J. Pseudo-Differ. Oper. Appl. (2023) 14:26
https://doi.org/10.1007/s11868-023-00520-y

The discrete spectrum of the Neumann-Poincaré operator in 3D elasticity

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Received: 3 December 2022 / Revised: 28 February 2023 / Accepted: 5 March 2023

Abstract
For the Neumann-Poincaré (double layer potential) operator in the three-dimensional elasticity we establish asymptotic formulas for eigenvalues converging to the points of the essential spectrum and discuss geometric and mechanical meaning of coefficients in these formulas. In particular, we establish that for any body, there are infinitely many eigenvalues converging from above to each point of the essential spectrum. On the other hand, if there is a point where the boundary is concave (in particular, if the body contains cavities) then for each point of the essential spectrum there exists a sequence of eigenvalues converging to this point from below. The reasoning is based upon the representation of the Neumann-Poincaré operator as a zero order pseudodifferential operator on the boundary and the earlier results by the author on the eigenvalue asymptotics for polynomially compact pseudodifferential operators.

Keywords Eigenvalue asymptotics · Pseudodifferential operators · Neumann-Poincaré operator · 3D elasticity

Mathematics Subject Classification Primary 47A75; Secondary 58J50

1 Introduction

The paper is devoted to the study of the spectrum of the Neumann-Poincaré (NP) operator in the 3D linear elasticity. It is based upon results of the previous paper [34], where we considered general polynomially compact pseudodifferential operators and have derived formulas describing for such operators the behavior of eigenvalues converging to the points of the essential spectrum. The motivating example, the Neumann-Poincaré

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Published online: 21 March 2023

(the double layer potential) operator $\mathfrak{K}$ in 3D elasticity was presented, and a discussion of spectral properties of this operator has started. The present paper continues the study of the eigenvalues of the Neumann-Poincaré elasticity operator for a homogeneous and isotropic 3D body $\mathcal{D}$ with smooth boundary Γ on the base of results in [34]. It is known, since [5, 32], that this operator possesses three points of essential spectrum, namely, the zero point and two symmetrical ones, $\pm\Bbbk$, where $\Bbbk$ is expressed via the Lamé constants of the material of the body and does not depend on its geometry. There may also exist finite or infinite sequences of eigenvalues, converging (in the latter case) to the points of the essential spectrum. In this paper we find sufficient geometrical conditions for these sequences to be infinite (above or below a point of the essential spectrum), and if this is the case, we study asymptotic properties of these sequences, following the general results in [34].

In fact, up to now, very little was known about the discrete spectrum of the operator $\mathfrak{K}$ beyond the case of the sphere (where the spectrum has been, rather recently, calculated explicitly, see [16]). In the general case, some estimates for the rate of convergence of these eigenvalues have been found in [5].

For the case of the ball with radius R and Lamé constants λ, μ, the eigenvalues of the NP operator, calculated in [16], form three series,

$$\begin{aligned}
\Lambda_n^0(\mathfrak{K}) &= \frac{3}{2(2n+1)} \sim \frac{3}{4n},\ n\to\infty \\
\Lambda_n^-(\mathfrak{K}) &= \frac{3\lambda - 2\mu(2n^2-2n-3)}{2(\lambda+2\mu)(4n^2-1)} \sim -\Bbbk + \frac{4\Bbbk}{n},\ n\to\infty \\
\Lambda_n^+(\mathfrak{K}) &= \frac{-3\lambda + 2\mu(2n^2+2n-3)}{2(\lambda+2\mu)(4n^2-1)}, \sim \Bbbk + \frac{4\Bbbk}{n},\ n\to\infty,
\end{aligned} \tag{1.1}$$

$\Bbbk = \frac{\mu}{2(2\mu+\lambda)}$, each of $\Lambda_n^-(\mathfrak{K})$, $\Lambda_n^0(\mathfrak{K})$, $\Lambda_n^+(\mathfrak{K})$ being a multiple eigenvalue with multiplicity $2n+1$. Additionally, it is found that no eigenvalues coincide with the points of the essential spectrum. One can see that the eigenvalues $\Lambda_n^{\pm}(\mathfrak{K})$ (as well as even their asymptotics) depend on the material of the body. A possible dependence on the geometry is concealed here, due to the fact that the spectrum of $\mathfrak{K}$ is invariant under the homotheties of the body, so the only geometric characteristic of the sphere, its radius, is not present in the formulas for eigenvalues. One can also notice that all three series converge to their limit values from above only; there are no eigenvalues that approach these points from below. This latter property is important, in particular, in the analysis of the plasmon resonance in elastic metamaterials, see, e.g., [2, 4, 7, 9, 10, 28, 29], therefore, it is interesting to determine, to what extent these properties persist in a more general case.

Recently, the spectral problem has been studied for the NP operator in electrostatics, where the operator is compact. There, in dimension 3, for a smooth boundary, the eigenvalue asymptotics, a power-like one, was found in [30, 31], with somewhat weaker results for the case of a finite smoothness. In dimension 2, the rate of convergence of eigenvalues to zero depends on the smoothness of the boundary; it is at least polynomial for a finite smoothness, super-polynomial for an infinitely smooth

boundary, and (again, at least) exponential for an analytic boundary. Only upper estimates for eigenvalues are known. In the only case where the eigenvalue asymptotics is found, namely, for an ellipse, the asymptotics is exponential (see [8]).

The situation is similar for the elastic NP operator in dimension 2, where it has two points of the essential spectrum ([4, 7]); it was found that the rate of convergence of eigenvalues to these points depends again on the smoothness of the boundary. Namely, the estimates obtained in these papers show that for a finitely smooth boundary, the eigenvalues converge to their limit points at least polynomially fast, for an infinitely smooth boundary these eigenvalues converge super-polynomially fast, while for an analytic boundary they converge at least exponentially. An exact asymptotics of eigenvalues was never found, even for an ellipse.

In the present paper we consider a body $\mathcal{D} \subset R^3$ made of a homogeneous isotropic elastic material with Lamé constants λ, μ and bounded by a smooth compact surface Γ. The NP operator is a polynomially compact zero order pseudodifferential operator, with three points $\omega_\iota : \omega_{-1} = -\Bbbk, \omega_0 = 0, \omega_1 = \Bbbk$, in the essential spectrum, according to the results of [1, 32]. The asymptotics of eigenvalues of such operator tending to a point ω_ι, $\iota = -1, 0, 1$, of the essential spectrum, as found in [34], is determined by a certain pseudodifferential operator $\mathfrak{M}_\iota$ of order -1. The procedure for calculating the principal symbol $\mathfrak{m}_\iota$ of $\mathfrak{M}_\iota$ is quite intricate. Moreover, the calculation of the coefficients in the eigenvalue asymptotic formulas involves the eigenvalues of the latter symbol, a 3×3 symbolic matrix; it presents the so-called irreducible case of the cubic equation. Therefore the symbolic expression for integrals of powers of these eigenvalues in the general case would be completely unwieldy and, even if found, be of no use for further analysis.

In our approach, using the qualitative analysis of the NP operator as a singular integral operator, we are able first to separate the dependence of the principal symbol $\mathfrak{m}_\iota$ on the geometric characteristics of the surface and the dependence on the material of the body. This development is achieved thanks to determining the qualitative structure of the symbol $\mathfrak{m}_\iota$ as described in [34]. Namely, we establish that, although the expression of the symbol $\mathfrak{m}_\iota(x, \xi)$, $(x, \xi) \in \mathrm{T}^*(\Gamma)$, contains 25 additive terms, each being the product of 5 symbolic matrices, it is one and only one factor in each such product that depends on the geometry of Γ, namely, it is a linear combination of principal curvatures of Γ at the point $x \in \Gamma$, with (matrix) coefficients depending universally on ξ and on the Lamé constants. This leads to our structure result: the combination of these terms, the symbol $\mathfrak{m}_\iota(x, \xi)$, is a linear form of the principal curvatures with universal (depending only on the Lamé constants and ξ) coefficients.

Then a miraculous circumstance helps us. The Birman-Solomyak formula ([11]) for coefficients in the eigenvalue asymptotics for a negative order pseudodifferential operator involves the trace of a certain, generally fractional, power of the principal symbol or of its positive or negative parts, in other words, the sum of powers of (all or a part of) eigenvalues of the matrix symbol $\mathfrak{m}_\iota(x, \xi)$. Calculating this sum requires, generally, knowing the eigenvalues themselves, and this, for 3×3 matrices, cannot be achieved in the symbolic way, as was just discussed. The situation seems hopeless. However, there are exceptions. If the power is *an integer*, the sum of powers of all eigenvalues of a Hermitian matrix (or a matrix similar to a Hermitian one, as it happens in our case) can be found without knowing the eigenvalues themselves, rather only by

using polynomial operations with entries of the matrix. And, luckily, for the case of the NP operator on a two-dimensional surface, this power equals exactly 2. Therefore, the integrand in the Birman-Solomyak formula is the sum of squares of linear forms in principal curvatures, therefore, it is a *quadratic form* of these curvatures, again, with universal coefficients depending on the Lamé constants only.

This way of reasoning enables us to determine the two-sided asymptotics for the eigenvalues of the NP operator, to say it more exactly, the asymptotics for the sum of the counting functions for eigenvalues above and below ω_ι. The reasoning explained above leads also to the fact that the geometrical characteristics entering in these formulas are the Euler characteristic of the surface Γ and its Willmore energy $W(\Gamma)$ (see, e.g., [36] for a discussion of classical and recent problems and results concerning this latter quantity which indicates how the surface in $\mathbb{R}^3$ is bent.) Such kind of formulas is similar to the case of the NP operator in 3D electrostatics, derived in [30, 31].

An important question concerns the infiniteness of the sequences of eigenvalues tending to the points of the essential spectrum separately from above and from below. The pattern obtained for the symbol $\mathfrak{m}_\iota(x,\xi)$, symmetry considerations, and the above asymptotic formulas show that the sequences of eigenvalues converging to ω_ι *from above* are always infinite. We also find a sufficient condition in geometric terms for the *infiniteness* of the sequences of eigenvalues converging to ω_ι *from below:* this happens, in particular, for sure, if there is at least one point on Γ where the surface is concave or, more generally, where the mean curvature in a special co-ordinates system is positive. It deserves noticing that for a body with a cavity, where Γ is not connected, this happens always.

In the last section a more detailed analysis which shows in what way the coefficients in the effective symbol $\mathfrak{m}_\iota$ should depend on the material characteristics, the Lamé constants. This task requires a more detailed analysis of the process of calculation of the symbol and subsymbol of the NP operator. The universality properties in the structure of the symbol $\mathfrak{m}_\iota$ enable us to reduce this calculation to a unique particular case, namely of the surface being a cylinder with circular cross-section, where one of the principal curvatures vanishes. The treatment of this case still requires a considerable calculational work, but it is at least more feasible as long, at least, as it concerns qualitative properties. As a result of our calculations, we establish that this symbol is a linear form of the quantities $\mathbb{k} = \frac{\mu}{2(2\mu+\lambda)}$ and $\mathbb{m} = \frac{1}{2} = \mathbb{k}$ with coefficients, now, depending only on ι and on $\xi \in S^1$. Some further properties of these coefficients are derived, using the universality, from the result for the case of the sphere.

We should mention that an alternative approach to the elastic NP spectral problem can, probably, be based upon recent results on diagonalizing matrix pseudodifferential operators, see [13, 14]; the important initial step, the global diagonalization of the principal symbol, is possible according to the results of [15]. We plan to explore this approach in the future.

The Author is appreciative to Miyanishi for introducing him to the NP problematic as well as for useful discussions.

2 Preliminaries

Let $\mathcal{D} \subset \mathbb{R}^3$ be a bounded (connected) body with smooth boundary Γ. It may happen that the surface Γ is not connected, namely, in the case when the body has some cavities.

We write the Lamé system for a homogeneous isotropic body $\mathcal{D}$ in the form

$$\mathcal{L}\mathbf{u} \equiv \mathcal{L}_{\lambda,\mu}\mathbf{u} \equiv -\mu \mathrm{div}\,(\mathrm{grad}\,\mathbf{u}) - (\lambda+\mu)\mathrm{grad}\,(\mathrm{div}\,\mathbf{u}) = 0,$$
$$\mathbf{x} = (x_1, x_2, x_3) \in \mathcal{D}, \mathbf{u} = (u_1, u_2, u_3)^\top,$$

where λ, μ are the Lamé constants. The fundamental solution $\mathcal{R}(\mathbf{x}, \mathbf{y}) = [\mathcal{R}(\mathbf{x}, \mathbf{y})]_{p,q}$, $p, q = 1, 2, 3$, for the Lamé equations, *the Kelvin matrix*, known since long ago, see, e.g., [25], equals

$$[\mathcal{R}(\mathbf{x}, \mathbf{y})]_{p,q} = \lambda' \frac{\delta_{p,q}}{|\mathbf{x}-\mathbf{y}|} + \mu' \frac{(x_p - y_p)(x_q - y_q)}{|\mathbf{x}-\mathbf{y}|^3},$$
$$\lambda' = \frac{\lambda+3\mu}{4\pi\mu(\lambda+2\mu)},\ \mu' = \frac{\lambda+\mu}{4\pi\mu(\lambda+2\mu)},\ \ p, q = 1, 2, 3,\ \ \mathbf{x}, \mathbf{y} \in \mathbb{R}^3.$$

This expression can be found, in particular, by inverting the Fourier transform of the symbol $\mathbf{r}(\boldsymbol{\xi})$ of $\mathcal{L}^{-1}$:

$$\mathcal{R}(\mathbf{x}, \mathbf{y}) = \mathcal{F}^{-1}[\mathbf{r}](\mathbf{x}-\mathbf{y}) \equiv (2\pi)^{-3} \int_{\mathbb{R}^3} e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}} \left(\mu\boldsymbol{\xi}\boldsymbol{\xi}^\top + (\lambda+\mu)|\boldsymbol{\xi}|^2\mathbf{E}\right)^{-1} d\boldsymbol{\xi}, \tag{2.1}$$

where $\boldsymbol{\xi}$ is treated as a column-vector, so that $\boldsymbol{\xi}\boldsymbol{\xi}^\top$ is a 3×3 square matrix; $\mathbf{E}$ is the unit 3×3 matrix.

The classical boundary problems for the Lamé system involve the coboundary (traction) operator

$$[\mathbf{T}(\mathbf{x}, \partial_{\boldsymbol{\nu}(\mathbf{x})})]_{p,q} = \lambda\nu_p\partial_q + \lambda\nu_q\partial_p + \mu\delta_{p,q}\partial_{\boldsymbol{\nu}},\ p, q = 1, 2, 3,$$

where $\boldsymbol{\nu} = \boldsymbol{\nu}(\mathbf{x}) = (\nu_1, \nu_2, \nu_3)$ is the outward normal unit vector to Γ at the point $\mathbf{x}$ and $\partial_{\boldsymbol{\nu}(\mathbf{x})}$ is the directional derivative along $\boldsymbol{\nu}(\mathbf{x})$.

