Liu et al. *Boundary Value Problems* (2023) 2023:28
https://doi.org/10.1186/s13661-023-01716-3

Boundary Value Problems
a SpringerOpen Journal

RESEARCH Open Access

A necessary and sufficient condition of blow-up for a nonlinear equation

Miao Liu[1], Jiangang Tang[2] and Shaoyong Lai[3*]

*Correspondence:
Laishaoy@swufe.edu.cn
[3]The School of Math., Southwestern University of Finance and Economics, Wenjiang, 611130, Chengdu, China
Full list of author information is available at the end of the article

Abstract
We investigate a nonlinear equation with quadratic nonlinearities, including a nonlinear model in Silva and Freire (J. Differ. Equ. 320:371–398, 2022). Using the classical energy estimate methods, we give a necessary and sufficient condition of blow-up of solutions to nonlinear equations. We answer a problem pointed out by Silva and Freire (J. Differ. Equ. 320:371–398, 2022).

MSC: 35G25; 35L05

Keywords: Local strong solutions; Nonlinear equation; Blow-up; Sufficient and necessary conditions

1 Introduction

Silva and Freire [1] investigated in detail the following equation:

$$W_t - W_{txx} = -WW_x + WW_{xxx}, \quad (t,x) \in \mathbb{R}_+ \times \mathbb{R}, \tag{1.1}$$

for which they considered continuation and persistence of solutions and necessary conditions for blow-up of a solution.

Equation (1.1) is related to the equation

$$W_t - W_{txx} + aW^k W_x = bW^{k-1} W_x W_{xx} + cW^k W_{xxx}, \tag{1.2}$$

where constants a, b, c satisfy $(ab, ac) \neq (0,0)$, and $k \neq 0$ (see [2]). Under certain restrictions on the parameters a, b, c, and k, the conserved currents, peakon solutions, and point symmetries are discussed in [2–4]. Obviously, when $a = 3, b = 2, c = 1$, and $k = 1$, Eq. (1.2) reduces to the standard Camassa–Holm equation [5]. If $a = 4, b = 3, c = 1$, and $k = 1$, then Eq. (1.2) becomes the Degasperis–Procesi model [6]. When $a = 4, b = 3, c = 1$, and $k = 2$, Eq. (1.2) reduces to the Novikov equation [7]. For $a = b + c$, $b \in \mathbb{R}$, $c \neq 0$, and $k > 0$, if the initial value belongs to a suitable Besov space, the well-posedness of short-time solutions for Eq. (1.2) is investigated in [8]. Under certain restrictions on the constants a, b, c, k, the global well-posedness for Eq. (1.2) is also established in Yan [8]. For real b, $c = 1$, and $a = b + 1$, the traveling wave solutions, the persistence properties, and unique continuation to Eq. (1.2) are considered by Guo et al. [9, 10] and Himonas and Thompson [11, 12].

Under different assumptions on the parameters a, b, c, k and the initial data, many useful dynamical properties for Eq. (1.2) can be found in [13–17].

We consider the following initial value problem:

$$\begin{cases} W_t - W_{txx} = -mWW_x + WW_{xxx}, \\ W(0,x) = W_0(x), \end{cases} \tag{1.3}$$

where the constant $m \in (-\infty, \infty)$. If $m = 1$, then the first equation in (1.3) becomes Eq. (1.1).

For problem (1.3) with $m = 1$, Silva and Freire [1] pointed out the following conjecture.

Conjecture *Let $m = 1, s > \frac{3}{2}$, $W_0(x) \in H^s(\mathbb{R})$, and lifespan $T > 0$. Then the solution $W(t,x)$ of problem (1.3) blows up at finite time if and only if*

$$\lim_{t\to T} \|W_x(t,\cdot)\|_{L^\infty} = \infty. \tag{1.4}$$

The conjecture is presented on p. 396 in [1]. We will derive several estimates from problem (1.3) itself. Using the obtained estimates, we obtain two results: (1) If $W_0(x) \in H^s(\mathbb{R}), s > \frac{3}{2}$, and the solution of problem (1.3) blows up, then $\int_0^T |W_x(t,x)|\,dx = \infty$, where T is the lifespan of $W(t,x)$ (2) If $W_0(x) \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, then $\lim_{t\to T} \|W(t,\cdot)\|_{H^s} = \infty$ if and only if (1.4) holds. Our Theorem 3.2 demonstrates that the conjecture is right for any constant $m \in (-\infty, \infty)$.

