 3.13 fall in the class of locality lattices which we now define, referring the reader to [19] for further details. They are a locality counterpart of ordinary lattices, see e.g. [3, 31].

We recall that a **lattice** is a poset $(L, \leq)$, any two-element subset $\{a, b\}$ of which has a least upper bound (also called a join) $a \vee b$, and a greatest lower bound (also called a meet) $a \wedge b$ such that

(a) both operations are associative and monotone with respect to the order;
(b) if $a_1 \leq b_1$ and $a_2 \leq b_2$, then $a_1 \wedge a_2 \leq b_1 \wedge b_2$ and $a_1 \vee a_2 \leq b_1 \vee b_2$.

We denote the lattice by $(L, \leq, \wedge, \vee)$.

A **locality lattice** is a lattice $(L, \leq, \wedge, \vee)$ with a compatibility condition between the partial order and the locality relation

$$a \leq b \Longrightarrow (\forall c \in L: \quad c\top b \Rightarrow c\top a), \tag{22}$$

as well as a compatibility condition between the locality and the join, namely

$$c\top a \text{ and } c\top b \Longrightarrow c\top(a \vee b) \quad \forall a, b, c \in L. \tag{23}$$

We denote the locality lattice by $(L, \leq, \wedge, \vee, \top)$. Examples of locality lattices are $(\mathbb{N}, |, \wedge, \vee, \top_{\mathrm{co}})$ (Example 3.9), $(\mathcal{P}(X), \subseteq, \cap, \cup, \top_\cap)$ (Example 3.11), and $\big(G(V), \preceq, \cap, +, \perp^Q\big)$ (Example 3.13).

The latter two locality lattices share another common feature, namely that they are orthocomplemented.

A poset $(P, \leq, 0)$ with bottom 0 is called **orthocomplemented** if it can be equipped with a map $\Psi : P \to P$ called the **orthocomplementation**, which assigns to $a \in P$ its **orthocomplement** $\Psi(a)$ such that

(a) (**Ψ antitone**) $b \leq a \Rightarrow \Psi(a) \leq \Psi(b) \quad \forall (a, b) \in P^2$;
(b) (**Ψ involutive**) $\Psi^2 = \mathrm{Id}$,
(c) (**Ψ separating**) For any $b \in P$, if $b \leq a$ and $b \leq \Psi(a)$, then $b = 0$.

The posets $(\mathcal{P}(X), \subseteq, \emptyset)$, resp. $(G(V), \preceq, \{0\})$ of Examples 3.11, resp. 3.13 are orthocomplemented with the orthocomplementation given by the map $\psi_\cap$, which sends a subset of X to its complement set in X, resp. the map Ψ_Q, which sends a subspace of V to its orthogonal complement space in V.

A lattice L bounded from below by 0 equipped with an orthocomplementation Ψ is clearly bounded from above with top $1 = \Psi(0)$ and one can show [14, Proposition 1.1 (iii), Proposition 1.7], that it satisfies the following **strongly separating** property

$$a \oplus \Psi(a) = 1 \quad \forall a \in L, \tag{24}$$

where $a \oplus b = c$ stands for $a \wedge b = 0$ and $a \vee b = c$.

An orthocomplementation Ψ on L gives rise to a strongly separating locality relation

$$a \top_\Psi b \Longleftrightarrow b \leq \Psi(a). \tag{25}$$

Example 3.14 The locality relations $\top_\cap$ and $\perp^Q$ of Examples 3.11 and 3.13 are of the type $\top_\Psi$ with Ψ given the orthocomplementation given respectively by the set complement map and the orthogonal complement map.

Equation (25) actually gives a one to one correspondence between a class of locality lattices and orthocomplementations [19, Theorem 5.16] and [14, Theorem 3.1].

The orthogonality and the disjointness relations underlie many other locality relations as we shall soon see.

3.2 Partial Product and Locality Semi-Group

We now give prototype examples which serve as a motivation to investigate locality structures, see [34, 36] for details. We shall see how locality can be useful to define partial operations on projections, on cones, on meromorphic germs and on distributions as well as morphisms compatible with these partial operations.

3.2.1 Locality Semigroups

Definition 3.15 We call a triple $(X, \top, m)$ a **locality semi-group** if $(X, \top)$ is a locality set equipped with a map $m : \top \to X$ which obeys the following properties:

- for any three elements a, b, c in X we have

$$(a \top b \ \wedge \ a \top c \ \wedge b \top c) \Longrightarrow a \top m(b, c); \tag{26}$$

- for any elements a, b, c in X such that $a \top b$, $b \top c$, $a \top c$, we have

$$m(m(a, b), c) = m(a, m(b, c)).$$

An **locality monoid** is a locality semi-group $(G, \top, m_G)$ together with a **unit element** $1_G \in G$ given by the defining property

$$\{1_G\}^\top = G \quad \text{ and } \quad m_G(x, 1_G) = m_G(1_G, x) = x \quad \forall x \in G.$$

We denote the locality monoid by $(G, \top, m_G, 1_G)$.

Example 3.16 With the notations of Example 3.9, the quadruple $(\mathbb{N}, \top_{\rm co}, \cdot, 1)$, where $\cdot$ is the product in $\mathbb{N}$ is a locality monoid. Indeed, for two coprime integers m and n, if for any $u \in \mathbb{N}$, $u \wedge m = 1$ and $u \wedge n = 1$, then $m \vee n = m \cdot n$ and u are coprime so that $m \vee n$ lies in $u^{\top_{\rm co}}$.

Example 3.17 With the notations of Example 3.11, the quadruple $(\mathcal{P}(X), \top_\cap, \sqcup, \emptyset)$, where $\sqcup$ is the disjoint union, is a locality monoid. Indeed, for U in $\mathcal{P}(X)$ such that $A \cap U = \emptyset$ and $B \cap U = \emptyset$, we have $(A \cup B) \cap U = \emptyset$.

Coexample 3.18 With the notations of Example 3.12, $\big(G(V), \top_{\{0\}}, \oplus, \{0\}\big)$ is not a locality semigroup. Indeed, in $\mathbb{R}^2$, let us consider the three vector spaces $W_1 = \mathbb{R} \times \{0\}$, $W_2 = \{0\} \times \mathbb{R}$, $W_3 = \{(\lambda, \lambda), \lambda \in \mathbb{R}\}$. Then $W_i \top_{\{0\}} W_j$ for any $i \neq j$ yet $(W_1 \oplus W_2) \cap W_3 = W_3 \neq \{0\}$.

Example 3.19 With the notations of Example 3.13, $\big(G(V), \perp^Q, \oplus, \{0\}\big)$ defines a locality monoid.

Remark 3.20 In the above three examples $(\mathbb{N}, \top_{\rm co}, \cdot)$, $(\mathcal{P}(X), \top_\cap, \sqcup)$, $\big(G(V), \perp^Q, \oplus\big)$ of locality semi-groups $(X, \top, m)$, one checks that the following equivalence holds for any $(x, y, z) \in X^3$:

$$x \top y \ \wedge \ m(x, y) \top z \Longleftrightarrow x \top y \ \wedge \ y \top z \ \wedge \ x \top z \Longleftrightarrow y \top z \ \wedge \ x \top m(y, z).$$

The second implication from right to left can be viewed as a decomposition property in the sense that if the product $m(y, z)$ is independent of x, then so are its factors y and z.

The paths in a quiver, together with their partial compositions, form a locality semigroup [72].

3.2.2 The Locality Monoid of Projections and K-Theory

An element p of a (unital) C^*-algebra A is a projection if $p^2 = p = p^*$ and the symmetric relation

$$p \perp q \Longleftrightarrow pq = qp = 0$$

defines a locality relation $\perp$ on the set $\mathrm{Proj}(A)$ of projections on A. The sum of two projections p and q is not in general a projection, yet it is if $p \perp q$, leading to the following straightforward result.

Proposition 3.21 $(\mathrm{Proj}(A), +, \perp)$ *defines a locality monoid*, *where* $+$ *is the addition in* $\mathrm{Proj}A$.

Note that two arbitrary projections p and q can be "localised" by taking the matrices $M_2(p) := \begin{bmatrix} p & 0 \\ 0 & 0 \end{bmatrix}$ and $\tilde{M}_2(q) := \begin{bmatrix} 0 & 0 \\ 0 & q \end{bmatrix}$, so that they satisfy the condition $M_2(p)^2 = M_2(p) = M_2(p)^*$ and similarly for $\tilde{M}_2(q)$ as well as the relation $M_2(p)\,\tilde{M}_2(q) = \tilde{M}_2(q)\,M_2(p) = 0$.

This can serve as a motivation to consider the matrix algebras $M_n(A)$, $n = 1, 2, \ldots$, with $M_n(A)$ canonically included in $M_{n+1}(A)$ via the injection $\iota_n : a \mapsto \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$, and their direct limit $M_\infty(A)$. There, such consideration gives rise to the addition in K-theory $K_0(A)$ [4, 55].

3.2.3 Multiplying Distributions

We saw that locality can serve as a separation device, with the disjointness relation in Example 3.11 separating sets and supports of functions. This can be useful to separate singularities arising from distributions, which are ubiquitous in quantum field theory. Feynman propagators are distributions, and renormalisation are essentially the problem of defining a product of distributions, as Stückelberg realised very early and later clarified by Bogoliubov, Shirkov, Epstein and Glaser (mentioned previously). Pioneering work by Radzikowski [51] later led to a complete reformulation of quantum field theory, where the wavefront set of distributions plays a central role, for example to determine the algebra of microcausal functions and to define a spectral condition for time-ordered products and quantum states.