The NP operator $\mathfrak{K}$ on Γ is defined as

$$(\mathfrak{K}[\psi])(\mathbf{x}) = \int_\Gamma \mathcal{K}(\mathbf{x}, \mathbf{y})\psi(\mathbf{y})d\mathcal{S}(\mathbf{y}) \equiv \int_\Gamma \mathbf{T}(\mathbf{y}, \partial_{\boldsymbol{\nu}(\mathbf{y})})\mathcal{R}(\mathbf{x}, \mathbf{y})^\top\psi(\mathbf{y})d\mathcal{S}(\mathbf{y}), \mathbf{x} \in \Gamma, \tag{2.2}$$

where $d\mathcal{S}$ is the natural surface measure on Γ, the Riemannian measure generated by the embedding of Γ in $\mathbb{R}^3$, and $\mathbf{T}(\mathbf{y}, \partial_{\boldsymbol{\nu}(\mathbf{y})})$ denotes the coboundary operator at the point $\mathbf{y} \in \Gamma$. The explicit expression for the kernel $\mathcal{K}(\mathbf{x}, \mathbf{y})$ is known since long ago as well; it is given, e.g., in [26], Sect. 4, Ch.2:

$$[\mathcal{K}(\mathbf{x}, \mathbf{y})]_{p,q} = \frac{\Bbbk}{2\pi} \frac{\nu_p(\mathbf{y})(x_q - y_q) - \nu_q(\mathbf{y})(x_p - y_p)}{|\mathbf{x}-\mathbf{y}|^3}$$

$$-\frac{1}{2\pi}\left(\Bbbk\delta_{p,q}+3\mathfrak{m}\frac{(x_p-y_p)(x_q-y_q)}{|\mathbf{x}-\mathbf{y}|^2}\right)$$
$$\sum_{l=1}^{3}\nu_l(\mathbf{y})\frac{x_l-y_l}{|\mathbf{x}-\mathbf{y}|^3};\ p,q=1,2,3, \tag{2.3}$$

where

$$\mathfrak{m}=\frac{\lambda+\mu}{2(\lambda+2\mu)}=\frac{1}{2}-\Bbbk.$$

Since the boundary is smooth, $\mathfrak{K}$ is a pseudodifferential operator of order zero on the surface Γ, i.e., a singular integral operator on Γ; the leading singularity at the diagonal $\mathbf{y}=\mathbf{x}\in\Gamma$ of the kernel, determined by off-diagonal terms in (2.3), is of order -2 and it is odd in $\mathbf{y}-\mathbf{x}$ as $\mathbf{y}\to\mathbf{x}$. To express the symbol of $\mathfrak{K}$ as a pseudodifferential operator, a local co-ordinate system on Γ and a frame in $\mathbb{R}^3$ are fixed. Following [1], for a fixed point $\mathbf{x}^\circ\in\Gamma$, two co-ordinate axes $x=(x_1,x_2)$ are orthogonally directed in the tangent plane to Γ at $\mathbf{x}^\circ$, and the third axis x_3 is directed orthogonally, in the *outward* direction, so that the surface Γ near $\mathbf{x}^\circ$ is described by the equation $x_3=F(x_1,x_2)$, and in these co-ordinates, $F(0,0)=0$, $\nabla F(0,0)=0$. These x_1,x_2 are chosen as local co-ordinates near $\mathbf{x}^\circ$ on Γ. The dual co-ordinates $\xi=(\xi_1,\xi_2)$ in the tangent plane are directed along the same spacial axes in $\mathtt{T}(\Gamma)$. The corresponding vectors will be accepted as the basis in the cotangent plane at $\mathbf{x}^\circ$ (identified naturally with the tangent plane.) The same vectors as well as the normal $\boldsymbol{\nu}(\mathbf{x}^\circ)$ are accepted as the frame in the fiber $\mathbb{C}^3$ over Γ near $\mathbf{x}^\circ$. In these co-ordinates, the principal symbol of $\mathfrak{K}$ is calculated in [1] to be equal to

$$\mathfrak{k}_0(x,\xi)=\frac{i\Bbbk}{|\xi|}\begin{pmatrix}0&0&-\xi_1\\0&0&-\xi_2\\\xi_1&\xi_2&0\end{pmatrix}, \tag{2.4}$$

where $\Bbbk=\frac{\mu}{2(2\mu+\lambda)}$, and the eigenvalues of this matrix, $\omega_{-1}=-\Bbbk$, $\omega_0=0$, $\omega_1=\Bbbk$, are the points of the essential spectrum of $\mathfrak{K}$. Note that these eigenvalues do not depend on the geometry of the body $\mathcal{D}$. At this point $\mathbf{x}^\circ$, dx_1dx_2 equals the area element for the surface measure on Γ generated from the Lebesgue measure in $\mathbb{R}^3$ by the embedding $\Gamma\subset\mathbb{R}^3$.

The eigenvectors of the principal symbol (2.4) equal $\mathbf{e}_\pm=2^{-\frac{1}{2}}|\xi|^{-1}(\xi_1,\xi_2,\pm\imath|\xi|)^\top$ for the eigenvalues $\pm\Bbbk$ and $\mathbf{e}_0=|\xi|^{-1}(\xi_1,-\xi_2,0)^\top$ for the eigenvalue 0. In the literature cited above, one can encounter some considerations concerning the mechanical meaning of such form of eigenvectors. In particular, there is an assumption that the eigenfunctions of the NP operator, corresponding to its eigenvalues close to 0, describe surface waves which are 'almost purely' compression ones, while two other eigenvectors describe 'almost incompressible' waves lying close to the subspace $\mathrm{div}\mathbf{u}=0$. We do not know to what extent our results support these speculations. An essential progress on this topic seems to have been made in a very recent preprint [18]. There, a splitting of the space of vector-functions on Γ into three

subspaces has been constructed, so that, on the one hand, the subspaces approximate spectral subspaces of $\mathfrak{K}$ corresponding to its spectrum around the points ω_ι, and, on the other hand, functions in these subspaces possess extensions inside or outside Γ to functions with special properties, divergence and/or rotor-free.

Due to the results of [34], in order to find the asymptotics of eigenvalues of $\mathfrak{K}$, we need also expressions for the subsymbol of the operator $\mathfrak{K}$ and the derivatives of the principal symbol. These objects depend essentially on the choice of local co-ordinates and frame. The convenient choice, refining the one described above, will be determined later on, enabling considerable simplification in our calculations. We use the notion of 'subsymbol' for the symbol of order -1 of a zero order pseudodifferential operator in a fixed co-ordinate system and a fixed frame. Unlike the classical notion of a 'subprincipal symbol' which is invariant under the change of local co-ordinates, the 'subsymbol' is not invariant, but the results of our eigenvalue calculations with the subsymbol turn out to be invariant.

As established in [34], the asymptotic behavior of the eigenvalues of $\mathfrak{K}$, as they approach the points ω_ι, is determined by the eigenvalue behavior for the compact operators $\mathfrak{M}_\iota = \mathbf{p}_\iota(\mathfrak{K})$, where $\mathbf{p}_\iota(\omega)$ are polynomials specially constructed according to the eigenvalues of the principal symbol of the operator, see Lemma 4.1 in [34]. In our case, the dimension $\mathbf{N}$ of the vector bundle, where the operator $\mathfrak{K}$ acts, equals 3 and all eigenvalues of the principal symbol are simple. Therefore, the number ι takes values $-1, 0, 1,$ and the degree of the polynomial $\mathbf{p}_\iota(\omega)$ equals 5. By Lemma 4.1 in [34], this means that the polynomial $\mathbf{p}_\iota(\omega)$ has the form

$$\mathbf{p}_\iota(\omega) = (\omega - \omega_\iota) \prod_{\iota' \neq \iota} (\omega - \omega_{\iota'})^2 \,, \ \iota = -1, 0, 1. \tag{2.5}$$

We are interested in the principal, order -1, symbol $\mathfrak{m}_\iota = \mathfrak{m}_{\iota,-1}(x, \xi)$ of the operator $\mathfrak{M}_\iota$; we call it the effective symbol. In [34], Proposition 4.2, the structure of this symbol was described. Due to this Proposition, $\mathfrak{m}_\iota$ is the sum of terms of 2 types. In order to write down them in a systematic way, we consider the sets $\mathbb{J}_\iota = \{\varsigma_j\}_{j=1,\dots,5}$, $\iota = -1, 0, 1$; the set $\mathbb{J}_\iota$ consists of the integers $-1, 0, 1$, in such way that each of them is repeated twice, except ι which is repeated only once, placed in the nondecreasing order, thus, $\mathbb{J}_{-1} = \{-1, 0, 0, 1, 1\}$, $\mathbb{J}_0 = \{-1, -1, 0, 1, 1\}$, $\mathbb{J}_1 = \{-1, -1, 0, 0, 1\}$. Then, by Proposition 4.2 in [34], the terms in $\mathfrak{m}_\iota$ of type 1 have the form

$$\mathbf{F}_{l,\iota} = \prod_{j<l} \left(\mathfrak{k}_0 - \omega_{\varsigma_j}\right) \mathfrak{k}_{-1} \prod_{j>l} \left(\mathfrak{k}_0 - \omega_{\varsigma_j}\right) \,, \ \varsigma_j \in \mathbb{J}_\iota, \ \iota = -1, 0, 1, \ \ l = 1, \dots, 5, \tag{2.6}$$

where $\mathfrak{k}_{-1}$ is the subsymbol of the operator $\mathfrak{K}$ – altogether 5 terms for each ι. The terms of type 2 have the form

$$\mathbf{G}_{\alpha,l,m,\iota} = \frac{1}{i} \prod_{j<l} (\mathfrak{k}_0 - \omega_{\varsigma_j}) (\partial_{\xi_\alpha} \mathfrak{k}_0) \prod_{l<j<m} (\mathfrak{k}_0 - \omega_{\varsigma_j}) (\partial_{x_\alpha} \mathfrak{k}_0) \prod_{j>m} (\mathfrak{k}_0 - \omega_{\varsigma_j}),$$
$$\alpha = 1, 2; \ 1 \leq l < m \leq 5, \tag{2.7}$$

altogether 20 terms for each ι.

One should keep in mind that all factors in (2.6), (2.7) are 3×3 matrices. The factors of the form $\mathfrak{k}_0 - \omega_{\varsigma_j}$ commute with each other but the ones containing the derivatives of $\mathfrak{k}_0$ and the subsymbol do not commute with $\mathfrak{k}_0 - \omega_{\varsigma_j}$. One can also see in (2.6), (2.7) that the terms of type 1 contain 4 factors $\mathfrak{k}_0 - \omega_{\varsigma_j}$ while terms of type 2 contain 3 such factors. Finally, the symbol $\mathfrak{m}_\iota$ is equal to

$$\mathfrak{m}_\iota = [\mathbf{p}'_\iota(\omega_\iota)]^{-1} \left(\sum_{l=1}^{5} \mathbf{F}_{l,\iota} + \sum_{\alpha=1,2} \sum_{1 \le l < m \le 5} \mathbf{G}_{\alpha,l,m,\iota} \right), \tag{2.8}$$

25 terms altogether, for each $\iota = -1, 0, 1$.

In Sect. 6, we discuss some economy approach for calculating the expression (2.8). This gives us a possibility to describe the dependence of the effective symbol on the Lamé constants.

As was emphasized earlier, the expressions in (2.6), (2.7) contain factors which are not invariant with respect to changes of local co-ordinates on the surface Γ and the frame. This is unavoidable but not important, since the *sum* of all such terms in (2.8), being the *principal*, order -1, symbol $\mathfrak{m}_{\iota,-1}(x,\xi)$ of the operator $\mathfrak{M}_\iota$, is in the usual sense invariant with respect to changes of local co-ordinates on Γ and may depend only on the frame; its eigenvalues are invariant. We will use this invariance essentially when choosing the co-ordinates and the frame in a convenient way. Moreover, the operator $\mathbf{p}_\iota(\mathfrak{K})$ and its principal symbol $\mathfrak{m}_{\iota,-1}$ as well do not depend on the order in which the terms $\mathfrak{K} - \omega_{\varsigma_j}$ are multiplied, although each particular term in (2.8) does.

A special feature of the operator $\mathfrak{K}$ is the fact that it is not self-adjoint in $L^2(\Gamma)$ (with the surface measure generated by the Lebesgue measure in $\mathbb{R}^3$). This operator is, however, *symmetrizable*. This property was discussed in [34], Sect. 6. Recall that the pseudodifferential operator $\mathfrak{K}$ is called symmetrizable if there exists a positive elliptic pseudodifferential operator $\mathfrak{S}$ such that

$$\mathfrak{A} = \mathfrak{S}^{-1/2} \mathfrak{K} \mathfrak{S}^{1/2},$$

is self-adjoint in $L^2(\Gamma)$ or, equivalently,

$$\mathfrak{S}\mathfrak{K}^* = \mathfrak{K}\mathfrak{S}. \tag{2.9}$$

In Sect. 3 we discuss the symmetrizability of our operator $\mathfrak{K}$ in more detail.

3 Symmetrization and general asymptotic formulas

3.1 Symmetrization

The fact that the operator $\mathfrak{K}$ is not self-adjoint in $L^2(\Gamma)$ is already easily visible from its definition (2.2)—the adjoint operator involves the normal derivative at the point $\mathbf{x}$,

and not at the point $\mathbf{y}$, as in (2.2). This shortcoming can be circumvented by showing that $\mathfrak{K}$ is symmetrizable.

Consider the *single layer potential operator* on Γ:

$$\mathfrak{S}[\psi](\mathbf{x}) = \int_\Gamma \mathcal{R}(\mathbf{x}, \mathbf{y})\psi(\mathbf{y})d\mathcal{S}(\mathbf{y}),\ \mathbf{x} \in \Gamma, \tag{3.1}$$

the kernel $\mathcal{R}$ being defined in (2.1). This is a self-adjoint operator in $L^2(\Gamma)$. It is well known, see, e.g. [1], that $\mathfrak{S}$ (it is denoted by A there) is an elliptic pseudodifferential operator of order -1. Therefore, $\mathfrak{S}$ maps the Sobolev space $H^s(\Gamma)$ into the space $H^{s+1}(\Gamma)$ for any $s \in (-\infty, \infty)$. The principal symbol of $\mathfrak{S}$ has been calculated in [1], Sect. 1.6. In the local co-ordinates and the frame just used above, it has the block-matrix form

$$\mathfrak{s}_{-1}(x, \xi) = \frac{1}{2\mu|\xi|}\left(\mathbf{E} - \mathbb{m}\begin{pmatrix}\mathbf{\Lambda}(\xi) & 0\\ 0 & 1\end{pmatrix}\right), \mathbb{m} = \frac{\lambda+\mu}{2(\lambda+2\mu)} = \frac{1}{2} - \mathbb{k}.$$

here $\mathbf{\Lambda}(\xi)$ is the 2×2 matrix

$$\mathbf{\Lambda}(\xi) = |\xi|^{-2}\begin{pmatrix}\xi_1^2 & \xi_1\xi_2\\ \xi_1\xi_2 & \xi_2^2\end{pmatrix},$$

$\mathbf{E}$ denotes the unit 3×3 matrix.

Matrix (3.1) is invertible, therefore, the operator $\mathfrak{S}$ is elliptic. We need some more, namely, that $\mathfrak{S}$ is positive in $L^2(\Gamma)$.

Proposition 3.1 *The single layer potential $\mathfrak{S}$ in (3.1) is a positive operator in $L^2(\Gamma)$, $\langle \mathfrak{S}\psi, \psi\rangle_{L^2(\Gamma)} > 0$.*

Proof In the scalar case, for the single layer electrostatic potential, this property is well-known, see, e.g., [27], Theorem 1.15. We failed to find an exact reasoning for the elastic case in the literature, therefore we present an elementary proof here. Denote by $\mathcal{Q}(\mathbf{x}, \mathbf{y})$ the fundamental solution for the square root of the Lamé operator $\mathcal{L}$. This function can be constructed as

$$\mathcal{Q}(\mathbf{x}, \mathbf{y}) = (2\pi)^{-3/2}\int_{\mathbb{R}^3} e^{i(\mathbf{x}-\mathbf{y})\boldsymbol{\xi}}\sqrt{\mathbf{r}(\boldsymbol{\xi})}d\boldsymbol{\xi},$$

with $\sqrt{\cdot}$ denoting here the positive square root of a positive matrix. Since $\sqrt{\mathbf{r}(\boldsymbol{\xi})} \times \sqrt{\mathbf{r}(\boldsymbol{\xi})} = \mathbf{r}(\boldsymbol{\xi})$, the kernel $\mathcal{Q}$ satisfies

$$\int_{\mathbb{R}^3} \mathcal{Q}(\mathbf{x}, \mathbf{z})\mathcal{Q}(\mathbf{z}, \mathbf{y})d\mathbf{z} = \mathcal{R}(\mathbf{x}, \mathbf{y}), \mathbf{x} \in \mathcal{D}. \tag{3.2}$$

Using (3.2), we can represent the single layer potential operator as $\mathfrak{S} = \mathbf{Q}^*\mathbf{Q}$, where $\mathbf{Q}$ is the operator acting from $L^2(\Gamma)$ to $L^2(\mathbb{R}^3)$ as

$$\mathbf{Q}[\psi](\mathbf{x}) = \int_\Gamma \mathcal{Q}(\mathbf{x}, \mathbf{y})\psi(\mathbf{y})d\mathcal{S}(\mathbf{y}).$$

This representation shows that the operator $\mathfrak{S}$ is nonnegative. Finally, in accordance with [1], Proposition 1.2, the null space of $\mathfrak{S}$ is trivial, so $\mathfrak{S}$ is positive. □

Taking into account the ellipticity of $\mathfrak{S}$, we know now that $\mathfrak{S}$ is an *isomorphism* of Sobolev spaces, $\mathfrak{S} : H^s(\Gamma) \to H^{s+1}(\Gamma)$, $-\infty < s < \infty$. Moreover, any power of $\mathfrak{S}$ is an isomorphism $\mathfrak{S}^\gamma : H^s(\Gamma) \to H^{s+\gamma}(\Gamma)$, $-\infty < \gamma < \infty$.