In Sect. 2, we present several lemmas, and in Sect. 3, we provide our main results and their proofs.

2 Several lemmas

Set $\Lambda^2 = 1 - \partial_x^2$. Then $\partial_x^2 = 1 - \Lambda^2$ and $\Lambda^{-2} = (1 - \partial_x^2)^{-1}$, and we have

$$\begin{aligned} W_t &= \Lambda^{-2}(WW_{xxx}) - m\Lambda^{-2}(WW_x) \\ &= \Lambda^{-2}\big((WW_{xx})_x - W_xW_{xx}\big) - m\Lambda^{-2}(WW_x) \\ &= \Lambda^{-2}\big(\big((WW_x)_x - W_x^2\big)_x - W_xW_{xx}\big) - m\Lambda^{-2}(WW_x) \\ &= \Lambda^{-2}\big((WW_x)_{xx} - 3W_xW_{xx}\big) - m\Lambda^{-2}(WW_x) \\ &= \Lambda^{-2}\big(1 - \Lambda^2\big)(WW_x) - 3\Lambda^{-2}(W_xW_{xx}) - m\Lambda^{-2}(WW_x) \\ &= -WW_x - 3\Lambda^{-2}(W_xW_{xx}) + \frac{1-m}{2}\Lambda^{-2}\big(W^2\big)_x. \end{aligned}$$

Thus problem (1.3) becomes

$$\begin{cases} W_t + WW_x = -3\Lambda^{-2}(W_xW_{xx}) + \frac{1-m}{2}\Lambda^{-2}(W^2)_x, \\ W(0,x) = W_0(x). \end{cases} \tag{2.1}$$

Lemma 2.1 *Let $W_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$. Then there is $T = T(W_0) > 0$ such that problem (2.1) has a unique solution $W(t,x)$, and*

$$W \in C\big([0,T); H^s(\mathbb{R})\big) \cap C^1\big([0,T); H^{s-1}(\mathbb{R})\big).$$

Using the Kato theorem [18], we can prove the well-posedness of local solutions for problem (2.1). In fact, the proof of well-posedness of a short-time solution for problem (2.1) is very similar to those of the famous Camassa–Holm and Degasperis–Procesi models (see [11, 15, 16]). Here we omit its proof.

Lemma 2.2 *Suppose that* $s \geq 3$ *and* $W(t,x) \in H^s(\mathbb{R})$. *Then*

$$\int_{\mathbb{R}} W W_x W_{xx}\,dx = -\frac{1}{2}\int_{\mathbb{R}} W_x^3\,dx, \tag{2.2}$$

$$\int_{\mathbb{R}} W W_{xx} W_{xxx}\,dx = -\frac{1}{2}\int_{\mathbb{R}} W_x W_{xx}^2\,dx. \tag{2.3}$$

Proof Since[1]

$$\begin{aligned}\int_{\mathbb{R}} W W_x W_{xx}\,dx &= \int_{\mathbb{R}} W W_x\,dW_x \\ &= \left(W W_x^2\right)\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} W_x\left(W_x^2 + W W_{xx}\right)dx, \\ &= -\int_{\mathbb{R}} W_x\left(W_x^2 + W W_{xx}\right)dx,\end{aligned}$$

we get (2.2). Similarly, we have

$$\begin{aligned}\int_{\mathbb{R}} W W_{xx} W_{xxx}\,dx &= \int_{\mathbb{R}} W W_{xx}\,dW_{xx} \\ &= \left(W W_{xx}^2\right)\Big|_{-\infty}^{\infty} - \int_{\mathbb{R}} W_{xx}(W_x W_{xx} + W W_{xxx})\,dx, \\ &= -\int_{\mathbb{R}} W_{xx}(W_x W_{xx} + W W_{xxx})\,dx,\end{aligned}$$