One of the main obstacles one comes across in that context is Laurent Schwartz' famous no-go theorem, which tells us that there is no general extension of multiplication to distributions. Indeed, one can multiply a distribution and a smooth function, but the product of two distributions is generally ill-defined. The disjointness locality relation described in Example 3.11 gives rise to locality relations on the space $\mathcal{D}'(U)$ of distributions on an open subset U of $\mathbb{R}^n$ corresponding to disjointness of supports, resp. singular supports, and on whose graphs there is a well-defined product of distributions. These can be revised to Hörmander's partial product on pairs of distributions whose wavefront sets are mutually well positioned. Let us first consider more naive ways of singling out multiplicable pairs. We refer the reader to the classical work of Hörmander [45] and to the pedagogical survey [11].

The disjointness locality relation on sets described in Example 3.11 induces the locality relation on the space $\mathcal{D}(U)$ of compactly supported smooth functions on U, given by disjointness of supports:

$$\varphi_1 \top \varphi_2 \Longleftrightarrow \mathrm{Supp}(\varphi_1) \cap \mathrm{Supp}(\varphi_2) = \emptyset \quad \forall \varphi_1, \varphi_2 \in \mathcal{D}(U), \tag{27}$$

which arises in Euclidean quantum field theory, see e.g. the locality for Schwinger distributions (12). It generalises to distributions. A distribution u in $\mathcal{D}'(U)$ is said to vanish on an open subset $V \subseteq U$ if $u(\varphi) = 0$ for any φ in $\mathcal{D}(U)$ with compact support in V, called an annihilation set of u. The support $\mathrm{Supp}(u)$ of a distribution u is the complement of the open union of all open annihilation sets of u and (27) generalises to

$$u_1 \top u_2 \Longleftrightarrow \mathrm{Supp}(u_1) \cap \mathrm{Supp}(u_2) = \emptyset \quad \forall u_1, u_2 \in \mathcal{D}'(U). \tag{28}$$

Like the product of two smooth functions with disjoint supports, the product of two distributions with disjoint supports vanishes. So we need to single out multiplicable pairs by means of a more refined notion of support.

Let us first consider the singular support of a distribution v in $\mathcal{D}'(U)$. For this purpose, we first recall that an element x in U is **regular**, if there is a smooth function $\chi \in \mathcal{D}(U)$ such that $\chi(x) = 1$ and the localised distribution $v\,\chi$ around x corresponding to the product of the function χ and the distribution v is regular i.e. $v\,\chi$ lies in $\mathcal{D}(U)$. A singular point is a non regular point and the **singular support** $\mathrm{Singsupp}(v)$ of v is the closure of the set of singular points. For example, the singular support of the Dirac distribution at zero, which coincides with its support, is $\{0\}$.

Note that $\mathrm{Singsupp}(v) \subseteq \mathrm{Supp}(v)$ and $\mathrm{Singsupp}(v) = \emptyset \Longleftrightarrow v \in C^\infty(U)$.

Example 3.22 (**Disjointness** of singular supports) Given an open subset U of $\mathbb{R}^n$, the space $\mathcal{D}'(U)$ of distributions on U can be equipped with the following locality relation

$$u \top^{\mathrm{sing}} v \Longleftrightarrow \mathrm{Singsupp}(u) \cap \mathrm{Singsupp}(v) = \emptyset.$$

The locality $\top^{\mathrm{sing}}$ is almost irreflexive in so far as the only elements independent of themselves are smooth functions and those are independent of any distribution.

The product of distributions is well defined on the graph of this relation since you can multiply two distributions with disjoint singular supports:

$$v_1\, v_2(f) = v_1\,(v_2\, \varphi\, f) + v_2\,(v_1\,(1-\varphi)\, f)$$

for any smooth function φ which vanishes on a neighborhood of the singular support of v_2 (in which case $v_2\,\varphi\, f$ defines a smooth function) and is identically 1 on a neighborhood of the singular support of v_1 (in which case $v_1\,(1-\varphi)\, f$ defines a smooth function).

To make sense of the product of two distributions with non disjoint singular supports, one can compare their wavefront sets. The wavefront set is a refinement of the singular support, in the sense that it describes in which direction the distribution is singular above each point of the singular support. More precisely, given an open subset U of $\mathbb{R}^n$, the singular support of v in $\mathcal{D}'(U)$ is the set of points x in U such that (x, ξ) lies in the wavefront set of v for some ξ in $\mathbb{R}^n \setminus \{0\}$. To make this precise, we first introduce the **frequency set** $\Sigma(v) \subseteq \mathbb{R}^n \setminus \{0\}$ of v defined as the complement of all directions $\xi \in \mathbb{R}^n \setminus \{0\}$ such that the Fourier transform $\hat{v}$ of v is sufficiently regular. Explicitly, an element $\xi_0 \in \mathbb{R}^n$ does not lie in $\Sigma(v)$ if there is a conical neighborhood $V(\xi_0)$ such that

$$\forall N \in \mathbb{N}, \exists C_N, \quad |\hat{v}(\xi)| \leq C_N\,(1+|\xi|)^{-N} \quad \forall k \in V(\xi_0).$$

The **singular fibre** $\Sigma_x(v)$ at a point x in U is a localised version at a point x. It is defined as the complement of all directions $\xi \in \mathbb{R}^n \setminus \{0\}$ such that the Fourier transform of v, localised at x, is sufficiently regular when restricted to a conical neighbourhood of ξ:

$$\Sigma_x(v) := \bigcap_{x \in \mathrm{Supp}(\chi),\, \chi \in \mathcal{D}(U)} \Sigma(\chi\, v).$$

The **wavefront set** of v is

$$\mathrm{WF}(v) := \{(x, \xi) \in T^*U \setminus \{0\} \mid \xi \in \Sigma_x(v)\}.$$

For example, the wavefront set of the Dirac distribution at 0 is $\mathrm{WF}(\delta_0) = \{0\} \times \mathbb{R}^n \setminus \{0\}$.

Similarly, for a distribution v on a smooth manifold M, the wave front set is a closed conical subset of the cotangent bundle without the zero part: $\mathrm{WF}(v) \subseteq T^*M \setminus \{0\}$.

Given two distributions v_1, v_2 in $\mathcal{D}'(\mathbb{R}^n)$, a smooth function χ on $\mathbb{R}^n$ with compact support which is one around a point x in U, for i in $\{1, 2\}$, the Fourier transform of the localised distributions $v_i\,\chi$ reads $\widehat{v_i\,\chi}(\phi) = (v_i\,\chi)\,(\hat{\phi}) = v_i(\chi\,\hat{\phi})$ and for ξ in $\mathbb{R}^n$, we would like to make sense of

$$\begin{aligned}(\widehat{v_1\,\chi)\,(v_2}\,\chi)(\xi) &= (\widehat{v_1\,\chi} \star \widehat{v_2\,\chi})\,(\xi)\\ &= \int_{\mathbb{R}^n} (\widehat{v_1\,\chi}(\xi-k)\,\widehat{v_2\,\chi}(k))\, dk.\end{aligned}$$

The distributions $v_i\,\chi$ have compact support from which it follows that their Fourier transform $\widehat{v_i\,\chi}$ have at most polynomial growth and hence lie in $\mathcal{S}'(\mathbb{R}^n)$. For x in $\mathrm{Singsupp}(v_1) \cap \mathrm{Singsupp}(v_2)$, the latter integral is absolutely convergent whenever either $\widehat{v_1\,\chi}(\xi-k)$ or $\widehat{v_2\,\chi}(k)$ is a Schwartz function. In particular (taking $\xi = 0$) if

(x,k) lies in $\mathrm{WF}(v_1)$ then $(x,-k)$ should not belong to $\mathrm{WF}(v_2)$ i.e. (x,k) should not lie in $\mathrm{WF}'(v_2)=\{(x,-\xi)\mid (x,\xi)\in \mathrm{WF}(v_2)\}$.

We are now ready to refine the locality relation $\top^{\mathrm{sing}}$ introduced in Example 3.22. (**Separation** of wavefront sets) The relation

$$v_1 \top^{\mathrm{WF}} v_2 \Longleftrightarrow \mathrm{WF}(v_1)\cap \mathrm{WF}'(v_2)=\emptyset \tag{29}$$

defines a locality relation on $\mathcal{D}'(U)$, where we have set

$$\mathrm{WF}'(v):=\{(x,-\xi)\in U\times(\mathbb{R}^n\setminus\{0\})\mid (x,\xi)\in \mathrm{WF}(v)\}.$$

Note that the wave front set of a compactly supported distribution is empty if and only if the distribution comes from an ordinary smooth function [45, below (8.1.1)] and these are independent of any distribution. So the relation is almost irreflexive.

Note that $v_1 \top^{\mathrm{sing}} v_2 \Longrightarrow v_1 \top^{\mathrm{WF}} v_2$.

The following counterexample inspired by [11, Example 19] and references therein, shows that two distributions can be independent for $\top^{\mathrm{WF}}$ and not for $\top^{\mathrm{sing}}$.