The matrix $\mathbf{\Lambda} = \mathbf{\Lambda}(\xi)$ satisfies $\mathbf{\Lambda}^2 = \mathbf{\Lambda}$, this property enables us to calculate explicitly principal symbols of some operators related with $\mathfrak{S}$. First, the inverse $\mathfrak{R} = \mathfrak{S}^{-1}$ is a pseudodifferential operator of order 1. Its principal symbol $\mathfrak{r}_1$ equals $\mathfrak{s}^{-1}$,

$$\mathfrak{r}_1(\xi) = 2\mu|\xi| \left(\mathbf{E} + \frac{\lambda+\mu}{\lambda+3\mu} \begin{pmatrix} \mathbf{\Lambda}(\xi) & 0 \\ 0 & 1 \end{pmatrix} \right) = (\mathfrak{s}_{-1}(\xi))^{-1}.$$

We will also need the (positive) square roots from the operators $\mathfrak{S}$ and $\mathfrak{R}$. The operator $\mathfrak{Q} = \mathfrak{S}^{\frac{1}{2}}$ is an elliptic pseudodifferential operator of order $-\frac{1}{2}$ and its principal symbol equals

$$\mathfrak{q}_{-\frac{1}{2}}(\xi) = (\mathfrak{s}_{-1}(\xi))^{\frac{1}{2}} = \frac{1}{(2\mu|\xi|)^{\frac{1}{2}}} \left(\mathbf{E} - (1-(1-\mathbb{m})^{\frac{1}{2}}) \begin{pmatrix} \mathbf{\Lambda}(\xi) & 0 \\ 0 & 1 \end{pmatrix} \right).$$

In its turn, the principal symbol of the order $\frac{1}{2}$ pseudodifferential operator $\mathfrak{Z} = \mathfrak{Q}^{-1} = \mathfrak{R}^{\frac{1}{2}}$ equals

$$\begin{aligned} \mathfrak{z}_{\frac{1}{2}}(\xi) &= (\mathfrak{r}_1(\xi))^{\frac{1}{2}} = (\mathfrak{q}_{-\frac{1}{2}}(\xi))^{-1} \\ &= (2\mu|\xi|)^{\frac{1}{2}} \left(\mathbf{E} + \left(\frac{1}{\sqrt{1-\mathbb{m}}} - 1 \right) \begin{pmatrix} \Lambda(\xi) & 0 \\ 0 & 1 \end{pmatrix} \right), \end{aligned} \tag{3.3}$$

We can show now that our operator $\mathfrak{K}$ is symmetrizable in $L^2(\Gamma)$, with the single layer potential operator $\mathfrak{S}$ acting as symmetrizer. In fact, the relation (2.9) has been known since long ago; for the Lamé system it was established in [17], p. 89, see also [1], Proposition 1.8. Moreover, we know now that $\mathfrak{S}$ is positive. Consequently, the operator $\mathfrak{A} = \mathfrak{S}^{\frac{1}{2}}\mathfrak{K}\mathfrak{S}^{-\frac{1}{2}} = \mathfrak{Z}\mathfrak{K}\mathfrak{Q}$ is self-adjoint in $L^2(\Gamma)$.

This operator $\mathfrak{A}$ is a zero order self-adjoint classical pseudodifferential operator, with the same spectrum as $\mathfrak{K}$. The latter statement is proved in [34], Proposition 6.1. By the composition rule for pseudodifferential operators, the principal symbol of $\mathfrak{A}$ is a matrix, similar to the principal symbol of $\mathfrak{K}$,

$$\mathfrak{a}_0(x,\xi) = \mathfrak{z}_{\frac{1}{2}}(x,\xi)\mathfrak{k}_0(x,\xi)\mathfrak{q}_{-\frac{1}{2}}(x,\xi).$$

Being the principal symbol of a self-adjoint operator, the symbol $\mathfrak{a}_0(x,\xi)$ has only real eigenvalues; it follows that the eigenvalues of the symbol $\mathfrak{k}_0(x,\xi)$, a matrix, similar to $\mathfrak{a}_0(x,\xi)$, are the same as the eigenvalues of $\mathfrak{a}_0(x,\xi)$; in particular, they are real as well. Moreover, we notice immediately that the matrices $\mathfrak{k}_0(\xi)$ and $\begin{pmatrix} \mathbf{\Lambda}(\xi) & 0 \\ 0 & 1 \end{pmatrix}$ commute.

Taking into account the expression for the principal symbols of $\mathfrak{S}^{\frac{1}{2}}$ and $\mathfrak{S}^{-\frac{1}{2}}$, we obtain

$$\mathfrak{a}_0(\xi) = \mathfrak{k}_0(\xi) = \frac{i\mathbb{k}}{|\xi|}\begin{pmatrix} 0 & 0 & -\xi_1 \\ 0 & 0 & -\xi_2 \\ \xi_1 & \xi_2 & 0 \end{pmatrix}. \tag{3.4}$$

So, the principal symbols of $\mathfrak{K}$ and $\mathfrak{A}$ coincide; in particular, it follows that $\mathfrak{k}_0$ is a Hermitian matrix. This property follows, of course, also from the fact, easily verified, that the difference, $\mathfrak{K} - \mathfrak{K}^*$, is a pseudodifferential operator of order -1.

We cannot declare here that the subsymbol of $\mathfrak{K}$ and $\mathfrak{s}_{-1}$ commute; in fact, they do not. However, the important property for our calculations is the following consequence of the composition rule.

Proposition 3.2 *Let $\mathfrak{K}$ be a polynomially compact zero order pseudodifferential operator and for given ι, $\mathbf{p}_\iota$ be the polynomial in (2.5). Suppose that $\mathfrak{K}$ is symmetrizable, with the pseudodifferential operator $\mathfrak{S}$ acting as symmetrizer. Then the principal symbol $\mathfrak{m}_{\iota,-1}(x,\xi)$ of the operator $\mathfrak{M}_\iota = \mathbf{p}_\iota(\mathfrak{K})$ is a matrix similar to a Hermitian one, namely to the principal symbol $\mathfrak{b}_{\iota,-1}(x,\xi)$ of the operator $\mathfrak{B}_\iota = \mathbf{p}_\iota(\mathfrak{A})$ equals*

$$\mathfrak{b}_{\iota,-1}(x,\xi) = \mathfrak{z}_{\frac{1}{2}}(x,\xi)\mathfrak{m}_{\iota,-1}(x,\xi)\mathfrak{q}_{-\frac{1}{2}}(x,\xi); \tag{3.5}$$

here $\mathfrak{q}_{-\frac{1}{2}}$, $\mathfrak{z}_{\frac{1}{2}}$ are principal symbols of operators $\mathfrak{Q} = \mathfrak{S}^{\frac{1}{2}}$, resp., $\mathfrak{Z} = \mathfrak{S}^{-\frac{1}{2}}$, see (3.1), (3.3).

Proof The pseudodifferential operator $\mathfrak{M}_\iota$ equals

$$\mathfrak{M}_\iota = \mathfrak{S}^{\frac{1}{2}}\mathfrak{B}_\iota\mathfrak{S}^{-\frac{1}{2}} = \mathfrak{S}^{\frac{1}{2}}\mathbf{p}_\iota(\mathfrak{A})\mathfrak{S}^{-\frac{1}{2}}, \tag{3.6}$$

therefore, for the principal symbols of $\mathfrak{M}_\iota$ and $\mathfrak{B}_\iota$, symbols of order -1 we have the equality,

$$\mathfrak{m}_{\iota,-1} = \mathfrak{q}_{-\frac{1}{2}}\mathfrak{b}_{\iota,-1}\mathfrak{z}_{\frac{1}{2}}. \tag{3.7}$$

It remains to recall that $\mathfrak{b}_{\iota,-1}$ is a principal symbol of a self-adjoint operator, and therefore it is a Hermitian matrix. □

3.2 General asymptotic formulas

The operator $\mathfrak{M}_\iota$ in (3.6) is a pseudodifferential operator of order -1, and its principal symbol, a 3×3 matrix with real eigenvalues, is denoted by $\mathfrak{m}_\iota(x,\xi)$ (sometimes the subscript -1 is added in this notation in order to recall that this is a symbol of order -1.) Having this symbol at hand, the asymptotics of the eigenvalues of $\mathfrak{K}$, tending to ω_ι, can be found by the general result obtained in [34], Theor. 6.2. We reproduce here this Theorem, adapted to our particular case.

First, we introduce proper notations. For a point ω_ι in the essential spectrum of $\mathfrak{K}$, $n_\pm(\mathfrak{K};\omega_\iota,\cdot)$ denotes the counting functions of eigenvalues of the operator $\mathfrak{K}$ in intervals near ω_ι. Namely, we fix some reference points $\tau_\pm$ in such way that the

interval $(\omega_\iota - \tau_-, \omega_\iota + \tau_+)$ contains no points of the essential spectrum of $\mathfrak{K}$, other than ω_ι. Then, for a small $\tau > 0$, $n_\pm(\mathfrak{K}; \omega_\iota, \tau)$ denotes the number of eigenvalues of $\mathfrak{K}$ in the interval $(\omega_\iota + \tau, \omega_\iota + \tau_+)$ for the '+' sign, resp., $(\omega_\iota - \tau_-, \omega_\iota - \tau)$ for the '-' sign. The object of our study is the behavior of $n_\pm(\mathfrak{K}; \omega_\iota, \tau)$ as $\tau \to 0$. It stands to reason that $n_\pm(\mathfrak{K}; \omega_\iota, \tau)$ is bounded if there are only finitely many eigenvalues of $\mathfrak{K}$ in the corresponding (upper or lower) neighborhood of ω_ι. On the other hand, if there are infinitely many eigenvalues of $\mathfrak{K}$ in such neighbourhood then these eigenvalues must converge to ω_ι and therefore $n_\pm(\mathfrak{K}; \omega_\iota, \tau)$ tends to infinity as $\tau \to 0$. The change in the choice of the reference points $\tau_\pm$ does not influence the rate of growth of $n_\pm(\mathfrak{K}; \omega_\iota, \tau)$ as $\tau \to 0$, therefore they are not reflected in our notations.

Next, for a diagonalizable matrix $\mathfrak{m}$ with real eigenvalues, the expression $\operatorname{Tr}^{(2)}_\pm(\mathfrak{m})$ denotes the *sum of squares* of positive, resp., negative, eigenvalues of the matrix $\mathfrak{m}$. Finally, $\boldsymbol{\omega}$ denotes the 1-form $\boldsymbol{\omega} = \xi_1 d\xi_2 - \xi_2 d\xi_1 = d\theta$ in the polar co-ordinates on the unit circle $S^1 \subset \mathbb{T}_x\Gamma$, $\xi_1 = \cos\theta, \xi_2 = \sin\theta, 0 \le \theta \le 2\pi$ and $d\mathcal{S}(x)$ denotes the area element for the surface measure induced by the embedding $\Gamma \subset \mathbb{R}^3$.

Theorem 3.3 *Let $\mathfrak{B}$ be an order -1 self-adjoint pseudodifferential operator on a manifold Γ of dimension 2, with principal symbol $\mathfrak{b}(x,\xi)$, a Hermitian matrix. Then for the eigenvalues of $\mathfrak{B}$ the asymptotic formulas hold*

$$n_\pm(\mathfrak{B}, 0, \tau) \sim C^\pm(\mathfrak{B}, 0)\tau^{-2}, \tau \to 0, \tag{3.8}$$

$$C^\pm(\mathfrak{B}, 0) = 2^{-1}(2\pi)^{-2} \int_{S^*\Gamma} \operatorname{Tr}^{(2)}_\pm(\mathfrak{b}(x,\xi))\boldsymbol{\omega} d\mathcal{S}(x), \tag{3.9}$$

(note that the zero in the notation $n_\pm(\mathfrak{B}, 0, \tau)$ refers to zero being the only point of essential spectrum for the compact operator $\mathfrak{B}$.) By the results of [34], for the zero order NP operator $\mathfrak{K}$, the asymptotics of $n_\pm(\mathfrak{K}; \omega_\iota, \cdot)$ is determined by the asymptotics of eigenvalues of the order -1 operator $\mathfrak{M}_\iota = \mathbf{p}_\iota(\mathfrak{K})$ with the principal symbol $\mathfrak{m}_\iota(x,\xi)$. Since the operator $\mathfrak{M}_\iota$ is symmetrizable, it has the same eigenvalues as the self-adjoint operator $\mathfrak{B}_\iota$. The principal symbol $\mathfrak{b}_\iota(x,\xi)$ of $\mathfrak{B}_\iota$ is a Hermitian matrix with the same eigenvalues as $\mathfrak{m}_\iota(x,\xi)$. Therefore, Theorem 3.3 can be made concrete in the following way.

Theorem 3.4 *Let $\mathfrak{K}$ be the elastic NP operator on a smooth closed surface $\Gamma \subset \mathbb{R}^3$. Then the asymptotics of eigenvalues of $\mathfrak{K}$, converging to the point ω_ι of the essential spectrum, is described by the formula*

$$n_\pm(\mathfrak{K}; \omega_\iota, \tau) \sim C^\pm(\omega_\iota)\tau^{-2}, \tau \to +0, \tag{3.10}$$

$$C^\pm(\omega_\iota) = 2^{-1}(2\pi)^{-2} \int_{S^*\Gamma} \operatorname{Tr}^{(2)}_\pm(\mathfrak{m}_\iota(x,\xi))\boldsymbol{\omega} d\mathcal{S}(x), \tag{3.11}$$

where $\mathfrak{m}_\iota = \mathfrak{m}_{\iota,-1}$ is the principal symbol of the order -1 operator $\mathfrak{M}_\iota = \mathbf{p}_\iota(\mathfrak{K})$, see (2.8).

In this way, the asymptotic formula (3.10), (3.11) is derived, as a special case, for our particular values of the dimension of the manifold and the order of the operator, of general results by Birman and Solomyak, [11] and [12]. The English translation

of the (rather technical) proof in [11] is virtually inaccessible now, but an interested Reader can be directed to a recent, soft analysis, proof by R. Ponge [33]. This proof concerns a more restricted version of the general result in [11], for a smooth surface and a homogeneous symbol, however it covers our situation. This result can also be found in [21], however the proof (based upon the general theory developed in this book) is left there as an exercise to readers.

There is a simple but important consequence of this general result.

Proposition 3.5 *Suppose that at some point* $(x_0, \xi_0) \in \mathrm{S}^*(\Gamma)$*, one of eigenvalues of the matrix* $\mathfrak{m}_\iota(x_0, \xi_0)$ *is positive (negative). Then there exists an infinite sequence of eigenvalues of* $\mathfrak{K}$ *converging to* ω_ι *from above (from below) satisfying the asymptotic formula* (3.10)*,* (3.11) *(with the corresponding sign), with a nonzero coefficient* $C^{\pm}(\omega_\iota)$.

Proof Eigenvalues of a continuous diagonalizable matrix-function of (x, ξ) depend continuously on the variables (x, ξ). Therefore the eigenvalue in question is positive (negative) in a neighborhood of (x_0, ξ_0). Consequently, the integrand in (3.11) with proper sign is positive on a set of positive measure. This means that the integral determining $C^{\pm}(\omega_\iota)$ is positive. □

Note that Proposition 3.5 uses, in fact, only that the coefficients $C^{\pm}(\omega_\iota)$ are integrals of sign-definite microlocal expressions involving the symbol, and not on the particular form of (3.11). In the next Section we express the conditions of Proposition 3.5 in geometrical terms.

Formally the basic theorem by Birman and Solomyak, as formulated in [11, 12], concerns only connected surfaces. However, obviously, in the case of the surface Γ consisting of several connected components, $\Gamma = \cup\Gamma_\varkappa$, which happens when the body $\mathcal{D}$ possesses cavities, the Neumann-Poincaré operator $\mathfrak{K}$ is the direct sum of pseudodifferential operators $\mathfrak{K}_\varkappa$ on the components $\Gamma_\varkappa$. The discrete spectrum of such sum is the union of the spectra of $\mathfrak{K}_\varkappa$.