which leads to (2.3). □

Lemma 2.3 *Let* $W_0(x) \in H^s(\mathbb{R})$ $(s > \frac{3}{2})$. *Then*

$$\int_{\mathbb{R}} \Lambda^{-2}\left(W^2\right)dx = \int_{\mathbb{R}} W^2\,dx, \qquad \int_{\mathbb{R}} \Lambda^{-2}\left(W_x^2\right)dx = \int_{\mathbb{R}} W_x^2\,dx. \tag{2.4}$$

Proof We only need to prove the first identity in (2.4). Since

$$\Lambda^{-2} W^2 = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-\eta|} W^2(t,\eta)\,d\eta \geq 0$$

and

$$\int_{\mathbb{R}} e^{-|x-\eta|}\,d\eta = 2,$$

[1] For any $f \in L^r(\mathbb{R})$ with $1 \leq r \leq \infty$, we have $\Lambda^{-2} f(x) = \frac{1}{2}\int_{\mathbb{R}} e^{-|x-\eta|} f(\eta)\,d\eta$ (see Constantin and Escher [14]). If a function $g \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, then $g(\pm\infty) = g'(\pm\infty) = g''(\pm\infty) = g^{[s]}(\pm\infty) = 0$, where $[s]$ denotes the integer part of s (see [18]).

by the Tonelli theorem we get

$$\begin{aligned}\int_{\mathbb{R}} \Lambda^{-2}\big(W^2\big)\,dx &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}} e^{-|x-\eta|} W^2(t,\eta)\,d\eta\,dx \\ &= \frac{1}{2}\int_{\mathbb{R}} W^2(t,\eta)\,d\eta \int_{\mathbb{R}} e^{-|x-\eta|}\,dx \\ &= \int_{\mathbb{R}} W^2(t,\eta)\,d\eta,\end{aligned}$$

which finishes the proof. □

Lemma 2.4 ([19]) *If $r \geq 0$ and $f_1, f_2 \in H^r \cap L^\infty$, then*

$$\|f_1 f_2\|_r \leq c\big(\|f_1\|_{L^\infty}\|f_2\|_r + \|f_1\|_r\|f_2\|_{L^\infty}\big),$$

where the constant $c > 0$ depends only on r.

Lemma 2.5 ([19]) *Let $f_1 \in H^r \cap W^{1,\infty}$ $(r > 0)$ and $f_2 \in H^{r-1} \cap L^\infty$. Then*

$$\big\|\big[\Lambda^r, f_1\big] f_2\big\|_{L^2} \leq c\big(\|\partial_x f_1\|_{L^\infty}\big\|\Lambda^{r-1} f_2\big\|_{L^2} + \big\|\Lambda^r f_1\big\|_{L^2}\|f_2\|_{L^\infty}\big),$$

where $[\Lambda^r, f_1] = \Lambda^r f_1 - f_1\Lambda^r$, and the constant $c > 0$ depends only on r.

Remark 1 Using the arguments in [8, 15], the lifespan T in Lemma 2.1 does not depend on the Sobolev index $s > \frac{3}{2}$. Namely, for arbitrary $s_1 > s > \frac{3}{2}$ or $s > s_1 > \frac{3}{2}$, the maximal existence time for $\|W\|_{H^s}$ and $\|W\|_{H^{s_1}}$ is the same.