Coexample 3.23 The distributions ν_1, ν_2 in $\mathcal{D}'(\mathbb{R}^2)$ defined by

$$\nu_1(\phi)=\int_{\mathbb{R}^2}\phi(0,y)\,dy=\int_{\mathbb{R}^2}\phi(x,y)\,\chi_{\{0\}\times\mathbb{R}}(x,y)\,dx\,dy$$

and

$$\nu_2(\phi)=\int\phi(x,0)\,dx=\int_{\mathbb{R}^2}\phi(x,y)\,\chi_{\mathbb{R}\times\{0\}}(x,y)\,dx\,dy,$$

where χ_S is the characteristic function of the set S, have wavefront sets

$$\mathrm{WF}(\nu_1)=\{((0,y);(\lambda,0))\mid y\in\mathbb{R},\lambda\in\mathbb{R}\setminus\{0\}\}$$

and

$$\mathrm{WF}(\nu_2)=\{((x,0);(0,\mu))\mid x\in\mathbb{R},\mu\in\mathbb{R}\setminus\{0\}\},$$

so $\nu_1 \top^{\mathrm{WF}} \nu_2$. Yet their singular supports intersect at the origin so ν_1 is not independent of ν_2 for the relation $\top^{\mathrm{sing}}$.

It follows from an important theorem by L. Hörmander that the product of distributions which is generally ill-defined, is well-defined on the graph of the locality relation $\top^{\mathrm{WF}}$. This leads to the concept of partial product viewed as a product on the graph of a locality relation.

Proposition 3.24 [45, *Theorem* 8.5.3], *see also* [11, *Theorem* 13] *For two distributions ν_1 and ν_2 such that $\nu_1 \top^{\mathrm{WF}} \nu_2$, the product $\nu_1\cdot\nu_2$ is well-defined and we have*

$$\mathrm{WF}(\nu_1\cdot\nu_2)\subseteq\Omega_+\cup\Omega_{12}\cup\Omega_{21},$$

with

$$\Omega_+ := \{(x, k_1 + k_2) \mid (x, k_1) \in \mathrm{WF}(\nu_1), (x, k_2) \in \mathrm{WF}(\nu_2)\},$$

$$\Omega_{ij} := \{(x, k) \in \mathrm{WF}(\nu_i) \mid x \in \mathrm{Supp}(\nu_j)\}.$$

However, $\left(\mathcal{D}'(U), \top^{\mathrm{WF}}, \cdot\right)$ is not a locality semi-group, as shown by the following counterexample.

Coexample 3.25 Let ν_1 and ν_2 be as in Counterexample 3.23, and let $\nu_3(\phi)) = \int_{\mathbb{R}^2} \phi(x, y)\, \chi_\Delta\, dx\, dy$ where $\Delta = \{(x, x), x \in \mathbb{R}\}$ is the first diagonal. Then $\nu_1 \top^{\mathrm{WF}} \nu_3$ and $\nu_2 \top^{\mathrm{WF}} \nu_3$ but $\nu_1 \cdot \nu_2$ is not independent of ν_3 for the relation $\top^{\mathrm{WF}}$ since

$$\begin{aligned}
&\mathrm{WF}(\nu_1 \cdot \nu_2)\\
&\quad = \{(0,0)\} \times (\mathbb{R} \setminus \{0\}) \times (\mathbb{R} \setminus \{0\}) \bigcup \{(0,0)\} \times (\mathbb{R} \setminus \{0\}) \times \{0\}\\
&\qquad \bigcup \{(0,0)\} \times \{0\} \times (\mathbb{R} \setminus \{0\})\\
&\quad = \{(0,0)\} \times \left(\mathbb{R}^2 \setminus \{(0,0)\}\right),
\end{aligned}$$

whereas

$$\mathrm{WF}(\nu_3) = \{(x, x); (\lambda, -\lambda)),\ x \in \mathbb{R},\ \lambda \in \mathbb{R} \setminus \{0\}\}$$

so that $\mathrm{WF}(\nu_1 \cdot \nu_2) \cap \mathrm{WF}'(\nu_3) = \{((0,0); (\lambda, -\lambda)), \lambda \in \mathbb{R} \setminus \{0\}\}$.

Note that this relates to Counterexample 3.18, which says that the set $G(V)$ of finite dimensional vector space equipped with the locality relation $\top_{\{0\}}$ and the direct sum is not a locality semigroup.

3.3 Locality Multiplicativity and Locality Morphisms

3.3.1 Locality Morphisms

Definition 3.26 A **locality map**, from a locality set $(X, \top_X)$ to a locality set $(Y, \top_Y)$, is a map $\phi : X \to Y$ such that $(\phi \times \phi)(\top_X) \subseteq \top_Y$.

Example 3.27 We equip $G(V)$ with the binary symmetric relation $U \top_{\{0\}} W \Leftrightarrow U \cap W = \{0\}$. The identity map $\mathrm{Id} : \left(G(V), \perp^Q\right) \longrightarrow \left(G(V), \top_{\{0\}}\right)$ then defines a locality map since $U \perp^Q W \Longrightarrow U \cap W = \{0\}$.

Definition 3.28 Let $(X, \top_X, \cdot_X)$ and $(Y, \top_Y, \cdot_Y)$ (resp. $(X, \top_X, \cdot_X, 1_X)$ and $(Y, \top_Y, \cdot_Y, 1_Y)$) be locality semi-groups (resp. locality monoids). A map $\phi : X \longrightarrow Y$ is called a **locality semi-group** (resp. **monoid**) **homomorphism**, if it

(a) is a locality map;
(b) is **locality multiplicative**: for $(a, b) \in \top_X$ we have $\phi(a \cdot_X b) = \phi(a) \cdot_Y \phi(b)$.
(c) (resp: $\phi(1_X) = 1_Y$.)

Example 3.29 Classical examples of locality monoid homomorphisms are given by **multiplicative functions** in number theory. Here a function $f : \mathbb{Z}_{\geq 1} \to \mathbb{Z}_{\geq 1}$ is multiplicative means $f(1) = 1$ and $f(mn) = f(m)f(n)$ if m and n are coprime. This means precisely that f is a locality monoid homomorphism from the locality monoid $(\mathbb{Z}_{\geq 1}, \top_{\rm co})$ where $\top_{\rm co}$ is the coprime relation of Example 3.9, to the locality monoid $(\mathbb{Z}_{\geq 1}, \top_{\rm triv})$, where $\top_{\rm triv}$ is the trivial locality relation $\mathbb{Z}_{\geq 1} \times \mathbb{Z}_{\geq 1}$.

Example 3.30 For a finite set X, the cardinal $\mathrm{card} : \mathcal{P}(X) \to \mathbb{Z}_{\geq 0}$, which to a set assigns its cardinal yields a locality monoid homomorphism from the locality monoid $(\mathcal{P}(X), \top_\cap, \sqcup, \emptyset)$ to the locality monoid $\left(\mathbb{Z}_{\geq 0}, \top_{\rm triv}, +\right)$, with $\top_{\rm triv} := \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ given by the trivial locality relation.

Example 3.31 The dimension $\dim : G(V) \to \mathbb{Z}_{\geq 0}$, which to a finite dimensional vector space assigns its dimension, yields a locality monoid homomorphism from the locality monoid $\left(G(V), \perp^Q, \oplus, \{0\}\right)$ considered in Example 3.19 to $\left(\mathbb{Z}_{\geq 0}, \top_{\rm triv}, +\right)$, with the above notations.

3.3.2 Convex Lattice Cones

The orthogonality relation $\perp^Q$ defined in Example 3.13 gives rise to a useful locality relation on convex polyhedral lattice cones. Consider the filtered rational Euclidean lattice space

$$\left(\mathbb{R}^\infty = \bigcup_{k\geq 1} \mathbb{R}^k, \mathbb{Z}^\infty = \bigcup_{k \geq 1} \mathbb{Z}^k, Q = (Q_k(\cdot,\cdot))_{k\geq 1}\right),$$

where $\mathbb{R}^\infty$ (resp. $\mathbb{Z}^\infty$) is obtained as the inductive limit of $\mathbb{R}^k$ (resp. $\mathbb{Z}^k$) with $\mathbb{R}^k$ (resp. $\mathbb{Z}^k$) embedded into $\mathbb{R}^{k+1}$ (resp. $\mathbb{Z}^{k+1}$) by adding a zero coordinate on the right, and

$$Q_k(\cdot,\cdot) : \mathbb{R}^k \otimes \mathbb{R}^k \to \mathbb{R}, \quad k \geq 1,$$

is an inner product in $\mathbb{R}^k$ such that $Q_{k+1}|_{\mathbb{R}^k \otimes \mathbb{R}^k} = Q_k$ and $Q_k(\mathbb{Z}^k \otimes \mathbb{Z}^k) \subseteq \mathbb{Q}$. A **lattice cone** is a pair (C, Λ_C) where C is a polyhedral rational cone in some $\mathbb{R}^k$, that is,

$$C = \langle u_1, \ldots, u_m \rangle := \left\{ \sum_{i=1}^m c_i u_i \,\middle|\, c_i \in \mathbb{R}_{\geq 0}, 1 \leq i \leq m \right\}$$

for some $u_1, \ldots, u_m \in \mathbb{Q}^k$, and Λ_C is a rational lattice of the linear subspace spanned by C. Here a rational lattice Λ of a rational subspace V of $\mathbb{R}^k$ is a free abelian subgroup generated by a rational basis of V. This (under the rationality assumption) is equivalent to the condition that Λ is a finitely generated abelian subgroup of $V \cap \mathbb{Q}^k$ that spans V.