3.3 The two-sided eigenvalue asymptotics

Although the expression in (3.11) depends formally on the choice of local co-ordinates on the surface Γ and the frame in the fiber, the integral in (3.11) and, moreover, the $\boldsymbol{\omega}$-integral in (3.11) over the cotangent circle are invariant under the change of an orthogonal co-ordinate system on Γ and under the change of the frame, as was established in [34].

The expression under the integral in (3.11) is rather hard to handle. In fact, for the matrix $\mathfrak{m}_{\iota,-1}$ depending on x, ξ—as we see later, determined by the material characteristics and by the principal curvatures—one needs to calculate the eigenvalues, separately the positive and negative ones, and then integrate over $\mathrm{S}^*(\Gamma)$ certain expressions containing the squares of positive, resp., negative, eigenvalues. It seems that the task of finding treatable analytical expressions here is very hard. On the first step above, namely, finding the eigenvalues of a 3×3 symbolic matrix, this means, solving a third degree algebraic equation with symbolic coefficients, is rather hard. This equation has three real roots, their expression involves cubic roots from complex

numbers—and formulas (3.11) require further separating positive and negative ones, and then integrate the result—altogether, this is highly impractical. Formulas (3.10), (3.11) may serve, probably, for numerical calculations as well as for the asymptotic analysis. These formulas can be, however, used for obtaining qualitative results for the properties of the NP eigenvalues under certain geometrical conditions, see Sects. 4, 5, 6. What we can, however, achieve more easily, is to find the explicit asymptotics for the distribution function for the *union* of the sequences of eigenvalues converging to ω_ι, in other terms, for the sum of functions $n_\pm(\mathfrak{K};\omega_\iota,\tau)$ counting the eigenvalues lying below and above ω_ι. It follows from the formulas for the asymptotics of the absolute values of the eigenvalues of an order -1 pseudodifferential operator to which the spectral problem for the NP operator is reduced. This possibility is explained by a very fortunate relations between the order of the operator involved and of the dimension of the space. Such relation leads to the exponent in (3.11) (which, in the general case, equals the dimension of the manifold divided by minus the order of the operator) to be equal to 2. And, fortunately, since the eigenvalues of the symbol $\mathfrak{m}_\iota$ are real, it is the trace of the square or the matrix symbol that expresses the sum of squares of its eigenvalues (an effect of similar kind is present also if we are interested in the sum of some even integer powers of the eigenvalues of the matrix, with a more complicated but still polynomial expression.) Therefore, Theorem 3.4 has as a consequence the following asymptotic formula.

Theorem 3.6 *Let the conditions of Theorem 3.4 be satisfied and let* $\mathfrak{m}_\iota = \mathfrak{m}_{\iota,-1}(x,\xi)$ *be the principal symbol of the operator* $\mathfrak{M}_\iota = \mathbf{p}_\iota(\mathfrak{K})$. *Then for the eigenvalues of* $\mathfrak{K}$ *the following asymptotics holds:*

$$n(\mathfrak{K};\omega_\iota,\tau) \equiv n_+(\mathfrak{K};\omega_\iota,\tau) + n_-(\mathfrak{K};\omega_\iota,\tau) \sim C(\omega_\iota)\tau^{-2},\ \tau\to 0, \tag{3.12}$$

where the coefficient $C(\omega_\iota)$ *equals*

$$C(\omega_\iota) = C^+(\omega_\iota) + C^-(\omega_\iota) = 2^{-1}(2\pi)^{-2}\int_{S^*\Gamma} \mathrm{Tr}\,(\mathfrak{m}_{\iota,-1}(x,\xi)^2)\boldsymbol{\omega} d\mathcal{S}(x). \tag{3.13}$$

The geometrical meaning of the expression in (3.13) will be discussed further on, in Sect. 4.

Thus, the problem remains of calculating the symbol $\mathfrak{m}_{\iota,-1}(x,\xi)$ (or its positive and negative eigenvalues) in *some* co-ordinate system and frame. This freedom will be used essentially in the reasoning to follow.

In [34], when discussing the reduction of the spectral problem for a polynomially compact pseudodifferential operator to the one for a compact operator, we considered also the case when the operator $\mathfrak{m}_{\iota,-1}$ has everywhere vanishing principal symbol, $\mathfrak{m}_{\iota,-1}(x,\xi)\equiv 0$. In this exceptional case, the coefficient in front of τ^{-2} in the asymptotic formula (3.10) vanishes for both signs; in this way (3.10) becomes non-informative. Such case was called 'degenerate' in [34], and obtaining eigenvalue asymptotic formulas required additional considerations, see Lemma 5.1 and Theorem 5.2 in [34]. However, in the only explicitly calculated case of the sphere, see (1.1), the results can be expressed as

$$
\begin{aligned}
&n_+(\mathfrak{K};0,\tau)\sim\frac{9}{16}\tau^{-2},\ \tau\to 0;\\
&n_-(\mathfrak{K};0,\tau)=0,;\\
&n_+(\mathfrak{K};-\Bbbk,\tau)\sim(4\Bbbk)^2\tau^{-2},\tau\to 0;\\
&n_-(\mathfrak{K};-\Bbbk,\tau)=0;\\
&n_+(\mathfrak{K};\Bbbk,\tau)\sim(4\Bbbk)^2\tau^{-2},\tau\to 0;\\
&n_-(\mathfrak{K};\Bbbk,\tau)=0.
\end{aligned}
\tag{3.14}
$$

so this case is non-degenerate. We will see further on in this study that for the NP operator, the nondegenerate case always occurs.

4 The structure of the symbol $\mathfrak{m}_{\iota,-1}$: geometry considerations

4.1 Formulation

Due to the results described in Sect. 3, in order to find the coefficients in the asymptotic formula for eigenvalues, we need to calculate the effective principal symbols $\mathfrak{m}_\iota(x,\xi)\equiv\mathfrak{m}_{\iota,-1}(x,\xi)$, $\iota=-1,0,1$, of the order -1 pseudodifferential operators $\mathfrak{M}_\iota=\mathbf{p}_\iota(\mathfrak{K})$. We aim now for avoiding the (very tedious) direct calculation of these symbols and their bulky and unwieldy expression using some *a prori* symmetry properties. More detailed calculations follow in Sect. 6.

We have already started to investigate the structure of the symbol $\mathfrak{m}_\iota$ in Sect. 2. Here we are going to determine the character of dependence of this symbol on the geometrical characteristics of the surface. We denote by $\mathbf{k}_1(\mathbf{x})$, $\mathbf{k}_2(\mathbf{x})$ the principal curvatures of the surface Γ at a point $\mathbf{x}\in\Gamma$. If a parametrization of Γ is chosen, the notation $\mathbf{k}_1(x)$, $\mathbf{k}_2(x)$ is used, $x\in\Omega\subset\mathbb{R}^2$, as long as it does not cause a misunderstanding. Recall that the product $\mathbf{k}_1(x)\mathbf{k}_2(x)$ is the Gaussian curvature, $\frac{\mathbf{k}_1(x)+\mathbf{k}_2(x)}{2}$ is the mean curvature and $\left(\frac{\mathbf{k}_1(x)+\mathbf{k}_2(x)}{2}\right)^2$ is the Willmore curvature of the surface. Note also that we have taken the co-ordinate x_3 directed along the outward normal to Γ, therefore the principal curvatures are *negative* at the points where the surface is convex.

We will call in this section a matrix-function $M(\xi)$ *universal* if it, in standard co-ordinates and frame, depends only on the covector ξ and the Lamé constants, but does not depend on the point on Γ where it is calculated. Later, in Sect. 6, the term 'universal' will refer to functions which depend only on ξ.

The crucial property we establish in this Section is the following.

Theorem 4.1 *For any point $\mathbf{x}^\circ\in\Gamma$, in the standardly chosen co-ordinates on the surface Γ in a neighborhood of $\mathbf{x}^\circ$ and the corresponding frame in $\mathbb{C}^3$, the symbol $\mathfrak{m}_\iota(x,\xi)$ has at the point $\mathbf{x}^\circ$ the structure*

$$
\mathfrak{m}_\iota(x,\xi)=\mathbf{k}_1(\mathbf{x}^\circ)M_\iota^{(1)}(\xi_1,\xi_2)+\mathbf{k}_2(\mathbf{x}^\circ)M_\iota^{(2)}(\xi_1,\xi_2),
\tag{4.1}
$$

with universal matrices $M_\iota^{(1)}(\xi_1,\xi_2)$, $M_\iota^{(2)}(\xi_1,\xi_2)$, order -1 positively homogeneous in ξ, depending on the Lamé constants λ,μ but not depending on the surface Γ.

We would like to stress that the representation (4.1) is valid only in the specially selected co-ordinates system and frame at $\mathbf{x}^\circ$. These are chosen depending on the geometry of Γ near the point $\mathbf{x}^\circ$. However, recall, the eigenvalues of the symbol $\mathfrak{m}_\iota(x,\xi)$ do not depend on the co-ordinates chosen or the frame, so the integrand in (3.11) is invariant under these changes.

We present the proof of Theorem 4.1 further on in this section.

4.2 C–co-ordinate systems

We choose near a point $\mathbf{x}^\circ \in \Gamma$ a special co-ordinate system, where the structure of the symbol is more treatable. We will call it the 'C–co-ordinates at $\mathbf{x}^{\circ\prime}$. It is in this system that the representation (4.1) is valid.

Suppose first that $\mathbf{x}^\circ$ is an umbilical point of the surface Γ. Recall that a point on a smooth surface in $\mathbb{R}^3$ is called umbilical if the principal curvatures at this point coincide.[1] For such a point, we direct the orthogonal x_1, x_2 axes arbitrarily in the tangent plane to Γ at $\mathbf{x}^\circ$ and direct the x_3 axis orthogonally to them, in the *outward* direction at $\mathbf{x}^\circ$.

If $\mathbf{x}^\circ$ is *not* an umbilical point, we direct x_1, x_2 axes along the lines of principal curvatures of Γ at $\mathbf{x}^\circ$ and direct the x_3 axis along the outward normal at $\mathbf{x}^\circ$.

In both cases, the surface Γ near $\mathbf{x}^\circ$ is described by the equation $x_3 = F(x_1, x_2) \equiv F(x)$ with $F(0,0) = 0$, so $\mathbf{x}^\circ = (0,0,0)$, and

$$\nabla F(0,0) = 0,\ F(x_1,x_2) = \frac{1}{2}\left(\mathbf{k}_1(\mathbf{x}^\circ)x_1^2 + \mathbf{k}_2(\mathbf{x}^\circ)x_2^2\right) + O\left(\left(x_1^2+x_2^2\right)^{3/2}\right), \tag{4.2}$$

where $\mathbf{k}_1(\mathbf{x}^\circ), \mathbf{k}_2(\mathbf{x}^\circ)$ are the principal curvatures of Γ at $\mathbf{x}^\circ$, while $\mathbf{k}_1(\mathbf{x}^\circ) = \mathbf{k}_2(\mathbf{x}^\circ)$ at an umbilical point. Note that in the non-umbilical case, the numbering of the principal curvatures matches the numbering of x_1, x_2 co-ordinates. This co-ordinate system will be called C–co-ordinates. The dual co-ordinates $\xi = (\xi_1, \xi_2)$ in the cotangent space are directed along the same axes. The frame in the fiber $\mathbb{R}^3$ at the point $\mathbf{x}^\circ$ is chosen along the axes x_1, x_2, x_3.

In a neighborhood of a non-umbilical point, the curvature lines are smooth and the C–co-ordinate systems chosen above depend smoothly on the point $\mathbf{x}^\circ$. On the other hand, near an umbilical point, the curvature lines may behave rather wildly, and the above co-ordinates system can depend on the base point fairly non-smoothly. Since we will need further on to trace the behavior of symbols under the change of the starting point $\mathbf{x}^\circ$, we adopt certain co-ordinate systems at points $\mathbf{x}^\bullet \in \Gamma$ near $\mathbf{x}^\circ$, arbitrarily, but consistent smoothly with the C–co-ordinate system at $\mathbf{x}^\circ$. Namely, for a point $\mathbf{x}^\bullet \in \Gamma$ with co-ordinates $(x_1, x_2, x_3) \equiv (x, x_3(= F(x)))$ with respect to the

[1] It is a long-standing problem, the Carathéodory conjecture, concerning the minimal possible number of umbilical points on a surface. According to this conjecture, for any smooth closed surface with nonzero Euler characteristic, there must exist at least two umbilical points. The topological torus may have no umbilical points at all. For surfaces of a different topological type, the existence of at least one umbilical point follows from simple topological considerations. For analytic surfaces the conjecture was settled not long ago, see [19] and [20] and the literature cited there. For a finite smoothness, and even for the infinite non-analytic one, the conjecture seems to be still unresolved.

C-co-ordinate system at $\mathbf{x}^\circ$, we consider the projection $P_{\mathbf{x}^\bullet}$ of the tangent plane at $\mathbf{x}^\circ$, $\mathrm{T}_{\mathbf{x}^\circ}(\Gamma)$, to the tangent plane $\mathrm{T}_{\mathbf{x}^\bullet}(\Gamma)$. The co-ordinates $y = (y_1, y_2)$ on $\mathrm{T}_{\mathbf{x}^\bullet}(\Gamma)$ will be generated on $\mathrm{T}_{\mathbf{x}^\bullet}(\Gamma)$ from $\mathrm{T}_{\mathbf{x}^\circ}(\Gamma)$ by this projection, with y_3 axis directed along the exterior normal at $\mathbf{x}^\bullet$ to Γ. What follows from this construction, is that the Jacobi matrix of this co-ordinate transformation is, up to higher order terms as $\mathbf{x}^\bullet \to \mathbf{x}^\circ$, the identity matrix $\mathbf{E}$ plus a term linear in first order derivatives of F. The first derivatives of this Jacobi matrix, by the chain rule, are matrices, linearly depending on the second derivatives of F. Therefore, due to (4.2), the derivatives of the Jacobi matrix are the principal curvatures $\mathbf{k}_1(\mathbf{x}^\circ)$, $\mathbf{k}_2(\mathbf{x}^\circ)$ at $\mathbf{x}^\circ$, which enter with universal coefficients. We direct the vectors in the frame of the fibre of the bundle at $\mathbf{x}^\bullet$ along the co-ordinate axes. The derivatives of the transformation matrix of the fiber to the standard frame at $\mathbf{x}^\circ$ are, again, linear forms of $\mathbf{k}_1(\mathbf{x}^\circ)$, $\mathbf{k}_2(\mathbf{x}^\circ)$ with universal coefficients, up to some higher order terms, as $\mathbf{x}^\bullet \to \mathbf{x}^\circ$.

4.3 The symbol $\mathfrak{m}_\iota$ and the principal curvatures

The key point in the reasoning to follow is the fact that the symbols we obtain are linear forms of principal curvatures with coefficients depending on the co-variables $\xi = (\xi_1, \xi_2)$ and the Lamé constants λ, μ but not on the point $\mathbf{x}^\circ$. We call such coefficients 'universal'. Our considerations will be based upon the analysis of the structure of various terms in the expansion of the principal symbol $\mathfrak{m}_{\iota,-1}$. The general idea is the following. If we have a function $\Phi(z)$, depending on some parameters λ, μ, ξ, and $z = F(x)$ is a function on $x \in \mathbb{R}^d$ such that $F(0) = 0, \nabla F(0) = 0$, then, according to the chain rule, the iterated gradient $\nabla_x^2 \Phi(F(x))|_{x=0}$ is a linear (matrix) form of second derivatives of F at zero, with coefficients depending only on λ, μ, ξ and $\Phi_z''(0)$, this means with universal coefficients.