3 Main results

Theorem 3.1 *Let $W_0 \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, and suppose W satisfies problem (1.3) or problem (2.1). If the lifespan T of W is finite and*

$$\lim_{t\to T}\big\|W(t,\cdot)\big\|_{H^s} = \infty, \tag{3.1}$$

then

$$\int_0^T \big\|W_x(\tau,\cdot)\big\|_{L^\infty}\,d\tau = \infty. \tag{3.2}$$

Proof If $s > \frac{3}{2}$, then using the operator $\Lambda^s W \Lambda^s$, from problem (2.1) we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\big(\Lambda^s W\big)^2\,dx \\ &\quad= \int_{\mathbb{R}}\big(\Lambda^s W\big)\Lambda^s W_t\,dx \\ &\quad= \int_{\mathbb{R}}\big(\Lambda^s W\big)\Lambda^s\bigg(-WW_x - \frac{3}{2}\Lambda^{-2}\partial_x\big(W_x^2\big) + \frac{1-m}{2}\Lambda^{-2}\big(W^2\big)_x\bigg)\,dx,\end{aligned}$$

which leads to

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\left(\Lambda^s W\right)^2 dx \\
&\quad\le \left|\int_{\mathbb{R}}\left(\Lambda^s W\right)\Lambda^s(WW_x)\,dx\right| + \frac{|m-1|}{2}\left|\int_{\mathbb{R}}\left(\Lambda^s W\right)\Lambda^{s-2}\left(W^2\right)_x dx\right| \\
&\qquad + \frac{3}{2}\left|\int_{\mathbb{R}}\Lambda^s W\Lambda^{s-2}\partial_x\left(W_x^2\right)dx\right| \\
&\quad= G_1 + G_2 + G_3.
\end{aligned} \tag{3.3}$$

In fact, we have

$$\begin{aligned}
\int_{\mathbb{R}} W\Lambda^s W\Lambda^s W_x\,dx &= \int_{\mathbb{R}} W\Lambda^s W\,d\left(\Lambda^s W\right) \\
&= -\int_{\mathbb{R}}\Lambda^s W\left(W_x\Lambda^s W + W\Lambda^s W_x\right)dx,
\end{aligned}$$

from which we obtain

$$\int_{\mathbb{R}} W\Lambda^s W\Lambda^s W_x\,dx = -\frac{1}{2}\int_{\mathbb{R}} W_x\Lambda^s W\Lambda^s W\,dx. \tag{3.4}$$

Employing the Cauchy–Schwarz inequality, (3.4), and Lemma 2.5, we acquire

$$\begin{aligned}
\left|\int_{\mathbb{R}}\left(\Lambda^s W\right)\Lambda^s(WW_x)\,dx\right| &= \left|\int_{\mathbb{R}}\left(\Lambda^s W\right)\left(\Lambda^s(WW_x) - W\Lambda^s W_x\right)dx\right. \\
&\quad \left. + \int_{\mathbb{R}}\left(\Lambda^s W\right)W\Lambda^s W_x\,dx\right| \\
&\le \left|\int_{\mathbb{R}}\left(\Lambda^s W\right)\left(\Lambda^s(WW_x) - W\Lambda^s W_x\right)dx\right| \\
&\quad + \left|\int_{\mathbb{R}}\left(\Lambda^s W\right)W\Lambda^s W_x\,dx\right| \\
&\le c\|W\|_{H^s}\left(\|W\|_{H^{s-1}}\|W_x\|_{L^\infty} + \|W\|_{H^s}\|W_x\|_{L^\infty}\right) \\
&\quad + \frac{1}{2}\|W_x\|_{L^\infty}\left\|\Lambda^s W\right\|_{L^2} \\
&\le c\|W_x\|_{L^\infty}\|W\|_{H^s}^2,
\end{aligned}$$

which leads to

$$G_1 \le c\|W_x\|_{L^\infty}\|W\|_{H^s}^2. \tag{3.5}$$

Similarly to the proof of (3.5), we have

$$\begin{aligned}
G_2 &\le \frac{|m-1|}{2}\left|\int_{\mathbb{R}}\left(\Lambda^{s-1}W\right)\Lambda^{s-1}\left(W^2\right)_x dx\right| \\
&\le c\left|\int_{\mathbb{R}}\left(\Lambda^{s-1}W\right)\Lambda^{s-1}(WW_x)\,dx\right|
\end{aligned}$$

$$\leq c\|W_x\|_{L^\infty}\|W\|_{H^{s-1}}^2$$
$$\leq c\|W_x\|_{L^\infty}\|W\|_{H^s}^2. \tag{3.6}$$

