Nonlinear Differ. Equ. Appl.
(2023) 30:40

https://doi.org/10.1007/s00030-023-00845-z

Nonlinear Differential Equations
and Applications NoDEA

Doubling the equatorial for the prescribed scalar curvature problem on $\mathbb{S}^N$

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Abstract. We consider the prescribed scalar curvature problem on $\mathbb{S}^N$

$$\Delta_{\mathbb{S}^N} v - \frac{N(N-2)}{2} v + \tilde{K}(y) v^{\frac{N+2}{N-2}} = 0 \quad \text{on } \mathbb{S}^N, \qquad v > 0 \qquad \text{in } \mathbb{S}^N,$$

under the assumptions that the scalar curvature $\tilde{K}$ is rotationally symmetric, and has a positive local maximum point between the poles. We prove the existence of infinitely many non-radial positive solutions, whose energy can be made arbitrarily large. These solutions are invariant under some non-trivial sub-group of $O(3)$ obtained doubling the equatorial. We use the finite dimensional Lyapunov–Schmidt reduction method.

Mathematics Subject Classification. 35A01, 35B09, 35B38.

Keywords. Prescribed scalar curvature problem, Finite dimensional Lyapunov–Schmidt reduction, Lyapunov–Schmidt reduction, Scalar curvature problem.

1. Introduction

Given the N-th sphere $(\mathbb{S}^N, g)$ equipped with the standard metric g and a fixed smooth function $\tilde{K}$, the prescribed scalar curvature problem on $\mathbb{S}^N$ consists in understanding whether it is possible to find another metric $\tilde{g}$ in the conformal class of g, such that the scalar curvature of $\tilde{g}$ is $\tilde{K}$. For some positive function $v : \mathbb{S}^N \to \mathbb{R}$, and a related conformal change of the metric

$$\tilde{g} = v^{\frac{4}{N-2}} g,$$

the scalar curvature with respect to $\tilde{g}$ is given by

$$v^{-\frac{N+2}{N-2}} \left(\Delta_{\mathbb{S}^N} v - \frac{N(N-2)}{2} v \right),$$

where $\Delta_{\mathbb{S}^N}$ is the Laplace–Beltrami operator on $\mathbb{S}^N$. Thus the prescribed scalar curvature problem on $\mathbb{S}^N$ can be addressed by studying the solvability of the

Published online: 21 March 2023

problem

$$\Delta_{\mathbb{S}^N} v - \frac{N(N-2)}{2} v + \tilde{K}(y) v^{\frac{N+2}{N-2}} = 0 \quad \text{on } \mathbb{S}^N, \qquad v > 0 \qquad \text{in } \mathbb{S}^N. \tag{1.1}$$

Testing the Eq. (1.1) against v and integrating on $\mathbb{S}^N$, we get that a necessary condition for the solvability of this problem is that $\tilde{K}(y)$ must be positive somewhere. There are other obstructions for the existence of solutions, which are said to be of topological type. For instance, a solution v must satisfy the following Kazdan-Warner type condition (see [15]):

$$\int_{\mathbb{S}^N} \nabla \tilde{K}(y) \cdot \nabla y \, v^{\frac{2N}{N-2}} \, d\sigma = 0. \tag{1.2}$$

This condition is a direct consequence of Theorem 5.17 in [16], where Kazdan and Warner proved that given a positive solution v to

$$\Delta_{\mathbb{S}^N} v - \frac{N(N-2)}{2} v + H(y) v^a = 0$$

on the standard sphere $\mathbb{S}^N$, $N \geq 3$, then

$$\int_{\mathbb{S}^N} v^{a+1} \nabla H \cdot \nabla F = \frac{1}{2}(N-2) \left(\frac{N+2}{N-2} - a \right) \int_{\mathbb{S}^N} v^{a+1} H \, F, \tag{1.3}$$

for any spherical harmonics F of degree 1. Taking $a = \frac{N+2}{N-2}$, $H = \tilde{K}$ and $F = y$ in (1.3), we can obtain condition (1.2). The problem of determining which $\tilde{K}(y)$ admits a solution has been the object of several studies in the past years. We refer the readers to [2–4, 6–8, 10, 14, 15, 19, 30], and the references therein.

By using the stereo-graphic projection $\pi_N : \mathbb{R}^N \to \mathbb{S}^N \backslash \{(0, 0, \ldots, 1)\}$, the prescribed scalar curvature problem on $\mathbb{S}^N$, i.e. (1.1), can be transformed into the following semi-linear elliptic equation

$$\Delta v + K(y) v^{2^*-1} = 0, \quad v > 0, \quad \text{in } \mathbb{R}^N, \, v \in D^{1,2}(\mathbb{R}^N). \tag{1.4}$$

Here $2^* = \frac{2N}{N-2}$, $K(y) = \tilde{K}(\pi_N y)$, and $D^{1,2}(\mathbb{R}^N)$ denote the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the norm $\int_{\mathbb{R}^N} |\nabla v|^2$. It is of interest to establish under what kind of assumptions on K problem (1.4) admits one or multiple solutions.

For $N = 3$, Li [17] showed problem (1.4) has infinitely many solutions provided that $K(y)$ is bounded below, and periodic in one of its variables, and the set $\{x \,|\, K(x) = \max_{y \in \mathbb{R}^3} K(y)\}$ is not empty and contains at least one bounded connected component.

If K has the form $K(y) = 1 + \epsilon h(y)$, namely it is a perturbation of the constant 1, Cao et al. [5] proved the existence of multiple solutions.

If $K(y)$ has a sequence of strictly local maximum points moving to infinity, Yan [32] constructed infinitely many solutions.

Wei and Yan [31] showed that problem (1.4) has infinitely many solutions provided K is radially symmetric $K(y) = K(r)$, $r = |y|$, and has a local maximum around a given $r_0 > 0$. More precisely, they ask that there are r_0, $c_0 > 0$ and $m \in [2, N-2)$ such that

$$K(s) = K(r_0) - c_0 |s - r_0|^m + O\big(|s - r_0|^{m+\sigma}\big), \quad s \in (r_0 - \delta, r_0 + \delta),$$

where σ, δ are small positive constants. In order to briefly discuss the main results in [31], we will recall the expression of Aubin-Talenti bubbles. It is well known (see [29]) that all solutions to the following problem

$$\Delta u + u^{2^*-1} = 0, \quad u > 0 \text{ in } \mathbb{R}^N, \tag{1.5}$$

are given by

$$U_{x,\Lambda}(y) = c_N\Big(\frac{\Lambda}{1+\Lambda^2|y-x|^2}\Big)^{\frac{N-2}{2}}, \ x \in \mathbb{R}^N, \ \Lambda > 0,$$

and $c_N = [N(N-2)]^{\frac{N-2}{4}}$. The solutions in [31] are obtained by gluing together a large number of Aubin-Talenti bubbles, which looks like

$$\tilde{u}_k \sim \sum_{j=1}^{k} U_{x_j,\bar{\Lambda}},$$

where $\bar{\Lambda}$ is a positive constant and the points x_j are distributed along the vertices of a regular polygon of k edges in the (y_1, y_2)-plane, with $|x_j| \to r_0$ as $k \to \infty$:

$$x_j = \left(\tilde{r}\cos\frac{2(j-1)\pi}{k}, \tilde{r}\sin\frac{2(j-1)\pi}{k}, 0, \dots, 0\right), \quad j = 1, \dots, k,$$

with $\tilde{r} \to r_0$ as $k \to \infty$.

Under a weaker symmetry condition for $K(y) = K(|y'|, y'')$ with $y = (|y'|, y'') \in \mathbb{R}^2 \times \mathbb{R}^{N-2}$, Peng et al. [27] constructed infinitely many bubbling solutions, which concentrate at the saddle points of the potential $K(y)$. Guo and Li [11] admitted infinitely many solutions for problems (1.4) with polyharmonic operators. For fractional case, we refer to [13,23].

The study of other aspects of problem (1.4), such as radial symmetry of their solutions, uniqueness of solutions, Liouville type theorem, a priori estimates, and bubbling analysis, have been the object of investigation of several researchers. We refer the readers to the papers [1,9,18,20–22,25,26,32] and the references therein.

Recently, Guo et al. [12] investigated the spectral property of the linearized problem associated to (1.4) around the solution $\tilde{u}_k$ found in [31]. They proved a non-degeneracy result for such operator by using a refined version of local Pohozaev identities. As an application of this non-degeneracy result, they built new type of solutions by gluing another large number of bubbles, whose centers lie near the circle $|y| = r_0$ in the (y_3, y_4)-plane.

All these results concern solutions made by gluing together Aubin-Talenti bubbles with centres distributed along the vertices of one or more planar polygons, thus of two-dimensional nature. The purpose of this paper is to present a different type of solution to (1.4) with a more complex concentration structure, which cannot be reduced to a two-dimensional one.

To present our result, we assume that K is radially symmetric and satisfies the following condition (**H**): There are r_0 and $c_0 > 0$ such that

$$K(s) = K(r_0) - c_0|s-r_0|^m + O\big(|s-r_0|^{m+\sigma}\big), \quad s \in (r_0-\delta, r_0+\delta),$$

where $\sigma, \delta > 0$ are small constants, and

$$m \in \begin{cases} [2, N-2) & \text{if} \quad N = 5 \text{ or } 6, \\ \left(\frac{(N-2)^2}{2N-3}, N-2\right) & \text{if} \quad N \geq 7. \end{cases} \tag{1.6}$$

There is a slight difference between our assumptions on $K(s)$ and the ones in [31]. We will comment on this issue later.

Without loss of generality, we assume $r_0 = 1$, $K(1) = 1$. For any integer k, we denote

$$\mathbf{r} = k^{\frac{N-2}{N-2-m}}, \tag{1.7}$$

and set $u(y) = \mathbf{r}^{-\frac{N-2}{2}} v\left(\frac{|y|}{\mathbf{r}}\right)$. Then the problem (1.4) can be rewritten, in terms of u, as

$$-\Delta u = K\left(\frac{|y|}{\mathbf{r}}\right) u^{2^*-1}, \quad u > 0, \quad \text{in } \mathbb{R}^N, \quad u \in D^{1,2}(\mathbb{R}^N). \tag{1.8}$$

We define

$$W_{r,h,\Lambda}(y) = \sum_{j=1}^{k} U_{\overline{x}_j,\Lambda}(y) + \sum_{j=1}^{k} U_{\underline{x}_j,\Lambda}(y), \quad y \in \mathbb{R}^N, \tag{1.9}$$

for k integer large, where

$$\begin{cases} \overline{x}_j = r\left(\sqrt{1-h^2}\cos\frac{2(j-1)\pi}{k}, \sqrt{1-h^2}\sin\frac{2(j-1)\pi}{k}, h, \mathbf{0}\right), & j = 1, \ldots, k, \\ \underline{x}_j = r\left(\sqrt{1-h^2}\cos\frac{2(j-1)\pi}{k}, \sqrt{1-h^2}\sin\frac{2(j-1)\pi}{k}, -h, \mathbf{0}\right), & j = 1, \ldots, k. \end{cases}$$

Here $\mathbf{0}$ is the zero vector in $\mathbb{R}^{N-3}$ and h, r are positive parameters.

We shall construct a family of solutions to problem (1.8) which are small perturbations of $W_{r,h,\Lambda}$. More precisely, the Aubin-Talenti bubbles are now centred at points lying on the top and the bottom circles of a cylinder and this configuration is now invariant under a non-trivial sub-group of $O(3)$ rather than $O(2)$.

Throughout of the present paper, we assume $N \geq 5$ and $(r, h, \Lambda) \in \mathscr{S}_k$, where

$$\mathscr{S}_k = \left\{ (r, h, \Lambda) \middle| \, r \in \left[k^{\frac{N-2}{N-2-m}} - \hat{\sigma}, k^{\frac{N-2}{N-2-m}} + \hat{\sigma}\right], \quad \Lambda \in \left[\Lambda_0 - \hat{\sigma}, \Lambda_0 + \hat{\sigma}\right], \right.$$
$$\left. h \in \left[\frac{B'}{k^{\frac{N-3}{N-1}}}\left(1 - \hat{\sigma}\right), \frac{B'}{k^{\frac{N-3}{N-1}}}\left(1 + \hat{\sigma}\right)\right] \right\}, \tag{1.10}$$

with Λ_0, B' being the constants in (3.7), (3.10) and $\hat{\sigma}$ a fixed small number, independent of k.

Since $h \to 0$ as $k \to \infty$, then the two circles where the points $\overline{x}_j$ and $\underline{x}_j$ are distributed become closer to each other as k increases.

In this paper, we shall prove that for any k large enough, problem (1.8) admits a family of solutions u_k with the approximate form

$$u_k(y) \sim W_{r,h,\Lambda}. \tag{1.11}$$

Moreover, these solutions are polygonal symmetry in the (y_1, y_2)-plane, even in the y_3 direction and radially symmetric in the variables $y_4, \dots, y_N$. Our solutions are thus different from the ones obtained in [31] and have strong analogies with the doubling construction of the entire finite energy sign-changing solutions for the Yamabe equation in [24].

Define the symmetric Sobolev space:

$$H_s = \Bigg\{ u : u \in H^1(\mathbb{R}^N), u \text{is even in} y_\ell,\ \ell = 2, 3, 4, \dots, N,$$
$$u\left(\sqrt{y_1^2 + y_2^2}\cos\theta, \sqrt{y_1^2 + y_2^2}\sin\theta, y_3, y''\right)$$
$$= u\left(\sqrt{y_1^2 + y_2^2}\cos\left(\theta + \frac{2j\pi}{k}\right), \sqrt{y_1^2 + y_2^2}\sin\left(\theta + \frac{2j\pi}{k}\right), y_3, y''\right)\Bigg\},$$

where $\theta = \arctan\frac{y_2}{y_1}$.

Let us define the following norms which capture the decay property of functions

$$\|u\|_* = \sup_{y\in\mathbb{R}^N}\left(\sum_{j=1}^{k}\left[\frac{1}{(1 + |y - \overline{x}_j|)^{\frac{N-2}{2}+\tau}} + \frac{1}{(1 + |y - \underline{x}_j|)^{\frac{N-2}{2}+\tau}}\right]\right)^{-1}|u(y)|, \tag{1.12}$$

and

$$\|f\|_{**} = \sup_{y\in\mathbb{R}^N}\left(\sum_{j=1}^{k}\left[\frac{1}{(1 + |y - \overline{x}_j|)^{\frac{N+2}{2}+\tau}} + \frac{1}{(1 + |y - \underline{x}_j|)^{\frac{N+2}{2}+\tau}}\right]\right)^{-1}|f(y)|, \tag{1.13}$$

where

$$\tau = \left(\frac{N - 2 - m}{N - 2}, \frac{N - 2 - m}{N - 2} + \epsilon_1\right), \tag{1.14}$$

for some ϵ_1 small. The main results of this paper are the following:

Theorem 1.1. *Let $N \geq 5$ and suppose that $K(|y|)$ satisfies* (**H**). *Then there exists a large integer k_0, such that for each integer $k \geq k_0$, problem* (1.8) *has a solution u_k of the form*

$$u_k(y) = W_{r_k,h_k,\Lambda_k}(y) + \phi_k(y), \tag{1.15}$$

where $\phi_k \in H_s$, $(r_k, h_k, \Lambda_k) \in \mathscr{S}_k$, and ϕ_k satisfies

$$\|\phi_k\|_* = o_k(1), \quad \text{as} \quad k \to \infty.$$

Equivalently, problem (1.4) *has solution $v_k(y)$ of the form*

$$v_k(y) = \mathbf{r}^{\frac{2-N}{2}}\Big[W_{r_k,h_k,\Lambda_k}(\mathbf{r}y) + \phi_k(\mathbf{r}y)\Big],$$

with $\mathbf{r}$ *as in* (1.7).

Let us sketch the proof of Theorem 1.1. The first step in our argument is to find ϕ so that $u = W_{r,h,\Lambda} + \phi$ solves the auxiliary problem

$$\begin{cases} -\Delta\big(W_{r,h,\Lambda} + \phi\big) = K\big(\frac{|y|}{\mathbf{r}}\big)\big(W_{r,h,\Lambda} + \phi\big)^{2^*-1} \\ \qquad\qquad + \sum\limits_{\ell=1}^{3}\sum\limits_{j=1}^{k} c_\ell \Big(U_{\overline{x}_j,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell j} + U_{\underline{x}_j,\Lambda}^{2^*-2}\underline{\mathbb{Z}}_{\ell j} \Big) \text{ in } \mathbb{R}^N, \\ \phi \in \mathbb{E}, \end{cases} \tag{1.16}$$

for some constants c_ℓ for $\ell = 1, 2, 3$. In (1.16), the functions $\overline{\mathbb{Z}}_{\ell j}$ and $\underline{\mathbb{Z}}_{\ell j}$ are given by

$$\overline{\mathbb{Z}}_{1j} = \frac{\partial U_{\overline{x}_j,\Lambda}}{\partial r}, \qquad \overline{\mathbb{Z}}_{2j} = \frac{\partial U_{\overline{x}_j,\Lambda}}{\partial h}, \qquad \overline{\mathbb{Z}}_{3j} = \frac{\partial U_{\overline{x}_j,\Lambda}}{\partial \Lambda},$$
$$\underline{\mathbb{Z}}_{1j} = \frac{\partial U_{\underline{x}_j,\Lambda}}{\partial r}, \qquad \underline{\mathbb{Z}}_{2j} = \frac{\partial U_{\underline{x}_j,\Lambda}}{\partial h}, \qquad \underline{\mathbb{Z}}_{3j} = \frac{\partial U_{\underline{x}_j,\Lambda}}{\partial \Lambda},$$

for $j = 1, \dots, k$. Moreover, the function ϕ belongs to the set $\mathbb{E}$ given by

$$\mathbb{E} = \Big\{ v : v \in H_s, \quad \int_{\mathbb{R}^N} U_{\overline{x}_j,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell j} v = 0 \quad \text{and} \\ \int_{\mathbb{R}^N} U_{\underline{x}_j,\Lambda}^{2^*-2}\underline{\mathbb{Z}}_{\ell j} v = 0, \quad j = 1, \dots, k, \quad \ell = 1, 2, 3 \Big\}. \tag{1.17}$$

From the linear theory developed in Sect. 2, problem (1.16) can be solved by means of the contraction mapping theorem. More precisely, we prove that, for any $(r, h, \Lambda) \in \mathscr{S}_k$ there exist $\phi = \phi_{r,h,\Lambda} \in \mathbb{E}$ and constants c_ℓ, $\ell = 1, 2, 3$ which solve the auxiliary problem (1.16).

After the correction ϕ has been found, we shall choose $(r, h, \Lambda) \in \mathscr{S}_k$ so that the multipliers $c_\ell = 0\,(\ell = 1, 2, 3)$ in (1.16). As a consequence, we can derive the results as in Theorem 1.1.

Equation (1.8) is the Euler–Lagrange equation associated to the energy functional

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^N} |\nabla u|^2 dy - \frac{1}{2^*}\int_{\mathbb{R}^N} K\Big(\frac{|y|}{\mathbf{r}}\Big)|u|^{2^*} dy. \tag{1.18}$$

Thus, roughly speaking, if (r, h, Λ) is a critical point of function

$$F(r, h, \Lambda) := I(W_{r,h,\Lambda} + \phi_{r,h,\Lambda}) \quad \text{for } \phi_{r,h,\Lambda} \in \mathbb{E},$$

then the constants c_ℓ, $\ell = 1, 2, 3$ would be zero. Thus finding solutions of problem (1.8) would be reduced to find a critical point of $F(r, h, \Lambda)$. This is the result in Proposition 3.1.