We pass to the study of the symbol $\mathfrak{m}_\iota(x, \xi)$ (the principal, order -1, symbol of the operator $\mathfrak{M}_\iota$). Recall that this symbol is constructed following the rules (2.6), (2.7), (2.8). The expression (2.8) is a sum of 25 terms. These terms involve the principal symbol $\mathfrak{k}_0$ of the operator $\mathfrak{K}$, its first order derivatives in x and ξ and, finally, the subsymbol $\mathfrak{k}_{-1}$ of $\mathfrak{K}$. It was explained in Sect. 2 that relations (2.6), (2.7), (2.8) show that each of 25 additive terms in the expression for $\mathfrak{m}_\iota$ contains only one factor of order -1, all the remaining factors having order 0. Since we may perform our calculations in any co-ordinate system by our choice, we will study $\mathfrak{m}_\iota$ in the C–co-ordinate system centered at the point $\mathbf{x}^\circ$.

The symbol $\mathfrak{k}_0$ does not depend on the geometry of Γ, as can be seen in (2.4). So, the only way how $\mathfrak{m}_\iota$ can depend on the geometry is via $\nabla_x \mathfrak{k}_0$, $\nabla_\xi \mathfrak{k}_0$, and $\mathfrak{k}_{-1}$.

4.4 Dependence on the geometry of Γ. 1. $\nabla_x \mathfrak{k}_0$, $\nabla_\xi \mathfrak{k}_0$

Here and in the next subsection we determine which characteristics of the surface may be present in the expression of the terms in (2.8). The unwieldy explicit formulas are not needed at the moment (they will be discussed in Sect. 6 in more detail). First, we can see in (2.4), that the expression for the symbol $\mathfrak{k}_0$ does not involve any dependence

on $\mathbf{x}^\circ$, therefore, the same is true for the ξ derivatives of $\mathfrak{k}_0$ (these derivatives can be calculated directly from (2.4), but we will not do this at the moment).

To evaluate $\nabla_x \mathfrak{k}_0$ at the point $\mathbf{x}^\circ \in \Gamma$, we take another point $\mathbf{x}^\bullet \in \Gamma$, in a neighborhood of $\mathbf{x}^\circ$. We consider also the C–co-ordinate system centered at $\mathbf{x}^\circ$ and the consistent system centered at $\mathbf{x}^\bullet$, as explained in Sect. 4.2. We will mark by the superscript $\circ$ the principal symbol $\mathfrak{k}_0$ and other objects expressed in the $\mathbf{x}^\circ$– centered system and by $\bullet$ the same objects, but expressed in the $\mathbf{x}^\bullet$- centered system.

In this notation, we are interested in the derivative $\nabla_x \mathfrak{k}_0^\circ(x,\xi)$ calculated at the point $\mathbf{x}^\circ$, i.e., at $x = 0$. Thus we study the behavior of the principal symbol as $\mathbf{x}^\bullet$ approaches $\mathbf{x}^\circ$. We denote by $Z = Z_{\mathbf{x}^\bullet}$ the variables change on Γ in a neighborhood of $\mathbf{x}^\circ$ from the $\mathbf{x}^\circ$-centered co-ordinates to the $\mathbf{x}^\bullet$-centered ones. The Jacobi matrix $DZ = DZ_{\mathbf{x}^\circ}$ of this transformation contains the first order derivatives of the function F at $\mathbf{x}^\bullet$. The transformation $U(\mathbf{x}^\bullet) \in GL(\mathbb{R}, 3)$ from the $\mathbf{x}^\circ$-frame to the $\mathbf{x}^\bullet$-frame depends linearly on the first order derivatives of F at $\mathbf{x}^\bullet$ as well.

We use now the classical rule of transformation of the symbol under the change of variables and the natural rule of transformation under the change of the basis in the fiber. Namely, we write the symbol in $\mathbf{x}^\bullet$-centered co-ordinates—it will have the same form as (2.4)—and then transform it to $\mathbf{x}^\circ$-centered co-ordinates. In this way, we have at the point $\mathbf{x}^\bullet$

$$\begin{aligned}\mathfrak{k}_0^\circ(x,\xi) &= U(\mathbf{x}^\bullet)\mathfrak{k}_0^\bullet(Z(\mathbf{x}^\bullet), ((DZ)^{-1})^\top \xi) U(\mathbf{x}^\bullet)^{-1} \\ &= U(\mathbf{x}^\bullet)\frac{i\Bbbk}{|\xi|}\begin{pmatrix} 0 & 0 & -\eta_1 \\ 0 & 0 & -\eta_2 \\ \eta_1 & \eta_2 & 0 \end{pmatrix} U(\mathbf{x}^\bullet)^{-1},\end{aligned}$$

with $\eta \equiv (\eta_1, \eta_2) = ((DZ)^{-1})^\top \xi$. We recall here that the variables change $Z(\mathbf{x}^\bullet)$, its differential $DZ(\mathbf{x}^\bullet)$, and the linear transformations $U(\mathbf{x}^\bullet)$ depend smoothly on the first order derivatives of the function $F(x_1, x_2)$, moreover they become identity maps as $\mathbf{x}^\bullet \to \mathbf{x}^\circ$, since the derivatives of F vanish ay $\mathbf{x}^\circ$. Therefore, the derivatives of $\mathfrak{k}_0^\circ(x,\xi)$ at $\mathbf{x}^\circ$, by the chain rule, depend linearly on the second derivatives of F at $\mathbf{x}^\circ$, with no more characteristics of F involved. Since the co-ordinates x_1, x_2 have been chosen along the curvature lines of Γ at $\mathbf{x}^\circ$, the mixed second derivative of F vanishes, while the pure second derivatives are equal to the principal curvatures of the surface at the point $\mathbf{x}^\circ$, moreover, this dependence is linear. Thus, we have established that

$$\nabla_x \mathfrak{k}_0^\circ(x,\xi)_{x=0} = \Bbbk(\mathbf{k}_1(\mathbf{x}^\circ) f_1(\xi) + \mathbf{k}_2(\mathbf{x}^\circ) f_2(\xi))$$

in the C–co-ordinate system centered at $\mathbf{x}^\circ$, with some (matrix) symbols f_1, f_2 of order 0 depending on ξ (and the Lamé constants) only. The same conclusion holds at umbilical points, where $\mathbf{k}_1(\mathbf{x}^\circ) = \mathbf{k}_2(\mathbf{x}^\circ)$, by a similar reasoning.

4.5 Dependence on the geometry of Γ. 2. $\mathfrak{k}_{-1}(x, \xi)$

In order to find the required representation for the subsymbol $\mathfrak{k}_{-1}(x,\xi)$ of the operator $\mathfrak{K}$, it is more convenient to work not with the symbol but with the kernel of the integral operator.

We consider the local expression (2.3) for the NP operator. Having the point $\mathbf{x} = \mathbf{x}^\circ$ (the point $x = 0$ in the C-co-ordinate system centered at $\mathbf{x}^\circ$) fixed, we expand all entries of the kernel $\mathcal{K}$ in the asymptotic series in terms, positively homogeneous in $y - x$. We are interested in the first two terms in this expansion in the form

$$\begin{aligned} \mathcal{K}(\mathbf{x}, \mathbf{y}) &= \mathcal{K}_0(x, x - y) + \mathcal{K}_{-1}(x, x - y) + O(1); \\ \mathbf{x} &= (x, F(x)) = 0 \in \Gamma, \ \mathbf{y} = (y, F(y)) \in \Gamma, \end{aligned} \tag{4.3}$$

where $\mathcal{K}_0(x, x - y)$ is order -2 positively homogeneous and odd in $(x - y)$ and $\mathcal{K}_{-1}(x, x - y)$ is order -1 positively homogeneous in $(x - y)$. In order to find these terms, we consider the expansion for separate terms in (2.3). Here we keep in mind the Taylor expansion for the function F near $\mathbf{x}^\circ$, $F(x) = \frac{1}{2}(H(\mathbf{x}^\circ)x, x) + O(|x|^3)$, $x \to 0$, where $H = \operatorname{diag}(\mathbf{k}_1(\mathbf{x}^\circ), \mathbf{k}_2(\mathbf{x}^\circ))$. Next, by our choice of co-ordinates, the co-ordinate axes x_1, x_2 lie along the eigenvectors of the matrix H (for an umbilical point, i.e., when H is a multiple of the unit matrix, any orthogonal directions may be chosen.) In this co-ordinate system, the first fundamental form of the surface Γ at $\mathbf{x}^\circ$ is the identity one,

$$\mathbf{I}[\Gamma]_{\mathbf{x}^\circ}(dx) = |dx|^2.$$

The second fundamental form for this surface at $\mathbf{x}^\circ$ is diagonal in this co-ordinate system:

$$\mathbf{II}[\Gamma]_{\mathbf{x}^\circ}(dx) = \mathbf{k}_1(\mathbf{x}^\circ)(dx_1)^2 + \mathbf{k}_2(\mathbf{x}^\circ)(dx_2)^2,$$

calculated, recall, with the direction of the normal vector chosen to be the outward one, so it is *negative* at those points where the surface is convex. For the entries in the kernel of the integral operator $\mathfrak{K}$, at the point $\mathbf{x}^\circ$ with co-ordinates $(x, F(x)) = (0, 0)$ in the chosen co-ordinate system, we use the standard relations for co-ordinates in this system. Namely, for the components of the normal vector, we have

$$\nu_\alpha(y) = \nu_\alpha(0) + \mathbf{k}_\alpha(\mathbf{x}^\circ)y_\alpha + O(|y|^2) = \mathbf{k}_\alpha(\mathbf{x}^\circ)y_\alpha + O(|y|^2), \alpha = 1, 2, \tag{4.4}$$

and

$$\nu_3(y) = 1 - O\left(|y|^2\right). \tag{4.5}$$

The distance between points, entering in (2.3), is found as

$$|\mathbf{x}^\circ - \mathbf{y}|^2 = |x - y|^2 \left(1 + 2\frac{\mathbf{II}[\Gamma]_{\mathbf{x}^\circ}(x - y)^2}{|x - y|^2}\right) + o(|x - y|^4). \tag{4.6}$$

We substitute (4.4), (4.5), (4.6) in (2.3) and obtain that the leading term $\mathcal{K}_{-1}$ in the singularity as $y \to x$ of the kernel $\mathcal{K}_{-1}(y, x - y)$ is a linear function of the principal curvatures.

4.6 Proof of Theorem 4.1

Finally, we take into account the structure properties of the symbol $\mathfrak{m}(x, \xi)$ as it depends on the principal symbol and the subsymbol of the operator $\mathfrak{K}$, see (2.6), (2.7),

(2.8). At a given point, in C–co-ordinates, for each summand in (2.8), the factors $\mathfrak{k}_0$ do not depend on the geometry of Γ. Exactly one factor is present in each summand that involves the geometry of Γ, and this term is linear in the principal curvatures. All other terms in the products in (2.8) do not involve geometric characteristics of the surface, and therefore are universal matrices (depending, of course, on the material constants and the direction of the covector $\xi \in \mathrm{T}^*_x\Gamma$.) Therefore, each summand, being the product of 5 factors, and further on, the whole symbol $\mathfrak{m}_\iota(x,\xi)$, depend linearly on the principal curvatures of Γ, with universal coefficients. This concludes the proof of Theorem 4.1.

5 Symmetries, reductions, and curvatures in asymptotic formulas

In this section we start applying the results about the spectrum of general polynomially compact pseudodifferential operators, obtained in [34], see Theorem 3.4, to study the asymptotics of eigenvalues of the NP operator. At this stage we will see that some properties of the spectrum can be derived by means of qualitative considerations, without calculating the symbols explicitly. Later, we will present more detailed spectral properties based upon explicit calculations.

5.1 Symmetries

The first property follows from the fact that the symbol $\mathfrak{k}(x,\xi)$ should transform in a definite way as soon as we permute the co-ordinate axes x_1 and x_2. We choose the principal curvatres in the convenient way. Namely, let $\mathbf{k}_1 = \mathbf{k} \neq 0$, $\mathbf{k}_2 = 0$. Then, by (4.1),

$$\mathfrak{m}_\iota(x,\xi_1,\xi_2) = \mathbf{k} M_\iota^{(1)}(\xi_1,\xi_2). \tag{5.1}$$

Now, let $\mathbf{k}_1 = 0$, $\mathbf{k}_2 = \mathbf{k}$. The corresponding symbol must be the same as (5.1), after the following transformations caused by the permutation of co-ordinates:

(1) The covariables (ξ_1,ξ_2) must be permuted, $(\xi_1,\xi_2) \leftrightarrows (\xi_2,\xi_1)$;
(2) $M_\iota^{(1)}$ should be replaced by $M_\iota^{(2)}$;
(3) The frame in $\mathbb{C}^3$ changes, therefore, the first two horizontal rows should be permuted as well as two first columns. This is obtained by the transformation $\mathfrak{m} \leftrightarrows V^{-1}\mathfrak{m}$, where V is the linear unitary transformation in $\mathbb{C}^3$ interchanging the first row with the second one, i.e., the matrix

$$V = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \ V = V^{-1}.$$

So, we obtain

$$\mathfrak{m}_\iota(x,\xi_1,\xi_2) = \mathbf{k} M_\iota^{(1)}(\xi_1,\xi_2) = \mathbf{k} V^{-1} M_\iota^{(2)}(\xi_2,\xi_1) V,$$

therefore, if we denote $\hat{\xi} \equiv \widehat{(\xi_1, \xi_2)} = (\xi_2, \xi_1)$

$$M_\iota^{(2)}(\xi_1, \xi_2) = V^{-1} M_\iota^{(1)}(\xi_2, \xi_1) V. \tag{5.2}$$

As a result, the symbol $\mathfrak{m}_\iota(x; \xi_2, \xi_1)$ must depend on *only one* universal matrix $M_\iota(\xi_1, \xi_2) = M_\iota^{(1)}(\xi_1, \xi_2)$, and we obtain, for a general body, with principal curvatures $\mathbf{k}_1(x), \mathbf{k}_2(x)$,

$$\mathfrak{m}_\iota(x, \xi) = \mathbf{k}_1(x) M_\iota(\xi) + \mathbf{k}_2(x) V^{-1} M_\iota(\hat{\xi}) V. \tag{5.3}$$

We consider now the special case of Γ being the unit sphere S^2 in $\mathbb{R}^3$. All points on Γ are umbilical, moreover, $\mathbf{k}_1(x) = \mathbf{k}_2(x) = -1$ everywhere on Γ. We can choose the local (orthogonal) co-ordinates in an arbitrary way. Therefore, the symbol $\mathfrak{m}_\iota(x, \xi)$ for the sphere equals

$$\mathfrak{m}_\iota(x, \xi) = -M_\iota(\xi) - V^{-1} M_\iota(\widehat{\xi}) V. \tag{5.4}$$

For the sphere, the eigenvalues of the symbol $\mathfrak{m}_\iota(x, \xi)$ are the same for all points x, and, accordingly, the integrand in (3.11) is independent of the point x. Therefore, the asymptotic formula (3.10), (3.11) gives the following expression for the coefficient $C^{\pm}(\omega_\iota)$ for the sphere:

$$\begin{aligned} C^{\pm}(\omega_\iota) &= 2^{-1}(2\pi)^{-2} \int_{S^*\Gamma} \operatorname{Tr} {}^{(2)}_{\mp}(M_\iota(\xi) + V^{-1} M_\iota(\hat{\xi}) V) \boldsymbol{\omega} d\mathcal{S}(x) \\ &= (2\pi)^{-1} \int_{S^1} \operatorname{Tr} {}^{(2)}_{\mp}(M_\iota(\xi) + V^{-1} M_\iota(\hat{\xi}) V) \boldsymbol{\omega} \end{aligned} \tag{5.5}$$

(note the sign: $\operatorname{Tr} {}^{(2)}_{\mp}$ enters in the formula, since the curvature $\mathbf{k}_1 = \mathbf{k}_2$ equals -1).

We recall now the explicit formulas for the eigenvalues of the NP operator on the sphere, see (1.1). These formulas show that the eigenvalues of $\mathfrak{K}$ approach each of three points of the essential spectrum *from above*. Therefore the coefficients $C^-(\omega_\iota)$ in (3.10) vanish. This, by Theorem 4.1, means that the integrand in the last line in (5.5) eigenvalue asymptotic formula (3.11), $\operatorname{Tr} {}^{(2)}_{\mp}(M_\iota(\xi) + V^{-1} M_\iota(\hat{\xi}) V)$ is always positive for $-$ sign and it is everywhere zero for the $+$ sign. Therefore, the matrix $(M_\iota(\xi) + V^{-1} M_\iota(\hat{\xi}) V)$ for each ι, is nonpositive for all $\xi \in S^1$, and at least for some ξ at least one of eigenvalues is strictly negative.