Now Lemma 2.4 yields

$$\begin{aligned} G_3 &\leq \left\|\Lambda^s W\right\|_{L^2}\left\|\Lambda^{s-2}\partial_x\left(W_x^2\right)\right\|_{L^2} \\ &\leq c\left\|\Lambda^s W\right\|_{L^2}\left\|W_x^2\right\|_{H^{s-1}} \\ &\leq c\left\|\Lambda^s W\right\|_{L^2}\|W_x\|_{L^\infty}\|W_x\|_{H^{s-1}} \\ &\leq c\|W_x\|_{L^\infty}\|W\|_{H^s}^2. \end{aligned} \tag{3.7}$$

Using inequalities (3.3), (3.5),(3.6), and (3.7) results in

$$\frac{1}{2}\frac{d}{dt}\int_{-\infty}^{\infty}\left(\Lambda^s W\right)^2 dx \leq c\|W_x\|_{L^\infty}\left\|\Lambda^s W\right\|_{L^2}^2, \tag{3.8}$$

where $c > 0$ is a constant. Using (3.8) yields

$$\|W\|_{H^s} \leq \|W_0\|_{H^s} e^{c\int_0^t \|W_x\|_{L^\infty}\,d\tau}. \tag{3.9}$$

Suppose that $\lim_{t\to T}\|W\|_{H^s} = \infty$. From (3.9) we have

$$\int_0^T \|W_x\|_{L^\infty}\,d\tau = \infty,$$

which ends the proof. □

Theorem 3.2 *Let $W_0(x) \in H^s(\mathbb{R})$ with $s > \frac{3}{2}$, and let T be the lifespan of solution $W(t,x)$ for problem (2.1). If T is finite, then*

$$\lim_{t\to T}\left\|W(t,\cdot)\right\|_{H^s(\mathbb{R})} = \infty \tag{3.10}$$

if and only if

$$\lim_{t\to T}\left\|W_x(t,\cdot)\right\|_{L^\infty(\mathbb{R})} = \infty. \tag{3.11}$$

Proof Let (3.10) hold. We will derive that (3.11) holds. Using Remark 1 and choosing $s = 3$, Lemma 2.1 ensures that there exists $W(t,x) \in C([0,T),H^3(\mathbb{R}))\cap C^1([0,T),H^2(\mathbb{R}))$. We will employ the classical energy estimates. From problem (2.1) we acquire

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} W^2\,dx &= \int_{\mathbb{R}} WW_t\,dx \\ &= \int_{\mathbb{R}} W\left(-WW_x - 3\Lambda^{-2}(W_xW_{xx})\right)dx + \frac{1-m}{2}\int_{\mathbb{R}} W\Lambda^{-2}\left(W^2\right)_x dx \\ &= -3\int_{\mathbb{R}} W\Lambda^{-2}(W_xW_{xx})\,dx + \frac{1-m}{2}\int_{\mathbb{R}} W\Lambda^{-2}\left(W^2\right)_x dx \\ &= -\frac{3}{2}\int_{\mathbb{R}} W\Lambda^{-2}\left(W_x^2\right)_x dx + \frac{1-m}{2}\int_{\mathbb{R}} W\Lambda^{-2}\left(W^2\right)_x dx \end{aligned}$$

$$= \frac{3}{2}\int_{\mathbb{R}} W_x \Lambda^{-2}\big(W_x^2\big)\,dx - \frac{1-m}{2}\int_{\mathbb{R}} W_x \Lambda^{-2}\big(W^2\big)\,dx. \tag{3.12}$$