Let $\mathcal{C}_k$ be the set of lattice cones of dimension k and

$$\mathcal{C} = \bigcup_{k\geq 1} \mathcal{C}_k \tag{30}$$

be the set of lattice cones in $(\mathbb{R}^\infty, \mathbb{Z}^\infty)$. Let $\mathbb{Q}\mathcal{C}_k$ and $\mathbb{Q}\mathcal{C}$ be the linear spans of $\mathcal{C}_k$ and $\mathcal{C}$ over $\mathbb{Q}$.

In $(\mathbb{R}^\infty, \mathbb{Z}^\infty, Q)$, we write $\perp^Q$ for the corresponding orthogonality relation (21).

Definition 3.32 We call two lattice cones (C, Λ_C) and (D, Λ_D) **orthogonal** (with respect to Q), if $Q(u, v) = 0$ for all $u \in \Lambda_C$, $v \in \Lambda_D$. Then we write $(C, \Lambda_C) \perp^Q (D, \Lambda_D)$.

This defines the locality set $\left(\mathbb{Q}\mathcal{C}, \perp^Q\right)$ of lattice cones.

The Minkowski sum defines an operation on lattice cones: for two convex cones $C := \langle u_1, \dots, u_m\rangle$ and $D := \langle v_1, \dots, v_n\rangle$ spanned by $u_1, \dots, u_m$ and $v_1, \dots, v_n$ respectively, their Minkowski sum is the convex cone

$$C \cdot D := \langle u_1, \dots, u_m, v_1, \dots, v_n\rangle.$$

This product can be extended to a product in $\mathcal{C}$:

$$(C, \Lambda_C) \cdot (D, \Lambda_D) := (C \cdot D, \Lambda_C + \Lambda_D), \tag{31}$$

where $\Lambda_C + \Lambda_D$ is the abelian group generated by Λ_C and Λ_D in $\mathbb{Q}^\infty$. This product endows $\mathcal{C}$ with a partial monoid structure with unit $(\{0\}, \{0\})$.

3.3.3 Meromorphic Germs

The orthogonality relation $\perp^Q$ defined in Example 3.13 also gives rise to a useful locality relation on multivariate meromorphic functions, which provide another fundamental example. Again in $(\mathbb{R}^\infty, \mathbb{Z}^\infty, Q)$, let $\mathcal{M}_\mathbb{Q}((\mathbb{R}^k)^* \otimes \mathbb{C})$ be the space of meromorphic germs at 0 with linear poles and rational coefficients [34, 36] and let

$$\mathcal{M}_\mathbb{Q} := \bigcup_{k\geq 1} \mathcal{M}_\mathbb{Q}((\mathbb{R}^k)^* \otimes \mathbb{C}). \tag{32}$$

An element of $\mathcal{M}_\mathbb{Q}$ can be written as a sum $h_0 + \sum_i f_i$ of a holomorphic germ h_0 and elements f_i of the form

$$f(z_1, \dots, z_k) = \frac{h(\ell_1, \dots, \ell_m)}{L_1^{s_1} \dots L_n^{s_n}}, \quad s_1, \dots, s_n \in \mathbb{Z}_{>0}, \tag{33}$$

where h is a holomorphic germ with rational coefficients in linear forms $\ell_1, \dots, \ell_m \in \mathbb{Q}^k$, and $L_1, \dots, L_n$ are linearly independent linear forms in $\mathbb{Q}^k$, $\ell_i \perp^Q L_j$ for all $i \in \{1, \dots, m\} \forall j \in \{1, \dots, n\}$, which is called a **polar germ.**

Definition 3.33 Two meromorphic germs with rational coefficients f and f' are ***Q*-orthogonal** which we denote by $f \perp^Q f'$ if there exist linear functions $L_1, \dots, L_m \in \mathbb{Q}^k$ and $L'_1, \dots, L'_n \in \mathbb{Q}^k$ satisfying $Q(L_i, L'_j) = 0$ for $i = 1, \dots, m$, $j = 1, \dots, n$, and meromorphic germs $g \in \mathcal{M}_\mathbb{Q}(\mathbb{R}^m \otimes \mathbb{C})$ and $g' \in \mathcal{M}_\mathbb{Q}(\mathbb{R}^n \otimes \mathbb{C})$, such that $f = g(L_1, \dots, L_m)$, $f' = g'(L'_1, \dots, L'_n)$. Let $(\mathcal{M}_\mathbb{Q}, \perp^Q)$ denote the resulting locality set.

3.3.4 Sums and Integrals on Cones as Locality Morphisms

We next give further useful examples of locality maps. Let (C, Λ_C) be a strongly convex lattice cone in $\mathbb{R}^k$ with interior C^o. For z in the dual cone

$$C^- := \{z \in (\mathbb{R}^k)^* \mid \langle x, z\rangle < 0, \forall x \in C\},$$

we define its **exponential sum** to be the sum

$$S(C, \Lambda_C)(z) := \sum_{n \in C^o \cap \Lambda_C} e^{\langle n, z\rangle}. \tag{34}$$

We also define its **exponential integral** $I(C, \Lambda_C)$ to be the integral

$$I(C, \Lambda_C)(z) := \int_C e^{\langle x, z\rangle} d\Lambda_x, \tag{35}$$

where $d\Lambda_x$ is the volume form induced by generators of Λ_C such that the polytope generated by a basis of Λ_C has volume 1.

The fact that z lies in C^- ensures the absolute convergence of both the sum (34) and the integral (35).

These assignments extend by subdivisions to maps:

$$S, I : \mathcal{C} \to \mathcal{M}_{\mathbb{Q}}. \tag{36}$$

Proposition 3.34 *For lattice cones (C, Λ_C) and (D, Λ_D), if $(C, \Lambda_C) \perp^Q (D, \Lambda_D)$, then $S(C, \Lambda_C) \perp^Q S(D, \Lambda_D)$ and $I(C, \Lambda_C) \perp^Q I(D, \Lambda_D)$ in the sense of Definition 3.33, that is, the exponential integral and exponential sum maps I and S are locality maps in the sense of Definition 3.26.*

Even though $\mathcal{C}$ and $\mathcal{M}_{\mathbb{Q}}$ both carry a natural multiplication defined on the full-fledged spaces, the importance of the locality structures on $\mathcal{C}$ and $\mathcal{M}_{\mathbb{Q}}$ becomes evident when studying the multiplicative property of the maps I and S from $\mathcal{C}$ to $\mathcal{M}_{\mathbb{Q}}$. Because of the idempotency $(C, \Lambda_C) \cdot (C, \Lambda_C) = (C, \Lambda_C)$ for $(C, \Lambda_C) \in \mathcal{C}$, the multiplicativity $I((C, \Lambda_C) \cdot (C, \Lambda_C)) = I(C, \Lambda_C)\, I(C, \Lambda_C)$ or $S((C, \Lambda_C) \cdot (C, \Lambda_C)) = S(C, \Lambda_C)\, S(C, \Lambda_C)$ does not hold in general since that would force the integral or the sum to be 0 or 1, which can not be the case, as can be seen from the one dimensional cone $(C, \Lambda_C) = (\langle e_1\rangle, \mathbb{Z}e_1)$.

Proposition 3.35 *For lattice cones (C, Λ_C) and (D, Λ_D), if $(C, \Lambda_C) \perp^Q (D, \Lambda_D)$, then*

$$S\big((C, \Lambda_C) \cdot (D, \Lambda_D)\big) = S\big((C, \Lambda_C)\big) \cdot S\big((D, \Lambda_D)\big),$$
$$I\big((C, \Lambda_C) \cdot (D, \Lambda_D)\big) = I\big((C, \Lambda_C)\big) \cdot I\big((D, \Lambda_D)\big),$$

that is, the exponential integral and exponential sum maps I and S are locality morphisms $\mathcal{C}$ to $\mathcal{M}_{\mathbb{Q}}$ in the sense of Definition 3.28.

The above proposition shows how locality provides a natural framework to express partial multiplicative properties.

3.4 Locality Algebra Morphisms

The notion of vector space generalises to a locality vector space. Roughly speaking, it is a vector space V equipped with a locality relation $\top$ compatible with the linear structure, whose precise definition requires the notion of polar set.

(a) On a locality set $(X, \top)$, for $U \subseteq X$, we call $U^\top := \{x \in X, \mid u \top x \ \forall u \in U\}$ the **polar set** of U in X. If U reduces to one element u in X, we simply write $u^\top$ for the polar set of $\{u\}$.
(b) A vector space V equipped with a locality relation $\top$ is called a **locality vector space** if $U^\top$ is a linear subspace of V for any subset U of V.

Example 3.36 With the notations of Eqn. (20), the couple $(V, \perp^Q)$ defines a locality vector space.

Example 3.37 [15, Proposition 3.9] The vector space $\mathcal{M}_\mathbb{Q}$ equipped with the relation $\perp^Q$ in Definition 3.33 is a locality vector space $\left(\mathcal{M}_\mathbb{Q}, \perp^Q\right)$.

We introduce the key concept of locality algebra. We use the notations of Definition 3.1.

(a) A **nonunitary locality algebra** over K is a locality vector space $(A, \top_A)$ over K together with a locality bilinear map

$$m_A : \top_A = A \times_\top A \to A$$

such that $(A, \top_A, m_A)$ is a locality semi-group in the sense of Definition 3.15.
(b) A **locality algebra** is a locality vector space $(A, \top_A)$ over K together with a locality bilinear map

$$m_A : A \times_\top A \to A$$

and a unit $1_A : K \to A$ such that $(A, \top_A, m_A, 1_A)$ is a locality monoid in the sense of Definition 3.15. We shall omit explicitly mentioning the unit 1_A and the product m_A unless this generates an ambiguity.