An important work of this paper is to give an accurate expression of $F(r, h, \Lambda)$ (see Proposition 3.2). Under the assumptions $r \sim k^{\frac{N-2}{N-2-m}}, h \to 0, \frac{1}{hk} \to 0$ as $k \to \infty$, we first get the expansion of the energy functional

$I(W_{r,h,\Lambda})$

$$\begin{aligned} F_1(r,h,\Lambda) &:= I(W_{r,h,\Lambda}) \\ &= kA_1 - \frac{k}{\Lambda^{N-2}}\Big[\frac{B_4 k^{N-2}}{(r\sqrt{1-h^2})^{N-2}} + \frac{B_5 k}{r^{N-2}h^{N-3}\sqrt{1-h^2}}\Big] \\ &\quad + k\Big[\frac{A_2}{\Lambda^m k^{\frac{(N-2)m}{N-2-m}}} + \frac{A_3}{\Lambda^{m-2}k^{\frac{(N-2)m}{N-2-m}}}(\mathbf{r}-r)^2\Big] + kO\Big(\frac{1}{k^{\frac{(N-2)m}{N-2-m}+\sigma}}\Big), \end{aligned}$$

where A_i for $i = 1, 2, 3$ and B_j for $j = 4, 5$ are constants. We denote

$$\mathcal{G}(h) := \frac{B_4 k^{N-2}}{(\sqrt{1-h^2})^{N-2}} + \frac{B_5 k}{h^{N-3}\sqrt{1-h^2}},$$

and let h be the solution of $\partial_h \mathcal{G}(h) = 0$, then

$$h = \frac{B'}{k^{\frac{N-3}{N-1}}}\big(1 + o(1)\big), \quad \text{as} \quad k \to \infty,$$

for some $B' > 0$. If $r \sim k^{\frac{N-2}{N-2-m}}, h \sim \frac{B'}{k^{\frac{N-3}{N-1}}}$, then

$$\frac{B_5 k}{r^{N-2}h^{N-3}\sqrt{1-h^2}} = \frac{\tilde{B}}{k^{\frac{(N-2)m}{N-2-m}+\frac{2(N-3)}{N-1}}}(1 + o(1)), \quad \text{as} \quad k \to \infty$$

for some constant $\tilde{B}$.

However, we now find that the term $O\Big(\frac{1}{k^{\frac{(N-2)m}{N-2-m}+\sigma}}\Big)$ in the expansion of $F_1(r, h, \Lambda)$ competes with the term $\frac{B_5 k}{r^{N-2}h^{N-3}\sqrt{1-h^2}}$. This makes it impossible to identify a critical point for $F_1(r, h, \Lambda)$. In reality, though the remainder $O\Big(\frac{1}{k^{\frac{(N-2)m}{N-2-m}+\sigma}}\Big)$ can be estimated in a more accurate way (see Proposition A.4) under our assumption (**H**).

We need to expand the full energy $F(r, h, \Lambda) = I(W_{r,h,\Lambda} + \phi_{r,h,\Lambda})$. We need a strong control on the size of $\phi_{r,h,\Lambda}$ in order not to destroy the critical point structure of $F_1(r, h, \Lambda)$ and to ensure the qualitative properties of the solutions as stated in Theorem 1.1. This is another delicate step of our construction, where we make full use of the assumption (**H**) on K.

Structure of the paper. The remaining part of this paper is devoted to the proof of Theorem 1.1, which will be organized as follows:

1. In Sect. 2, we will establish the linearized theory for the linearized projected problem. We will give estimates for the error terms in this Section.
2. In Sect. 3, we shall prove Theorem 1.1 by showing there exists a critical point of reduction function $F(r, h, \Lambda)$.
3. Some tedious computations and some useful Lemmas will be given in Appendices 3–4.

Notation and preliminary results. For the readers' convenience, we will provide a collection of notation. Throughout this paper, we employ C, C_j to denote certain constants and σ, τ, σ_j to denote some small constants or functions. We also note that δ_{ij} is Kronecker delta function:

$$\delta_{ij} = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{if } i \neq j. \end{cases}$$

Furthermore, we also employ the common notation by writing $O(f(r, h)), o(f(r, h))$ for the functions which satisfy

$$\text{if} \quad g(r,h) \in O(f(r,h)) \quad \text{then} \quad \lim_{k\to+\infty} \Big| \frac{g(r,h)}{f(r,h)} \Big| \le C < +\infty,$$

and

$$\text{if} \quad g(r,h) \in o(f(r,h)) \quad \text{then} \quad \lim_{k\to+\infty} \frac{g(r,h)}{f(r,h)} = 0.$$

2. Finite dimensional reduction

For $j = 1, \dots, k$, we divide $\mathbb{R}^N$ into k parts:

$$\Omega_j := \Big\{ y = (y_1, y_2, y_3, y'') \in \mathbb{R}^3 \times \mathbb{R}^{N-3} : \\ \Big\langle \frac{(y_1, y_2)}{|(y_1, y_2)|}, \Big(\cos\frac{2(j-1)\pi}{k}, \sin\frac{2(j-1)\pi}{k} \Big) \Big\rangle_{\mathbb{R}^2} \ge \cos\frac{\pi}{k} \Big\}.$$

where $\langle , \rangle_{\mathbb{R}^2}$ denote the dot product in $\mathbb{R}^2$. For Ω_j, we further divide it into two parts:

$$\Omega_j^+ = \Big\{ y : y = (y_1, y_2, y_3, y'') \in \Omega_j, y_3 \ge 0 \Big\},$$
$$\Omega_j^- = \Big\{ y : y = (y_1, y_2, y_3, y'') \in \Omega_j, y_3 < 0 \Big\}.$$

We can know that

$$\mathbb{R}^N = \cup_{j=1}^k \Omega_j, \quad \Omega_j = \Omega_j^+ \cup \Omega_j^-$$

and

$$\Omega_j \cap \Omega_i = \emptyset, \quad \Omega_j^+ \cap \Omega_j^- = \emptyset, \qquad \text{if} \quad i \neq j.$$

We consider the following linearized problem

$$\begin{cases} -\Delta\phi - (2^*-1)K\big(\frac{|y|}{\mathbf{r}}\big)W_{r,h,\Lambda}^{2^*-2}\phi = f \\ \quad + \sum_{i=1}^{k}\sum_{\ell=1}^{3} \Big(c_\ell U_{\overline{x}_i,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell i} + c_\ell U_{\underline{x}_i,\Lambda}^{2^*-2}\underline{\mathbb{Z}}_{\ell i} \Big) \ \text{in } \mathbb{R}^N, \\ \phi \in \mathbb{E}, \end{cases} \tag{2.1}$$

for some constants c_ℓ.

Coming back to Eq. (1.5), we recall that the functions

$$Z_i(y) := \frac{\partial U}{\partial y_i}(y), \quad i = 1, \dots, N, \quad Z_{N+1}(y) := \frac{N-2}{2}U(y) + y\cdot\nabla U(y). \tag{2.2}$$

belong to the null space of the linearized problem associated to (1.5) around an Aubin-Talenti bubble, namely they solve

$$\Delta\phi + (2^*-1)U^{2^*-2}\phi = 0, \text{ in } \mathbb{R}^N, \quad \phi \in D^{1,2}(\mathbb{R}^N). \tag{2.3}$$

It is known [28] that these functions span the set of the solutions to (2.3). This fact will be used in the following crucial lemma which concerns the linearized problem (2.1).

Lemma 2.1. *Suppose that ϕ_k solves (2.1) for $f = f_k$. If $\|f_k\|_{**}$ tends to zero as k tends to infinity, so does $\|\phi_k\|_*$.*

The norms $\|\cdot\|_*$ and $\|\cdot\|_{**}$ are defined respectively in (1.12) and (1.13).

Proof. We prove the Lemma by contradiction. Suppose that there exists a sequence of $(r_k, h_k, \Lambda_k) \in \mathscr{S}_k$, and for ϕ_k satisfies (2.1) with $f = f_k, r = r_k, h = h_k, \Lambda = \Lambda_k$, with $\|f_k\|_{**} \to 0$, and $\|\phi_k\|_* \geq c' > 0$. Without loss of generality, we can assume that $\|\phi_k\|_* = 1$. For convenience, we drop the subscript k.

From (2.1), we know that

$$\begin{aligned}\phi(y) &= (2^*-1)\int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} K\Big(\frac{|z|}{\mathbf{r}}\Big) W_{r,h,\Lambda}^{2^*-2}\phi(z)\,\mathrm{d}z + \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} f(z)\mathrm{d}z \\ &\quad + \int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} \sum_{j=1}^{k}\sum_{\ell=1}^{3} \Big(c_\ell U_{\overline{x}_j,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell j} + c_\ell U_{\underline{x}_j,\Lambda}^{2^*-2}\underline{\mathbb{Z}}_{\ell j}\Big)\mathrm{d}z \\ &:= M_1 + M_2 + M_3.\end{aligned}$$

For the first term M_1, we make use of Lemma B.5, so that

$$\begin{aligned} M_1 &\leq C\|\phi\|_* \int_{\mathbb{R}^N} \frac{K\Big(\frac{|z|}{\mathbf{r}}\Big)}{|z-y|^{N-2}} W_{r,h,\Lambda}^{2^*-2} \\ &\quad \times \left(\sum_{j=1}^{k}\Big[\frac{1}{(1+|z-\overline{x}_j|)^{\frac{N-2}{2}+\tau}} + \frac{1}{(1+|z-\underline{x}_j|)^{\frac{N-2}{2}+\tau}}\Big]\right)\mathrm{d}z \\ &\leq C\|\phi\|_* \sum_{j=1}^{k}\Big[\frac{1}{(1+|z-\overline{x}_j|)^{\frac{N-2}{2}+\tau+\sigma}} + \frac{1}{(1+|z-\underline{x}_j|)^{\frac{N-2}{2}+\tau+\sigma}}\Big].\end{aligned}$$

For the second term M_2, we make use of Lemma B.4, so that

$$\begin{aligned} M_2 &\leq C\|f\|_{**}\int_{\mathbb{R}^N} \frac{1}{|z-y|^{N-2}} \sum_{j=1}^{k}\Big[\frac{1}{(1+|z-\overline{x}_j|)^{\frac{N+2}{2}+\tau}} + \frac{1}{(1+|z-\underline{x}_j|)^{\frac{N+2}{2}+\tau}}\Big]\mathrm{d}z \\ &\leq C\|f\|_{**}\sum_{j=1}^{k}\Big[\frac{1}{(1+|y-\overline{x}_j|)^{\frac{N-2}{2}+\tau}} + \frac{1}{(1+|y-\underline{x}_j|)^{\frac{N-2}{2}+\tau}}\Big].\end{aligned}$$

In order to estimate the term M_3, we will first give the estimates of $\overline{\mathbb{Z}}_{1j}$ and $\underline{\mathbb{Z}}_{1j}$

$$|\overline{\mathbb{Z}}_{1j}| \le \frac{C}{(1+|y-\overline{x}_j|)^{N-2}}, \quad |\overline{\mathbb{Z}}_{2j}| \le \frac{Cr}{(1+|y-\overline{x}_j|)^{N-2}}, \quad |\overline{\mathbb{Z}}_{3j}| \le \frac{C}{(1+|y-\overline{x}_j|)^{N-2}},$$
$$|\underline{\mathbb{Z}}_{1j}| \le \frac{C}{(1+|y-\underline{x}_j|)^{N-2}}, \quad |\underline{\mathbb{Z}}_{2j}| \le \frac{Cr}{(1+|y-\underline{x}_j|)^{N-2}}, \quad |\underline{\mathbb{Z}}_{3j}| \le \frac{C}{(1+|y-\underline{x}_j|)^{N-2}}. \tag{2.4}$$

Combining estimates (A.26) and Lemma B.4, we have

$$\begin{aligned}\sum_{j=1}^{k}\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}\,U_{\overline{x}_j,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell j}\,\mathrm{d}z &\le C\sum_{j=1}^{k}\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}\,\frac{(1+r\delta_{\ell 2})}{(1+|z-\overline{x}_j|)^{N+2}}\,\mathrm{d}z\\ &\le C\sum_{j=1}^{k}\frac{(1+r\delta_{\ell 2})}{(1+|y-\overline{x}_j|)^{\frac{N-2}{2}+\tau}}, \quad \text{for } \ell=1,2,3,\end{aligned}$$

where $\delta_{\ell 2}=0$ if $\ell\neq 2$, $\delta_{\ell 2}=1$ if $\ell=2$. Similarly, we have

$$\sum_{j=1}^{k}\int_{\mathbb{R}^N}\frac{1}{|z-y|^{N-2}}\,U_{\underline{x}_j,\Lambda}^{2^*-2}\underline{\mathbb{Z}}_{\ell j}\,\mathrm{d}z \le C\sum_{j=1}^{k}\frac{(1+r\,\delta_{\ell 2})}{(1+|y-\underline{x}_j|)^{\frac{N-2}{2}+\tau}}, \quad \text{for } \ell=1,2,3.$$

Next, we will give the estimates of $c_\ell, \ell=1,2,3$. Multiply both sides of (2.1) by $\overline{\mathbb{Z}}_{q1}, q=1,2,3$, then we obtain that

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big[-\Delta\phi-(2^*-1)K\Big(\frac{|y|}{\mathbf{r}}\Big)W_{r,h,\Lambda}^{2^*-2}\phi\Big]\overline{\mathbb{Z}}_{q1}\\ &\quad=\int_{\mathbb{R}^N} f\,\overline{\mathbb{Z}}_{q1}+\sum_{j=1}^{k}\sum_{\ell=1}^{3}\int_{\mathbb{R}^N}\Big(c_\ell U_{\overline{x}_j,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell j}+c_\ell U_{\underline{x}_j,\Lambda}^{2^*-2}\underline{\mathbb{Z}}_{\ell j}\Big)\overline{\mathbb{Z}}_{q1}.\end{aligned} \tag{2.5}$$

Using Lemma B.3, we can get

$$\begin{aligned}\int_{\mathbb{R}^N} f\,\overline{\mathbb{Z}}_{q1} &\le C\|f\|_{**}\sum_{j=1}^{k}\int_{\mathbb{R}^N}\frac{1+r\,\delta_{\ell 2}}{(1+|y-\overline{x}_1|)^{N-2}}\\ &\quad\Bigg[\frac{1}{(1+|y-\overline{x}_j|)^{\frac{N+2}{2}+\tau}}+\frac{1}{(1+|y-\underline{x}_j|)^{\frac{N+2}{2}+\tau}}\Bigg]\\ &\le C(1+r\,\delta_{\ell 2})\|f_k\|_{**}.\end{aligned}$$

The discussion on the left side of (2.5) may be more tricky, in fact, we have

$$\begin{aligned}&\int_{\mathbb{R}^N}\Big[-\Delta\phi-(2^*-1)K\Big(\frac{|y|}{\mathbf{r}}\Big)W_{r,h,\Lambda}^{2^*-2}\phi\Big]\overline{\mathbb{Z}}_{q1}\\ &=\int_{\mathbb{R}^N}\Big[-\Delta\overline{\mathbb{Z}}_{q1}-(2^*-1)K\Big(\frac{|y|}{\mathbf{r}}\Big)W_{r,h,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{q1}\Big]\phi\\ &=(2^*-1)\int_{\mathbb{R}^N}\Big[1-K\Big(\frac{|y|}{\mathbf{r}}\Big)\Big]W_{r,h,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{q1}\phi+\Big(U_{\overline{x}_1,\Lambda}^{2^*-2}-W_{r,h,\Lambda}^{2^*-2}\Big)\overline{\mathbb{Z}}_{q1}\phi\\ &:=J_1+J_2.\end{aligned}$$

Using the property of $K(s)$, similar to the proof of Lemma B.5, we can get

$$
\begin{aligned}
J_1 \le\ & C\|\phi\|_* \int_{\mathbb{R}^N} \left|1-K\Big(\frac{|y|}{\mathbf{r}}\Big)\right| W_{r,h,\Lambda}^{2^*-2}\mathbb{Z}_{q1} \sum_{j=1}^{k} \\
& \left[\frac{1}{(1+|y-\overline{x}_j|)^{\frac{N-2}{2}+\tau}}+\frac{1}{(1+|y-\underline{x}_j|)^{\frac{N-2}{2}+\tau}}\right] \\
=\ & C\|\phi\|_* \int\limits_{||y|-\mathbf{r}|\le\sqrt{\mathbf{r}}} \left|1-K\left(\frac{|y|}{\mathbf{r}}\right)\right| W_{r,h,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{q1} \\
& \sum_{j=1}^{k}\left[\frac{1}{(1+|y-\overline{x}_j|)^{\frac{N-2}{2}+\tau}}+\frac{1}{(1+|y-\underline{x}_j|)^{\frac{N-2}{2}+\tau}}\right] \\
& + C\|\phi\|_* \int\limits_{||y|-\mathbf{r}|\ge\sqrt{\mathbf{r}}} \left|1-K\left(\frac{|y|}{\mathbf{r}}\right)\right| W_{r,h,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{q1} \\
& \sum_{j=1}^{k}\left[\frac{1}{(1+|y-\overline{x}_j|)^{\frac{N-2}{2}+\tau}}+\frac{1}{(1+|y-\underline{x}_j|)^{\frac{N-2}{2}+\tau}}\right] \\
\le\ & \frac{C}{\sqrt{\mathbf{r}}}\int_{\mathbb{R}^N} W_{r,h,\Lambda}^{2^*-2}(y)\frac{1+r\,\delta_{\ell 2}}{(1+|y-\overline{x}_1|)^{N-2}}\sum_{j=1}^{k}\frac{1}{(1+|y-\underline{x}_j|)^{\frac{N-2}{2}+\tau}} \\
& + \frac{C^{\sigma}}{\mathbf{r}}\int_{\mathbb{R}^N} W_{r,h,\Lambda}^{2^*-2}(y)\frac{1+r\,\delta_{\ell 2}}{(1+|y-\overline{x}_1|)^{N-2}}\sum_{j=1}^{k}\frac{1}{(1+|y-\overline{x}_j|)^{\frac{N-2}{2}+\tau-2\sigma}} \\
\le\ & \frac{C^{\sigma}}{\mathbf{r}}(1+r\,\delta_{\ell 2}).
\end{aligned}
$$

For J_2, it is easy to derive that

$$
\begin{aligned}
J_2 \le\ & \int_{\mathbb{R}^N}\left|U_{\overline{x}_1,\Lambda}^{2^*-2}-W_{r,h,\Lambda}^{2^*-2}\right|\frac{1+r\,\delta_{\ell 2}}{(1+|y-\overline{x}_1|)^{N-2}} \\
& \times\sum_{j=1}^{k}\left[\frac{1}{(1+|y-\overline{x}_j|)^{\frac{N-2}{2}+\tau}}+\frac{1}{(1+|y-\underline{x}_j|)^{\frac{N-2}{2}+\tau}}\right] \\
\le\ & \frac{C^{\sigma}}{\mathbf{r}}(1+r\,\delta_{\ell 2}).
\end{aligned}
$$

Then, we get

$$
\int_{\mathbb{R}^N}\Big[-\Delta\phi-(2^*-1)K\Big(\frac{|y|}{\mathbf{r}}\Big)W_{r,h,\Lambda}^{2^*-2}\,\phi\Big]\overline{\mathbb{Z}}_{q1}\le\frac{C^{\sigma}}{\mathbf{r}}(1+r\,\delta_{\ell 2})\,\|\phi\|_*.
$$

On the other hand, there holds

$$
\sum_{j=1}^{k}\int_{\mathbb{R}^N}\left(U_{\overline{x}_j,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell j}+U_{\underline{x}_j,\Lambda}^{2^*-2}\underline{\mathbb{Z}}_{\ell j}\right)\overline{\mathbb{Z}}_{q1}=\bar{c}_\ell\delta_{\ell q}(1+\delta_{q2}r^2)+o(1),\quad \text{as}\quad k\to\infty.
$$

Note that

$$\int_{\mathbb{R}^N} U_{\overline{x}_1,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell 1}\overline{\mathbb{Z}}_{q1} = \begin{cases} 0, & \text{if } \quad \ell \neq q, \\ \bar{c}_q(1+\delta_{q2}r^2), & \text{if } \quad \ell = q, \end{cases}$$

for some constant $\bar{c}_q > 0$. Then we can get

$$c_\ell = \frac{1+r\delta_{\ell 2}}{1+\delta_{\ell 2}r^2} O\Big(\frac{1}{\mathbf{r}}^\sigma \|\phi\|_* + \|f\|_{**}\Big) = o(1), \quad \text{as} \quad k \to \infty. \tag{2.6}$$

Then we have

$$\begin{aligned} |\phi| \leq \Bigg(& \|f\|_{**} \sum_{j=1}^k \Big[\frac{1}{(1+|y-\overline{x}_j|)^{\frac{N-2}{2}+\tau}} + \frac{1}{(1+|y-\underline{x}_j|)^{\frac{N-2}{2}+\tau}}\Big] \\ & + \sum_{j=1}^k \Bigg[\frac{1}{(1+|y-\overline{x}_j|)^{\frac{N-2}{2}+\tau+\sigma}} + \frac{1}{(1+|y-\underline{x}_j|)^{\frac{N-2}{2}+\tau+\sigma}}\Bigg]\Bigg). \end{aligned} \tag{2.7}$$

Combining this fact and $\|\phi\|_* = 1$, we have the following claim:
Claim 1: There exist some positive constants $\bar{R}, \delta_1$ such that

$$\|\phi\|_{L^\infty(B_{\bar{R}}(\overline{x}_l))} \geq \delta_1 > 0, \tag{2.8}$$

for some $l \in \{1, 2, \ldots, k\}$.