Applying the first equation in (2.1) yields

$$\begin{aligned}
W_{tx} &= -W_x^2 - WW_{xx} - \frac{3}{2}\Lambda^{-2}\big(W_x^2\big)_{xx} + \frac{1-m}{2}\Lambda^{-2}\big(W^2\big)_{xx} \\
&= -W_x^2 - WW_{xx} - \frac{3}{2}\Lambda^{-2}\big(1-\Lambda^2\big)\big(W_x^2\big) \\
&\quad + \frac{1-m}{2}\Lambda^{-2}\big(1-\Lambda^2\big)\big(W^2\big) \\
&= -W_x^2 - WW_{xx} - \frac{3}{2}\Lambda^{-2}W_x^2 + \frac{3}{2}W_x^2 \\
&\quad + \frac{1-m}{2}\Lambda^{-2}\big(W^2\big) - \frac{1-m}{2}W^2 \\
&= \frac{1}{2}W_x^2 - WW_{xx} - \frac{1-m}{2}W^2 - \frac{3}{2}\Lambda^{-2}W_x^2 + \frac{1-m}{2}\Lambda^{-2}\big(W^2\big).
\end{aligned} \tag{3.13}$$

Using Lemma 2.2 and (3.13), we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} W_x^2\,dx &= \int_{\mathbb{R}} W_x\bigg(\frac{1}{2}W_x^2 - WW_{xx} - \frac{1-m}{2}W^2 - \frac{3}{2}\Lambda^{-2}W_x^2 \\
&\quad + \frac{1-m}{2}\Lambda^{-2}W^2\bigg)\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}} W_x^3\,dx - \int_{\mathbb{R}} WW_xW_{xx}\,dx - \frac{3}{2}\int_{\mathbb{R}} W_x\Lambda^{-2}W_x^2\,dx \\
&\quad + \frac{1-m}{2}\int_{\mathbb{R}} W_x\Lambda^{-2}W^2\,dx \\
&= \int_{\mathbb{R}} W_x^3\,dx - \frac{3}{2}\int_{\mathbb{R}} W_x\Lambda^{-2}W_x^2\,dx + \frac{1-m}{2}\int_{\mathbb{R}} W_x\Lambda^{-2}W^2\,dx.
\end{aligned} \tag{3.14}$$

Using (3.13) gives rise to

$$\begin{aligned}
W_{txx} &= W_xW_{xx} - W_xW_{xx} - WW_{xxx} - (1-m)WW_x \\
&\quad - \frac{3}{2}\Lambda^{-2}\big(W_x^2\big)_x + \frac{1-m}{2}\Lambda^{-2}\big(W^2\big)_x \\
&= -WW_{xxx} - (1-m)WW_x - \frac{3}{2}\Lambda^{-2}\big(W_x^2\big)_x + \frac{1-m}{2}\Lambda^{-2}\big(W^2\big)_x.
\end{aligned} \tag{3.15}$$

Applying integration by parts, (3.15), and Lemma 2.2, we have

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}} W_{xx}^2\,dx \\
&= -\int_{\mathbb{R}} WW_{xx}W_{xxx}\,dx - (1-m)\int_{\mathbb{R}} WW_xW_{xx}\,dx \\
&\quad - \frac{3}{2}\int_{\mathbb{R}} W_{xx}\Lambda^{-2}\big(W_x^2\big)_x\,dx + \frac{1-m}{2}\int_{\mathbb{R}} W_{xx}\Lambda^{-2}\big(W^2\big)_x\,dx \\
&= \frac{1}{2}\int_{\mathbb{R}} W_xW_{xx}^2\,dx + \frac{1-m}{2}\int_{\mathbb{R}} W_x^3\,dx
\end{aligned}$$