5.2 Curvatures and the eigenvalue asymptotics

The property of $(M_\iota(\xi) + V^{-1} M_\iota(\hat{\xi}) V)$ being non-positive does not automatically imply that the matrix $M_\iota(\xi)$ is non-positive – it is easy to construct a counterexample. Nevertheless, certain important results can be derived from it.

First of all, we note that the trace $\operatorname{tr}(M_\iota(\xi) + V^{-1} M_\iota(\hat{\xi}) V)$ is non-positive for all $\xi \in S^1$ and strictly negative for some ξ. Since $\operatorname{tr}(V^{-1} M_\iota(\hat{\xi}) V)) = \operatorname{tr} M_\iota(\hat{\xi})$, we have

$$\operatorname{tr} M_\iota(\xi) + \operatorname{tr} M_\iota(\hat{\xi}) \le 0.$$

and at least at one point $\xi \in S^1$ this trace is negative. It follows that for such ξ at least one of eigenvalues of the matrix $M_\iota(\xi)$ is strictly negative.

Now we use the fact that for any bounded body $\mathcal{D} \subset \mathbb{R}^3$ with smooth boundary Γ there exists a point $\mathbf{x}^\circ$ where *both principal curvatures are strictly negative*—see Sect. 5.4. Then the trace

$$\operatorname{tr}(\mathfrak{m}_\iota(x,\xi)) = \mathbf{k}_1(\mathbf{x}^\circ)\operatorname{tr}\left(M_\iota^{(1)}(\xi)\right) + \mathbf{k}_2(\mathbf{x}^\circ)\operatorname{tr}\left(M_\iota^{(2)}(\xi)\right) \tag{5.6}$$

is strictly positive for some ξ and, consequently, at least one of eigenvalues of $\mathfrak{m}_\iota(x,\xi)$ is strictly negative. Therefore, as explained in Proposition 3.5, the coefficient C_ι^+ does not vanish. Since on every compact smooth surface in $\mathbb{R}^3$, such point $\mathbf{x}^\circ$ exists, this gives us the following result for the eigenvalues of the NP operator.

Theorem 5.1 *For any body $\mathcal{D}$, the coefficients $C^+(\omega_\iota)$ in (3.11) are strictly positive; therefore there exist infinitely many eigenvalues of the NP operator $\mathfrak{K}$ approaching the points ω_ι of the essential spectrum* from above*, and they satisfy the asymptotic law (3.10), (3.11) with 'plus' sign.*

On the other hand, if there exists a point at the boundary where it is concave, the existence of eigenvalues converging to ω_ι from below is granted. More exactly,

Theorem 5.2 *Let the boundary Γ of the body $\mathcal{D}$ have at least one point such that both principal curvatures are non-negative while at least one is strictly positive. Then the coefficients $C^-(\omega_\iota)$ in (3.11) is positive, this means that there are infinitely many eigenvalues of $\mathfrak{K}$ approaching ω_ι from below, and they satisfy the asymptotic formula (3.10) with '-' sign.*

An interesting particular case of this theorem is the following.

Corollary 5.3 *Suppose that the body $\mathcal{D}$ contains a cavity inside. Then the coefficients $C^-(\omega_\iota)$ in (3.11) are positive, there are infinitely many eigenvalues of $\mathfrak{K}$ approaching ω_ι from below, and they satisfy the asymptotic formula (3.10) with '-' sign.*

It is clear that at the point where the boundary of the cavity is convex, the surface Γ, considered as the boundary of $\mathcal{D}$, is concave and Theorem 5.2 applies.

5.3 Two-sided asymptotics

Using our results on the structure of the symbol $\mathfrak{m}_\iota(x,\xi)$ we can find a visual expression for the two-sided asymptotics of eigenvalues of the NP operator $\mathfrak{K}$, following Theorem 3.6.

Let Γ be a smooth compact surface and $\mathbf{k}_1(\mathbf{x}), \mathbf{k}_2(\mathbf{x})$ be the principal curvatures at the point $\mathbf{x} \in \Gamma$. We suppose that they are calculated in C-co-ordinate systems, discussed above, where $d\mathcal{S}(x)$ equals the area element for the measure on Γ induced by the Lebesgue measure in $\mathbb{R}^3$.

We calculate the integrand in (3.13); it gives

$$\operatorname{tr}\left((\mathfrak{m}_\iota(x,\xi))^2\right) = \operatorname{tr}\left(\left[\mathbf{k}_1(x)M_\iota(\xi) + \mathbf{k}_2(x)V^{-1}M_\iota(\hat{\xi})V\right]^2\right)$$

$$= \mathbf{k}_1(x)^2 \operatorname{tr} M_\iota^2(\xi) + 2\mathbf{k}_1(x)\mathbf{k}_2(x) \operatorname{tr} \left[M_\iota(\xi) V^{-1} M_\iota\left(\hat{\xi}\right) V \right] + \mathbf{k}_2^2 \operatorname{tr} M_\iota^2(\hat{\xi}),$$

(here, we used the fact that $\operatorname{tr}((V^{-1}MV)^2) = \operatorname{tr}(M^2)$). We substitute (5.3) into (3.13) and use the fact that $\int_{S^1} M_\iota^2(\hat{\xi})\boldsymbol{\omega} = \int_{S^1} M_\iota^2(\xi)\boldsymbol{\omega}$ and obtain

$$\begin{aligned} C(\omega_\iota) &= 2^{-1} \int_\Gamma (\mathbf{k}_1^2 + \mathbf{k}_2^2) d\mathcal{S}(x) \int_{S^1} \operatorname{tr}(M_\iota^2(\xi))\boldsymbol{\omega} \\ &\quad + \int_\Gamma \mathbf{k}_1(x)\mathbf{k}_2(x) d\mathcal{S}(x) \int \operatorname{tr}[M_\iota(\xi) V^{-1} M_\iota(\hat{\xi}) V]\boldsymbol{\omega} \\ &\equiv \mathscr{A}_\iota W(\Gamma) + \mathscr{B}_\iota \chi(\Gamma), \end{aligned} \tag{5.7}$$

where $\chi(\Gamma)$ is the Euler characteristic of the surface Γ,

$$\chi(\Gamma) = (2\pi)^{-1} \int_\Gamma \mathbf{k}_1(x)\mathbf{k}_2(x) d\mathcal{S}(x),$$

by the Gauss-Bonnet formula, and $W(\Gamma)$ is the Willmore energy of Γ,

$$W(\Gamma) = \int_\Gamma \frac{(\mathbf{k}_1(x) + \mathbf{k}_2(x))^2}{4} d\mathcal{S}(x).$$

The coefficients $\mathscr{A}_\iota$, $\mathscr{B}_\iota$ depend only on the Lamé constants λ, μ and they are equal to

$$\mathscr{A}_\iota = 2 \int_{S^1} \operatorname{tr}(M_\iota^2(\xi))\boldsymbol{\omega},$$

and

$$\mathscr{B}_\iota = 2\pi \int_{S^1} \left[\operatorname{tr}[M_\iota(\xi) V^{-1} M_\iota(\hat{\xi}) V] - \operatorname{tr}[M_\iota^2(\xi)] \right] \boldsymbol{\omega}$$

We formulate this, rather esthetic, result as a theorem.

Theorem 5.4 *Let Γ be a compact smooth surface in $\mathbb{R}^3$. Then for each point $\Bbbk_\iota$, $\iota = -1, 0, 1$, of the NP operator, there exist infinite sequences of eigenvalues converging to $\Bbbk_\iota$ satisfying the two-sided asymptotic law (3.12) with nonzero coefficients given by (5.7). In particular, the degenerate case in the spectral analysis of polynomially compact pseudodifferential operators never happens for the elastic NP operator.*

Remark 5.5 One can note a similarity of our Theorem 5.4 and, especially, formula (5.7), with the results of the papers [30, 32] concerning the eigenvalues, tending to zero, of the compact NP operator in 3D electrostatics. There, the asymptotics of the, separately considered, positive and negative eigenvalues contains, respectively, integral of positive and negative parts of a certain rather complicated expression involving the curvatures of the surface, while the coefficient in the two-sided asymptotics of these eigenvalues is expressed in a linear way via the global geometric characteristics of the surface, namely, its Euler charateristic and Willmore energy.

5.4 Some geometry

In the considerations above we used the following geometrical fact: for a smooth closed surface $\Gamma \subset \mathbb{R}^3$, there exists a point $\mathbf{x}^\circ \in \Gamma$ such that the surface is strictly convex at $\mathbf{x}^\circ$, this means, both principal curvatures of Γ at $\mathbf{x}^\circ$ are negative. We were not able to locate a proof of this, probably, folklore, result in the literature, therefore we give an elementary proof here (not pretending that it is a novel one).

Let $\mathbf{d} = \operatorname{diam}(\Gamma)$ be the diameter of Γ, the largest distance between a pair of points in Γ. By compactness, such pair must exist (probably, not a unique one, but this does not matter.) Let A, B be such points. Let $\mathbf{k}_1(B), \mathbf{k}_2(B)$ be the principal curvatures of Γ at B in a C-co-ordinate system.

Proposition 5.6 *The principal curvatures at the point B satisfy $\mathbf{k}_1(B), \mathbf{k}_2(B) \le -\mathbf{d}^{-1}$.*

Proof In a C-co-ordinate system centered at B, the surface is described near B by

$$x_3 = \frac{\mathbf{k}_1(B)}{2} x_1^2 + \frac{\mathbf{k}_2(B)}{2} {x_2}^2 + O\left(|x|^3\right).$$

Therefore, the distance from A to the point $\mathbf{x}$ of Γ near B, with co-ordinates x_1, x_2, x_3 satisfies

$$\operatorname{dist}^2(A, \mathbf{x}) = \left(x_1^2 + x_2^2 + \left(\mathbf{d} + \frac{\mathbf{k}_1(B)}{2} x_1^2 + \frac{\mathbf{k}_2(B)}{2} {x_2}^2\right)\right)^2 + O(|x|^3).$$

If we suppose, for example, that $\mathbf{k}_1(B) > -\mathbf{d}^{-1}$, we have for $x_2 = 0$,

$$\operatorname{dist}^2(A, \mathbf{x}) = \left(x_1^2 + \left(\mathbf{d} + \frac{\mathbf{k}_1(B)}{2} x_1^2\right)\right)^2 + O\left(x_1^3\right) = \mathbf{d}^2 + x_1^2 - \mathbf{d}\mathbf{k}_1(B) x_1^2 + O\left(x_1^3\right),$$

therefore $\operatorname{dist}(A, \mathbf{x}) > \mathbf{d}$ for small x_1, and this contradicts the fact that $\mathbf{d}$ is the diameter of Γ. □

6 Explicit calculation of the symbol $\mathfrak{m}_\iota$: dependence on the Lamé constants

As marked above, the expression for the coefficients in (3.11) is highly complicated since it involves integration of the eigenvalues of a definite sign of the symbolic matrices $\mathfrak{m}_\iota(x, \xi)$. In this section we demonstrate, nevertheless, how the symbol $\mathfrak{m}_\iota$ can be calculated. In order to find the symbol $\mathfrak{m}_{\iota,-1}(x, \xi)$ we need to find just one matrix-function $M_\iota(\xi)$. This matrix depends on the Lamé constants of the material and is a function of the covector $\xi \in \mathbb{R}^2$.

6.1 The standard surface

Since the universal symbol $M_\iota(\xi)$ is the same for all surfaces, it is sufficient to find it for just one, specially chosen, surface, where the calculation of M_ι is less troublesome. As such a surface we take the one with only one of principal curvatures nonvanishing. As such standard surface, we select the cylinder Γ with radius $R = -\mathbf{k}^{-1}$, $\mathbf{k} < 0$ and we perform calculation in more detail than it was done in Sect. 4. This enables us to determine the dependence of the effective symbol on the Lamé constants. (It does not matter that the cylinder is non-compact since the symbol $M_\iota(\xi)$ is a local quantity.)

We consider such cylinder Γ as being described by the equation $x_1^2 + (x_3 + R)^2 = R^2$; in a neighborhood of'the point $\mathbf{x}^\circ$ placed in the origin, $\mathbf{x}^\circ : x_1 = x_2 = x_3 = 0$, we can write

$$x_3 = F(x) = -R + \sqrt{R^2 - x_1^2} = \frac{1}{2}\mathbf{k}x_1^2 + O\left(x_1^4\right),\ x_1 \to 0,\ x_2 \in \mathbb{R}^1, \qquad (6.1)$$

with the normal vector directed outside the body, i.e., upward along the x_3 axis at the point $\mathbf{x}^\circ = (0, 0, 0)$. The C-co-ordinates lines are, due to our construction in Sect. 4.2, directed along the curvature lines, i.e., along the orthogonal cross-section of the cylinder and along its axis. The frame vectors are directed along x_1, x_2, x_3 axes in $\mathbb{R}^3 \subset \mathbb{C}^3$. In the chosen co-ordinates, $\mathbf{k}_1(\mathbf{x}^\circ) = \mathbf{k}$, $\mathbf{k}_2(\mathbf{x}^\circ) = 0$, and derivatives of all entries in the direction x_2 vanish.

The first and second fundamental forms of the surface G at the point $\mathbf{x}^\circ$ are equal to

$$\mathbf{I}(\mathbf{x}^\circ, dx) = dx_1^2 + dx_2^2;\ \mathbf{II}(\mathbf{x}^\circ, dx) = \mathbf{k}dx_1^2.$$

The normal vector $\boldsymbol{\nu}(\mathbf{y}) = \boldsymbol{\nu}(y_1, y_2, F(y_1))$ at the point $\mathbf{y} = (y, F(y)) = (y_1, y_2, F(y_1, y_2))$ lying close to $\mathbf{x}^\circ$ equals

$$\boldsymbol{\nu}(\mathbf{y}) = \left(-\mathbf{k}y_1, 0, \sqrt{1 - \mathbf{k}^2 y_1^2}\right)^\top = (-\mathbf{k}y_1, 0, 1)^\top + O(y_1^2),\ y_1 \to 0. \qquad (6.2)$$

Therefore, its derivatives are

$$\frac{\partial \boldsymbol{\nu}(\mathbf{y})}{\partial y_1} = (-\mathbf{k}, 0, 0)^\top + O(|y_1|);\ \frac{\partial \boldsymbol{\nu}(\mathbf{y})}{\partial y_2} = 0.$$

6.2 The expansion of the kernel

Recall that the kernel $\mathcal{K}(\mathbf{x}, \mathbf{y})$ of the NP operator $\mathfrak{K}$ is given in (2.3). We will express it in co-ordinates x, y and find the first two power terms in the expansion of $\mathcal{K}(y, y - x)$, as $y \to x$.

In the C–co-ordinate system centered at $\mathbf{x} = \mathbf{x}^\circ = 0$, we calculate contributions to the singularities of the symbol of $\mathfrak{K}$ coming from the terms in (2.3) separately. First, we find the expansion of the distance $|\mathbf{x} - \mathbf{y}| = |\mathbf{y}|$ and its powers. We have:

$|\mathbf{y}|^2 = y_1^2 + y_2^2 + F(y_1)^2$, therefore

$$|\mathbf{y}|^{-2} = |y|^{-2}\left(1 + \frac{F(y_1)^2}{|y|^2}\right)^{-1} = |y|^{-2} + O(1).$$

Similarly,

$$|\mathbf{y}|^{-3} = |y|^{-3}\left(1 + O(|y|^2)\right) = |y|^{-3} + O\left(|y|^{-1}\right),\ y \to 0.$$

Thus, when calculating the leading two terms in the expansion in (4.3), we may replace $|\mathbf{y}|$ by $|y|$.