Remark 3.38 A locality algebra $(A, \top_A)$ with the trivial locality relation $\top_A = A \times A$ amounts to an ordinary algebra.

Example 3.39 The locality space $\left(\mathbb{Q}\mathcal{C}, \perp^Q\right)$, with the multiplication obtained from the linear extension of the locality monoid structure on $\mathcal{C}$ by the Minkowski sum in Eq. (31), is a locality commutative algebra.

Example 3.40 The locality vector space $\left(\mathcal{M}_\mathbb{Q}, \perp^Q\right)$ equipped with the product of functions is also a commutative locality algebra.

We call a locality linear map $f : (A, \top_A, \cdot_A) \to (B, \top_B, \cdot_B)$ between two (not necessarily unital) locality algebras a **locality algebra homomorphism** if it is a locality morphism of semi-groups

$$f(u \cdot_A v) = f(u) \cdot_B f(v) \quad \forall (u, v) \in \top_A. \tag{37}$$

A locality algebra $(A, \top_A, m_A)$ is called a **locality subalgebra** of a locality algebra $(B, \top_B, m_B)$ if $\top_A = \top_B \cap (A \times A)$ and $m_A | \top_B$ so that the inclusion map $\iota : A \to B$ is a locality algebra morphism.

Example 3.41 $\left(\mathcal{M}_{\mathbb{Q}+}, \perp^Q, \cdot\right)$ is a locality subalgebra of $\left(\mathcal{M}_{\mathbb{Q}}, \perp^Q, \cdot\right)$.

Let $\mathcal{M}_{\mathbb{Q}-}^Q$ denote the subspace spanned by polar germs defined by Eq. (33). The one variable counterpart of $\mathcal{M}_{\mathbb{Q}-}^Q$, namely the space $\varepsilon^{-1}\mathbb{C}[\varepsilon^{-1}]$, is a subalgebra in the space of meromorphic functions in one variable. In contrast, the space $\mathcal{M}_{\mathbb{Q}-}^Q$ is not a subalgebra of $\mathcal{M}_{\mathbb{Q}}$.

Proposition 3.42 [37, *Corollary* 4.16] *There is a direct sum decomposition*

$$\mathcal{M}_{\mathbb{Q}} = \mathcal{M}_{\mathbb{Q}+} \oplus \mathcal{M}_{\mathbb{Q}-}^Q, \tag{38}$$

where $\mathcal{M}_{\mathbb{Q}+}$ is the subspace of holomorphic functions. The space $\mathcal{M}_{\mathbb{Q}-}^Q$ is a locality subalgebra of $\mathcal{M}_{\mathbb{Q}}$.

A locality subalgebra I of a locality commutative algebra $(A, \top, m_A)$ is called a **locality ideal** of A if for any $b \in I$ we have

$$\forall (a, b) \in A \times I : \quad a \top b \Longrightarrow a \cdot b \in I \quad \text{or equivalently} \quad a \cdot b \subseteq I \ \forall a \in b^\top. \tag{39}$$

Example 3.43 Take $A = A_+ \oplus A_-$ a commutative $\mathbb{Z}_2$-graded algebra equipped with the locality relation $a \top b \Leftrightarrow (a, b) \in A_\pm \times A_\mp$. Note that $0 \top A$. Then A_- is not a subalgebra since the product maps $A_- \times A_-$ to A_+ yet it is a locality ideal (and hence a locality subalgebra) since the product maps $A \times_\top A_- := (A \times A_-) \cap \top$ to A_-.

Proposition 3.44 [15, *Corollary* 3.23], [37, *Corollary* 4.18] *The locality subalgebra $\mathcal{M}_{\mathbb{Q}-}^Q$ is actually a locality ideal of $\mathcal{M}_{\mathbb{Q}}$ and hence the projection $\pi_+^Q : \mathcal{M}_{\mathbb{Q}} \to \mathcal{M}_{\mathbb{Q}+}$ is a locality algebra homomorphism.*

The fact that the projection $\pi_+^Q : \mathcal{M}_{\mathbb{Q}} \to \mathcal{M}_{\mathbb{Q}+}$ is a locality algebra homomorphism in our multivariable setup does not hold in the usual single variable framework used for renormalisation purposes, which speaks for the relevance of working in the locality framework when considering the multivariable setup.

4 Locality and Renormalization

As already mentioned, perturbative quantum field theory often involves Feynman integrals which diverge, and sophisticated methods were developed by physicists to 'cure' divergences occurring in renormalisable theories, while preserving the locality property (in the sense of (1)) of the Lagrangian defining the theory. The question we address here is how the locality framework can be useful for renormalisation. For this purpose, we first briefly review (leaving out many technical steps) the general lines of a renormalisation procedure to shed light on the differences between the usual approach which uses a one parameter regularisation procedure and our multivariable approach. We then discuss the algebraic Birkhoff factorisation used in Connes and Kreimer's approach to renormalisation and extend it to a locality framework. Theorem 4.16 shows that the locality constraints enable to circumvent Birkhoff factorisation in so far as the renormalised (locality character) can be directly obtained as the composition of the locality projection map with the original map.

4.1 One Variable Versus Multivariable Regularisations

Renormalisation can be carried out by means of a one parameter regularisation procedure, typically dimensional regularisation (dim. reg) developed by 't Hooft and Veltman [63], [64]. It is based on an analytic continuation of the Feynman integrals corresponding to the Feynman graphs entering the perturbative expansion to complex dimension $d(z)$ in a neighborhood of the integral dimension $d = d(0)$ at which ultra-violet divergences occur. When letting the complex parameter z tend to zero, so when the complex dimension $d(z)$ tends to the integer dimension d, the analytically continued integrals become singular and the expression is a meromorphic function at zero, giving rise to a map

$$\phi: \text{ Feynman graphs } \underset{\textbf{dim. reg.}}{\longmapsto} \text{ meromorphic germs in one variable,}$$

which sends graphs to meromorphic germs at zero in one variable resulting from dimensional regularisation. In accordance with the locality principle, the map is extended to a multiplicative map on the free algebra generated by Feynman graphs with the product corresponding to the concatenation $F_1 \bullet F_2$ of two Feynman graphs F_1, F_2. To simplify notations, we denote the extension by the same symbol ϕ, which by construction fulfills the following **factorisation property**:

$$\phi(F_1 \bullet F_2) = \phi(F_1)\,\phi(F_2), \tag{40}$$

with the pointwise product of functions on the r.h.s. Preserving this factorisation property throughout the renormalisation procedure is a challenge.

One approach to renormalisation uses a **minimal subtraction** scheme at each order of the expansion, in subtracting the singular part of the Laurent series in z while keeping the regular part. Given a splitting $\phi(F) = \phi^{\text{reg}}(F) + \phi^{\text{sing}}(F)$ of the image $\phi(F)$ of a Feynman graph F, with $\phi^{\text{reg}}(F)$, resp. $\phi^{\text{sing}}(F)$ corresponding to the regular, resp. singular part in the Laurent expansion, then in general

$\phi^{\rm reg}(F_1 \bullet F_2) \neq \phi^{\rm reg}(F_1)\, \phi^{\rm reg}(F_2)$, so the factorisation property (40) is not preserved when bluntly subtracting out singular terms.

The Bogoliubov-Parasiuk preparation, or BPHZ method (for Bogoliubov-Parasiuk-Hepp-Zimmermann) [6, 42, 73] (also known as the "forest formula") implemented in QFT is an inductive procedure on the number of loops of the graphs, which circumvents this difficulty in adding to the original Lagrangian an infinite series of counterterms relative to the singular parts, labelled by the Feynman graphs. By introducing appropriate counterterms counteracting the divergences at each step of the induction, it takes care of subdivergences in the renormalisation process while avoiding the occurrence of non-local terms in the counterterms which would not match and hence spoil the locality of the original Lagrangian in the sense of (1).

This sophisticated BPHZ inductive procedure, which ensures that the factorisation property (40) is preserved after renormalisation, was later reinterpreted in a coalgebraic language as an algebraic-combinatorial factorisation in the group of characters over the Hopf algebra of Feynman graphs of the map ϕ. As we shall see, this strongly uses the coalgebraic structure on the source algebra of graphs [20–22].

An alternative renormalisation to the BPHZ method uses a **multiparameter regularisation procedure** inspired by Speer's analytic renormalisation (anal. reg) [57] for amplitudes in QFT, recently implemented in [24] to describe Feynman amplitudes on a closed manifold. It is based on an analytic continuation in several variables. For this, a Riesz type regularisation is used to decorate the Feynman diagrams, associating a parameter z_i with each line of the Feynman graph. The poles of the resulting meromorphic germs at zero turn out to be linear in the variables $z_1, \dots, z_i, \dots$, more precisely, they are linear combinations with zero or one coefficients i.e., $\sum_{i\in I} z_i$ for some finite subset $I \subseteq \mathbb{N}$, which reflect the structure of the graphs they come from, giving rise to a map

$$\begin{aligned} \phi :\ & \text{Feynman graphs (edges decorated with } z_i) \\ & \underset{\textbf{anal. reg.}}{\longmapsto} \text{ meromorphic germs in several variables.} \end{aligned} \tag{41}$$

The inductive BPHZ renormalisation procedure applies to one variable regularisations. It uses the Bogoliubov-Parasiuk preparation that works through the subdivergences of Feynman integrals. The recursive nature of this procedure is reflected in the coalgebra structure of Feynman graphs in the Connes and Kreimer's approach. In contrast, Speer renormalises the Feynman amplitudes using multivariable regularisations without the use of recursion, by means of generalised evaluators operating on the target algebra of meromorphic germs. As we shall see, this seemingly oversimplifying approach compared to the BPHZ procedure turns out to be natural for multivariable renormalisations once they are put in our algebraic approach of locality.