$$
\begin{aligned}
&-\frac{3}{2}\int_{\mathbb{R}} W\Lambda^{-2}\big(W_x^2\big)_{xxx}\,dx-\frac{1-m}{2}\int_{\mathbb{R}} W_x\Lambda^{-2}\big(W^2\big)_{xx}\,dx\\
&=\frac{1}{2}\int_{\mathbb{R}} W_xW_{xx}^2\,dx+\frac{1-m}{2}\int_{\mathbb{R}} W_x^3\,dx-\frac{3}{2}\int_{\mathbb{R}} W\Lambda^{-2}\big(1-\Lambda^2\big)\big(W_x^2\big)_x\,dx\\
&\quad-\frac{1-m}{2}\int_{\mathbb{R}} W_x\Lambda^{-2}\big(1-\Lambda^2\big)\big(W^2\big)\,dx\\
&=\frac{1}{2}\int_{\mathbb{R}} W_xW_{xx}^2\,dx+\frac{1-m}{2}\int_{\mathbb{R}} W_x^3\,dx+3\int_{\mathbb{R}} WW_xW_{xx}\,dx\\
&\quad-\frac{3}{2}\int_{\mathbb{R}} W\Lambda^{-2}\big(W_x^2\big)_x\,dx-\frac{1-m}{2}\int_{\mathbb{R}} W_x\Lambda^{-2}W^2\,dx\\
&=\frac{1}{2}\int_{\mathbb{R}} W_xW_{xx}^2\,dx-\frac{m+2}{2}\int_{\mathbb{R}} W_x^3\,dx\\
&\quad+\frac{3}{2}\int_{\mathbb{R}} W_x\Lambda^{-2}\big(W_x^2\big)\,dx-\frac{1-m}{2}\int_{\mathbb{R}} W_x\Lambda^{-2}W^2\,dx.
\end{aligned}
\tag{3.16}
$$

Using (3.12), (3.14), and (3.16), we have

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\int_{\mathbb{R}}\big(W^2+W_x^2+W_{xx}^2\big)\,dx\\
&\quad=-\frac{m}{2}\int_{\mathbb{R}} W_x^3\,dx+\frac{1}{2}\int_{\mathbb{R}} W_xW_{xx}^2\,dx\\
&\qquad+\frac{m-1}{2}\int_{\mathbb{R}} W_x\Lambda^{-2}W^2\,dx+\frac{3}{2}\int_{\mathbb{R}} W_x\Lambda^{-2}\big(W_x^2\big)\,dx.
\end{aligned}
\tag{3.17}
$$

If (3.10) holds, then suppose that we can choose a positive constant M satisfying

$$
\big|W_x(t,x)\big|<M,\quad t\in[0,T),x\in\mathbb{R}.
\tag{3.18}
$$

Employing (3.17), (3.18), Lemma 2.3, $\Lambda^{-2}(W^2)\geq 0$, and $\Lambda^{-2}(W_x)^2\geq 0$, we have

$$
\begin{aligned}
&\frac{1}{2}\left[\frac{d}{dt}\int_{\mathbb{R}}\big(W^2+W_x^2+W_{xx}^2\big)\,dx\right]\\
&\quad<\frac{M|m|}{2}\int_{\mathbb{R}} W_x^2\,dx+\frac{M}{2}\int_{\mathbb{R}} W_{xx}^2\,dx+\frac{|m-1|M}{2}\int_{\mathbb{R}} W^2\,dx+\frac{3M}{2}\int_{\mathbb{R}} W_x^2\,dx\\
&\quad<\max\left\{\frac{M|m|}{2},\frac{3M}{2},\frac{|m-1|M}{2}\right\}\int_{\mathbb{R}}\big(W^2+W_x^2+W_{xx}^2\big)\,dx.
\end{aligned}
\tag{3.19}
$$

Let

$$
H(t)=\int_{\mathbb{R}}\big(W^2+W_x^2+W_{xx}^2\big)\,dx,\qquad K=\max\big\{M|m|,3M,|m-1|M\big\}.
$$

From (3.19) we obtain

$$
H(t)\leq H(0)+K\int_0^t H(\tau)\,d\tau,
$$

which, together with the Gronwall inequality, yields

$$
H(t)\leq H(0)e^{Kt}.
\tag{3.20}
$$

From (3.20) we obtain $W(t,x) \in H^2(\mathbb{R})$, which, combined with Remark 1, is a contradiction to (3.10). Therefore we conclude that assumption (3.18) is not right.

Conversely, using $\|W_x\|_{L^\infty} < c\|W\|_{H^s}$, if

$$\lim_{t\to T}\big\|W_x(t,\cdot)\big\|_{L^\infty(\mathbb{R})} = \infty,$$

then we derive that

$$\lim_{t\to T}\big\|W(t,\cdot)\big\|_{H^s} = \infty.$$

The proof is completed. □

Funding
This work is supported by the National Natural Science Foundation of China (No. 11471263).