Next we evaluate the expression $S_1(\mathbf{y}) = -\sum_{l=1}^3 \nu_l(\mathbf{y}) y_l$ entering in the second line (2.3), for $\mathbf{y} = (y, F(y_1)) \in \Gamma$. Since $\boldsymbol{\nu}(\mathbf{y}) = (-\mathbf{k}y_1, 0, 1)^\top + O(y_1^2)$, we have

$$S_1 = -(-\mathbf{k}y_1, 0, 1)\left(y_1, y_2, \frac{1}{2}\mathbf{k}y_1^2\right)^\top + O\left(|y|^3\right) = \frac{1}{2}\mathbf{k}y_1^2 + O\left(|y|^3\right),\ y \to 0.$$

Further on, we calculate the expression $S_{2,p,q}(y) = (x_p - y_p)(x_q - y_q)$ in (2.3), with $x_p = x_q = 0$. On the matrix diagonal, i.e., for $p = q$,

$$S_{2,1,1} = y_1^2;\ S_{2,2,2} = y_2^2;\ S_{2,3,3} = F(y_1)^2 = O\left(y_1^4\right),$$

while off-diagonal, for $p \neq q$,

$$S_{2,1,2} = S_{2,2,1} = y_1 y_2;\ S_{2,1,3} = S_{2,3,1} = F(y_1)y_1 = O\left(|y|^3\right);$$
$$S_{2,3,2} = S_{2,2,3} = F(y_1)^2 = O\left(|y|^4\right).$$

Therefore, the expression on the second line in (2.3) equals, for $\mathbf{x} = 0$,

$$[\mathcal{K}^{(2)}(y)]_{p,q} = \frac{1}{2\pi}\left[\Bbbk\delta_{pq} + \frac{3\mathfrak{m}}{2}\frac{y_p y_q}{|y|^2}\right]\frac{\mathbf{k}y_1^2}{|y|^3} + O(1),\ p, q = 1, 2,$$
$$[\mathcal{K}^{(2)}(y)]_{p,q} = O(1),\ \text{for } p = 3, q = 2 \text{ or } p = 2, q = 3,$$

and, finally,

$$[\mathcal{K}^{(2)}(y)]_{3,3} = -\frac{1}{2\pi}\Bbbk\mathbf{k}\frac{y_1^2}{|y|^3} + O(1)$$

(recall that $\Bbbk = \frac{\mu}{2(\lambda+2\mu)}$).

In particular, we can see that the expression on the second line in (2.3) has singularity of order -1 in $|x - y|$, and, therefore, contributes only to the subsymbol of the pseudodifferential operator $\mathfrak{K}$, but not to its principal symbol.

Next we calculate the expansion on the first line in (2.3) (recall, we set $\mathbf{x} = \mathbf{x}^\circ = 0$ here.) These terms may contribute to both principal and subsymbol of $\mathfrak{K}$. Note first

that the matrix defined by this line is antisymmetric, therefore, the diagonal terms, $p = q$, vanish.

For the off-diagonal terms, using (6.2), we obtain

$$\begin{aligned}
\nu_1(\mathbf{y})y_2 - \nu_2(\mathbf{y})y_1 &= -\mathbf{k}y_1y_2;\\
\nu_1(\mathbf{y})y_3 - \nu_3(\mathbf{y})y_1 &= -y_1 + O(|y|^3);\\
\nu_2(\mathbf{y})y_3 - \nu_3(\mathbf{y})y_2 &= -y_2 + O(|y|^3).
\end{aligned}$$

Therefore, the expression on the first line in (2.3) equals

$$\begin{aligned}
\mathcal{K}^{(1)}(y) = \frac{1}{2\pi}\Bbbk|y|^{-3}&\begin{pmatrix} 0 & 0 & -y_1\\ 0 & 0 & -y_2\\ y_1 & y_2 & 0\end{pmatrix}\\
+\frac{1}{2\pi}\Bbbk\mathbf{k}|y|^{-3}&\begin{pmatrix} 0 & -y_1y_2 & 0\\ y_1y_2 & 0 & 0\\ 0 & 0 & 0\end{pmatrix},
\end{aligned}\tag{6.3}$$

and the one on the second line is

$$\begin{aligned}
\mathcal{K}^{(2)}(y) &= \frac{\mathbf{k}}{2\pi}\begin{pmatrix} \frac{3\Bbbm}{2}\frac{y_1^4}{|y|^5} + \Bbbk\frac{y_1^2}{|y|^3} & \frac{3\Bbbm}{2}\frac{y_1^3y_2}{|y|^5} & 0\\ \frac{3\Bbbm}{2}\frac{y_1^3y_2}{|y|^5} & \frac{3\Bbbm}{2}\frac{y_1^2y_2^2}{|y|^5} + \Bbbk\frac{y_1^2}{|y|^3} & 0\\ 0 & 0 & \Bbbk\frac{y_1^2}{|y|^3}\end{pmatrix}\\
&= \frac{1}{2\pi}\mathbf{k}\Bbbk\frac{y_1^2}{|y|^3}\mathbf{E} + \frac{3}{4\pi}\mathbf{k}\Bbbm y_1^2|y|^{-5}\begin{pmatrix} y_1^2 & y_1y_2 & 0\\ y_1y_2 & y_2^2 & 0\\ 0 & 0 & 0\end{pmatrix}.
\end{aligned}\tag{6.4}$$

The first term in (6.3) corresponds to the principal part of the symbol of the NP operator only, while (6.4) and the second term in (6.3) contribute to the subsymbol.

6.3 The expansion of the symbol

Next we transform our formulas for the kernels (6.3), (6.4) of the integral operators to the corresponding expressions for the components of the symbol of the pseudodifferential operator $\mathfrak{K}$. Recall that this symbol is the Fourier transform of the kernel $\mathcal{K}$ of the integral operator in $x - y \in \mathbb{R}^2$ variable. Some of formulas we use can be found in standard tables of the Fourier transform of distributions, other ones need to be calculated by hand.

We start with recalling that the kernel $(2\pi|x-y|)^{-1}$ in $\mathbb{R}^2$ corresponds, by means of the Fourier transform $\mathscr{F}$ in $\mathbb{R}^2$, to the symbol $|\xi|^{-1}$,

$$\mathscr{F}[(2\pi|y|)^{-1}] = |\xi|^{-1}.$$

Further on, since $\frac{y_p}{|y|^3} = -\partial_p(|y|^{-1})$, $p = 1, 2$,

$$\mathscr{F}\left[(2\pi)^{-1}\frac{y_p}{|y|^3}\right] = i\xi_p|\xi|^{-1},$$

in the sense of distributions.

Next, we consider the kernel $|y|^{-5}$. Its Fourier transform in the sense of distributions equals $\frac{2\pi}{9}|\xi|^3$. By the properties of the Fourier transform, we have

$$\mathscr{F}\left[\frac{y_1^3 y_2}{2\pi|y|^5}\right] = \frac{1}{9}\partial_{\xi_2}\partial_{\xi_1}^3(|\xi|^3) = \xi_2^3\xi_1|\xi|^{-5}.$$

In the same way,

$$\mathscr{F}\left[\frac{y_1^4}{2\pi|y|^5}\right] = \frac{1}{9}\partial_{\xi_1}^4(|\xi|^3) = \xi_2^4|\xi|^{-5}.$$

Finally,

$$\mathscr{F}\left[\frac{y_1^2}{2\pi|y|^3}\right] = \xi_2^2|\xi|^{-3}, \tag{6.5}$$

$$\mathscr{F}\left[\frac{y_1 y_2}{2\pi|y|^3}\right] = \xi_1\xi_2|\xi|^{-3}, \tag{6.6}$$

and

$$\mathscr{F}\left[\frac{y_2^2}{2\pi|y|^3}\right] = \xi_1^2|\xi|^{-3}. \tag{6.7}$$

As a result, the symbol of the NP operator $\mathfrak{K}$ on the surface Γ equals, in C-coordinates centered at the point $\mathbf{x}^\circ = (0, 0, 0)$,

$$\begin{aligned}\mathfrak{k}(\mathbf{x}^\circ, \xi) = i\Bbbk &\begin{pmatrix} 0 & 0 & -\xi_1|\xi|^{-1} \\ 0 & 0 & -\xi_2|\xi|^{-1} \\ \xi_1|\xi|^{-1} & \xi_2|\xi|^{-1} & 0 \end{pmatrix} \\ &+\frac{1}{2}\mathbf{k}\Bbbk\xi_2^2|\xi|^{-3}\mathbf{E} + \frac{3}{2}\mathbf{k}\mathbb{m}|\xi|^{-5}\xi_2^2\begin{pmatrix} \xi_2^2 & \xi_1\xi_2 & 0 \\ \xi_1\xi_2 & \xi_1^2 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\ &+\mathbf{k}\mathbb{m}|\xi|^{-3}\begin{pmatrix} 0 & \xi_1\xi_2 & 0 \\ -\xi_1\xi_2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + O(1). \end{aligned} \tag{6.8}$$

So, we have calculated the leading two terms in the expansion of the symbol of $\mathfrak{K}$ for the case of a cylinder. In the expression (6.8), the first term is order -0 homogeneous

and represents the leading symbol $\mathfrak{k}_0$, as we already know, while the remaining terms, the ones on the second and third lines, represent the subprincipal symbol $\mathfrak{k}_{-1}$. We note that both $\mathfrak{k}_0$ and $\mathfrak{k}_{-1}$ are Hermitian. Moreover, $\mathfrak{k}_0$ does not depend on the geometry of the surface Γ, while $\mathfrak{k}_{-1}$ depends linearly on the curvature $\mathbf{k}$, as we have already found from less detailed considerations in Sect. 4. Therefore, for a general, non-cylindrical surface, by the curvature linearity established in Sect. 4, a similar term containing the second principal curvature should be added to the subsymbol. Additionally, we pinpoint that the subsymbol terms in (6.8) depend linearly on the material characteristics $\mathbb{k} = \frac{\mu}{2(2\mu+\lambda)}$ and $\mathbb{m} = \frac{\lambda+\mu}{2(2\mu+\lambda)} = \frac{1}{2} - \mathbb{k}$. These properties of the symbol of $\mathfrak{K}$ will be discussed further on.

6.4 The gradient of the principal symbol

Next, we need to find an explicit expression of one more object, namely the gradient of the principal symbol $\mathfrak{k}_0(x, \xi)$ that enters in the expression for $\mathfrak{m}_\iota$. We note, for a further reference, that the principal symbol $\mathfrak{k}_0$ depends linearly on $\mathbb{k}$ but does not depend on $\mathbb{m}$. The same property is valid for all derivatives of $\mathfrak{k}_0$.

Formula (6.8) gives us the representation of this symbol at the point $\mathbf{x}^\circ = 0$ in the co-ordinates and the frame associated with this point. What we need now is to calculate this symbol at a different point $\mathbf{x}^\bullet = (x, F(x)) \neq \mathbf{x}^\circ$ and then find derivatives $\partial_{x_\alpha} \mathfrak{k}_0(x, \xi)$ and $\partial_{\xi_\alpha} \mathfrak{k}_0(x, \xi)$ for $x = 0$ by means of making $\mathbf{x}^\bullet$ approaching $\mathbf{x}^\circ$. Again, a condense description of this calculatoin is presented in Sect. 4. This calculation is needed for $\alpha = 1$ only, since, on the cylinder, all derivatives in x_2 variable vanish. We find the derivative in ξ_1 first. By a direct calculation, we obtain from (6.8):

$$\partial_{\xi_1} \mathfrak{k}_0(x, \xi) = \frac{i\mathbb{k}}{|\xi|^3} \begin{pmatrix} 0 & 0 & \xi_2^2 \\ 0 & 0 & \xi_1\xi_2 \\ -\xi_2^2 & -\xi_1\xi_2 & 0 \end{pmatrix}.$$

Next we find the derivative $\partial_{x_1} \mathfrak{k}_0(x, \xi)$. For a given x_1, we take a point $\mathbf{x}^\bullet = (x_1, 0, F(x_1)) \in \Gamma$ and consider the C–co-ordinates system (y_1, y_2, y_3) centered at $\mathbf{x}^\bullet$ and the corresponding frame in $\mathbb{R}^3$. These co-ordinates are rotated compared with the system centered at $\mathbf{x}^\circ$, the rotation realized by the matrix

$$U(\mathbf{x}^\bullet) = \begin{pmatrix} \cos(\theta) & 0 & \sin(\theta) \\ 0 & 1 & 0 \\ -\sin(\theta) & 0 & \cos(\theta) \end{pmatrix}, \tag{6.9}$$

where the angle θ equals

$$\theta = \arcsin F'(x_1) = \arcsin\left(\frac{x_1}{\mathbf{k}}\right). \tag{6.10}$$

This is a rotation around the y_2 axis, directed, recall, along the directrix of the cylinder Γ, i.e., parallel to the x_2 axis. The frame in $\mathbb{R}^3$ at the point $\mathbf{x}^\bullet$ has directions along the axes y_1, y_2, y_3; the covariables η_j are directed along the corresponding y_j axes.

Further on, since we need to trace the dependence of the symbols of our operator on the co-ordinate system, we will mark it by the superscript: thus, $\mathfrak{k}_0^\circ$ denotes the expression of the symbol in the C–co-ordinate system and frame centered at $\mathbf{x}^\circ$ etc.

Now, the principal symbol of the operator $\mathfrak{K}$ at the point $\mathbf{x}^\bullet$, calculated in $(\mathbf{y}, \eta)$-C–co-ordinate system, is given by the same expression as in (6.8), just with ξ replaced by η :

$$\mathfrak{k}_0^\bullet(\mathbf{x}^\bullet, \eta) = i\mathbb{k}|\eta|^{-1}\begin{pmatrix} 0 & 0 & -\eta_1 \\ 0 & 0 & -\eta_2 \\ \eta_1 & \eta_2 & 0 \end{pmatrix}. \tag{6.11}$$

Now we apply the rule of the variables change in the principal symbol of pseudodifferential operators;

$$\mathfrak{k}_0^{\mathbf{x}}(\mathbf{x}^\bullet, \xi) = U(x_1)^{-1}\mathfrak{k}_0^\bullet(\mathbf{x}^\bullet, U(\mathbf{y})^*\xi)U(x_1),$$

where $U = U(x_1)$ is the linear transformation (6.9). Note, that in addition to the standard formula for the change of variables in pseudodifferential operators, which is reflected by the presence of U in the argument in $\mathfrak{k}$, formula (6.11) takes into account the circumstance that the symbols at the points $\mathbf{x}^\bullet$ and $\mathbf{x}^\circ$ are represented in different frames, related, again, by means of the same matrix $U(x_1)$. This gives us

$$\begin{aligned}\mathfrak{k}_0^{\mathbf{x}^\bullet}(\mathbf{x}^\circ, \xi) &= \frac{i\mathbb{k}}{|\xi|}U^{-1}\begin{pmatrix} \xi_1\sin\theta & 0 & -\xi_1\cos\theta \\ 0 & 0 & -\xi_2 \\ \xi_1\cos\theta & \xi_2 & \xi_1\sin\theta \end{pmatrix}U \\ &= \frac{i\mathbb{k}}{|\xi|}\begin{pmatrix} \xi_1\sin\theta & 0 & -\xi_1\cos\theta \\ 0 & 0 & -\xi_2 \\ \xi_1\cos\theta & \xi_2 & \xi_1\sin\theta \end{pmatrix}. \end{aligned} \tag{6.12}$$

The last equality uses the fact that, in our case, the matrix U commutes with the matrix $\mathfrak{k}_0$. Thus, we have obtained the expression for the principal symbol of $\mathfrak{K}$ written in one and the same co-ordinate system and the same frame. Now we can differentiate the expression (6.12) in x_1 variable, taking into account (6.10):

$$\frac{\partial}{\partial x_1}\mathfrak{k}_0^{\mathbf{x}^\circ}(x_1, 0, F(x_1), \xi)\,|_{x_1=0} = \frac{i\mathbb{k}\mathbf{k}}{|\xi|}\begin{pmatrix} \xi_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \xi_1 \end{pmatrix}. \tag{6.13}$$

Finally, we collect all terms we calculated in the symbols; it is more graphical to represent them in homogeneous variables $\phi_\beta = \xi_\beta/|\xi|$, $\beta = 1, 2$, so that $\phi_1^2 + \phi_2^2 = 1$. In this way we have

$$\mathfrak{k}_0(\xi) = i\mathbb{k}\begin{pmatrix} 0 & 0 & -\phi_1 \\ 0 & 0 & -\phi_2 \\ \phi_1 & \phi_2 & 0 \end{pmatrix};\ \mathfrak{k}_{-1}(\xi) = |\xi|^{-1}\mathbf{k}(\mathbb{k}\mathfrak{u}(\xi) + \mathbb{m}\mathfrak{v}(\xi)),$$

$$\mathfrak{u}(\xi) = -\phi_2^2\mathbf{E};\ \mathfrak{v}(\xi) = -\frac{1}{2}\phi_2^2\begin{pmatrix}\phi_2^2 & \phi_1\phi_2 & 0\\ \phi_1\phi_2 & \phi_1^2 & 0\\ 0 & 0 & 0\end{pmatrix} - \frac{3}{2}\begin{pmatrix}0 & \phi_1\phi_2 & 0\\ -\phi_1\phi_2 & 0 & 0\\ 0 & 0 & 0\end{pmatrix};$$

$$\partial_{\xi_1}\mathfrak{k}_0(x,\xi) = i\mathbb{k}|\xi|^{-1}\begin{pmatrix}0 & 0 & -\phi_2^2\\ 0 & 0 & -\phi_1\phi_2\\ \phi_2^2 & \phi_1\phi_2 & 0\end{pmatrix};$$

$$\partial_{x_1}\mathfrak{k}_0(x,\xi) = i\mathbb{k}\mathbf{k}|\xi|^{-1}\begin{pmatrix}\phi_1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & \phi_1\end{pmatrix}.$$

6.5 The symbol $\mathfrak{m}_\iota$

By the reasons discussed earlier, the explicit expression of the symbol of the order -1 pseudodifferential operator $\mathfrak{M}_\iota$ is rather wild. Even in the case of a cylinder, the task of calculating the eigenvalues of the symbol remains quite irrational. This calculation would involves 15 products of matrices, leading to a incomprehensible expression. It would become even more obscure and senseless after adding 10 more terms present if *both* principal curvatures are nonzero.To illustrate the above, we write down all terms in the terms for the cylinder case in the symbol $\mathfrak{m}_{-1}$.