Since $\phi \in H_s$, we assume that $l = 1$. By using local elliptic estimates and (2.7), we can get, up to subsequence, $\tilde{\phi}(y) = \phi(y - \overline{x}_1)$ converge uniformly in any compact set to a solution

$$-\Delta u - (2^* - 1)U_{0,\Lambda}^{2^*-2} u = 0, \quad \text{in } \mathbb{R}^N,$$

for some $\Lambda \in [L_1, L_2]$. Since ϕ is even in $y_d, d = 2, 4, \ldots, N$, we know that u is also even in $y_d, d = 2, 4, \ldots, N$. Then we know that u must be a linear combination of the functions

$$\frac{\partial U_{0,\Lambda}}{\partial y_1}, \qquad \frac{\partial U_{0,\Lambda}}{\partial y_3}, \qquad y \cdot \nabla U_{0,\Lambda} + (N-2)U_{0,\Lambda}.$$

From the assumptions

$$\int_{\mathbb{R}^N} U_{\overline{x}_1,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell 1}\,\tilde{\phi} = 0 \quad \text{for } \ell = 1, 2, 3,$$

we can get

$$\begin{aligned} \sqrt{1-h^2}\int_{\mathbb{R}^N} U_{0,\Lambda}^{2^*-2}\frac{\partial U_{0,\Lambda}}{\partial y_1}\,\tilde{\phi} + h\int_{\mathbb{R}^N} U_{0,\Lambda}^{2^*-2}\frac{\partial U_{0,\Lambda}}{\partial y_3}\,\tilde{\phi} = 0, \\ \sqrt{1-h^2}\int_{\mathbb{R}^N} U_{0,\Lambda}^{2^*-2}\frac{\partial U_{0,\Lambda}}{\partial y_1}\,\tilde{\phi} - h\int_{\mathbb{R}^N} U_{0,\Lambda}^{2^*-2}\frac{\partial U_{0,\Lambda}}{\partial y_3}\,\tilde{\phi} = 0, \end{aligned}$$

and

$$\int_{\mathbb{R}^N} U_{0,\Lambda}^{2^*-2}\Big[y \cdot \nabla U_{0,\Lambda} + (N-2)U_{0,\Lambda}\Big]\tilde{\phi} = 0.$$

By taking limit, we have

$$\int_{\mathbb{R}^N} U_{0,\Lambda}^{2^*-2} \frac{\partial U_{0,\Lambda}}{\partial y_1} u$$
$$= \int_{\mathbb{R}^N} U_{0,\Lambda}^{2^*-2} \frac{\partial U_{0,\Lambda}}{\partial y_3} u = \int_{\mathbb{R}^N} U_{0,\Lambda}^{2^*-2} \Big[y \cdot \nabla U_{0,\Lambda} + (N-2) U_{0,\Lambda} \Big] u = 0.$$

So we have $u = 0$. This is a contradiction to (2.8). □

For the linearized problem (2.1), we have the following existence, uniqueness results. Furthermore, we can give the estimates of ϕ and $c_\ell, \ell = 1, 2, 3$.

Proposition 2.2. *There exist $k_0 > 0$ and a constant $C > 0$ such that for all $k \geq k_0$ and all $f \in L^\infty(\mathbb{R}^N)$, problem (2.1) has a unique solution $\phi \equiv \mathbf{L}_k(f)$. Besides,*

$$\|\phi\|_* \leq C\|f\|_{**}, \qquad |c_\ell| \leq \frac{C}{1+\delta_{\ell 2} r}\|f\|_{**}, \quad \ell = 1, 2, 3. \tag{2.9}$$

Proof. Recall the definition of $\mathbb{E}$ as in (1.17), we can rewrite problem (2.1) in the form

$$-\Delta\phi = f + (2^*-1) K\Big(\frac{|y|}{\mathbf{r}}\Big) W_{r,h,\Lambda}^{2^*-2}\phi \quad \text{for all } \phi \in \mathbb{E}, \tag{2.10}$$

in the sense of distribution. Furthermore, by using Riesz's representation theorem, Eq. (2.10) can be rewritten in the operational form

$$(\mathbb{I} - \mathbb{T}_k)\phi = \tilde{f}, \quad \text{in } \mathbb{E}, \tag{2.11}$$

where $\mathbb{I}$ is identity operator and $\mathbb{T}_k$ is a compact operator. Fredholm's alternative yields that problem (2.11) is uniquely solvable for any $\tilde{f}$ when the homogeneous equation

$$(\mathbb{I} - \mathbb{T}_k)\phi = 0, \quad \text{in } \mathbb{E}, \tag{2.12}$$

has only the trivial solution. Moreover, problem (2.12) can be rewritten as following

$$\begin{cases} -\Delta\phi - (2^*-1) K\Big(\frac{|y|}{\mathbf{r}}\Big) W_{r,h,\Lambda}^{2^*-2}\phi \\ \quad = \sum\limits_{i=1}^{k}\sum\limits_{\ell=1}^{3} \Big(c_\ell U_{\overline{x}_i,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell i} + c_\ell U_{\underline{x}_i,\Lambda}^{2^*-2}\underline{\mathbb{Z}}_{\ell i}\Big) \text{ in } \mathbb{R}^N, \\ \phi \in \mathbb{E}. \end{cases} \tag{2.13}$$

Suppose that (2.13) has nontrivial solution ϕ_k and satisfies $\|\phi_k\|_* = 1$. From Lemma 2.1, we know $\|\phi_k\|_*$ tends to zero as $k \to +\infty$, which is a contradiction. Thus problem (2.12) (or (2.13)) only has trivial solution. So we can get unique solvability for problem (2.1). Using Lemma 2.1, the estimates (2.9) can be proved by a standard method. □

We can rewrite problem (1.16) as following

$$\begin{cases} -\Delta\phi - (2^*-1) K\Big(\frac{|y|}{\mathbf{r}}\Big) W_{r,h,\Lambda}^{2^*-2}\phi = \mathbf{N}(\phi) + \mathbf{l}_k \\ \qquad\qquad + \sum\limits_{j=1}^{k}\sum\limits_{\ell=1}^{3} \Big(c_\ell U_{\overline{x}_j,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell j} + c_\ell U_{\underline{x}_j,\Lambda}^{2^*-2}\underline{\mathbb{Z}}_{\ell j}\Big) \text{ in } \mathbb{R}^N, \\ \phi \in \mathbb{E}, \end{cases} \tag{2.14}$$

where

$$\mathbf{N}(\phi) = K\Big(\frac{|y|}{\mathbf{r}}\Big)\Big[\big(W_{r,h,\Lambda}+\phi\big)^{2^*-1} - W_{r,h,\Lambda}^{2^*-1} - (2^*-1)W_{r,h,\Lambda}^{2^*-2}\phi\Big],$$

and

$$\mathbf{l}_k = K\Big(\frac{|y|}{\mathbf{r}}\Big)W_{r,h,\Lambda}^{2^*-1} - \sum_{j=1}^{k}\Big(U_{\overline{x}_j,\Lambda}^{2^*-1} + U_{\underline{x}_j,\Lambda}^{2^*-1}\Big).$$

Next, we will use the Contraction Mapping Principle to show that problem (2.14) has a unique solution in the set that $\|\phi\|_*$ is small enough. Before that, we will give the estimate of $\mathbf{N}(\phi)$ and $\mathbf{l}_k$.

Lemma 2.3. *Suppose $N \geq 5$. There exists $C > 0$ such that*

$$\|\mathbf{N}(\phi)\|_{**} \leq C\|\phi\|_*^{\min\{2^*-1,2\}},$$

for all $\phi \in \mathbb{E}$.

Proof. The proof is similar to that of Lemma 2.4 in [31]. Here we omit it. □

We next give the estimate of $\mathbf{l}_k$.

Lemma 2.4. *Suppose $K(|y|)$ satisfies* (**H**) *and $N \geq 5$, $(r,h,\Lambda) \in \mathscr{S}_k$. There exists k_0 and $C > 0$ such that for all $k \geq k_0$*

$$\|\ \mathbf{l}_k\ \|_{**} \leq C\max\Big\{\frac{1}{k^{(\frac{m}{N-2-m})(\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1)}}, \frac{1}{k^{(\frac{N-2}{N-2-m})\min\{m,\frac{m+3}{2}\}}}\Big\}, \quad (2.15)$$

where ϵ_1 is small constant given in (1.14).

Proof. We can rewrite $\mathbf{l}_k$ as

$$\begin{aligned}\mathbf{l}_k &= K\Big(\frac{|y|}{\mathbf{r}}\Big)\Big[\ W_{r,h,\Lambda}^{2^*-1} - \sum_{j=1}^{k}\Big(U_{\overline{x}_j,\Lambda}^{2^*-1} + U_{\underline{x}_j,\Lambda}^{2^*-1}\Big)\Big] \\ &\quad + \sum_{j=1}^{k}\Big[\ K\Big(\frac{|y|}{\mathbf{r}}\Big) - 1\Big]\Big(U_{\overline{x}_j,\Lambda}^{2^*-1} + U_{\underline{x}_j,\Lambda}^{2^*-1}\Big) := S_1 + S_2.\end{aligned}$$

Assume that $y \in \Omega_1^+$, then we get

$$\begin{aligned}S_1 &= K\Big(\frac{|y|}{\mathbf{r}}\Big)\Big[\Big(\sum_{j=1}^{k}U_{\overline{x}_j,\Lambda} + U_{\underline{x}_j,\Lambda}\Big)^{2^*-1} - \sum_{j=1}^{k}\Big(U_{\overline{x}_j,\Lambda}^{2^*-1} + U_{\underline{x}_j,\Lambda}^{2^*-1}\Big)\Big] \\ &\leq CK\Big(\frac{|y|}{\mathbf{r}}\Big)\Big[U_{\overline{x}_1,\Lambda}^{2^*-2}\Big(\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda} + \sum_{j=1}^{k}U_{\underline{x}_j,\Lambda}\Big) + \Big(\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda} + \sum_{j=1}^{k}U_{\underline{x}_j,\Lambda}\Big)^{2^*-1}\Big].\end{aligned}$$

Thus, we have

$$\begin{aligned} S_1 \leq & C\frac{1}{(1+|y-\overline{x}_1|)^4}\sum_{j=2}^{k}\frac{1}{(1+|y-\overline{x}_j|)^{N-2}} \\ & + C\frac{1}{(1+|y-\overline{x}_1|)^4}\sum_{j=1}^{k}\frac{1}{(1+|y-\underline{x}_j|)^{N-2}} \\ & + C\Big(\sum_{j=2}^{k}\frac{1}{(1+|y-\overline{x}_j|)^{N-2}}\Big)^{2^*-1} := S_{11}+S_{12}+S_{13}. \end{aligned}$$

We first consider the case $N=5$. It is easy to get that

$$\begin{aligned} S_{11}\Big|_{N=5} & \leq C\frac{1}{(1+|y-\overline{x}_1|)^{\frac{7}{2}+\tau}}\sum_{j=2}^{k}\frac{1}{|\overline{x}_j-\overline{x}_1|^3} \\ & \leq C\frac{1}{(1+|y-\overline{x}_1|)^{\frac{7}{2}+\tau}}\Big(\frac{k}{\mathbf{r}}\Big)^3. \end{aligned} \tag{2.16}$$

When $N\geq 6$, similar to the proof of Lemma B.1, for any $1<\alpha_1<N-2$, we have

$$\sum_{j=2}^{k}\frac{1}{(1+|y-\overline{x}_j|)^{N-2}} \leq \frac{C}{(1+|y-\overline{x}_1|)^{N-2-\alpha_1}}\sum_{j=2}^{k}\frac{1}{|\overline{x}_j-\overline{x}_1|^{\alpha_1}}.$$

Since $\tau\in(\frac{N-2-m}{N-2},\frac{N-2-m}{N-2}+\epsilon_1)$, we can choose α_1 satisfies

$$\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1<\alpha_1=\frac{N+2}{2}-\tau<N-2.$$

Then

$$\begin{aligned} S_{11}\Big|_{N\geq 6} & \leq \frac{C}{(1+|y-\overline{x}_1|)^{N+2-\alpha_1}}\sum_{j=2}^{k}\frac{1}{|\overline{x}_j-\overline{x}_1|^{\alpha_1}} \\ & \leq \frac{C}{(1+|y-\overline{x}_1|)^{N+2-\alpha_1}}\Big(\frac{k}{\mathbf{r}\sqrt{1-h^2}}\Big)^{\alpha_1} \\ & \leq C\frac{1}{(1+|y-\overline{x}_1|)^{\frac{N+2}{2}+\tau}}\Big(\frac{k}{\mathbf{r}}\Big)^{\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1}. \end{aligned} \tag{2.17}$$

Then combining (2.16) and (2.17), we can get

$$\|S_{11}\|_{**} \leq \begin{cases} C\Big(\frac{k}{\mathbf{r}}\Big)^{\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1}, & \text{if } \ N\geq 6, \\ C\Big(\frac{k}{\mathbf{r}}\Big)^3, & \text{if } \ N=5. \end{cases} \tag{2.18}$$

For S_{12}, we can rewrite it as following

$$S_{12} = C\frac{1}{(1+|y-\overline{x}_1|)^4}\Big[\frac{1}{(1+|y-\underline{x}_1|)^{N-2}} + \sum_{j=2}^{k}\frac{1}{(1+|y-\underline{x}_j|)^{N-2}}\Big]$$
$$\le C\frac{1}{(1+|y-\overline{x}_1|)^4}\Big[\frac{1}{(1+|y-\underline{x}_1|)^{N-2}} + \sum_{j=2}^{k}\frac{1}{(1+|y-\bar{x}_j|)^{N-2}}\Big].$$

Similarly to (2.16), we can obtain

$$S_{12}\Big|_{N=5} \le C\frac{1}{(1+|y-\overline{x}_1|)^{\frac{7}{2}+\tau}}\Big(\frac{k}{\mathbf{r}}\Big)^3.$$

For $N \ge 6$ and the same α_1 as in (2.18), it is easy to derive that

$$\frac{1}{(1+|y-\overline{x}_1|)^4}\frac{1}{(1+|y-\underline{x}_1|)^{N-2}}$$
$$\le \Big[\frac{1}{(1+|y-\overline{x}_1|)^{N+2-\alpha_1}} + \frac{1}{(1+|y-\underline{x}_1|)^{N+2-\alpha_1}}\Big]\frac{1}{|\underline{x}_1-\overline{x}_1|^{\alpha_1}}$$
$$\le \frac{C}{(1+|y-\overline{x}_1|)^{N+2-\alpha_1}}\frac{1}{(hr)^{\alpha_1}}$$
$$\le C\frac{1}{(1+|y-\overline{x}_1|)^{\frac{N+2}{2}+\tau}}\Big(\frac{k}{\mathbf{r}}\Big)^{\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1},$$

where we have used the fact $hr > C\frac{\mathbf{r}}{k}$. Thus, we can obtain that

$$\|S_{12}\|_{**} \le \begin{cases} C\Big(\frac{k}{\mathbf{r}}\Big)^{\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1}, & \text{if } N \ge 6, \\ C\Big(\frac{k}{\mathbf{r}}\Big)^3, & \text{if } N = 5. \end{cases} \tag{2.19}$$

Next, we consider S_{13}. For $y \in \Omega_1^+$,

$$\sum_{j=2}^{k}\frac{1}{(1+|y-\overline{x}_j|)^{N-2}} \le \sum_{j=2}^{k}\frac{1}{(1+|y-\overline{x}_1|)^{\frac{N-2}{2}}}\frac{1}{(1+|y-\overline{x}_j|)^{\frac{N-2}{2}}}$$
$$\le \sum_{j=2}^{k}\frac{C}{|\overline{x}_j-\overline{x}_1|^{\frac{N-2}{2}-\frac{N-2}{N+2}\tau}}\frac{1}{(1+|y-\overline{x}_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}$$
$$\le C\Big(\frac{k}{\mathbf{r}\sqrt{1-h^2}}\Big)^{\frac{N-2}{2}-\frac{N-2}{N+2}\tau}\frac{1}{(1+|y-\overline{x}_1|)^{\frac{N-2}{2}+\frac{N-2}{N+2}\tau}}.$$

Thus we have

$$S_{13} \le \Big(\frac{k}{\mathbf{r}\sqrt{1-h^2}}\Big)^{\frac{N+2}{2}-\tau}\frac{C}{(1+|y-\overline{x}_1|)^{\frac{N+2}{2}+\tau}}$$
$$\le \frac{C}{(1+|y-\overline{x}_1|)^{\frac{N+2}{2}+\tau}}\Big(\frac{k}{\mathbf{r}}\Big)^{\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1}.$$

Since $\left(\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1\right)\Big|_{N=5}>3$ for $m\in[2,3)$, then we have

$$\|S_{13}\|_{**}\leq\begin{cases}C\left(\frac{k}{\mathbf{r}}\right)^{\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1}, & \text{if}\quad N\geq 6,\\ C\left(\frac{k}{\mathbf{r}}\right)^{3}, & \text{if}\quad N=5.\end{cases}\tag{2.20}$$

Combining (2.18), (2.19), (2.20), we obtain

$$\|S_{1}\|_{**}\leq\begin{cases}C\left(\frac{k}{\mathbf{r}}\right)^{\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1}, & \text{if}\quad N\geq 6,\\ C\left(\frac{k}{\mathbf{r}}\right)^{3}, & \text{if}\quad N=5.\end{cases}\tag{2.21}$$

We now consider the estimate of S_2. For $y\in\Omega_1^+$, we have

$$\begin{aligned}S_2&\leq 2\sum_{j=1}^{k}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]U_{\overline{x}_j,\Lambda}^{2^*-1}\\&=2\,U_{\overline{x}_1,\Lambda}^{2^*-1}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]+2\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda}^{2^*-1}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]\\&:=S_{21}+S_{22}.\end{aligned}$$

- If $|\frac{|y|}{\mathbf{r}}-1|\geq\delta_1$, where $\delta>\delta_1>0$, then

$$|y-\overline{x}_1|\geq\big||y|-\mathbf{r}\big|-\big|\mathbf{r}-|\overline{x}_1|\big|\geq\frac{1}{2}\delta_1\mathbf{r}.$$

As a result, we get

$$\begin{aligned}U_{\overline{x}_1,\Lambda}^{2^*-1}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]&\leq\frac{C}{(1+|y-\overline{x}_1|)^{\frac{N+2}{2}+\tau}}\frac{1}{\mathbf{r}}^{\frac{N+2}{2}-\tau}\\&\leq\frac{C}{(1+|y-\overline{x}_1|)^{\frac{N+2}{2}+\tau}}\Big(\frac{k}{\mathbf{r}}\Big)^{\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1}.\end{aligned}$$

- If $|\frac{|y|}{\mathbf{r}}-1|\leq\delta_1$, then

$$\begin{aligned}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]&\leq C\Big|\frac{|y|}{\mathbf{r}}-1\Big|^m=\frac{C}{\mathbf{r}}^m\big||y|-\mathbf{r}\big|^m\\&\leq\frac{C}{\mathbf{r}}^m\Big[\big||y|-|\overline{x}_1|\big|^m+\big||\overline{x}_1|-\mathbf{r}\big|^m\Big]\\&\leq\frac{C}{\mathbf{r}}^m\Big[\big||y|-|\overline{x}_1|\big|^m+\frac{1}{k^{\bar{\theta}m}}\Big].\end{aligned}$$

Thus, we can get, if $m > 3$,

$$
\begin{aligned}
&U_{\overline{x}_1,\Lambda}^{2^*-1}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]\\
&\quad\le \frac{C}{\mathbf{r}}^m\Big[\big||y|-|\overline{x}_1|\big|^m+\frac{1}{k^{\bar\theta m}}\Big]\frac{C}{\big(1+|y-\overline{x}_1|\big)^{N+2}}\\
&\quad\le \frac{C}{\mathbf{r}}^{\frac{m+3}{2}}\Big[\frac{\big||y|-|\overline{x}_1|\big|^{\frac{m+3}{2}}}{\big(1+|y-\overline{x}_1|\big)^{N+2}}\\
&\qquad+\frac{1}{\mathbf{r}}^{\frac{m-3}{2}}\frac{1}{k^{\bar\theta m}}\frac{1}{\big(1+|y-\overline{x}_1|\big)^{N+2}}\Big]\\
&\quad\le \frac{C}{\mathbf{r}}^{\frac{m+3}{2}}\Big[\frac{1}{\big(1+|y-\overline{x}_1|\big)^{\frac{N+2}{2}+\tau}}\frac{1}{\big(1+|y-\overline{x}_1|\big)^{\frac{N+2}{2}-\tau-\frac{m+3}{2}}}+\frac{1}{\big(1+|y-\overline{x}_1|\big)^{N+2}}\Big]\\
&\quad\le \frac{1}{\mathbf{r}}^{\frac{m+3}{2}}\frac{C}{\big(1+|y-\overline{x}_1|\big)^{\frac{N+2}{2}+\tau}},
\end{aligned}
$$

the last inequality holds due to $\frac{N+2}{2}-\tau-\frac{m+3}{2}>0$.