Generalised evaluators are linear forms $\mathcal{E}$ on the algebra of meromorphic germs at zero in several variables, which composed with ϕ, give rise to a scalar valued renormalised map $\phi^{\rm ren} := \mathcal{E} \circ \phi$ that fulfills the factorisation property. The renormalised map $\phi^{\rm ren}$ indeed factorises on the concatenation of graphs (denoted here by $\bullet$) decorated by disjoint sets of variables:

$$F_1 \text{ and } F_2 \text{ have different sets of decorations} \Longrightarrow \phi^{\text{ren}}(F_1 \bullet F_2) = \phi^{\text{ren}}(F_1)\,\phi^{\text{ren}}(F_2) \tag{42}$$

with the function product on the right hand side. In our language, the binary relation $F_1 \top^{\text{Speer}} F_2$ if F_1 and F_2 have different sets of decorations, defines a locality structure on decorated graphs. Taking the locality structure into consideration, we interpret the locality principle as the condition that the regularisation be a locality morphism. This can be best carried out in the multivariable setup which naturally keeps track of the levels of subdivergences.

Multivariable regularisation of integrals associated with Feynman graphs usually gives rise to a specific class of meromorphic germs at zero in several complex variables, namely those that have **linear poles**, so germs in $\mathcal{M}_{\mathbb{Q}}$. As briefly alluded above, Speer shows [57, Theorem 1] that Feynman amplitudes built from a Feynman diagram whose propagators are regularised by a Riesz procedure, give rise to a class of meromorphic germs at zero with linear poles $L_I := \sum_{i\in I} z_i$ for some subset I of $\{1, \ldots, k\}$, which form a subalgebra of $\mathcal{M}_{\mathbb{Q}}$.

4.2 Algebraic Birkhoff Factorisation

The Connes-Kreimer renormalisation scheme [20–22], known as algebraic Birkhoff factorisation and which encodes the forest formula in QFT, is based on the factorisation of the above mentioned character ϕ on a Hopf algebra of Feynman graphs with values in an algebra of meromorphic germs, as a convolution product of two characters. Traditionally, Birkhoff factorization [3] refers to representations of an invertible matrix valued function $z \mapsto M(z)$ on the unit circle S^1 of the form $M = M_+ \,\Delta\, M_-$, where $M_\pm$ are the boundary values of invertible matrix-functions holomorphic inside (respectively, outside) the circle S^1, and Δ is a diagonal matrix-function. In Connes and Kreimer's approach, it takes the form of a factorisation of the map ϕ as a convolution product involving a regular part ϕ_+ and a singular part ϕ_-. Its precise formulation requires several algebraic and coalgebraic concepts such as that of a Hopf algebra, Rota-Baxter algebra and convolution product. The principle of locality, in the form of additional locality structures described in this paper, suggests the study of a locality version of the algebraic and coalgebraic structures underlying the algebraic Birkhoff factorisation, in order to pass from one parameter renormalisations to multivariable renormalisations.

Precisely, we want a locality counterpart of the subsequent statement. Rather than expliciting the various notions of (connected) Hopf algebra, Rota-Baxter algebra and convolution product arising in the statement of Theorem 4.1, we shall define their locality counterpart as we go along. Forgetting requirements related to locality gives back the usual definitions, which one can find e.g. in the very nice survey [48]. What follows is mostly taken from [15, 16].

Theorem 4.1 [21, *Theorem* 2], [48, *Theorem II*.5.1.] (***Algebraic Birkhoff factorisation***) *Let H be a connected Hopf algebra and let (A, P) be a commutative Rota-*

Baxter algebra of weight -1 *with an idempotent Rota-Baxter operator* P. *Any algebra homomorphism* $\phi : H \to A$ *factors uniquely as the convolution product*

$$\phi = \phi_-^{\star(-1)} \star \phi_+ \tag{43}$$

of algebra homomorphisms $\phi_- : H \to K + P(A)$ *and* $\phi_+ : H \to K + (Id - P)(A)$. *Here the convolution of two linear maps* $f, g : B \to B$ *is defined by*

$$f \star g = m(f \otimes g)\Delta. \tag{44}$$

An important instance of Hopf algebra (resp. Rota-Baxter algebra) in quantum field renormalisation is the Connes-Kreimer Hopf algebra of Feynman graphs (resp. the algebra of Laurent series $\mathbb{C}[\varepsilon^{-1}, \varepsilon]]$ hosting meromorphic germs at zero from one parameter regularisations). The locality setting becomes necessary for multivariable regularisations.

4.3 Ingredients for a Locality Algebraic Birkhoff Factorisation

We introduce a few more definitions along the way to a locality counterpart of the above statement. This paragraph uses results of [15].

A locality algebra $(A, \top_A, m_A)$ with a linear grading $A = \oplus_{n\geq 0} A_n$ is called a **locality graded algebra** if $m_A((A_m \times A_n) \cap \top_A) \subseteq A_{m+n}$ for all $m, n \in \mathbb{Z}$.

Example 4.2 [15, Example 3.18] Given a finite set X, the triple $(K\mathcal{P}(X), \top_\cap, \sqcup)$ where $\top_\cap$ defined in (18) has been extended by bilinearity, defines a locality graded algebra for the grading given by the cardinality.

Example 4.3 With the grading induced from $\mathcal{C} = \sqcup_{n\geq 0}\mathcal{C}_n$ in Eq. (30), $\mathbb{Q}\mathcal{C} = \oplus_{n\geq 0}\mathbb{Q}\mathcal{C}_n$ is a graded locality algebra.

Recall that the locality product $m : (a, b) \mapsto a \cdot b$ in a locality semigroup $(G, \top)$ is defined on pairs (a, b) such that $a \top b$. We therefore expect a locality coproduct $\Delta : c \mapsto \sum_{(c)} c_1 \otimes c_2$ which roughly speaking undoes such a product, to break an element c into pairs (c_1, c_2) of mutually independent elements $c_1 \top c_2$. The precise definition actually requires a more stringent condition, which uses a locality tensor product:

The **locality tensor product** of two subspaces V_1 and V_2 of a locality vector space $(V, \top)$ is defined as the subspace $V_1 \otimes_\top V_2 \subseteq V_1 \otimes V_2$ of the ordinary tensor product, linearly generated by tensor products of the form $v_1 \otimes v_2$ with $v_1 \top v_2$.

If $\top = V \times V$ is the trivial locality relation, then $V_1 \otimes_\top V_2 = V_1 \otimes V_2$.

We are now ready to define the locality coproduct.

Definition 4.4 Let $(C, \top)$ be a locality vector space and let $\Delta : C \to C \otimes C$ be a linear map. $(C, \top, \Delta)$ is a **locality (noncounital) coalgebra** if it satisfies the following two conditions

(a) for any $U \subseteq C$

$$\Delta(U^\top) \subseteq U^\top \otimes_\top U^\top. \tag{45}$$

In particular, $\Delta(C) \subseteq C \otimes_\top C$;

(b) the following **coassociativity** holds:

$$(\mathrm{Id}_C \otimes_\top \Delta)\,\Delta = (\Delta \otimes_\top \mathrm{Id}_C)\,\Delta.$$

- If in addition, there is a **counit**, namely a linear map $\varepsilon : C \to K$ such that $(\mathrm{Id}_C \otimes \varepsilon)\,\Delta = (\varepsilon \otimes \mathrm{Id}_C)\,\Delta = \mathrm{Id}_C$, then $(C, \top, \Delta, \varepsilon)$ is called a **locality coalgebra.**

Remark 4.5 For the trivial locality relation $\top = C \times C$, this gives back the usual notion of locality (nonunital) coalgebra, see e.g. [48].

Example 4.6 Given a finite set X, the quadruple $(K\mathcal{P}(X), \top_\cap, \Delta, \epsilon)$ where $\top_\cap$ has been extended by bilinearity and $\Delta C = \sum_{A\subseteq C} A \otimes C \setminus A$ also extended by linearity, defines a locality coalgebra with counit the map ϵ which takes all sets to zero except for the empty set which is mapped to 1.

A **connected locality coalgebra** is a locality coalgebra $(C, \top, \Delta)$ with a grading $C = \oplus_{n\geq 0} C_n$ such that, for any $U \subseteq C$,

$$\Delta(C_n \cap U^\top) \subseteq \bigoplus_{p+q=n} (C_p \cap U^\top) \otimes_\top (C_q \cap U^\top), \qquad \bigoplus_{n\geq 1} C_n = \ker \varepsilon. \tag{46}$$

We denote by J the unique element of C_0 with $\varepsilon(J) = 1_K$, giving $C_0 = K\, J$.