To shorten the notations, we write $\mathfrak{k}$ instead of $\mathfrak{k}_0$, $\mathfrak{j}$ instead of $\mathfrak{k}_{-1}$, $\mathfrak{g}$ instead of $\partial_{\xi_1}\mathfrak{k}$, and $\mathfrak{h}$ instead of $\partial_{x_1}\mathfrak{k}$, and also omit the subscript α in (2.7).

So, for calculating $\mathfrak{m}_{-1}$ (recall, this is the effective symbol for the point $\omega_{-1} = -\mathbb{k}$), we compose terms of type 1, see (2.6) and find their sum:

$$\begin{aligned}\mathbf{F} := F_1 + F_2 + F_3 + F_4 + F_5 &= \mathfrak{j}\mathfrak{k}^2(\mathfrak{k}-\mathbb{k})^2 + (\mathfrak{k}+\mathbb{k})\mathfrak{j}\mathfrak{k}(\mathfrak{k}-\mathbb{k})^2\\ &+(\mathfrak{k}+\mathbb{k})\mathfrak{k}\mathfrak{j}(\mathfrak{k}-\mathbb{k})^2 + (\mathfrak{k}+\mathbb{k})\mathfrak{k}^2\mathfrak{j}(\mathfrak{k}-\mathbb{k}) + (\mathfrak{k}+\mathbb{k})\mathfrak{k}^2\mathfrak{k}-\mathbb{k})\mathfrak{j}. \qquad (6.14)\end{aligned}$$

Then we determine the terms of type 2, see (2.7), and write down their sum:

$$\begin{aligned}\mathbf{G} &:= -\iota[G_{1,2}+G_{1,3}+G_{1,4}+G_{1,5}+G_{2,3}+G_{2,4}+G_{2,5}+G_{3,4}+G_{3,5}+G_{4,5}]\\ &= -\iota[\mathfrak{g}\mathfrak{h}\mathfrak{k}(\mathfrak{k}-\mathbb{k})^2 + \mathfrak{g}\mathfrak{k}\mathfrak{h}(\mathfrak{k}-\mathbb{k})^2 + \mathfrak{g}\mathfrak{k}(\mathfrak{k}-\mathbb{k})\mathfrak{h}(\mathfrak{k}-\mathbb{k}) + \mathfrak{g}\mathfrak{k}(\mathfrak{k}-\mathbb{k})^2\mathfrak{h}\\ &\quad +(\mathfrak{k}+\mathbb{k})\mathfrak{g}\mathfrak{h}(\mathfrak{k}-\mathbb{k})^2 + (\mathfrak{k}+\mathbb{k})\mathfrak{g}\mathfrak{k}\mathfrak{h}(\mathfrak{k}-\mathbb{k}) + (\mathfrak{k}+\mathbb{k})\mathfrak{g}\mathfrak{k}(\mathfrak{k}-\mathbb{k})\mathfrak{h}\\ &\quad +(\mathfrak{k}+\mathbb{k})\mathfrak{k}\mathfrak{g}\mathfrak{h}(\mathfrak{k}-\mathbb{k}) + (\mathfrak{k}+\mathbb{k})\mathfrak{k}\mathfrak{g}(\mathfrak{k}-\mathbb{k})\mathfrak{h} + (\mathfrak{k}+\mathbb{k})\mathfrak{k}^2\mathfrak{g}\mathfrak{h}]. \qquad (6.15)\end{aligned}$$

Finally, $\mathfrak{m}_{-1} = (\mathbf{p}'(\omega_\iota))^{-1}(\mathbf{F}+\mathbf{G})$, it is the sum of 15 terms, for a cylinder with radius $\mathbf{k}$, and the coefficient $M_{-1}(\xi)$ is equal to $M_{-1}(\xi) = \mathbf{k}^{-1}\mathfrak{m}_{-1}(\xi)$. This coefficient is universal, and for a general surface with principal curvatures $\mathbf{k}_1(\mathbf{x})$, $\mathbf{k}_2(x)$, the symbol is calculated using the linearity in the principal curvatures, see (5.3). The integrand in the asymptotic formulas is, up to some constant,

For calculating $\mathfrak{m}_\iota$ for $\iota = 0$ and for $\iota = 1$, the formulas are analogous to (6.14), (6.15), but with different combinations of $\mathfrak{k}-\omega_\iota$, in accordance with (2.6), (2.7).

6.6 Dependence on the material

The explicit expression for $\mathfrak{m}(x,\xi)$ is extremely cumbersome; if typed, it would fill several unreadable pages. We, however, have already determined its dependence on the geometry of the body $\mathcal{D}$. What we can do now is to describe its dependence on the material of the body, namely on the Lamé constants.

When analysing the expression (6.14), we see that in each of 5 summands (the last one vanishes, but this does not matter) there are 4 factors containing the principal symbol $\mathfrak{k} = \mathfrak{k}_0(\xi)$, each of them contains the factor $\mathbb{k}$. Additionally, one of factors is the symbol j, *linear in* $\mathbb{k}$ *and* $\mathbb{m}$. The above 4 entries of $\mathbb{k}$ are cancelled by the factor $(\mathbf{p}'_\iota(\omega_\iota))^{-1}$ in (2.8) which contains $\mathbb{k}^{-4}$. What remains, is that all summands in (6.14) produce a contribution to $\mathfrak{m}_\iota$, being linear form linearly of $\mathbb{k}$ and $\mathbb{m}$ with universal coefficients which are some matrices depending only on ξ (and, of course, on ι). Let us pass to the terms of type 2 in (6.15). Each summand in $\mathbf{G}$ is a product of five terms, of which 3 contain the principal symbol $\mathfrak{k} = \mathfrak{k}_0(x,\xi)$, the other two are the gradient in ξ and the gradient in x variables of $\mathfrak{k}$. As follows from our calculations above, each of these factors in (2.7) contain the factor $\mathbb{k}$, altogether 5 of them. Four of them are canceled, again, by the coefficient $(\mathbf{p}'_\iota(\omega_i))^{-1}$, therefore, again, a linear dependence of the parameter $\mathbb{k}$ remains in $\mathfrak{m}_\iota$ but not a dependence on $\mathbb{m}$ any more. We sum this reasoning by stating that the contribution of the term $\mathbf{G}$ in (2.8) to $\mathfrak{m}_\iota$ is a linear function of quantities $\mathbb{k} = \frac{\mu}{2(\lambda+2\mu)}$ with a universal coefficient.

Altogether, we obtain that the matrix $M_\iota(\xi)$ is a linear form of $\mathbb{k}$ and $\mathbb{m} = \frac{1}{2} - \mathbb{k}$ with universal coefficients. Therefore, (3.10) involves the eigenvalues of a matrix depending linearly on the material parameters $\mathbb{k}$ and $\mathbb{m}$ and linearly on the principal curvatures. We arrive at the following representation for the effective symbol.

Theorem 6.1 *The effective symbol* $\mathfrak{m}_\iota$ *satisfies*

$$\mathfrak{m}_\iota(x,\xi) = \mathbf{k}_1(x)(\mathbb{k}, X_\iota(\xi) + \mathbb{m}Y_\iota(\xi)) + \mathbf{k}_2(x)V^{-1}(\mathbb{k}X_\iota(\hat{\xi}) + \mathbb{m}Y_\iota(\hat{\xi}))V^{-1}. \quad (6.16)$$

with universal matrix functions $X_\iota(\xi)$, $Y_\iota(\xi)$, *order* -1 *positively homogeneous in* $\xi \in \mathbb{R}^2$.

We obtain more knowledge on the matrices $\mathbb{k}X_i(\xi)$, $Y_\iota(\xi)$ after some further analysis of the spherical case.

6.7 Returning to the sphere

Having the expression (6.16) for the symbol $\mathfrak{m}_\iota$, we return to the result (5.7) for the two-sided asymptotics of eigenvalues, We obtain the following expressions for the coefficients $\mathcal{A}_\iota$ and $\mathcal{B}_\iota$ in (5.7).

$$\begin{aligned}\mathcal{A}_\iota &= 2\pi^{-1}\left[\int_{S^1} \mathbb{k}^2 \operatorname{tr}(X_\iota(\xi))^2 \boldsymbol{\omega} + 2\int_{S^1} \mathbb{k}\mathbb{m} \operatorname{tr}(X_\iota(\xi)Y_\iota(\xi))\boldsymbol{\omega} + \mathbb{m}^2 \int_{S^1} \operatorname{tr} Y_\iota^2(\xi)\boldsymbol{\omega}\right] \\ &= \boldsymbol{\Omega}_{\iota,W}(\mathbb{k},\mathbb{m}), \end{aligned} \quad (6.17)$$

and a similar expression for $\mathcal{B}_\iota$,

$$\mathcal{B}_\iota = 2\left(\int_{S^1} \mathrm{tr}\,\left[(\Bbbk X_\iota(\xi) + \mathrm{m} Y_\iota(\xi))\, V^{-1}\, (\Bbbk X_\iota(\xi) + \mathrm{m} Y_\iota(\xi))\, V\right]\boldsymbol{\omega} - \mathcal{A}_\iota\right.$$
$$= \boldsymbol{\Omega}_{\iota,\chi}(\Bbbk, \mathrm{m}). \tag{6.18}$$

In these formulas $\boldsymbol{\Omega}_{\iota,W}(\Bbbk, \mathrm{m})$, $\boldsymbol{\Omega}_{\iota,\chi}(\Bbbk, \mathrm{m})$ are quadratic forms of the quantities $\Bbbk$, m, with universal coefficients depending only on ι.

In particular, these coefficients have the same form for the body being the unit ball; the Euler characteristic and the Willmore energy for the sphere equal, correspondingly, $\chi(S^2) = 2$, $W(S^2) = 4\pi$. We compare (6.17), (6.18) with the asymptotic formulas (3.14). Due to the fact that both formulas (3.14) and (5.7) with coefficients (6.17), (6.18) must give the same result for all combinations of the Lamé constants λ, μ, we derive from this comparison the following. In particular, the coefficient in the eigenvalue asymptotics for a sphere, see (3.14), for $\iota = 0$ does not depend on the Lamé constants, therefore, all their entries for $\iota = 0$ cancel due to $\Bbbk + \mathrm{m} = \frac{1}{2}$.

Theorem 6.2 *The quadratic form* $\Upsilon_\iota(\Bbbk, \mathrm{m}) = 2\pi^{-1}\mathcal{A}_\iota(\Bbbk, \mathrm{m}) + 2\mathcal{B}_\iota$ *possesses the following properties*

- *For* $\iota = \pm 1$, *the form* $\Upsilon_\iota(\Bbbk, \mathrm{m})$ *equals* $\gamma_\iota \Bbbk^2$,
- *For* $\iota = 0$, $\Upsilon_\iota(\Bbbk, \mathrm{m}) = \gamma_0$,

Here γ_ι *are absolute numeric coefficients obtained by integrating expressions containing the matrix-functions* X_ι, Y_ι, *see* (6.17), (6.18).

We can also note that for the sphere, there are no eigenvalues approaching ω_ι from below. This means that the symbol (6.16) is non-negative for all $\iota, \Bbbk, \xi$.

Acknowledgements The Author is grateful to Y. Miyanishi for introducing him to the NP problematic and for useful discussions.

Author Contributions Not applicable

Funding Open access funding provided by Chalmers University of Technology. The work was performed at the Saint Petersburg Leonhard Euler International Mathematical Institute and supported by the Ministry of Science and Higher Education (Agreement No. 075–15–2022–287).

Data availability Not applicable.

Code availability Not applicable.

Declarations

Conflict of interest The Author declares no conflict of interest.

Consent to participate Not applicable.

Consent for publication Not applicable.

Appendix A: Some remarks on the nonhomogeneous material

In the paper [32] an approach was made to consider a mathematical model of a *nonhomogeneous* elastic body (with the Lamé) parameters $\lambda(\mathbf{x})$, $\mu(\mathbf{x})$ being smooth, possibly nonconstant, functions of the point $\mathbf{x} \in \mathcal{D}$. A possible version of the NP operator was constructed. It was found that the essential spectrum of this operator $\mathfrak{K}$ may fill nontrivial intervals of the real line, namely, the set of values of the, now functions, $\Bbbk(\mathbf{x}) = \frac{\mu(\mathbf{x})}{2(2\mu(\mathbf{x})+\lambda(\mathbf{x}))}$ and $-\Bbbk(\mathbf{x})$ for $\mathbf{x} \in \Gamma$, as well as the point 0. This fact was based upon the representation, found in [32], of $\mathfrak{K}$ as an order zero pseudodifferential operator with principal symbol

$$\mathfrak{k}_0(x,\xi) = \frac{i\Bbbk(x)}{|\xi|}\begin{pmatrix} 0 & 0 & -\xi_1 \\ 0 & 0 & -\xi_2 \\ \xi_1 & \xi_2 & 0 \end{pmatrix}, \tag{A.1}$$

$(x,\xi) \in T^*\Gamma$, in the same local co-ordinate system and local frame, as we use here for a homogeneous material. A question arises about the eigenvalues converging to the tips of the essential spectrum, namely, to the boundary points of the above intervals. It is natural to expect that the character of this convergence should depend on the structure of these boundary points.

The starting point here is the case of a nondegenerate extremal point of $\Bbbk(x)$ at the boundary point. If $\Bbbk(x)$, say, has a nondegenerate maximal point at $\mathbf{x}^\circ \in \Gamma$ then the corresponding eigenvalue $\sigma(x,\xi)$ of the principal symbol $\mathfrak{k}_0(x,\xi)$ is equal to $\Bbbk(\mathbf{x}^\circ)$ and therefore does not depend on ξ. It has its extremal value, moreover a nondegenerate one, for all ξ.

Another special (and quite convenient) property concerns the subsymbol of $\mathfrak{K}$. As the reasoning in the present paper shows, the subsymbol of $\mathfrak{K}$ is constructed using the Taylor expansion of the integral kernel of the operator $\mathfrak{K}$. This reasoning, based on the explicit formulas in [32], can be performed analogously to the one in Sect. 5 in our present paper. Unlike our present case, for a nonhomogeneous material, this expansion would involve not only geometrical characteristics (principal curvatures) of Γ, but also the derivatives of the function $\Bbbk(x)$ along directions on Γ. Fortunately, in the extremal point of $\Bbbk(x)$, the first order derivatives of this function vanish, therefore, the expression for the subsymbol of $\mathfrak{K}$ at this point turns out to be the same as for the homogeneous case, thus given by (6.8). A construction has been performed in [34], showing that under the above conditions, the knowledge of the second order jet of the eigenvectors of $\mathfrak{k}_0(x,\xi)$ at the point $\mathbf{x}^\circ$, together of the subsymbol $\mathfrak{k}_{-1}$ at this point suffices to find the asymptotics of eigenvalues converging to $\Bbbk(\mathbf{x}^\circ)$.

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