On the other hand, if $m \le 3$, we have

$$
\begin{aligned}
&U_{\overline{x}_1,\Lambda}^{2^*-1}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]\\
&\quad\le \frac{C}{\mathbf{r}}^m\Big[\big||y|-|\overline{x}_1|\big|^m+\frac{1}{k^{\bar\theta m}}\Big]\frac{C}{\big(1+|y-\overline{x}_1|\big)^{N+2}}\\
&\quad\le \frac{C}{\mathbf{r}}^m\Big[\frac{1}{\big(1+|y-\overline{x}_1|\big)^{\frac{N+2}{2}+\tau}}\frac{1}{\big(1+|y-\overline{x}_1|\big)^{\frac{N+2}{2}-\tau-m}}\\
&\qquad+\frac{1}{\big(1+|y-\overline{x}_1|\big)^{N+2}}\Big]\\
&\quad\le \frac{C}{\mathbf{r}}^m\frac{1}{\big(1+|y-\overline{x}_1|\big)^{\frac{N+2}{2}+\tau}},
\end{aligned}
$$

since $\frac{N+2}{2}-\tau-m>0$. Thus we have

$$
U_{\overline{x}_1,\Lambda}^{2^*-1}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]\le \frac{C}{\mathbf{r}}^{\min\{m,\frac{m+3}{2}\}}\frac{1}{\big(1+|y-\overline{x}_1|\big)^{\frac{N+2}{2}+\tau}}.
$$

As a result,

$$
S_{21}\le C\max\Big\{\Big(\frac{k}{\mathbf{r}}\Big)^{\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1},\frac{1}{\mathbf{r}}^{\min\{m,\frac{m+3}{2}\}}\Big\}\frac{1}{\big(1+|y-\overline{x}_1|\big)^{\frac{N+2}{2}+\tau}}. \tag{2.22}
$$

Since $y\in\Omega_1^+$, then for $j=2,\dots,k$, there holds

$$
|\overline{x}_1-\overline{x}_j|\le|y-\overline{x}_1|+|y-\overline{x}_j|\le 2|y-\overline{x}_j|.
$$

Therefore, it is easy to derive that

$$\begin{aligned} S_{22} &\leq C\frac{1}{(1+|y-\overline{x}_1|)^{\frac{N+2}{2}}}\sum_{j=2}^{k}\frac{1}{(1+|y-\overline{x}_j|)^{\frac{N+2}{2}}} \\ &\leq C\frac{1}{(1+|y-\overline{x}_1|)^{\frac{N+2}{2}+\tau}}\sum_{j=2}^{k}\frac{1}{|\overline{x}_1-\overline{x}_j|^{\frac{N+2}{2}-\tau}} \\ &\leq \frac{C}{(1+|y-\overline{x}_1|)^{\frac{N+2}{2}+\tau}}\Big(\frac{k}{\mathbf{r}}\Big)^{\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1}. \end{aligned} \tag{2.23}$$

Combining (2.22) with (2.23), we obtain

$$\|S_2\|_{**} \leq C\max\Big\{\Big(\frac{k}{\mathbf{r}}\Big)^{\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1}, \frac{1}{\mathbf{r}}^{\min\{m,\frac{m+3}{2}\}}\Big\}.$$

If $N=5$, we can check that $\frac{1}{\mathbf{r}}^{m} = \Big(\frac{k}{\mathbf{r}}\Big)^3$. Thus, we can rewrite (2.21) as

$$\|S_1\|_{**} \leq C\max\Big\{\Big(\frac{k}{\mathbf{r}}\Big)^{\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1}, \frac{1}{\mathbf{r}}^{\min\{m,\frac{m+3}{2}\}}\Big\}.$$

Therefore, we showed (2.15). □

The solvability theory for the projected problem (2.14) can be provided in the following:

Proposition 2.5. *Suppose that $K(|y|)$ satisfies* $(\mathbf{H})$ *and $N \geq 5$, $(r,h,\Lambda) \in \mathscr{S}_k$. There exists an integer k_0 large enough, such that for all $k \geq k_0$ problem* (2.14) *has a unique solution ϕ_k which satisfies*

$$\|\phi_k\|_* \leq C\max\Big\{\frac{1}{k^{(\frac{m}{N-2-m})(\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1)}}, \frac{1}{k^{(\frac{N-2}{N-2-m})\min\{m,\frac{m+3}{2}\}}}\Big\}, \tag{2.24}$$

and

$$|c_\ell| \leq \frac{C}{(1+\delta_{\ell 2}\mathbf{r})} \max\Big\{\frac{1}{k^{(\frac{m}{N-2-m})(\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1)}}, \frac{1}{k^{(\frac{N-2}{N-2-m})\min\{m,\frac{m+3}{2}\}}}\Big\}, \quad for\ \ell = 1,2,3. \tag{2.25}$$

Proof. We first denote

$$\mathcal{B} := \Big\{v : v \in \mathbb{E} \quad \|v\|_* \leq C\max\Big\{\frac{1}{k^{(\frac{m}{N-2-m})(\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1)}}, \frac{1}{k^{(\frac{N-2}{N-2-m})\min\{m,\frac{m+3}{2}\}}}\Big\}\Big\}.$$

From Proposition 2.2, we know that problem (2.14) is equivalent to the following fixed point problem

$$\phi = \mathbf{L}_k\big(\mathbf{N}(\phi)+\mathbf{l}_k\big) =: \mathbf{A}(\phi),$$

where $\mathbf{L}_k$ is the linear bounded operator defined in Proposition 2.2.

From Lemmas 2.3 and 2.4, we know, for $\phi \in \mathcal{B}$

$$\begin{aligned}\|\mathbf{A}(\phi)\|_* &\le C\Big(\|\mathbf{N}(\phi)\|_{**} + \|\mathbf{l}_k\|_{**}\Big)\\ &\le O(\|\phi\|_*^{1+\sigma})\\ &\quad + \max\left\{\frac{1}{k^{(\frac{m}{N-2-m})(\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1)}}, \frac{1}{k^{(\frac{N-2}{N-2-m})\min\{m,\frac{m+3}{2}\}}}\right\}\\ &\le \max\left\{\frac{1}{k^{(\frac{m}{N-2-m})(\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1)}}, \frac{1}{k^{(\frac{N-2}{N-2-m})\min\{m,\frac{m+3}{2}\}}}\right\}.\end{aligned}$$

So the operator $\mathbf{A}$ maps from $\mathcal{B}$ to $\mathcal{B}$. Furthermore, we can show that $\mathbf{A}$ is a contraction mapping. In fact, for any $\phi_1, \phi_2 \in \mathcal{B}$, we have

$$\|\mathbf{A}(\phi_1) - \mathbf{A}(\phi_2)\|_* \le C\|\mathbf{N}(\phi_1) - \mathbf{N}(\phi_2)\|_{**}.$$

Since $\mathbf{N}(\phi)$ has a power-like behavior with power greater than one, then we can easily get

$$\|\mathbf{A}(\phi_1) - \mathbf{A}(\phi_2)\|_* \le o(1)\|\phi_1 - \phi_2\|_*.$$

A direct application of the contraction mapping principle yields that problem (2.14) has a unique solution $\phi \in \mathcal{B}$. The estimates for $c_\ell, \ell = 1, 2, 3$ can be got easily from (2.6). □

3. Proof of Theorem 1.1

Proposition 3.1. *Let $\phi_{r,h,\Lambda}$ be a function obtained in Proposition 2.5 and*

$$F(r, h, \Lambda) := I(W_{r,h,\Lambda} + \phi_{r,h,\Lambda}).$$

If (r, h, Λ) is a critical point of $F(r, h, \Lambda)$, then

$$u = W_{r,h,\Lambda} + \phi_{r,h,\Lambda}$$

is a critical point of $I(u)$ in $H^1(\mathbb{R}^N)$. □

We will give the expression of $F(r, h, \Lambda)$. We first note that we employ the notation $\mathcal{C}(r, \Lambda)$ to denote functions which are independent of h and uniformly bounded.

Proposition 3.2. *Suppose that $K(|y|)$ satisfies* (**H**) *and $N \ge 5$, $(r, h, \Lambda) \in \mathscr{S}_k$. We have the following expansion as $k \to \infty$*

$$\begin{aligned}&F(r, h, \Lambda)\\ &= I(W_{r,h,\Lambda}) + kO\Big(\frac{1}{k^{\left(\frac{m(N-2)}{N-2-m}+\frac{2(N-3)}{N-1}+\sigma\right)}}\Big)\\ &= kA_1 - \frac{k}{\Lambda^{N-2}}\Big[\frac{B_4 k^{N-2}}{(r\sqrt{1-h^2})^{N-2}} + \frac{B_5 k}{r^{N-2}h^{N-3}\sqrt{1-h^2}}\Big]\\ &\quad + k\Big[\frac{A_2}{\Lambda^m k^{\frac{(N-2)m}{N-2-m}}} + \frac{A_3}{\Lambda^{m-2} k^{\frac{(N-2)m}{N-2-m}}}(\mathbf{r} - r)^2\Big] + k\frac{\mathcal{C}(r,\Lambda)}{k^{\frac{m(N-2)}{N-2-m}}}(\mathbf{r} - r)^{2+\sigma}\\ &\quad + k\frac{\mathcal{C}(r,\Lambda)}{k^{\frac{m(N-2)}{N-2-m}+\sigma}} + kO\Big(\frac{1}{k^{\left(\frac{m(N-2)}{N-2-m}+\frac{2(N-3)}{N-1}+\sigma\right)}}\Big),\end{aligned}$$

where A_1, A_2, A_3, B_4, B_5 are positive constants.

Proof. The proof of Proposition 3.2 is similar to that of Proposition 3.1 in [31]. We omit it here. □

Next, we will give the expansions of $\frac{\partial F(r,h,\Lambda)}{\partial \Lambda}$ and $\frac{\partial F(r,h,\Lambda)}{\partial h}$.

Proposition 3.3. *Suppose that $K(|y|)$ satisfies* (**H**) *and $N \geq 5$, $(r,h,\Lambda) \in \mathscr{S}_k$. We have the following expansion for $k \to \infty$*

$$\begin{aligned}\frac{\partial F(r,h,\Lambda)}{\partial \Lambda} &= \frac{k(N-2)}{\Lambda^{N-1}}\Big[\frac{B_4 k^{N-2}}{(r\sqrt{1-h^2})^{N-2}} + \frac{B_5 k}{r^{N-2}h^{N-3}\sqrt{1-h^2}}\Big] \\ &\quad - k\Big[\frac{mA_2}{\Lambda^{m+1}k^{\frac{(N-2)m}{N-2-m}}} + \frac{(m-2)A_3}{\Lambda^{m-1}k^{\frac{(N-2)m}{N-2-m}}}(\mathbf{r}-r)^2\Big] + kO\Big(\frac{1}{k^{\frac{(N-2)m}{N-2-m}+\sigma}}\Big), \qquad (3.1)\end{aligned}$$

where A_2, A_3, B_4, B_5 are positive constants.

Proof. The proof of this proposition can be found in [31]. We omit it here. □

Proposition 3.4. *Suppose that $K(|y|)$ satisfies* (**H**) *and $N \geq 5$, $(r,h,\Lambda) \in \mathscr{S}_k$. We have the following expansion*

$$\begin{aligned}\frac{\partial F(r,h,\Lambda)}{\partial h} &= -\frac{k}{\Lambda^{N-2}}\Big[(N-2)\frac{B_4 k^{N-2}}{r^{N-2}(\sqrt{1-h^2})^{N}}h - (N-3)\frac{B_5 k}{r^{N-2}h^{N-2}\sqrt{1-h^2}}\Big] \\ &\quad + kO\Big(\frac{1}{k^{\left(\frac{m(N-2)}{N-2-m}+\frac{(N-3)}{N-1}+\sigma\right)}}\Big), \qquad (3.2)\end{aligned}$$

where B_4, B_5 are positive constants.

Proof. Notice that $F(r,h,\Lambda) = I(W_{r,h,\Lambda} + \phi_{r,h,\Lambda})$, there holds

$$\begin{aligned}\frac{\partial F(r,h,\Lambda)}{\partial h} &= \Big\langle I'(W_{r,h,\Lambda}+\phi_{r,h,\Lambda}), \frac{\partial (W_{r,h,\Lambda}+\phi_{r,h,\Lambda})}{\partial h}\Big\rangle \\ &= \Big\langle I'(W_{r,h,\Lambda}+\phi_{r,h,\Lambda}), \frac{\partial W_{r,h,\Lambda}}{\partial h}\Big\rangle + \Big\langle I'(W_{r,h,\Lambda}+\phi_{r,h,\Lambda}), \frac{\partial \phi_{r,h,\Lambda}}{\partial h}\Big\rangle \\ &= \Big\langle I'(W_{r,h,\Lambda}+\phi_{r,h,\Lambda}), \frac{\partial W_{r,h,\Lambda}}{\partial h}\Big\rangle \\ &\quad + \Big\langle \sum_{j=1}^{k}\sum_{\ell=1}^{3}\Big(c_\ell U_{\overline{x}_j,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell j} + c_\ell U_{\underline{x}_j,\Lambda}^{2^*-2}\underline{\mathbb{Z}}_{\ell j}\Big), \frac{\partial \phi_{r,h,\Lambda}}{\partial h}\Big\rangle. \qquad (3.3)\end{aligned}$$

Since $\int_{\mathbb{R}^N} U_{\overline{x}_j,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell j}\phi_{r,h,\Lambda} = \int_{\mathbb{R}^N} U_{\underline{x}_j,\Lambda}^{2^*-2}\underline{\mathbb{Z}}_{\ell j}\phi_{r,h,\Lambda} = 0$, we can get easily

$$\Big\langle U_{\overline{x}_j,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell j}, \frac{\partial \phi_{r,h,\Lambda}}{\partial h}\Big\rangle = -\Big\langle \frac{\partial (U_{\overline{x}_j,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell j})}{\partial h}, \phi_{r,h,\Lambda}\Big\rangle,$$

$$\Big\langle U_{\underline{x}_j,\Lambda}^{2^*-2}\underline{\mathbb{Z}}_{\ell j}, \frac{\partial \phi_{r,h,\Lambda}}{\partial h}\Big\rangle = -\Big\langle \frac{\partial (U_{\underline{x}_j,\Lambda}^{2^*-2}\underline{\mathbb{Z}}_{\ell j})}{\partial h}, \phi_{r,h,\Lambda}\Big\rangle.$$

Then

$$\begin{aligned}
&\Big\langle \sum_{j=1}^{k}\Big(c_\ell U_{\overline{x}_j,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell j} + c_\ell U_{\underline{x}_j,\Lambda}^{2^*-2}\underline{\mathbb{Z}}_{\ell j}\Big), \frac{\partial \phi_{r,h,\Lambda}}{\partial h}\Big\rangle \\
&\le C|c_\ell|\|\phi_{r,h,\Lambda}\|_* \sum_{i=1}^{k}\int_{\mathbb{R}^N} \frac{\partial (U_{\overline{x}_i,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell i})}{\partial h} \\
&\quad \Bigg(\sum_{j=1}^{k}\Big[\frac{1}{(1+|y-\overline{x}_j|)^{\frac{N-2}{2}+\tau}} + \frac{1}{(1+|y-\underline{x}_j|)^{\frac{N-2}{2}+\tau}}\Big]\Bigg) \\
&\le C|c_\ell|\|\phi_{r,h,\Lambda}\|_* \\
&\quad \times \sum_{i=1}^{k}\int_{\mathbb{R}^N} \frac{r(1+\delta_{\ell 2}\mathbf{r})}{(1+|y-\overline{x}_i|)^{N+3}} \\
&\quad \Bigg(\sum_{j=1}^{k}\Big[\frac{1}{(1+|y-\overline{x}_j|)^{\frac{N-2}{2}+\tau}} + \frac{1}{(1+|y-\underline{x}_j|)^{\frac{N-2}{2}+\tau}}\Big]\Bigg) \\
&\le C\mathbf{r}\max\Big\{\frac{1}{k^{(\frac{m}{N-2-m})(N+2-2\frac{N-2-m}{N-2}-2\epsilon_1)}}, \frac{1}{k^{(\frac{N-2}{N-2-m})\min\{2m,m+3\}}}\Big\},
\end{aligned} \tag{3.4}$$

where we used the estimates (2.24)-(2.25) and the inequalities

$$\Big|\frac{\partial\big(U_{\overline{x}_i,\Lambda}^{2^*-2}\overline{\mathbb{Z}}_{\ell i}\big)}{\partial h}\Big| \le C\frac{\mathbf{r}(1+\delta_{\ell 2}r)}{(1+|y-\overline{x}_i|)^{N+3}} \quad \text{for } i=1,\dots,k, \ell=1,2,3.$$

On the other hand, we have

$$\begin{aligned}
&\Big\langle I'(W_{r,h,\Lambda}+\phi_{r,h,\Lambda}), \frac{\partial W_{r,h,\Lambda}}{\partial h}\Big\rangle \\
&= \int_{\mathbb{R}^N} \nabla(W_{r,h,\Lambda}+\phi_{r,h,\Lambda})\nabla W_{r,h,\Lambda} - \int_{\mathbb{R}^N} K\Big(\frac{|y|}{\mathbf{r}}\Big)(W_{r,h,\Lambda}+\phi_{r,h,\Lambda})^{2^*-1}\frac{\partial W_{r,h,\Lambda}}{\partial h} \\
&= \int_{\mathbb{R}^N} \nabla W_{r,h,\Lambda}\nabla \frac{\partial W_{r,h,\Lambda}}{\partial h} - \int_{\mathbb{R}^N} K\Big(\frac{|y|}{\mathbf{r}}\Big)(W_{r,h,\Lambda}+\phi_{r,h,\Lambda})^{2^*-1}\frac{\partial W_{r,h,\Lambda}}{\partial h} \\
&= \frac{\partial I(W_{r,h,\Lambda})}{\partial h} + (2^*-1)\int_{\mathbb{R}^N} K\Big(\frac{|y|}{\mathbf{r}}\Big)W_{r,h,\Lambda}^{2^*-2}\frac{\partial W_{r,h,\Lambda}}{\partial h}\phi_{r,h,\Lambda} + O\Big(\int_{\mathbb{R}^N}\phi_{r,h,\Lambda}^2\Big).
\end{aligned} \tag{3.5}$$

For the second term in (3.5), using the decay property of $K(|y|)$ and orthogonality of $\phi_{r,h,\Lambda}$, we can show this term is small. In fact, we have

$$\begin{aligned}
&\int_{\mathbb{R}^N} K\Big(\frac{|y|}{\mathbf{r}}\Big) W_{r,h,\Lambda}^{2^*-2} \frac{\partial W_{r,h,\Lambda}}{\partial h} \phi_{r,h,\Lambda} \\
&= \int_{\mathbb{R}^N} K\Big(\frac{|y|}{\mathbf{r}}\Big) \Big[W_{r,h,\Lambda}^{2^*-2} \frac{\partial W_{r,h,\Lambda}}{\partial h} - \sum_{i=1}^{k} \big(U_{\overline{x}_i,\Lambda}^{2^*-2} \overline{\mathbb{Z}}_{2i} + U_{\underline{x}_i,\Lambda}^{2^*-2} \underline{\mathbb{Z}}_{2i} \big) \Big] \phi_{r,h,\Lambda} \\
&\quad + \sum_{i=1}^{k} \int_{\mathbb{R}^N} \Big[K\Big(\frac{|y|}{\mathbf{r}}\Big) - 1 \Big] \big(U_{\overline{x}_i,\Lambda}^{2^*-2} \overline{\mathbb{Z}}_{2i} + U_{\underline{x}_i,\Lambda}^{2^*-2} \underline{\mathbb{Z}}_{2i} \big) \phi_{r,h,\Lambda} \\
&= 2k \int_{\Omega_1^+} K\Big(\frac{|y|}{\mathbf{r}}\Big) \Big[W_{r,h,\Lambda}^{2^*-2} \frac{\partial W_{r,h,\Lambda}}{\partial h} - \sum_{i=1}^{k} \big(U_{\overline{x}_i,\Lambda}^{2^*-2} \overline{\mathbb{Z}}_{2i} + U_{\underline{x}_i,\Lambda}^{2^*-2} \underline{\mathbb{Z}}_{2i} \big) \Big] \phi_{r,h,\Lambda} \\
&\quad + 2k \int_{\mathbb{R}^N} \Big[K\Big(\frac{|y|}{\mathbf{r}}\Big) - 1 \Big] U_{\overline{x}_1,\Lambda}^{2^*-2} \overline{\mathbb{Z}}_{21} \phi_{r,h,\Lambda}.
\end{aligned}$$