Example 4.7 [15, Lemma 4.5] We equip $\mathbb{Q}\mathcal{C} = \sqcup_{n\geq 0} \mathbb{Q}\mathcal{C}_n$ introduced in Eq. (30) with a coproduct defined as follows:

$$\Delta^Q(C, \Lambda_C) := \sum_{(F,\Lambda_F)\preceq(C,\Lambda_C)} (t^Q(C, F), \Lambda_{t(C,F)}) \otimes (F, \Lambda_F), \tag{47}$$

where the sum on the right hand side is taken over the faces $F \preceq C$ of C, and $t^Q(C, F)$ is the transverse cone to F defined as the projection of the cone C onto the orthogonal space to the face F for the inner product Q. Since $t^Q((C, \Lambda_C), (F, \Lambda_F)) \perp^Q (F, \Lambda_F)$ by definition, the quadruple $(\mathbb{Q}\mathcal{C}, \perp^Q, \Delta^Q, \varepsilon)$ is a locality coalgebra with the locality counit given by the linear extension of the map

$$\varepsilon : \mathcal{C} \to \mathbb{Q}, (C, \Lambda_C) \mapsto \begin{cases} 1, & (C, \Lambda_C) = (\{0\}, \{0\}), \\ 0, & (C, \Lambda_C) \neq (\{0\}, \{0\}). \end{cases}$$

Further the connectedness conditions in Eq. (46) are satisfied. This gives a connected locality coalgebra $\left(\mathbb{Q}\mathcal{C}, \perp^Q, \Delta\right)$.

The following definition generalises the notion of (connected) bialgebra which merges the notions of algebra and coalgebra, see e.g. [48, §I.5].

(a) A **locality bialgebra** is a sextuple $(B, \top, m, u, \Delta, \varepsilon)$ consisting of a locality algebra $(B, m, u, \top)$ and a locality coalgebra $(B, \Delta, \top, \varepsilon)$ that are locality compatible in the sense that Δ and ε are locality algebra homomorphisms.
(b) A locality bialgebra B is called **connected** if there is a $\mathbb{Z}_{\geq 0}$-grading $B = \oplus_{n\geq 0} B_n$ with respect to which B is both a locality graded algebra in the sense that $m_B((B_m \times B_n) \cap \top_B) \subseteq B_{m+n}$ and a connected locality coalgebra in the sense of Definition 4.4. Then $J = 1_B$.

Let us go back to the space $\mathbb{Q}\mathcal{C}$ of lattice cones with the Minkowski product and the coproduct Δ defined in Eq. (47). We observe that the idempotency $(C, \Lambda_C) \cdot (C, \Lambda_C) = (C, \Lambda_C)$ hinders the compatibility between the product and the coproduct. For example, taking $(C, \Lambda_C) = (\langle e_1 \rangle, \mathbb{Z}e_1)$, then $\Delta^Q(C \cdot C) = \Delta^Q(C) \cdot \Delta^Q(C)$ does not hold. However, this compatibility can be recovered in the context of locality bialgebras:

Example 4.8 [15, Proposition 5.2] $(\mathbb{Q}\mathcal{C}, \perp^Q, \cdot, u, \Delta^Q, \varepsilon)$ is a connected locality bialgebra.

We now specialise to Hopf algebras, named after Heinz Hopf [44], namely bialgebras equipped with an antipode map, which in the case of a group algebra, comes down to taking inverses of group elements. Hopf algebras occur naturally in algebraic topology, where they originated, and are used in many other areas of mathematics such as Lie groups, algebraic groups and Galois theory, as well as in quantum field theory, as the main protagonists of Connes and Kreimer's coalgebraic approach to renormalisation [20]. As before, we define their locality counterpart.

Definition 4.9 A **locality Hopf algebra** is a locality bialgebra $(B, \top, m, \Delta, u, \varepsilon)$ with an antipode, defined to be a linear map $S : B \to B$ such that S and Id_B are mutually independent in the sense that $(S \times \mathrm{Id}_B)(\top) \subseteq \top$ in which case we require that:

$$S \star \mathrm{Id} = \mathrm{Id} \star S = u\,\varepsilon,$$

where as in (44), $\star$ stands for the convolution product.

Remark 4.10 When $\top = B \times B$ is the trivial locality relation, this gives back the usual notion of Hopf algebra, see e.g. [48, §1.5] and references therein.

Example 4.11 Given a finite set X, the sextuple $(K\mathcal{P}(X), \top_\cap, \cup, \emptyset, \Delta, \epsilon)$ defines a locality bialgebra. If we grade $\mathcal{P}(X)$ by cardinality, this is a graded bialgebra thus a locality Hopf algebra.

As in the usual setup (see e.g. [48, Corollary II.3.2.]), a connected locality bialgebra B turns out to be a locality Hopf algebra with the antipode map $S : B \to B$ given by $S = \sum_{k=0}^{\infty}(u\varepsilon - \mathrm{Id})^{\star k}$, see [15, Proposition 5.6].

Algebraic Birkhoff factorisation uses the notion of Rota-Baxter algebra, which is an associative algebra with a linear operator that generalises the algebra of continuous functions with the integral operator, see e.g. [32, 33] and references therein. We give the definition of a Rota-Baxter algebra in the locality setup.

Definition 4.12 A linear operator $P : A \to A$ on a locality algebra $(A, \top)$ over a field K is called a **locality Rota-Baxter operator of weight** $\lambda \in K$, if it is a locality map, independent of Id_A, and satisfies the following locality Rota-Baxter relation:

$$P(a)\,P(b) = P(P(a)\,b) + P(a\,P(b)) + \lambda\,P(a\,b) \quad \forall (a, b) \in \top. \tag{48}$$

We call the triple $(A, \top, P)$ a locality Rota-Baxter algebra.

Remark 4.13 A locality Rota-Baxter algebra with trivial locality $\top = A \times A$ yields back an ordinary Rota-Baxter algebra.

Example 4.14 With the notations of Example 3.43, the projection P onto A_+ along A_- is a locality Rota-Baxter operator but not a Rota-Baxter operator.

Example 4.15 [15, Corollary 3.23] The locality projection $\pi_+^Q : \mathcal{M}_{\mathbb{Q}} \to \mathcal{M}_{\mathbb{Q}+}$ defines a locality Rota-Baxter operator on $\left(\mathcal{M}_{\mathbb{Q}}, \perp^Q\right)$, but not an ordinary Rota-Baxter operator on $\mathcal{M}_{\mathbb{Q}}$.

These two examples are specific instances of a more general fact. Let $A = A_+ \oplus A_-$ be a commutative locality algebra equipped with a locality linear idempotent operator $P : A \longrightarrow A$ given by the projection onto A_+ along A_-. The following statements are equivalent [15, Proposition 3.22]:

(a) P or $\mathrm{Id} - P$ (and hence both) is a locality Rota-Baxter operator of weight -1;
(b) $A_\pm$ are locality subalgebras of A, and $A_+ \top A_-$.

If one of the conditions holds, then P is locality multiplicative if and only if A_- is a locality ideal of A.

4.4 Locality Algebraic Birkhoff Factorisation Versus Minimal Subtraction

We are now ready to state the locality version of Algebraic Birkhoff Factorisation (see e.g. [48, Theorem II.5.1.]).

Theorem 4.16 [17, *Theorem* 4.4], *see also* [15, *Theorem* 4.10] (***Locality Algebraic Birkhoff Factorisation***) *Let* $(H, \top_H)$ *be a locality connected Hopf algebra,* $H = \oplus_{n\geq 0} H_n$, $H_0 = Ke$. *Let* $(A, \top_A, \cdot, P)$ *be a commutative locality Rota-Baxter algebra of weight* -1 *with* P *idempotent.*

Denote $A_+ = P(A)$ *and* $A_- = (\mathrm{Id} - P)(A)$. *Let*

$$\phi : (H, \top_H) \longrightarrow (A, \top_A)$$

be a locality algebra homomorphism. Then there are unique locality algebra homomorphisms $\phi_\pm : H \to K + A_\pm$ *with* $\phi_\pm(\mathrm{Ker}\varepsilon) \subseteq A_\pm$, *such that* $u \top_H v \Rightarrow$

$\phi_+(u)\top_A\phi_-(v)$ (in which case we say ϕ_+ and ϕ_- are independent and write $\phi_+\top\phi_-$) and

$$\phi = \phi_+^{\star(-1)} \star \phi_-. \tag{49}$$

The map $\phi_+^{\star(-1)}$ is also a locality algebra homomorphism and $\phi_+\top\phi$ i.e., $u\top_H v \Rightarrow \phi_+(u)\top_A\phi(v)$.

If in addition A_- is a locality ideal of A, then $\phi_+^{\star(-1)} = P \circ \phi$.

The last statement shows that the locality constraints on $\phi_\pm$ enable to circumvent Birkhoff factorisation in so far as the (locality character) ϕ_+ can be directly obtained as the composition of the locality projection map P with the original map ϕ.

We have seen how the multivariable regularisation approach to a field theory (see 41) gives rise to a map ϕ on Feynman graphs with values in meromorphic germs in several variables with linear poles at zero. Assuming that the theory under consideration gives rise to a regularised map defined on a locality Hopf algebra $(H, \top_H)$ which is a locality algebra homomorphism:

$$\phi : (H, \top_H) \to (\mathcal{M}_{\mathbb{Q}}, \perp^Q), \tag{50}$$

then by Theorem 4.16, the composition $\pi_+^Q \circ \phi$ is the renormalisation of ϕ. The latter can be interpreted as a multivariable minimal subtraction scheme in that it extracts the holomorphic part of $\phi(h)$ for ay h in H by projecting it onto the subalgebra of holomorphic germs. This can be summarised in the form of the subsequent slogan: for a locality morphism (50), we have the following equivalence:

Locality algebraic Birkhoff Factorisation of ϕ

$\Longleftrightarrow$ Minimal subtraction scheme applied to ϕ.