Availability of data and materials
Not applicable.

Declarations

Competing interests
The authors declare no competing interests.

Author contributions
All authors contributed equally in this work. All authors read and approved the final manuscript.

Author details
[1]The School of Math. and Statis., Yili Normal University, 835000, Yining, China. [2]Institute of Appl. Math., Yili Normal Uiversity, 835000, Yining, China. [3]The School of Math., Southwestern University of Finance and Economics, Wenjiang, 611130, Chengdu, China.

Publisher's Note

Received: 10 September 2022 Accepted: 10 March 2023 Published online: 21 March 2023

References

1. Silva, P.L., Freire, I.L.: Existence, persistence, and continuation of solutions for a generalized 0-Holm–Staley equation. J. Differ. Equ. **320**, 371–398 (2022)
2. Anco, S.C., Silva, P.L., Freire, I.L.: A family of wave-breaking equations generalizing the Camassa–Holm and Novikov equations. J. Math. Phys. **56**, 091506 (2015)
3. Freire, I.L.: A look on some results about Camassa–Holm type equations. Commun. Math. **29**, 115–130 (2021)
4. Freire, I.L.: Conserved quantities, continuation and compactly supported solutions of some shallow water models. J. Phys. A, Math. Theor. **54**, 015207 (2021)
5. Camassa, R., Holm, D.: An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. **71**, 1661–1664 (1993)
6. Degasperis, A., Procesi, M.: Asymptotic integrability. In: Degasperis, A., Gaeta, G. (eds.) Symmetry and Perturbation Theory, vol. 1, pp. 23–37. World Scientific, Singapore (1999)
7. Novikov, V.: Generalizations of the Camassa–Holm equation. J. Phys. A **42**, 342002 (2009)
8. Yan, K.: Wave-breaking and global existence for a family of peakon equations with high order nonlinearity. Nonlinear Anal., Real World Appl. **45**, 721–735 (2019)
9. Guo, Z.G., Li, K., Xu, C.: On generalized Camassa–Holm type equation with $(k + 1)$-degree nonlinearities. Z. Angew. Math. Mech. **98**, 1567–1573 (2018)
10. Guo, Z.G., Li, X.G., Yu, C.: Some properties of solutions to the Camassa–Holm-type equation with higher order nonlinearities. J. Nonlinear Sci. **28**, 1901–1914 (2018)
11. Himonas, A.A., Holliman, C.: The Cauchy problem for a generalized Camassa–Holm equation. Adv. Differ. Equ. **19**, 161–200 (2014)
12. Himonas, A.A., Holliman, C., Kenig, C.: Construction of 2-peakon solutions and ill-posedness for the Novikov equation. SIAM J. Math. Anal. **50**, 2968–3006 (2018)
13. Himonas, A.A., Misiolek, G., Ponce, G., Zhou, Y.: Persistence properties and unique continuation of solutions of the Camassa–Holm equation. Commun. Math. Phys. **271**, 511–522 (2007)
14. Constantin, A., Escher, J.: Wave breaking for nonlinear nonlocal shallow water equations. Acta Math. **181**, 229–243 (1998)

15. Yin, Z.Y.: On the Cauchy problem for an integrable equation with peakon solutions. Ill. J. Math. **47**, 649–666 (2003)
16. Zhou, Y.: On solutions to the Holm–Staley b-family of equations. Nonlinearity **23**, 369–381 (2010)
17. Ming, S., Lai, S.Y., Su, Y.Q.: Well-posedness and behaviors of solutions to an integrable evolution equation. Bound. Value Probl. **2020**, 165 (2020)
18. Kato, T.: Quasi-linear equations of evolution with applications to partial differential equations. In: Spectral Theory and Differential Equations. Lecture Notes in Math, vol. 448, pp. 25–70. Springer, Berlin (1975)
19. Kato, T., Ponce, G.: Commutator estimates and the Euler and Navier–Stokes equations. Commun. Pure Appl. Math. **41**, 891–907 (1998)