According to the expression of $W_{r,h,\Lambda}$, we can obtain that

$$\begin{aligned}
&\int_{\Omega_1^+} K\left(\frac{|y|}{\mathbf{r}}\right) \left[W_{r,h,\Lambda}^{2^*-2} \frac{\partial W_{r,h,\Lambda}}{\partial h} - \sum_{i=1}^{k} (U_{\overline{x}_i,\Lambda}^{2^*-2} \overline{\mathbb{Z}}_{2i} + U_{\underline{x}_i,\Lambda}^{2^*-2} \underline{\mathbb{Z}}_{2i} \right] \phi_{r,h,\Lambda} \\
&\le C \int_{\Omega_1^+} \left[U_{\overline{x}_1,\Lambda}^{2^*-2} \left(\sum_{j=2}^{k} \overline{\mathbb{Z}}_{2j} + \sum_{j=1}^{k} \underline{\mathbb{Z}}_{2j} \right) + \left(\sum_{i=2}^{k} U_{\overline{x}_i,\Lambda}^{2^*-2} \overline{\mathbb{Z}}_{2i} + \sum_{i=1}^{k} U_{\underline{x}_i,\Lambda}^{2^*-2} \underline{\mathbb{Z}}_{2i} \right) \right] \phi_{r,h,\Lambda} \\
&\le C \Big(\frac{k}{\mathbf{r}}\Big)^{\frac{N+2}{2}-\tau} \int_{\Omega_1^+} \frac{\mathbf{r}}{(1+|y-\overline{x}_1|)^{\frac{N}{2}+2+\tau}} \phi_{r,h,\Lambda} \\
&\le C \Big(\frac{k}{\mathbf{r}}\Big)^{\frac{N+2}{2}-\tau} \|\phi_{r,h,\Lambda}\|_* \int_{\Omega_1^+} \frac{\mathbf{r}}{(1+|y-\overline{x}_1|)^{\frac{N}{2}+2+\tau}} \\
&\quad \times \Big(\sum_{j=1}^{k} \Big[\frac{1}{(1+|y-\overline{x}_j|)^{\frac{N-2}{2}+\tau}} + \frac{1}{(1+|y-\underline{x}_j|)^{\frac{N-2}{2}+\tau}} \Big] \Big) \\
&\le C\mathbf{r} \Big(\frac{k}{\mathbf{r}}\Big)^{\frac{N+2}{2}-\tau} \|\phi_{r,h,\Lambda}\|_* \le C\mathbf{r} \Big(\frac{k}{\mathbf{r}}\Big)^{\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1} \\
&\quad \max\Big\{ \Big(\frac{k}{\mathbf{r}}\Big)^{\frac{N+2}{2}-\frac{N-2-m}{N-2}-\epsilon_1}, \frac{1}{\mathbf{r}}^{\min\{m,\frac{m+3}{2}\}} \Big\} \\
&\le C\mathbf{r} \max\Big\{ \frac{1}{k^{(\frac{m}{N-2-m})(N+2-2\frac{N-2-m}{N-2}-2\epsilon_1)}}, \frac{1}{k^{(\frac{N-2}{N-2-m})\min\{2m,m+3\}}} \Big\}.
\end{aligned}$$

And it's easy to show that

$$\begin{aligned}
&\int_{\mathbb{R}^N} \Big[K\Big(\frac{|y|}{\mathbf{r}}\Big) - 1 \Big] U_{\overline{x}_1,\Lambda}^{2^*-2} \overline{\mathbb{Z}}_{21} \phi_{r,h,\Lambda} \\
&\quad \le C\mathbf{r} \max\Big\{ \frac{1}{k^{(\frac{m}{N-2-m})(N+2-2\frac{N-2-m}{N-2}-2\epsilon_1)}}, \frac{1}{k^{(\frac{N-2}{N-2-m})\min\{2m,m+3\}}} \Big\}.
\end{aligned}$$

Combining all above, we can get

$$\frac{\partial F(r,h,\Lambda)}{\partial h} = \frac{\partial I(W_{r,h,\Lambda})}{\partial h} + kO\Big(\mathbf{r} \max\Big\{\frac{1}{k^{(\frac{m}{N-2-m})(N+2-2\frac{N-2-m}{N-2}-2\epsilon_1)}}, \frac{1}{k^{(\frac{N-2}{N-2-m})\min\{2m,m+3\}}}\Big\}\Big). \tag{3.6}$$

Combing (3.6), Proposition A.6 and Lemma B.6, we can get (A.28) □

Remark 3.5. The expansions of $\frac{\partial F(r,h,\Lambda)}{\partial h}$ and $\frac{\partial F(r,h,\Lambda)}{\partial \Lambda}$ would be applied in the proof of Proposition 3.6, which is essential for proving the existence critical point of $F(r,h,\Lambda)$. In order to get a proper expansion of $\frac{\partial F(r,h,\Lambda)}{\partial h}$, we need accurate estimates for $\phi_{r,h,\Lambda}$. □

Rewritten the expansion of the energy functional.

Let Λ_0 be

$$\Lambda_0 = \Big[\frac{(N-2)B_4}{A_2 m}\Big]^{\frac{1}{N-2-m}}. \tag{3.7}$$

Then it solves

$$\frac{B_4(N-2)}{\Lambda^{N-1}} - \frac{A_2 m}{\Lambda^{m+1}} = 0.$$

Denote

$$\mathcal{G}(h) := \frac{B_4 k^{N-2}}{(\sqrt{1-h^2})^{N-2}} + \frac{B_5 k}{h^{N-3}\sqrt{1-h^2}},$$

then

$$\begin{aligned}\mathcal{G}'(h) &= (N-2)\frac{B_4 k^{N-2} h}{(\sqrt{1-h^2})^{N}} - (N-3)\frac{B_5 k}{h^{N-2}\sqrt{1-h^2}} + h\frac{B_5 k}{h^{N-4}(1-h^2)^{\frac{3}{2}}}\\ &= (N-2)B_4 k^{N-2} h\big[1+O(h^2)\big]\\ &\quad - (N-3)\frac{B_5 k}{h^{N-2}}\big[1+O(h^2)\big] + \frac{B_5 k}{h^{N-4}}\big[1+O(h^2)\big]\\ &= \Big[(N-2)B_4 k^{N-2} h - (N-3)\frac{B_5 k}{h^{N-2}}\Big] + O\Big(\frac{k}{h^{N-4}}\Big),\end{aligned}$$

and

$$\begin{aligned}\mathcal{G}''(h) &= (N-2)\frac{B_4 k^{N-2}}{(\sqrt{1-h^2})^{N}} + (N-2)N\frac{B_4 k^{N-2} h^2}{(\sqrt{1-h^2})^{N+2}}\\ &\quad + (N-3)(N-2)\frac{B_5 k}{h^{N-1}\sqrt{1-h^2}} - (N-3)\frac{B_5 k}{h^{N-3}(1-h^2)^{\frac{3}{2}}}\\ &\quad - (N-4)\frac{B_5 k}{h^{N-3}(1-h^2)^{\frac{3}{2}}} + \frac{3B_5 k}{h^{N-5}(1-h^2)^{\frac{5}{2}}}\\ &= (N-2)B_4 k^{N-2} + (N-3)(N-2)\frac{B_5 k}{h^{N-1}} + O\big(h^2 k^{N-2}\big) + O\Big(\frac{k}{h^{N-3}}\Big),\end{aligned} \tag{3.8}$$

and

$$\mathcal{G}'''(h) = O\Big(\frac{k}{h^N}\Big). \tag{3.9}$$

Let $\mathbf{h}$ be a solution of

$$\Big[(N-2)B_4 k^{N-2}h - (N-3)\frac{B_5 k}{h^{N-2}}\Big] = 0,$$

then

$$\mathbf{h} = \frac{B'}{k^{\frac{N-3}{N-1}}}, \quad \text{with } B' = \Big[\frac{(N-3)B_5}{(N-2)B_4}\Big]^{\frac{1}{N-1}}. \tag{3.10}$$

Define

$$\mathbf{S}_k = \Big\{(r,h,\Lambda)\Big| \, r \in \Big[k^{\frac{N-2}{N-2-m}} - \frac{1}{k^{\bar\theta}}, k^{\frac{N-2}{N-2-m}} + \frac{1}{k^{\bar\theta}}\Big], \quad \Lambda \in \Big[\Lambda_0 - \frac{1}{k^{\frac{3\bar\theta}{2}}}, \Lambda_0 + \frac{1}{k^{\frac{3\bar\theta}{2}}}\Big], \\ h \in \Big[\frac{B'}{k^{\frac{N-3}{N-1}}}\Big(1 - \frac{1}{k^{\bar\theta}}\Big), \frac{B'}{k^{\frac{N-3}{N-1}}}\Big(1 + \frac{1}{k^{\bar\theta}}\Big)\Big]\Big\},$$

for $\bar\theta$ is a small constant such that $\bar\theta \le \frac{\sigma}{100}$. In fact, $\mathbf{S}_k$ is a subset of $\mathscr{S}_k$. We will find a critical point of $F(r,h,\Lambda)$ in $\mathbf{S}_k$.

A direct Taylor expansion gives that

$$\mathcal{G}(h) = \mathcal{G}(\mathbf{h}) + \mathcal{G}'(\mathbf{h})(h-\mathbf{h}) + \frac{1}{2}\mathcal{G}''(\mathbf{h})(h-\mathbf{h})^2 + O\Big(\mathcal{G}'''\big(\mathbf{h} + (1-\iota)h\big)\Big)(h-\mathbf{h})^3, \tag{3.11}$$

where

$$\mathcal{G}(\mathbf{h}) = B_4 k^{N-2}\Big[1 + \frac{N-2}{2}\mathbf{h}^2 + O(\mathbf{h}^4)\Big] + \frac{B_5 k}{\mathbf{h}}^{N-3}\Big[1 + \frac{1}{2}\mathbf{h}^2 + O(\mathbf{h}^4)\Big],$$

$$\mathcal{G}'(\mathbf{h}) = O\Big(\frac{k}{\mathbf{h}}^{N-4}\Big)$$

and

$$\mathcal{G}''(\mathbf{h}) = \frac{(N-2)}{2}\Big[B_4 k^{N-2} + (N-3)\frac{B_5 k}{\mathbf{h}}^{N-1}\Big] + O\big(\mathbf{h}^2 k^{N-2}\big).$$

Since $\mathcal{G}(\mathbf{h}), \mathcal{G}''(\mathbf{h})$ are independent of h, r, Λ, for simplicity, in the following, we will denote

$$\mathcal{G}(\mathbf{h}) = B_4 k^{N-2} + \frac{(N-2)B_4}{2}k^{N-2}\mathbf{h}^2 + \frac{B_5 k}{\mathbf{h}^{N-3}}, \tag{3.12}$$

$$\mathcal{G}''(\mathbf{h}) = \frac{(N-2)}{2}\Big[B_4 k^{N-2} + (N-3)\frac{B_5 k}{\mathbf{h}}^{N-1}\Big]. \tag{3.13}$$

Then combining (3.11), (3.12), (3.13), we can get

$$\mathcal{G}(h) = B_4 k^{N-2} + \frac{(N-2)B_4}{2}k^{N-2}\mathbf{h}^2 + \frac{B_5 k}{\mathbf{h}}^{N-3} + O\Big(\frac{k}{\mathbf{h}}^{N-4}\Big)(h-\mathbf{h}) \\ + \frac{(N-2)}{2}\Big[B_4 k^{N-2} + (N-3)\frac{B_5 k}{\mathbf{h}}^{N-1}\Big](h-\mathbf{h})^2 + O\Big(\frac{k}{\mathbf{h}}^{N}\Big)(h-\mathbf{h})^3.$$

Therefore, we get

$$\begin{aligned}\mathcal{G}(h) &= B_4k^{N-2} + \Big[\frac{(N-2)B_4{B'}^2}{2} + \frac{B_5}{{B'}^{N-3}}\Big]\frac{k^{N-2}}{k^{\frac{2(N-3)}{N-1}}}\\ &\quad + \frac{(N-2)}{2}\Big[B_4{B'}^2 + \frac{(N-3)B_5}{{B'}^{N-3}}\Big]\frac{k^{N-2}}{k^{\frac{2(N-3)}{N-1}}}(1-\mathbf{h}^{-1}h)^2\\ &\quad + O\Big(\frac{k^{N-2}}{k^{\frac{2(N-3)}{N-1}}}\Big)(1-\mathbf{h}^{-1}h)^3\\ &= B_4k^{N-2} + B_6\frac{k^{N-2}}{k^{\frac{2(N-3)}{N-1}}} + B_7\frac{k^{N-2}}{k^{\frac{2(N-3)}{N-1}}}(1-\mathbf{h}^{-1}h)^2\\ &\quad + O\Big(\frac{k^{N-2}}{k^{\frac{2(N-3)}{N-1}}}\Big)(1-\mathbf{h}^{-1}h)^3, \end{aligned} \tag{3.14}$$

where

$$B_6 = \frac{(N-2)B_4{B'}^2}{2} + \frac{B_5}{{B'}^{N-3}}, \quad B_7 = \frac{(N-2)}{2}\Big[B_4{B'}^2 + \frac{(N-3)B_5}{{B'}^{N-3}}\Big].$$

Since

$$r \in \Big[k^{\frac{N-2}{N-2-m}} - \frac{1}{k^{\bar\theta}}, \quad k^{\frac{N-2}{N-2-m}} + \frac{1}{k^{\bar\theta}}\Big],$$

then

$$r^{N-2} = k^{\frac{(N-2)^2}{N-2-m}}\Big(1 + \frac{\mathcal{C}(r,\Lambda)}{k^{\frac{(N-2)}{N-2-m}+\bar\theta}}\Big).$$

We now rewrite

$$\begin{aligned}&\frac{B_4k^{N-2}}{(r\sqrt{1-h^2})^{N-2}} + \frac{B_5k}{r^{N-2}h^{N-3}\sqrt{1-h^2}}\\ &\quad = \frac{B_4}{k^{\frac{(N-2)m}{N-2-m}}} + \frac{B_6}{k^{\frac{(N-2)m}{N-2-m}+\frac{2(N-3)}{N-1}}} + \frac{\mathcal{C}(r,\Lambda)}{k^{\frac{(N-2)m}{N-2-m}+\sigma}}\\ &\qquad + \frac{B_7}{k^{\frac{(N-2)m}{N-2-m}+\frac{2(N-3)}{N-1}}}(1-\mathbf{h}^{-1}h)^2 + O\Big(\frac{1}{k^{\frac{(N-2)m}{N-2-m}+\frac{2(N-3)}{N-1}}}\Big)(1-\mathbf{h}^{-1}h)^3.\end{aligned}$$

Then we can express $F(r,h,\Lambda)$ as

$$\begin{aligned}F(r,h,\Lambda) &= kA_1 - k\Big[\frac{B_4}{\Lambda^{N-2}k^{\frac{(N-2)m}{N-2-m}}} + \frac{B_6}{\Lambda^{N-2}k^{\frac{(N-2)m}{N-2-m}+\frac{2(N-3)}{N-1}}}\\ &\quad + \frac{B_7}{\Lambda^{N-2}k^{\frac{(N-2)m}{N-2-m}+\frac{2(N-3)}{N-1}}}(1-\mathbf{h}^{-1}h)^2\Big]\\ &\quad + k\Big[\frac{A_2}{\Lambda^m k^{\frac{(N-2)m}{N-2-m}}} + \frac{A_3}{\Lambda^{m-2}k^{\frac{(N-2)m}{N-2-m}}}(\mathbf{r}-r)^2\Big] + k\frac{\mathcal{C}(r,\Lambda)}{k^{\frac{(N-2)m}{N-2-m}}}(\mathbf{r}-r)^{2+\sigma}\\ &\quad + kO\Big(\frac{1}{k^{\frac{(N-2)m}{N-2-m}+\frac{2(N-3)}{N-1}}}\Big)(1-\mathbf{h}^{-1}h)^3 + kO\Big(\frac{1}{k^{\left(\frac{m(N-2)}{N-2-m}+\frac{2(N-3)}{N-1}+\sigma\right)}}\Big). \end{aligned} \tag{3.15}$$

And similarly, we have

$$\frac{\partial F(r,h,\Lambda)}{\partial \Lambda} = k\Big[\frac{(N-2)B_4}{\Lambda^{N-1}k^{\frac{(N-2)m}{N-2-m}}} - \frac{mA_2}{\Lambda^{m+1}k^{\frac{(N-2)m}{N-2-m}}}\Big] + \frac{(m-2)A_3}{\Lambda^{m-1}k^{\frac{(N-2)m}{N-2-m}}}(\mathbf{r}-r)^2 + kO\Big(\frac{1}{k^{\frac{(N-2)m}{N-2-m}}}(\mathbf{r}-r)^{2+\sigma}\Big);$$

and from (A.28), by using some calculations, we have

$$\begin{aligned}\frac{\partial F(r,h,\Lambda)}{\partial h} = \frac{k}{\Lambda^{N-2}}\Big[\frac{2B_7}{\Lambda^{N-2}k^{\frac{(N-2)m}{N-2-m}+\frac{(N-3)}{N-1}}}(1-\mathbf{h}^{-1}h)\Big] &+ kO\Big(\frac{1}{k^{\left(\frac{m(N-2)}{N-2-m}+\frac{(N-3)}{N-1}\right)}}\Big)(1-\mathbf{h}^{-1}h)^2 \\ &+ kO\Big(\frac{1}{k^{\left(\frac{m(N-2)}{N-2-m}+\frac{(N-3)}{N-1}+\sigma\right)}}\Big).\end{aligned} \tag{3.16}$$

Now define

$$\bar{F}(r,h,\Lambda) = -F(r,h,\Lambda), \tag{3.17}$$

and

$$\mathbf{t}_2 = k(-A_1+\eta_1), \quad \mathbf{t}_1 = k\left(-A_1 - \left(\frac{A_2}{\Lambda_0^m} - \frac{B_4}{\Lambda_0^{N-2}}\right)\frac{1}{k^{\frac{(N-2)m}{N-2-m}}} - \frac{1}{k^{\frac{(N-2)m}{N-2-m}+\frac{5\bar{\theta}}{2}}}\right),$$

where $\eta_1 > 0$ small. We also define the energy level set

$$\bar{F}^{\mathbf{t}} = \Big\{(r,h,\Lambda)\big|\ (r,h,\Lambda)\in \mathbf{S}_k,\ \bar{F}(r,h,\Lambda)\le \mathbf{t}\Big\}.$$

We consider the following gradient flow system

$$\begin{cases}\frac{\mathrm{d}r}{\mathrm{d}t} = -\bar{F}_r, & t>0;\\ \frac{\mathrm{d}h}{\mathrm{d}t} = -\bar{F}_h, & t>0;\\ \frac{\mathrm{d}\Lambda}{\mathrm{d}t} = -\bar{F}_\Lambda, & t>0;\\ (r,h,\Lambda)\big|_{t=0}\in \bar{F}_2^{\mathbf{t}}. \end{cases}$$

The next proposition would play an important role in the proof of Theorem 1.1.