4.5 Euler-Maclaurin Formula for Lattice Cones

As an example, let us revisit Euler-Maclaurin formula for lattice cones. For a discussion of the baby model of integrals on rooted trees, see [18].

For a lattice cone (C, Λ_C), let us view the exponential sum

$$S(C, \Lambda_C)(z) := \sum_{x \in C^o \cap \Lambda_C} e^{\langle z, x\rangle}$$

introduced in (34) as the regularisation of the number of lattice points in a lattice cone and the corresponding integral

$$I(C, \Lambda_C)(z) := \int_C e^{\langle z, x\rangle} d\Lambda_x$$

introduced in (35) as the regularisation of the volume of the cone. They map cones to meromorphic functions with linear poles, and by Proposition 3.35, they give rise to locality algebra homomorphisms $S, I : \left(\mathbb{Q}, \mathcal{C}, \perp^Q\right) \to \left(\mathcal{M}_\mathbb{Q}, \perp^Q\right)$ for the locality relations $\perp^Q$ on $\mathcal{C}$ as described in Definition 3.32, and on $\mathcal{M}_\mathbb{Q}$ as in Definition 3.33.

Berline and Vergne [2] showed the existence and uniqueness of a map

$$\mu : \mathbb{Q}\mathcal{C} \to \mathcal{M}_\mathbb{Q}$$

which is regular at zero and which interpolates the meromorphic germs $z \mapsto S(C, \Lambda_C)(z)$ given by the sum and those $z \mapsto I(C, \Lambda_C)(z)$ corresponding to the integral on the lattice cone (C, Λ_C):

$$S(\Lambda_C) = \sum_{(F,\Lambda_F)\preceq(C,\Lambda_C)} \mu(t(C, F), \Lambda_{t(C,F)})\, I(F, \Lambda_F), \tag{51}$$

where we have used the notations of (47). On the one dimensional cone, this amounts to an Euler-Maclaurin formula.

Example 4.17 For the one dimensional cone $C = [0, \infty[$ and lattice $\Lambda = \mathbb{Z}$, for $\mathrm{Re}\,(z) < 0$ we have

$$S(C, \Lambda)(z) = \sum_{n=1}^{\infty} e^{n\,z} = \frac{e^z}{1 - e^z}$$

and

$$I(C)(z) = \int_0^\infty e^{x\,z}\, dx = -\frac{1}{z}.$$

Eqn. (51) reduces to the ordinary Euler-Maclaurin formula

$$\sum_{n=0}^{\infty} f(n) = \int_0^\infty f(x)\, dx + \text{ remainder term.}$$

It relates the sum and the integral of the function $x \mapsto f(x) = e^{x\,z}$, by means of the following identity of meromorphic germs at zero:

$$S(C, \Lambda)(z) = \mu(z) + I(C)(z),$$

$$\text{with} \quad z \mapsto \mu(z) := \frac{e^z}{1 - e^z} + \frac{1}{z} \quad \text{holomorphic at zero.}$$

Formula (51) can therefore be viewed as a generalised Euler-Maclaurin formula on lattice cones, in that it relates a discrete sum with the corresponding integral of the exponential generating function on lattice cones.

In [35, Corollary 4.5 and Theorem 4.7], the Euler-Maclaurin formula (51) was interpreted as a locality algebraic Birkhoff factorisation formula of the locality morphism

$$S : \left(\mathbb{Q}\mathcal{C}, \perp^Q\right) \to \left(\mathcal{M}_\mathbb{Q}, \perp^Q\right),$$

namely

$$S = \mu \star^Q I, \tag{52}$$

where $\phi \star^Q \psi := \sum_{(F,\Lambda_F)\preceq(C,\Lambda_C)} \phi(t^Q(C,F), \Lambda_{t(C,F)})\, \psi(F, \Lambda_F)$ is the convolution product induced by the coproduct Δ^Q.

In the context of Theorem 4.16, μ is the renormalisation of S. This can be summarised in the following equivalence:

Locality algebraic Birkhoff Factorisation of S

$\Longleftrightarrow$ Euler Maclaurin formula for the exponential map $x \mapsto e^{\langle z,x\rangle}$.

5 Appendices

5.1 Poincaré Group

We recall the definition of the Poincaré and proper orthochronous Lorentz group, referring the reader to e.g. [65] for further details.

Relativistic invariance of classical field theory on M is described by the Poincaré group $P(1,d-1)$ of affine isometries of $\mathbb{R}^{1,d-1}$ for the metric g. It corresponds to the semi-direct product $\mathbb{R}^{1,d-1} \rtimes O(1,d-1)$ of the group $\mathbb{R}^{1,d-1}$ of translations with the Lorentz group $O(1,d-1)$ consisting of linear isometries of $\mathbb{R}^{1,d-1}$ for the metric g. Here, the product is given by $(v,\Lambda)\cdot(v',\Lambda') = (v+\Lambda\cdot v', \Lambda\cdot\Lambda')$, for any vectors v, v' in $\mathbb{R}^{1,d-1}$ and any Λ, Λ' in $O(1,d-1)$. Note that the Poincaré group acts continuously on the space $\mathcal{S}(\mathbb{R}^d)$ of Schwartz functions from the left by $p\cdot f(x) = f(p^{-1}x)$ for any x in M and p in $P(1,d-1)$.

The identity component of the Lorentz group called the **proper orthochronous Lorentz group**, is the subgroup of $O(1,d)$ defined by $SO^+(1,d) = O^+(1,d-1) \cap SO(1,d-1)$ consisting of all matrices Λ in $O(1,d-1)$ with determinant one and such that $\Lambda_{00} > 0$, which amounts to Λ mapping $u = (1,0,\ldots,0)$ to $(a,0,\ldots,0)$ with $a > 0$. Note that these preserve the future cone, mapping time like vectors to time like vectors, hence the name orthochronous.

In the bulk of the paper we use the notation $\hat{P}(1,d-1) := \mathbb{R}^{1,d-1} \rtimes \widetilde{SO^+(1,d-1)}$, where $\widetilde{SO^+(1,d-1)}$ stands for the simply connected universal cover of $SO^+(1,d-1)$, which is a simply connected group with the same Lie algebra as $O(1,d-1)$.

5.2 Causal Sets and Causality Relations

As observed in Sects. 2 and 3, locality in physics builds on causality. Here we give a summary of the various notions of causality in the framework of causal sets and indicate their compatibility.

Causality has been defined in terms of certain ordered sets in several approaches to quantum gravity and quantum field theory, from the school of Sorkin on causal sets (causets) and causal set theory (CST) [8] to Borcherds' notion of causality relations [9]. In [54], a locality relation is defined to be the symmetrising intersection of a certain (a)causality relation. This agrees with locality property of quantum field theories in the sense that operators with spacelike-separated supports commute, as studied in [9].

There are different conventions on causality, even on the notion of causal set. Their consistency comes from the following notions and properties of partial order sets.

For a set X, a relation $\prec$ on X is called

(a) a **partial order** if $\prec$ is reflexive, skewsymmetric and transitive;
(b) a **strict partial order** if $\prec$ is irreflexive, asymmetric and transitive.

It is easily seen that a transitive relation is irreflexive if and only if it is asymmetric.

We now put together various notions of causal sets, before clarifying the relationship among them.

(a) The original definition of a causal set is from [8] (also [10, 56]): a **causal set** is set with a relation $\prec$ that is strictly partial and locally finite (in the sense that for any $a \prec b$, the interval $[a, b]$ is finite). The notions in [41] and [61], even if in different terminologies, agree with [8].
(b) The references [23, 25, 61] use instead the conditions transitivity, non-circularity ($a \prec b$ and $b \prec a$ imply $a = b$, but this is the usual antisymmetry), and local finiteness. In [23] it is still called acyclicity, with the notation $\preceq$.
(c) In more recent literature [52, 68, 70, 71], a causal set is simply taken to be a set with a locally finite partial order.
(d) [9] gives another notion of causality relation, defined to be a closed (as a subset of $X \times X$ for a manifold X [43]), transitive and reflexive relation $\preceq$. Two elements are called **spacelike** if they are not comparable in the sense that neither $x \preceq y$ nor $y \preceq x$ holds.
(e) [54] gives still another notion of causality relation, to be a relation that is irreflexive and not symmetric.

The consistency of the notions of causal sets in (a) – (c) is secured by the following fact [69, 70].

Proposition 5.1 *Let X be a set. Let* $\mathrm{PO}(X)$ *be the set of partial orders on X and let* $\mathrm{SPO}(X)$ *be the set of strict partial orders on X. Let $\Delta = \{(x, x) \,|\, x \in X\} \subseteq X \times X$ denote the diagonal. There are mutually invertible maps*

$$\phi : \mathrm{PO}(X) \to \mathrm{SPO}(X), \quad \prec \mapsto \lhd := \lhd_{\prec} := \prec \setminus \Delta, \tag{53}$$

and

$$\psi : \mathrm{SPO}(X) \to \mathrm{PO}(X), \quad \lhd \mapsto \prec := \prec_{\lhd} = \lhd \cup \Delta. \tag{54}$$

Thus the two sets $\mathrm{PO}(X)$ *and* $\mathrm{SPO}(X)$ *are in bijection with each other.*

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