Proposition 3.6. *The flow would not leave $\mathbf{S}_k$ before it reaches $\bar{F}_1^{\mathbf{t}}$.*

Proof. There are three positions that the flow tends to leave $\mathbf{S}_k$:
position 1. $|r-\mathbf{r}| = \frac{1}{k^{\bar\theta}}$ and $|1-\mathbf{h}^{-1}h| \le \frac{1}{k^{\bar\theta}}$, $\quad |\Lambda-\Lambda_0| \le \frac{1}{k^{\frac{3\bar\theta}{2}}}$;
position 2. $|1-\mathbf{h}^{-1}h| = \frac{1}{k^{\bar\theta}}$ when $|r-\mathbf{r}| \le \frac{1}{k^{\bar\theta}}$, $\quad |\Lambda-\Lambda_0| \le \frac{1}{k^{\frac{3\bar\theta}{2}}}$;
position 3. $|\Lambda-\Lambda_0| = \frac{1}{k^{\frac{3\bar\theta}{2}}}$ when $|r-\mathbf{r}| \le \frac{1}{k^{\bar\theta}}$, $\quad |1-\mathbf{h}^{-1}h| \le \frac{1}{k^{\bar\theta}}$.
♠ We now consider **position 1**. Since $|\Lambda-\Lambda_0| \le \frac{1}{k^{\frac{3\bar\theta}{2}}}$, it is easy to derive that

$$\begin{aligned}\Big(\frac{B_4}{\Lambda^{N-2}} - \frac{A_2}{\Lambda^m}\Big) &= \Big(\frac{B_4}{\Lambda_0^{N-2}} - \frac{A_2}{\Lambda_0^m}\Big) + O(|\Lambda-\Lambda_0|^2)\\ &= \Big(\frac{B_4}{\Lambda_0^{N-2}} - \frac{A_2}{\Lambda_0^m}\Big) + O\Big(\frac{1}{k^{3\bar\theta}}\Big).\end{aligned} \tag{3.18}$$

Combining (3.15), (3.17), (3.18), we can obtain that, if (r,h,Λ) lies in **position 1**,

$$\bar F(r,h,\Lambda)=-kA_1+k\Big[\frac{B_4}{\Lambda_0^{N-2}k^{\frac{(N-2)m}{N-2-m}}}-\frac{A_2}{\Lambda_0^{m}k^{\frac{(N-2)m}{N-2-m}}}\Big] -k\frac{A_3}{\Lambda_0^{m-2}k^{\frac{(N-2)m}{N-2-m}+2\bar\theta}}+O\Big(\frac{1}{k^{\frac{(N-2)m}{N-2-m}+\frac{5\bar\theta}{2}}}\Big)<\mathbf{t}_1.$$

♠ On the other hand, we claim that it's impossible for the flow $\big(r(t),h(t),\Lambda(t)\big)$ leaves $\mathbf{S}_k$ when it lies in **position 2**. If $1-\mathbf{h}^{-1}h=\frac{1}{k^{\bar\theta}}$, then from (3.16) and (3.17), we have

$$\frac{\partial\bar F(r,h,\Lambda)}{\partial h}=-\frac{k}{\Lambda^{N-2}}\Big[\frac{2B_7}{\Lambda^{N-2}k^{\frac{(N-2)m}{N-2-m}+\frac{(N-3)}{N-1}+\bar\theta}}\Big]+O\Big(\frac{1}{k^{\frac{(N-2)m}{N-2-m}+\frac{N-3}{N-1}+2\bar\theta}}\Big)<0. \tag{3.19}$$

On the other hand, if $1-\mathbf{h}^{-1}h=-\frac{1}{k^{\bar\theta}}$

$$\frac{\partial\bar F(r,h,\Lambda)}{\partial h}=\frac{k}{\Lambda^{N-2}}\Big[\frac{2B_7}{\Lambda^{N-2}k^{\frac{(N-2)m}{N-2-m}+\frac{(N-3)}{N-1}+\bar\theta}}\Big]+O\Big(\frac{1}{k^{\frac{(N-2)m}{N-2-m}+\frac{N-3}{N-1}+2\bar\theta}}\Big)>0. \tag{3.20}$$

So it's impossible for the flow leaves $\mathbf{S}_k$ when it lies in **position 2**.
♠ Finally, we consider **position 3**. If $\Lambda=\Lambda_0+\frac{1}{k^{\frac{3\bar\theta}{2}}}$, from (3.1) and (3.17), there exists a constant C_1 such that

$$\frac{\partial\bar F(r,h,\Lambda)}{\partial\Lambda}=k\Big[C_1\frac{1}{k^{\frac{(N-2)m}{N-2-m}+\frac{3}{2}\bar\theta}}+O\Big(\frac{1}{k^{\frac{(N-2)m}{N-2-m}+2\bar\theta}}\Big)\Big]>0.$$

On the other hand, if $\Lambda=\Lambda_0-\frac{1}{k^{\frac{3\bar\theta}{2}}}$, there exists a constant C_2 such that

$$\frac{\partial\bar F(r,h,\Lambda)}{\partial\Lambda}=k\Big[-C_2\frac{1}{k^{\frac{(N-2)m}{N-2-m}+\frac{3}{2}\bar\theta}}+O\Big(\frac{1}{k^{\frac{(N-2)m}{N-2-m}+2\bar\theta}}\Big)\Big]<0.$$

Hence the flow $\big(r(t),h(t),\Lambda(t)\big)$ does not leave $\mathbf{S}_k$ when $|\Lambda-\Lambda_0|=\frac{1}{k^{\frac{3\bar\theta}{2}}}$.

Combining above results, we conclude that the flow would not leave $\mathbf{S}_k$ before it reach $\bar F_1^{\mathbf{t}}$. □

Now we give the proof of Theorem 1.1.

Proof of Theorem 1.1. : According to Proposition 3.1, in order to show Theorem 1.1, we only need to show that function $\bar F(r,h,\Lambda)$, and thus $F(r,h,\Lambda)$, has a critical point in $\mathbf{S}_k$.

Define

$$\Gamma=\Big\{\gamma:\quad \gamma(r,h,\Lambda)=\big(\gamma_1(r,h,\Lambda),\gamma_2(r,h,\Lambda),\gamma_3(r,h,\Lambda)\big)\in\mathbf{S}_k,(r,h,\Lambda)\in\mathbf{S}_k;\quad \gamma(r,h,\Lambda)=(r,h,\Lambda),\ \text{if } |r-\mathbf{r}|=\frac{1}{k^{\bar\theta}}\Big\}.$$

Let

$$\mathbf{c}=\inf_{\gamma\in\Gamma}\max_{(r,h,\Lambda)\in\mathbf{S}_k}\bar F\big(\gamma(r,h,\Lambda)\big).$$

We claim that $\mathbf{c}$ is a critical value of $\bar{F}(r,h,\Lambda)$ and can be achieved by some $(r,h,\Lambda) \in \mathbf{S}_k$. By the minimax theory, we need to show that

(i) $\mathbf{t}_1 < \mathbf{c} < \mathbf{t}_2$;

(ii) $\sup_{|r-\mathbf{r}|=\frac{1}{k^\theta}} \bar{F}\big(\gamma(r,h,\Lambda)\big) < \mathbf{t}_1, \ \forall\, \gamma \in \Gamma$.

Using the results in Proposition 3.6 we can prove (i) and (ii) easily.

Finally, for every k large enough, we get the critical point (r_k, h_k, Λ_k) of $F(r,h,\Lambda)$. □

Acknowledgements

L. Duan was supported by the China Scholarship Council and NSFC grant (No.11771167, No.12201140), Technology Foundation of Guizhou Province ([2001]ZK008) and Guangdong Basic and Applied Basic Research Foundation (No. 2022A1515111131). M. Musso was supported by EPSRC research Grant EP/T008458/1. S. Wei was supported by the NSFC Grant (No.12001203) and Guangdong Basic and Applied Basic Research Foundation (No. 2020A1515 110622). Some part of the work was done during the visit of L. Duan to Prof. M. Musso at the University of Bath. L. Duan would like to thank the Department of Mathematical Sciences for its warm hospitality and supports.

Appendix A: Expansions for the energy functional

This section is devoted to the computation of the expansion for the energy functional $I(W_{r,h,\Lambda})$. We first give the following Lemma.

Lemma A.1. *$N \geq 5$ and $(r,h,\Lambda) \in \mathscr{S}_k$. We have the following expansions for $k \to \infty$:*

$$\sum_{i=2}^{k} \frac{1}{|\overline{x}_1 - \overline{x}_i|^{N-2}} = \frac{k^{N-2}}{\big(r\sqrt{1-h^2}\big)^{N-2}} \big(B_1 + \sigma_1(k)\big), \tag{A.1}$$

$$\sum_{i=1}^{k}\frac{1}{|\overline{x}_1-\underline{x}_i|^{N-2}}=\frac{B_2k}{r^{N-2}h^{N-3}\sqrt{1-h^2}}\big(1+\sigma_2(k)\big)+\frac{\sigma_1(k)k^{N-2}}{\big(r\sqrt{1-h^2}\big)^{N-2}},\tag{A.2}$$

where

$$B_1=\frac{2}{(2\pi)^{N-2}}\sum_{i=1}^{\infty}\frac{1}{i^{N-2}},\quad B_2=\frac{1}{2^{N-3}\pi}\int_0^{+\infty}\frac{1}{\big(s^2+1\big)^{\frac{N-2}{2}}}\,\mathrm{d}s,\tag{A.3}$$

and

$$\sigma_1(k)=\begin{cases}O\big(\frac{1}{k^2}\big), & N\geq 6,\\ O\big(\frac{\ln k}{k^2}\big), & N=5,\end{cases}\qquad \sigma_2(k)=O\big((hk)^{-1}\big).\tag{A.4}$$

Proof. In fact, for $\frac{1}{2}<c_3\leq c_4\leq 1$, we have

$$c_3\frac{i\pi}{k}\leq \sin\frac{i\pi}{k}\leq c_4\frac{i\pi}{k},\quad \text{for } i\in\big\{1,\ldots,\frac{k}{2}\big\}.$$

Without loss of generality, we can assume k is even. It is easy to derive that

$$\sum_{i=2}^{k}\frac{1}{|\overline{x}_1-\overline{x}_i|^{N-2}}=\sum_{i=1}^{k}\Big(\frac{1}{2r\sqrt{1-h^2}\sin\frac{i\pi}{k}}\Big)^{N-2}$$
$$=\sum_{i=1}^{\frac{k}{2}}\Big(\frac{1}{2r\sqrt{1-h^2}\sin\frac{i\pi}{k}}\Big)^{N-2}+\sum_{i=\frac{k}{2}+1}^{k}\Big(\frac{1}{2r\sqrt{1-h^2}\sin\frac{i\pi}{k}}\Big)^{N-2}.$$

Direct computations show that

$$\begin{aligned}&\sum_{i=1}^{\frac{k}{2}}\Big(\frac{1}{2r\sqrt{1-h^2}\sin\frac{i\pi}{k}}\Big)^{N-2}\\ &=\sum_{i=1}^{[\frac{k}{6}]}\Big(\frac{1}{2r\sqrt{1-h^2}\sin\frac{i\pi}{k}}\Big)^{N-2}+\sum_{i=[\frac{k}{6}]+1}^{\frac{k}{2}}\Big(\frac{1}{2r\sqrt{1-h^2}\sin\frac{i\pi}{k}}\Big)^{N-2}\\ &=\sum_{i=1}^{[\frac{k}{6}]}\Big(\frac{1}{2r\sqrt{1-h^2}\frac{i\pi}{k}}\Big)^{N-2}\Big(1+O\Big(\frac{i^2}{k^2}\Big)\Big)+O\Big(\frac{k}{\big(2r\sqrt{1-h^2}\big)^{N-2}}\Big)\\ &=\Big(\frac{k}{r\sqrt{1-h^2}}\Big)^{N-2}\Big(D_1+\sigma_1(k)\Big),\end{aligned}\tag{A.5}$$

where $D_1=\frac{1}{2\pi^{N-2}}\sum_{i=1}^{\infty}\frac{1}{i^{N-2}}$ and $\sigma_1(k)$ is defined in (A.4). Using symmetry of function $\sin x$, we can easily show

$$\sum_{i=\frac{k}{2}+1}^{k}\Big(\frac{1}{2r\sqrt{1-h^2}\sin\frac{i\pi}{k}}\Big)^{N-2}=\Big(\frac{k}{r\sqrt{1-h^2}}\Big)^{N-2}\Big(D_1+\sigma_1(k)\Big).$$

Thus we proved (A.1).

Similarly, we can obtain

$$\sum_{i=1}^{k}\frac{1}{|\overline{x}_1-\underline{x}_i|^{N-2}}=\sum_{i=1}^{k}\frac{1}{\Big(2r\big[(1-h^2)\sin^2\frac{(i-1)\pi}{k}+h^2\big]^{\frac{1}{2}}\Big)^{N-2}}$$

$$=\frac{2}{(2rh)^{N-2}}\sum_{i=1}^{\frac{k}{2}}\frac{1}{\Big(\frac{(1-h^2)}{h^2}\frac{(i-1)^2\pi^2}{k^2}+1\Big)^{\frac{N-2}{2}}}+\sigma_1(k)O\Big(\Big(\frac{k}{r\sqrt{1-h^2}}\Big)^{N-2}\Big).$$

Consider $O\big((hk)^{-1}\big)=o(1)$ as $k\to\infty$. Since

$$\sum_{j=1}^{\frac{k}{2}}\frac{1}{\Big(\frac{(1-h^2)}{h^2}\frac{(j-1)^2\pi^2}{k^2}+1\Big)^{\frac{N-2}{2}}}\geq\int_0^{\frac{k}{2}}\frac{1}{\Big(\frac{(1-h^2)}{h^2}\frac{x^2\pi^2}{k^2}+1\Big)^{\frac{N-2}{2}}}\,\mathrm{d}x$$

$$\geq\int_0^{2}\frac{1}{\Big(\frac{(1-h^2)}{h^2}\frac{x^2\pi^2}{k^2}+1\Big)^{\frac{N-2}{2}}}\,\mathrm{d}x+\sum_{j=4}^{\frac{k}{2}+1}\frac{1}{\Big(\frac{(1-h^2)}{h^2}\frac{(j-1)^2\pi^2}{k^2}+1\Big)^{\frac{N-2}{2}}},$$

then we have

$$\sum_{j=1}^{\frac{k}{2}}\frac{1}{\Big(\frac{(1-h^2)}{h^2}\frac{(j-1)^2\pi^2}{k^2}+1\Big)^{\frac{N-2}{2}}}$$

$$=\int_0^{\frac{k}{2}}\frac{1}{\Big(\frac{(1-h^2)}{h^2}\frac{x^2\pi^2}{k^2}+1\Big)^{\frac{N-2}{2}}}\,\mathrm{d}x+1+o(1)$$

$$=\frac{hk}{\sqrt{1-h^2}\pi}\int_0^{\frac{(1-h^2)}{4h^2}\pi^2}\frac{1}{\big(s^2+1\big)^{\frac{N-2}{2}}}\,\mathrm{d}s+1+o(1)$$

$$=\frac{hk}{\sqrt{1-h^2}\pi}\int_0^{+\infty}\frac{1}{\big(s^2+1\big)^{\frac{N-2}{2}}}\,\mathrm{d}s\Big(1+O\big((kh)^{-1}\big)\Big).$$

Combining above calculations, we can obtain that

$$\sum_{i=1}^{k}\frac{1}{|\overline{x}_1-\underline{x}_i|^{N-2}}=\frac{1}{(rh)^{N-2}}\frac{B_2hk}{\sqrt{1-h^2}}\Big(1+\sigma_2(k)\Big)+O\Big(\frac{\sigma_1(k)k^{N-2}}{\big(r\sqrt{1-h^2}\big)^{N-2}}\Big),$$

where B_2 and σ_2 are defined in (A.3), (A.4). □

Lemma A.2. *We have the expansion, for $k\to\infty$*

$$\int_{\mathbb{R}^N}U_{\overline{x}_1,\Lambda}^{2^*-1}U_{\overline{x}_i,\Lambda}=\frac{B_0}{\Lambda^{N-2}|\overline{x}_1-\overline{x}_i|^{N-2}}+O\Big(\frac{1}{|\overline{x}_1-\overline{x}_i|^{N-\epsilon_0}}\Big),$$

and

$$\int_{\mathbb{R}^N}U_{\overline{x}_1,\Lambda}^{2^*-1}U_{\underline{x}_i,\Lambda}=\frac{B_0}{\Lambda^{N-2}|\overline{x}_1-\underline{x}_i|^{N-2}}+O\Big(\frac{1}{|\overline{x}_1-\underline{x}_i|^{N-\epsilon_0}}\Big),$$

where $B_0=\int_{\mathbb{R}^N}\frac{1}{(1+z^2)^{\frac{N+2}{2}}}$ and ϵ_0 is constant small enough.

Proof. Let $\overline{d}_j = |\overline{x}_1 - \overline{x}_j|$, $\underline{d}_j = |\overline{x}_1 - \underline{x}_j|$ for $j = 1, \ldots, k$. We consider

$$\begin{aligned}
&\int_{\mathbb{R}^N} U^{2^*-1}_{\overline{x}_1,\Lambda} U_{\overline{x}_i,\Lambda} \\
&\quad = \int_{\mathbb{R}^N} \frac{\Lambda^{\frac{N+2}{2}}}{(1+\Lambda^2|y-\overline{x}_1|^2)^{\frac{N+2}{2}}} \frac{\Lambda^{\frac{N-2}{2}}}{(1+\Lambda^2|y-\overline{x}_i|^2)^{\frac{N-2}{2}}} \\
&\quad = \left\{ \int_{B_{\frac{\overline{d}_i}{4}}(\overline{x}_1)} + \int_{\mathbb{R}^N \setminus B_{\frac{\overline{d}_i}{4}}(\overline{x}_1)} \right\} \\
&\quad\quad \frac{\Lambda^{\frac{N+2}{2}}}{(1+\Lambda^2|y-\overline{x}_1|^2)^{\frac{N+2}{2}}} \frac{\Lambda^{\frac{N-2}{2}}}{(1+\Lambda^2|y-\overline{x}_i|^2)^{\frac{N-2}{2}}}.
\end{aligned} \tag{A.6}$$

First, we have

$$\begin{aligned}
&\int_{B_{\frac{\overline{d}_i}{4}}(\overline{x}_1)} \frac{\Lambda^{\frac{N+2}{2}}}{(1+\Lambda^2|y-\overline{x}_1|^2)^{\frac{N+2}{2}}} \frac{\Lambda^{\frac{N-2}{2}}}{(1+\Lambda^2|y-\overline{x}_i|^2)^{\frac{N-2}{2}}} \\
&= \int_{B_{\frac{\Lambda\overline{d}_i}{4}}(0)} \frac{1}{(1+z^2)^{\frac{N+2}{2}}} \frac{1}{(1+z^2+2\Lambda\langle z, \overline{x}_1-\overline{x}_i\rangle + \Lambda^2|\overline{x}_1-\overline{x}_i|^2)^{\frac{N-2}{2}}} \\
&= \frac{1}{\Lambda^{N-2}|\overline{x}_1-\overline{x}_i|^{N-2}} \int_{B_{\frac{\Lambda\overline{d}_i}{4}}(0)} \frac{1}{(1+z^2)^{\frac{N+2}{2}}} \Bigg(1 - \frac{N-2}{2} \frac{1+z^2+2\Lambda\langle z, \overline{x}_1-\overline{x}_i\rangle}{\Lambda^2|\overline{x}_1-\overline{x}_i|^2} \\
&\qquad + O\Big(\Big(\frac{1+z^2+2\Lambda\langle z, \overline{x}_1-\overline{x}_i\rangle}{\Lambda^2|\overline{x}_1-\overline{x}_i|^2}\Big)^2\Big)\Bigg).
\end{aligned} \tag{A.7}$$

It is easy to check that

$$\begin{aligned}
&\frac{1}{\Lambda^{N-2}|\overline{x}_1-\overline{x}_i|^{N-2}} O\Big(\int_{B_{\frac{\Lambda\overline{d}_i}{4}}(0)} \frac{1}{(1+z^2)^{\frac{N+2}{2}}} \Big(\frac{1+z^2+2\Lambda\langle z, \overline{x}_1-\overline{x}_i\rangle}{\Lambda^2|\overline{x}_1-\overline{x}_i|^2}\Big)^2\Big) \\
&\quad = O\Big(\frac{1}{|\overline{x}_1-\overline{x}_i|^N}\Big),
\end{aligned} \tag{A.8}$$

and

$$\begin{aligned}
&\frac{1}{\Lambda^{N}|\overline{x}_1-\overline{x}_i|^{N}} \int_{B_{\frac{\Lambda\overline{d}_i}{4}}(0)} \frac{1}{(1+z^2)^{\frac{N+2}{2}}} \Big(1+z^2+2\Lambda\langle z, \overline{x}_1-\overline{x}_i\rangle\Big) \\
&\qquad = O\Big(\frac{1}{|\overline{x}_1-\overline{x}_i|^{N-\epsilon_0}}\Big).
\end{aligned} \tag{A.9}$$

Standard calculation implies that

$$\begin{aligned}
&\frac{1}{\Lambda^{N-2}|\overline{x}_1-\overline{x}_i|^{N-2}} \int_{B_{\frac{\Lambda\overline{d}_i}{4}}(0)} \frac{1}{(1+z^2)^{\frac{N+2}{2}}} = \frac{B_0}{\Lambda^{N-2}|\overline{x}_1-\overline{x}_i|^{N-2}} \\
&\qquad + O\Big(\frac{1}{|\overline{x}_1-\overline{x}_i|^N}\Big),
\end{aligned} \tag{A.10}$$

where $B_0 = \int_{\mathbb{R}^N} \frac{1}{(1+z^2)^{\frac{N+2}{2}}}$.

From (A.7)–(A.10), we get

$$\int_{B_{\frac{\bar{d}_i}{4}}(\overline{x}_1)} \frac{\Lambda^{\frac{N+2}{2}}}{(1+\Lambda^2|y-\overline{x}_1|^2)^{\frac{N+2}{2}}} \frac{\Lambda^{\frac{N-2}{2}}}{(1+\Lambda^2|y-\overline{x}_i|^2)^{\frac{N-2}{2}}}$$
$$= \frac{B_0}{\Lambda^{N-2}|\overline{x}_1-\overline{x}_i|^{N-2}} + O\Big(\frac{1}{|\overline{x}_1-\overline{x}_i|^{N-\epsilon_0}}\Big). \tag{A.11}$$

When $y \in \mathbb{R}^N \backslash B_{\frac{\bar{d}_i}{4}}(\overline{x}_1)$, there holds

$$|y-\overline{x}_1| \geq \frac{1}{4}|\overline{x}_1-\overline{x}_i|.$$

It's easy to get

$$\int_{\mathbb{R}^N \backslash B_{\frac{\bar{d}_i}{4}}(\overline{x}_1)} \frac{\Lambda^{\frac{N+2}{2}}}{(1+\Lambda^2|y-\overline{x}_1|^2)^{\frac{N+2}{2}}} \frac{\Lambda^{\frac{N-2}{2}}}{(1+\Lambda^2|y-\overline{x}_i|^2)^{\frac{N-2}{2}}} = O\Big(\frac{1}{|\overline{x}_1-\overline{x}_i|^{N-\epsilon_0}}\Big). \tag{A.12}$$

Combining (A.6), (A.11) and (A.12), we can get

$$\int_{\mathbb{R}^N} U_{\overline{x}_1,\Lambda}^{2^*-1} U_{\overline{x}_i,\Lambda} = \frac{B_0}{\Lambda^{N-2}|\overline{x}_1-\overline{x}_i|^{N-2}} + O\Big(\frac{1}{|\overline{x}_1-\overline{x}_i|^{N-\epsilon_0}}\Big). \tag{A.13}$$

Similarly, we can get

$$\int_{\mathbb{R}^N} U_{\overline{x}_1,\Lambda}^{2^*-1} U_{\underline{x}_i,\Lambda} = \frac{B_0}{\Lambda^{N-2}|\overline{x}_1-\underline{x}_i|^{N-2}} + O\Big(\frac{1}{|\overline{x}_1-\underline{x}_i|^{N-\epsilon_0}}\Big),$$

for $i = 1, \dots, k$. □

Lemma A.3. *Suppose that $K(|y|)$ satisfies* (**H**) *and $N \geq 5$, $(r, h, \Lambda) \in \mathscr{S}_k$. We have the expansion for $k \to \infty$*

$$\begin{aligned} I(W_{r,h,\Lambda}) =& kA_1 - k\int_{\mathbb{R}^N} U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^k U_{\overline{x}_j,\Lambda} + \sum_{j=1}^k U_{\underline{x}_j,\Lambda}\Big) \\ &+ k\Big[\frac{A_2}{\Lambda^m \mathbf{r}^m} + \frac{A_3}{\Lambda^{m-2}\mathbf{r}^m}(\mathbf{r}-r)^2\Big] + k\frac{\mathcal{C}(r,\Lambda)}{\mathbf{r}}^m(\mathbf{r}-r)^{2+\sigma} \\ &+ k\frac{\mathcal{C}(r,\Lambda)}{\mathbf{r}}^{m+\sigma} + kO\Big(\Big(\frac{k}{\mathbf{r}}\Big)^{N-\epsilon_0}\Big) + kO\Big(\frac{1}{k^m}\Big(\frac{k}{\mathbf{r}}\Big)^{N-2}\Big), \end{aligned}$$

where $\mathcal{C}(r, \Lambda)$ denotes function independent of h and should be order of $O(1)$,

$$A_1 = \Big(1-\frac{2}{2^*}\Big)\int_{\mathbb{R}^N} |U_{0,1}|^{2^*}, \quad A_2 = \frac{2c_0}{2^*}\int_{\mathbb{R}^N} |y_1|^m U_{0,1}^{2^*}, \tag{A.14}$$

$$A_3 = \frac{c_0 m(m-1)}{2^*}\int_{\mathbb{R}^N} |y_1|^{m-2} U_{0,1}^{2^*}, \tag{A.15}$$

and ϵ_0 is constant can be chosen small enough.

Proof. Recalling the definition of $I(u)$ as in (1.18), then we obtain that

$$\begin{aligned} I(W_{r,h,\Lambda}) &= \frac{1}{2}\int_{\mathbb{R}^N} |\nabla W_{r,h,\Lambda}|^2 - \frac{1}{2^*}\int_{\mathbb{R}^N} K\Big(\frac{|y|}{\mathbf{r}}\Big) W_{r,h,\Lambda}^{2^*} \\ &:= I_1 - I_2. \end{aligned} \tag{A.16}$$

According to the expression of $W_{r,h,\Lambda}$, we have

$$\begin{aligned} I_1 &= \frac{1}{2}\sum_{j=1}^{k}\sum_{i=1}^{k}\int_{\mathbb{R}^N} -\Delta\Big(U_{\overline{x}_j,\Lambda}+U_{\underline{x}_j,\Lambda}\Big)\Big(U_{\overline{x}_i,\Lambda}+U_{\underline{x}_i,\Lambda}\Big) \\ &= k\sum_{j=1}^{k}\int_{\mathbb{R}^N}\Big(U_{\overline{x}_1,\Lambda}^{2^*-1}U_{\overline{x}_j,\Lambda}+U_{\underline{x}_1,\Lambda}^{2^*-1}U_{\overline{x}_j,\Lambda}\Big) \\ &= k\int_{\mathbb{R}^N}\Big(U_{0,1}^{2^*}+\sum_{j=2}^{k}U_{\overline{x}_1,\Lambda}^{2^*-1}U_{\overline{x}_j,\Lambda}\Big) + k\int_{\mathbb{R}^N}\sum_{j=1}^{k}U_{\underline{x}_1,\Lambda}^{2^*-1}U_{\overline{x}_j,\Lambda} \\ &= k\int_{\mathbb{R}^N}U_{0,1}^{2^*}+k\int_{\mathbb{R}^N}U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda}+\sum_{i=1}^{k}U_{\underline{x}_j,\Lambda}\Big). \end{aligned} \tag{A.17}$$

For I_2, using the symmetry of function $W_{r,h,\Lambda}$, we have

$$\begin{aligned} I_2 &= \frac{2k}{2^*}\int_{\Omega_1^+}K\Big(\frac{|y|}{\mathbf{r}}\Big)W_{r,h,\Lambda}^{2^*} \\ &= \frac{2k}{2^*}\int_{\Omega_1^+}K\Big(\frac{|y|}{\mathbf{r}}\Big)\Bigg\{U_{\overline{x}_1,\Lambda}^{2^*}+2^*U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda}+\sum_{j=1}^{k}U_{\underline{x}_j,\Lambda}\Big) \\ &\quad + O\left(U_{\overline{x}_1,\Lambda}^{\frac{2^*}{2}}\Big(\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda}+\sum_{j=1}^{k}U_{\underline{x}_j,\Lambda}\right)^{\frac{2^*}{2}}\Big)\Bigg\} \\ &:= \frac{2k}{2^*}\Big(I_{21}+I_{22}+I_{23}\Big). \end{aligned} \tag{A.18}$$

For $y\in\Omega_1^+$, from Lemma B.1, we have

$$\Big(\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda}+\sum_{j=1}^{k}U_{\underline{x}_j,\Lambda}\Big)\le\frac{C}{\big(1+|y-\overline{x}_1|\big)^{\frac{(N-2)\epsilon_0}{N}}}\Big(\frac{k}{\mathbf{r}}\Big)^{N-2-\frac{(N-2)\epsilon_0}{N}},$$

with $\epsilon_0>0$ can be chosen small enough. Then we can get

$$I_{23} = O\left(\int_{\Omega_1^+}K\left(\frac{|y|}{\mathbf{r}}\right)U_{\overline{x}_1,\Lambda}^{\frac{2^*}{2}}\left(\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda}+\sum_{j=1}^{k}U_{\underline{x}_j,\Lambda}\right)^{\frac{2^*}{2}}\right)=O\left(\left(\frac{k}{\mathbf{r}}\right)^{N-\epsilon_0}\right). \tag{A.19}$$

For I_{21}, we can rewrite it as following

$$\begin{aligned} I_{21} &= \int_{\Omega_1^+}U_{\overline{x}_1,\Lambda}^{2^*}+\int_{\Omega_1^+}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]U_{\overline{x}_1,\Lambda}^{2^*} \\ &= \int_{\mathbb{R}^N}U_{0,1}^{2^*}+\int_{\Omega_1^+}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]U_{\overline{x}_1,\Lambda}^{2^*}+O\Big(\Big(\frac{k}{\mathbf{r}}\Big)^{N}\Big). \end{aligned}$$

Furthermore, we obtain

$$\int_{\Omega_1^+}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]U_{\overline{x}_1,\Lambda}^{2^*}=\int_{\Omega_1^+\cap\left\{y:\,|\frac{|y|}{\mathbf{r}}-1|\geq\delta\right\}}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]U_{\overline{x}_1,\Lambda}^{2^*}+\int_{\Omega_1^+\cap\left\{y:\,|\frac{|y|}{\mathbf{r}}-1|\leq\delta\right\}}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]U_{\overline{x}_1,\Lambda}^{2^*}.$$

When $|\frac{|y|}{\mathbf{r}}-1|\geq\delta$, there holds

$$|y-\overline{x}_1|\geq\big||y|-\mathbf{r}\big|-\big|\mathbf{r}-|\overline{x}_1|\big|\geq\frac{1}{2}\delta\mathbf{r}.$$

Thus we can easily get

$$\int_{\Omega_1^+\cap\left\{y:\,|\frac{|y|}{\mathbf{r}}-1|\geq\delta\right\}}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]U_{\overline{x}_1,\Lambda}^{2^*}\leq\frac{C}{\mathbf{r}}^{N-\epsilon_0}.$$

If $|\frac{|y|}{\mathbf{r}}-1|\leq\delta$, recalling the decay property of K, we can obtain that

$$\begin{aligned}
&\int_{\Omega_1^+\cap\left\{y:\,|\frac{|y|}{\mathbf{r}}-1|\leq\delta\right\}}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]U_{\overline{x}_1,\Lambda}^{2^*}\\
&\quad=-c_0\frac{1}{\mathbf{r}}^m\int_{\Omega_1^+\cap\left\{y:\,|\frac{|y|}{\mathbf{r}}-1|\leq\delta\right\}}\big||y|-\mathbf{r}\big|^m U_{\overline{x}_1,\Lambda}^{2^*}\\
&\qquad+O\Big(\frac{1}{\mathbf{r}}^{m+\sigma}\int_{\Omega_1^+\cap\left\{y:\,|\frac{|y|}{\mathbf{r}}-1|\leq\delta\right\}}\big||y|-\mathbf{r}\big|^{m+\sigma}U_{\overline{x}_1,\Lambda}^{2^*}\Big)\\
&\quad=-c_0\frac{1}{\mathbf{r}}^m\int_{\mathbb{R}^N}\big||y|-\mathbf{r}\big|^m U_{\overline{x}_1,\Lambda}^{2^*}+O\Big(\int_{\mathbb{R}^N\setminus B_{\frac{\mathbf{r}}{k}}(\overline{x}_1)}\Big(\frac{|y|^m}{\mathbf{r}}^m+1\Big)U_{\overline{x}_1,\Lambda}^{2^*}\Big)\\
&\qquad+O\Big(\frac{1}{\mathbf{r}}^{m+\sigma}\int_{\Omega_1^+\cap\left\{y:\,|\frac{|y|}{\mathbf{r}}-1|\leq\delta\right\}}\big||y|-\mathbf{r}\big|^{m+\sigma}U_{\overline{x}_1,\Lambda}^{2^*}\Big)\\
&\quad=-c_0\frac{1}{\mathbf{r}}^m\int_{\mathbb{R}^N}\big||y+\overline{x}_1|-\mathbf{r}\big|^m U_{0,\Lambda}^{2^*}\\
&\qquad+O\Big(\frac{1}{\mathbf{r}}^{m+\sigma}\int_{\Omega_1^+\cap\left\{y:\,|\frac{|y|}{\mathbf{r}}-1|\leq\delta\right\}}\big||y|-\mathbf{r}\big|^{m+\sigma}U_{\overline{x}_1,\Lambda}^{2^*}\Big)+O\Big(\Big(\frac{k}{\mathbf{r}}\Big)^{N-\epsilon_0}\Big).
\end{aligned}$$

Furthermore, recalling $|\overline{x}_1|=r$ and using the symmetry property, we have

$$\int_{\mathbb{R}^N}\big||y+\overline{x}_1|-\mathbf{r}\big|^m U_{0,\Lambda}^{2^*}=\int_{\mathbb{R}^N}\big||y+e_1r|-\mathbf{r}\big|^m U_{0,\Lambda}^{2^*},$$

where $e_1=(1,0,\ldots,0)$.

We get

$$\begin{aligned}
&\int_{\mathbb{R}^N}||y+\overline{x}_1|-\mathbf{r}|^m U_{0,\Lambda}^{2^*}\\
&\quad=\int_{\mathbb{R}^N}|y_1|^m U_{0,\Lambda}^{2^*}+\frac{1}{2}m(m-1)\int_{\mathbb{R}^N}|y_1|^{m-2}U_{0,\Lambda}^{2^*}(\mathbf{r}-r)^2+\mathcal{C}(r,\Lambda)(\mathbf{r}-r)^{2+\sigma},
\end{aligned}$$

here $\mathcal{C}(r,\Lambda)$ denote functions which are independent of h and can be absorbed in $O(1)$.

Similarly, we can also have the following expression

$$\begin{aligned}
&O\Big(\frac{1}{\mathbf{r}}^{m+\sigma}\int_{\Omega_1^+\cap\big\{y:\,|\frac{|y|}{\mathbf{r}}-1|\le\delta\big\}}\big||y|-\mathbf{r}\big|^{m+\sigma}U_{\overline{x}_1,\Lambda}^{2^*}\Big)\\
&\quad=O\Big(\frac{1}{\mathbf{r}}^{m+\sigma}\int_{\mathbb{R}^N}\big||y|-\mathbf{r}\big|^{m+\sigma}U_{\overline{x}_1,\Lambda}^{2^*}\Big)+O\Big(\Big(\frac{k}{\mathbf{r}}\Big)^{N-\epsilon_0}\Big)\\
&\quad=\frac{\mathcal{C}(r,\Lambda)}{\mathbf{r}}^{m+\sigma}+O\Big(\Big(\frac{k}{\mathbf{r}}\Big)^{N-\epsilon_0}\Big).
\end{aligned}$$

Then, we can obtain that

$$\begin{aligned}
I_{21}&=\int_{\mathbb{R}^N}|U_{0,1}|^{2^*}-\frac{c_0}{\Lambda^m\mathbf{r}^m}\int_{\mathbb{R}^N}|y_1|^mU_{0,1}^{2^*}\\
&\quad-\frac{1}{2}m(m-1)\frac{c_0}{\Lambda^{m-2}\mathbf{r}^m}\int_{\mathbb{R}^N}|y_1|^{m-2}U_{0,1}^{2^*}(\mathbf{r}-r)^2\\
&\quad+\frac{\mathcal{C}(r,\Lambda)}{\mathbf{r}}^m(\mathbf{r}-r)^{2+\sigma}+\frac{\mathcal{C}(r,\Lambda)}{\mathbf{r}}^{m+\sigma}+O\Big(\Big(\frac{k}{\mathbf{r}}\Big)^{N-\epsilon_0}\Big).
\end{aligned}\tag{A.20}$$

Finally, we consider I_{22}

$$\begin{aligned}
I_{22}&=2^*\int_{\Omega_1^+}U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^kU_{\overline{x}_j,\Lambda}+\sum_{j=1}^kU_{\underline{x}_j,\Lambda}\Big)\\
&\qquad+2^*\int_{\Omega_1^+}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^kU_{\overline{x}_j,\Lambda}+\sum_{j=1}^kU_{\underline{x}_j,\Lambda}\Big)\\
&=2^*\int_{\mathbb{R}^N}U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^kU_{\overline{x}_j,\Lambda}+\sum_{j=1}^kU_{\underline{x}_j,\Lambda}\Big)\\
&\qquad-2^*\int_{\mathbb{R}^N\setminus\Omega_1^+}U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^kU_{\overline{x}_j,\Lambda}+\sum_{j=1}^kU_{\underline{x}_j,\Lambda}\Big)\\
&\qquad+2^*\int_{\Omega_1^+}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^kU_{\overline{x}_j,\Lambda}+\sum_{j=1}^kU_{\underline{x}_j,\Lambda}\Big)\\
&:=I_{221}+I_{222}+I_{223}.
\end{aligned}$$

For I_{222}, it is easy to derive that

$$\sum_{j=1}^k\int_{\mathbb{R}^N\setminus\Omega_1^+}U_{\overline{x}_1,\Lambda}^{2^*-1}U_{\underline{x}_j,\Lambda}=O\Big(\frac{k^N}{\mathbf{r}}^N\Big).$$

Moreover, we know that

$$\begin{aligned}
&\sum_{j=2}^{k}\int_{\mathbb{R}^N\setminus\Omega_1^+} U_{\overline{x}_1,\Lambda}^{2^*-1}U_{\overline{x}_j,\Lambda}\\
&\quad=\sum_{j=2}^{k}\int_{\left(\mathbb{R}^N\setminus\Omega_1^+\right)\cap B_{\overline{d}_j/2}(\overline{x}_1)} U_{\overline{x}_1,\Lambda}^{2^*-1}U_{\overline{x}_j,\Lambda}+\sum_{j=2}^{k}\int_{\left(\mathbb{R}^N\setminus\Omega_1^+\right)\setminus B_{\overline{d}_j/2}(\overline{x}_1)} U_{\overline{x}_1,\Lambda}^{2^*-1}U_{\overline{x}_j,\Lambda}\\
&\quad=\sum_{j=2}^{k}\int_{\left(\mathbb{R}^N\setminus\Omega_1^+\right)\cap B_{\overline{d}_j/2}(\overline{x}_1)} U_{\overline{x}_1,\Lambda}^{2^*-1}U_{\overline{x}_j,\Lambda}+O\Big(\sum_{j=2}^{k}\frac{1}{|\overline{x}_1-\overline{x}_j|^{N-\epsilon_0}}\Big)\\
&\quad\le C\sum_{j=2}^{k}\int_{B_{\overline{d}_j/2}(\overline{x}_1)\setminus B_{\overline{d}_2/2}(\overline{x}_1)} U_{\overline{x}_1,\Lambda}^{2^*-1}U_{\overline{x}_j,\Lambda}+O\Big(\sum_{j=2}^{k}\frac{1}{|\overline{x}_1-\overline{x}_j|^{N-\epsilon_0}}\Big)\\
&=C\sum_{j=2}^{k}\frac{1}{|\overline{x}_1-\overline{x}_j|^{N-2}}\int_{B_{\Lambda\overline{d}_j/2}(0)\setminus B_{\Lambda\overline{d}_2/2}(0)}\frac{1}{(1+z^2)^{\frac{N+2}{2}}}+O\Big(\sum_{j=2}^{k}\frac{1}{|\overline{x}_1-\overline{x}_j|^{N-\epsilon_0}}\Big)\\
&\le C\sum_{j=2}^{k}\frac{1}{|\overline{x}_1-\overline{x}_j|^{N-2}}O\Big(\frac{1}{\overline{d}_2^2}\Big)+O\Big(\sum_{j=2}^{k}\frac{1}{|\overline{x}_1-\overline{x}_j|^{N-\epsilon_0}}\Big)\\
&=O\Big(\frac{k^2}{r^2}\Big)\sum_{j=2}^{k}\frac{1}{|\overline{x}_1-\overline{x}_j|^{N-2}}+O\Big(\sum_{j=2}^{k}\frac{1}{|\overline{x}_1-\overline{x}_j|^{N-\epsilon_0}}\Big)\\
&=O\Big(\frac{k^N}{\mathbf{r}}^N\Big)+O\Big(\sum_{j=2}^{k}\frac{1}{|\overline{x}_1-\overline{x}_j|^{N-\epsilon_0}}\Big),
\end{aligned}$$

where $\overline{d}_j=|\overline{x}_1-\overline{x}_j|$ for $j=2,\dots,k$ and $\overline{d}_2=|\overline{x}_1-\overline{x}_2|=2r\sqrt{1-h^2}\sin\frac{\pi}{k}=O\big(\frac{r}{k}\big)$. Then we get

$$I_{222}=O\Big(\Big(\frac{k}{\mathbf{r}}\Big)^{N-\epsilon_0}\Big). \tag{A.21}$$

Next, we consider the term I_{223}. In fact, we have

$$\begin{aligned}
I_{223}=&\int_{\Omega_1^+\cap\left\{y:\,|\frac{|y|}{\mathbf{r}}-1|\ge\delta\right\}}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda}+\sum_{j=1}^{k}U_{\underline{x}_j,\Lambda}\Big)\\
&+\int_{\Omega_1^+\cap\left\{y:\,|\frac{|y|}{\mathbf{r}}-1|\le\delta\right\}}\Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda}+\sum_{j=1}^{k}U_{\underline{x}_j,\Lambda}\Big).
\end{aligned}$$

When $|\frac{|y|}{\mathbf{r}}-1|\ge\delta$, there hold

$$|y-\overline{x}_1|\ge\big||y|-\mathbf{r}\big|-\big|\mathbf{r}-|\overline{x}_1|\big|\ge\frac{1}{2}\delta\mathbf{r}.$$

And for $y\in\Omega_1^+$ and $|\frac{|y|}{\mathbf{r}}-1|\ge\delta$, we have

$$\Big(\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda}+\sum_{j=1}^{k}U_{\underline{x}_j,\Lambda}\Big)\le C\Big(\frac{k}{\mathbf{r}}\Big)^{\alpha}\frac{1}{\big(1+|y-\overline{x}_1|\big)^{N-2-\alpha}}, \tag{A.22}$$

with $\alpha = (\frac{N-2-m}{N-2}, \frac{N-2}{2})$. Then we can get easily

$$\int_{\Omega_1^+\cap\left\{y:\,|\frac{|y|}{\mathbf{r}}-1|\geq\delta\right\}} \Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda}+\sum_{j=1}^{k}U_{\underline{x}_j,\Lambda}\Big)$$
$$\leq \frac{C}{\mathbf{r}}^{N-\alpha-\epsilon_0}\Big(\frac{k}{\mathbf{r}}\Big)^{\alpha}\leq C\Big(\frac{k}{\mathbf{r}}\Big)^{N}.$$

If $|\frac{|y|}{\mathbf{r}}-1|\leq\delta$, then

$$\int_{\Omega_1^+\cap\left\{y:\,|\frac{|y|}{\mathbf{r}}-1|\leq\delta\right\}} \Big[K\Big(\frac{|y|}{\mathbf{r}}\Big)-1\Big]U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda}+\sum_{j=1}^{k}U_{\underline{x}_j,\Lambda}\Big)$$
$$\leq \frac{C}{\mathbf{r}}^{m}\int_{\Omega_1^+\cap\left\{y:\,|\frac{|y|}{\mathbf{r}}-1|\leq\delta\right\}}\big||y|-\mathbf{r}\big|^{m}U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda}+\sum_{j=1}^{k}U_{\underline{x}_j,\Lambda}\Big)$$
$$=\frac{C}{\mathbf{r}}^{m}\int_{\Omega_1^+\cap\left\{y:\,|\frac{|y|}{\mathbf{r}}-1|\leq\delta\right\}\cap\left\{y:\,|y-\overline{x}_1|\leq\frac{\delta_1\mathbf{r}}{k}\right\}}\big||y|-\mathbf{r}\big|^{m}U_{\overline{x}_1,\Lambda}^{2^*-1}$$
$$\Big(\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda}+\sum_{j=1}^{k}U_{\underline{x}_j,\Lambda}\Big)$$
$$+\frac{C}{\mathbf{r}}^{m}\int_{\Omega_1^+\cap\left\{y:\,|\frac{|y|}{\mathbf{r}}-1|\leq\delta\right\}\cap\left\{y:\,|y-\overline{x}_1|\geq\frac{\delta_1\mathbf{r}}{k}\right\}}\big||y|-\mathbf{r}\big|^{m}U_{\overline{x}_1,\Lambda}^{2^*-1}$$
$$\Big(\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda}+\sum_{j=1}^{k}U_{\underline{x}_j,\Lambda}\Big),$$

where δ_1 is small constant. If $|y-\overline{x}_1|\leq\frac{\delta_1\mathbf{r}}{k}$, it is easy to derive

$$\big||y|-\mathbf{r}\big|\leq|y-\overline{x}_1|+||\overline{x}_1|-\mathbf{r}|\leq\frac{\delta_2\mathbf{r}}{k},$$

for some small δ_2. Therefore,

$$\frac{C}{\mathbf{r}}^{m}\big||y|-\mathbf{r}\big|^{m}\leq\frac{C}{k^m}.$$

Hence

$$\frac{C}{\mathbf{r}}^{m}\int_{\Omega_1^+\cap\left\{y:\,|\frac{|y|}{\mathbf{r}}-1|\leq\delta\right\}\cap\left\{y:\,|y-\overline{x}_1|\leq\frac{\delta_1\mathbf{r}}{k}\right\}}\big||y|-\mathbf{r}\big|^{m}U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda}+\sum_{j=1}^{k}U_{\underline{x}_j,\Lambda}\Big)$$
$$\leq\frac{C}{k^m}\int_{\mathbb{R}^N}U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^{k}U_{\overline{x}_j,\Lambda}+\sum_{j=1}^{k}U_{\underline{x}_j,\Lambda}\Big)$$
$$\leq\frac{C}{k^m}\Big(\frac{k}{\mathbf{r}}\Big)^{N-2}.$$

When $|y-\overline{x}_1|\geq\frac{\delta_1\mathbf{r}}{k}$, combing (A.22), we can get easily,

$$\frac{C^m}{\mathbf{r}}\int_{\Omega_1^+\cap\left\{y:\,|\frac{|y|}{\mathbf{r}}-1|\leq\delta\right\}\cap\left\{y:\,|y-\overline{x}_1|\geq\frac{\delta_1\mathbf{r}}{k}\right\}}\big||y|-\mathbf{r}\big|^m U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^k U_{\overline{x}_j,\Lambda}+\sum_{j=1}^k U_{\underline{x}_j,\Lambda}\Big)$$
$$\leq C\Big(\frac{k}{\mathbf{r}}\Big)^{N-\epsilon_0}.$$

Thus we can get

$$I_{223}=O\Big(\Big(\frac{k}{\mathbf{r}}\Big)^{N-\epsilon_0}\Big)+O\Big(\frac{1}{k^m}\Big(\frac{k}{\mathbf{r}}\Big)^{N-2}\Big). \tag{A.23}$$

Combining (A.17), (A.18), (A.20), (A.19), (A.21) and (A.23), we can get

$$\begin{aligned}
&I(W_{r,h,\Lambda})\\
&=k\Big(1-\frac{2}{2^*}\Big)\int_{\mathbb{R}^N}|U_{0,1}|^{2^*}-k\int_{\mathbb{R}^N}U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^k U_{\overline{x}_j,\Lambda}+\sum_{j=1}^k U_{\underline{x}_j,\Lambda}\Big)\\
&\quad+\frac{2k}{2^*}\Big[\frac{c_0}{\Lambda^m\mathbf{r}^m}\int_{\mathbb{R}^N}|y_1|^m U_{0,1}^{2^*}+\frac{c_0 m(m-1)}{2\Lambda^{m-2}\mathbf{r}^m}\int_{\mathbb{R}^N}|y_1|^{m-2}U_{0,1}^{2^*}(\mathbf{r}-r)^2\Big]\\
&\quad+k\frac{\mathcal{C}(r,\Lambda)}{k^{\frac{m(N-2)}{N-2-m}}}(\mathbf{r}-r)^{2+\sigma}+k\frac{\mathcal{C}(r,\Lambda)}{\mathbf{r}}^{m+\sigma}+kO\Big(\Big(\frac{k}{\mathbf{r}}\Big)^{N-\epsilon_0}\Big)+kO\Big(\frac{1}{k^m}\Big(\frac{k}{\mathbf{r}}\Big)^{N-2}\Big).
\end{aligned}$$

□

Combining Lemma A.1–A.3, we can get the following Proposition which gives the expression of $I(W_{r,h,\Lambda})$.

Proposition A.4. *Suppose that $K(|y|)$ satisfies* (**H**) *and $N\geq 5$, $(r,h,\Lambda)\in\mathscr{S}_k$. Then we have*

$$\begin{aligned}
I(W_{r,h,\Lambda})&=kA_1-\frac{k}{\Lambda^{N-2}}\Big[\frac{B_4k^{N-2}}{(r\sqrt{1-h^2})^{N-2}}+\frac{B_5k}{r^{N-2}h^{N-3}\sqrt{1-h^2}}\Big]\\
&\quad+k\Big[\frac{A_2}{\Lambda^m k^{\frac{(N-2)m}{N-2-m}}}+\frac{A_3}{\Lambda^{m-2}k^{\frac{(N-2)m}{N-2-m}}}(\mathbf{r}-r)^2\Big]+k\frac{\mathcal{C}(r,\Lambda)}{k^{\frac{m(N-2)}{N-2-m}}}(\mathbf{r}-r)^{2+\sigma}\\
&\quad+k\frac{\mathcal{C}(r,\Lambda)}{k^{\frac{m(N-2)}{N-2-m}+\sigma}}+kO\Big(\frac{1}{k^{\left(\frac{m(N-2)}{N-2-m}+\frac{2(N-3)}{N-1}+\sigma\right)}}\Big),
\end{aligned} \tag{A.24}$$

as $k\to\infty$, where $A_i,(i=1,2,3),B_4,B_5$ are positive constants.

Proof. A direct result of Lemma A.1–A.3 is

$$\begin{aligned}
I(W_{r,h,\Lambda})&=kA_1-\frac{k}{\Lambda^{N-2}}\Big[\frac{B_4k^{N-2}}{(r\sqrt{1-h^2})^{N-2}}+\frac{B_5k}{r^{N-2}h^{N-3}\sqrt{1-h^2}}\Big]\\
&\quad+k\Big[\frac{A_2}{\Lambda^m\mathbf{r}^m}+\frac{A_3}{\Lambda^{m-2}\mathbf{r}^m}(\mathbf{r}-r)^2\Big]+k\frac{\mathcal{C}(r,\Lambda)}{k^{\frac{m(N-2)}{N-2-m}}}(\mathbf{r}-r)^{2+\sigma}\\
&\quad+k\frac{\mathcal{C}(r,\Lambda)}{k^{\frac{(N-2)m}{N-2-m}+\sigma}}+kO\Big(\Big(\frac{k}{\mathbf{r}}\Big)^{N-\epsilon_0}\Big)+kO\Big(\frac{1}{k^m}\Big(\frac{k}{\mathbf{r}}\Big)^{N-2}\Big)\\
&\quad+kO\Big(\frac{\sigma_1(k)k^{N-2}}{\big(r\sqrt{1-h^2}\big)^{N-2}}\Big)+kO\Big(\frac{\sigma_2(k)k}{r^{N-2}h^{N-3}\sqrt{1-h^2}}\Big),
\end{aligned}$$

with $B_4 = B_0B_1, B_5 = B_0B_2$ are positive constants. From the expressions of $\sigma_1(k), \sigma_2(k)$ and asymptotic expression of h, r as in (A.4), (1.10) , we can show that

$$\frac{\sigma_1(k)k^{N-2}}{\big(r\sqrt{1-h^2}\big)^{N-2}}, \qquad \frac{\sigma_2(k)k}{r^{N-2}h^{N-3}\sqrt{1-h^2}},$$

can be absorbed in $O\Big(\frac{1}{k^{\left(\frac{m(N-2)}{N-2-m}+\frac{2(N-3)}{N-1}+\sigma\right)}}\Big)$.

Noting that $m > \frac{N-2}{2}$ implies

$$\frac{N-3}{N-1} < \frac{m}{N-2-m},$$

thus provided with ϵ_0, σ small enough, we can get

$$\Big(\frac{k}{\mathbf{r}}\Big)^{N-\epsilon_0} = \frac{1}{k^{\frac{m(2-\epsilon_0)}{N-2-m}}}\frac{1}{k^{\frac{m(N-2)}{N-2-m}}} \le C\frac{1}{k^{\left(\frac{m(N-2)}{N-2-m}+\frac{2(N-3)}{N-1}+\sigma\right)}}.$$

Since $m \ge 2$, we can check that

$$\frac{1}{k^m}\Big(\frac{k}{\mathbf{r}}\Big)^{N-2} \le \frac{C}{k^{\left(\frac{m(N-2)}{N-2-m}+\frac{2(N-3)}{N-1}+\sigma\right)}}.$$

Thus we can get (A.24). □

To get the expansions of $\frac{F(r,h,\Lambda)}{\partial\Lambda}, \frac{F(r,h,\Lambda)}{\partial h}$, we need the following expansions for $\frac{\partial I(W_{r,h,\Lambda})}{\partial\Lambda}, \frac{\partial I(W_{r,h,\Lambda})}{\partial h}$.

Proposition A.5. *Suppose that $K(|y|)$ satisfies* (**H**) *and $N \ge 5$, $(r, h, \Lambda) \in \mathscr{S}_k$. We have*

$$\begin{aligned}\frac{\partial I(W_{r,h,\Lambda})}{\partial\Lambda} &= \frac{k(N-2)}{\Lambda^{N-1}}\Big[\frac{B_4k^{N-2}}{(r\sqrt{1-h^2})^{N-2}} + \frac{B_5k}{r^{N-2}h^{N-3}\sqrt{1-h^2}}\Big]\\ &\quad - k\Big[\frac{mA_2}{\Lambda^{m+1}k^{\frac{(N-2)m}{N-2-m}}} + \frac{(m-2)A_3}{\Lambda^{m-1}k^{\frac{(N-2)m}{N-2-m}}}(\mathbf{r}-r)^2\Big]\\ &\quad + kO\Big(\frac{1}{k^{\frac{(N-2)m}{N-2-m}+\sigma}}\Big),\end{aligned}$$

as $k \to \infty$, where the constants $B_i, i = 4, 5$ and $A_i, i = 2, 3$ are defined in Proposition A.4.

Proof. The proof of this proposition is standard and the reader can refer to [31] for details. □

Proposition A.6. *Suppose that $K(|y|)$ satisfies* (**H**) *and $N \ge 5$, $(r, h, \Lambda) \in \mathscr{S}_k$. Then we have*

$$\begin{aligned}\frac{\partial I(W_{r,h,\Lambda})}{\partial h} &= -\frac{k}{\Lambda^{N-2}}\Big[(N-2)\frac{B_4k^{N-2}}{r^{N-2}(\sqrt{1-h^2})^N}h - (N-3)\\ &\qquad \frac{B_5k}{r^{N-2}h^{N-2}\sqrt{1-h^2}}\Big]\\ &\qquad + kO\Big(\frac{1}{k^{\left(\frac{m(N-2)}{N-2-m}+\frac{(N-3)}{N-1}+\sigma\right)}}\Big)\end{aligned} \tag{A.25}$$

as $k \to \infty$.

Proof. Recall

$$\overline{\mathbb{Z}}_{2j} \le C\frac{r}{(1+|y-\overline{x}_j|)^{N-1}}, \quad \underline{\mathbb{Z}}_{2j} \le C\frac{r}{(1+|y-\underline{x}_j|)^{N-1}}. \tag{A.26}$$

We know that

$$\begin{aligned}\frac{\partial I(W_{r,h,\Lambda})}{\partial h} &= \frac{1}{2}\frac{\partial}{\partial h}\int_{\mathbb{R}^N}|\nabla W_{r,h,\Lambda}|^2 - \frac{1}{2^*}\frac{\partial}{\partial h}\int_{\mathbb{R}^N}K\Big(\frac{|y|}{\mathbf{r}}\Big)W_{r,h,\Lambda}^{2^*}\\ &= k\frac{\partial}{\partial h}\int_{\mathbb{R}^N}U_{\overline{x}_1,\Lambda}^{2^*-1}\Big(\sum_{j=2}^k U_{\overline{x}_j,\Lambda} + \sum_{i=1}^k U_{\underline{x}_j,\Lambda}\Big)\\ &\quad - \int_{\mathbb{R}^N}K\Big(\frac{|y|}{\mathbf{r}}\Big)W_{r,h,\Lambda}^{2^*-1}\Big(\overline{\mathbb{Z}}_{21} + \sum_{j=2}^k\overline{\mathbb{Z}}_{2j} + \sum_{j=1}^k\underline{\mathbb{Z}}_{2j}\Big).\end{aligned} \tag{A.27}$$

From (A.27), similar to the calculations in the proof of Proposition A.3, we can get

$$\frac{\partial I(W_{r,h,\Lambda})}{\partial h} = -k\frac{\partial}{\partial h}\int_{\mathbb{R}^N}U_{\overline{x}_1,\Lambda}^{2^*-1}\left(\sum_{j=2}^k U_{\overline{x}_j,\Lambda} + \sum_{i=1}^k U_{\underline{x}_j,\Lambda}\right) + k^2 O\left(\left(\frac{k}{\mathbf{r}}\right)^{N-\epsilon_0}\right). \tag{A.28}$$

Then by some tedious but straightforward analysis, we can get

$$\begin{aligned}&\frac{\partial I(W_{r,h,\Lambda})}{\partial h}\\ &= -\frac{k}{\Lambda^{N-2}}\Big[(N-2)\frac{B_4k^{N-2}}{r^{N-2}(\sqrt{1-h^2})^N}h - (N-3)\frac{B_5k}{r^{N-2}h^{N-2}\sqrt{1-h^2}}\\ &\quad + h\frac{B_5k}{r^{N-2}h^{N-3}(1-h^2)^{\frac{3}{2}}}\Big] + k^2O\Big(\Big(\frac{k}{\mathbf{r}}\Big)^{N-\epsilon_0}\Big),\end{aligned} \tag{A.29}$$

for some ϵ_0 small enough. In fact, we know that $k\Big(\frac{k}{\mathbf{r}}\Big)^{N-\epsilon_0}$ and $h\frac{B_5k}{r^{N-2}h^{N-3}(1-h^2)^{\frac{3}{2}}}$ can be absorbed in $O\Big(\frac{1}{k^{\left(\frac{m(N-2)}{N-2-m}+\frac{(N-3)}{N-1}+\sigma\right)}}\Big)$ provided with m satisfying (1.6) and ϵ_0, σ small enough. In fact, this is the reason why we need the assumption (1.6). Then we can get (A.25) directly. □

4. Appendix B. Some basic estimates and lemmas

Lemma B.1. *Under the condition* $(r,h,\Lambda) \in \mathscr{S}_k$, *for* $y \in \Omega_1^+$ *there exists a constant* C *such that*

$$\left(\sum_{j=2}^k U_{\overline{x}_j,\Lambda} + \sum_{j=1}^k U_{\underline{x}_j,\Lambda}\right) \le C\left(\frac{k}{\mathbf{r}}\right)^{\alpha}\frac{1}{(1+|y-\overline{x}_1|)^{N-2-\alpha}},$$

with $\alpha = (1, N-2)$.

Proof. For $y \in \Omega_1^+$ and $j = 2, \dots, k$, we have

$$|y - \overline{x}_j| \geq |\overline{x}_1 - \overline{x}_j| - |y - \overline{x}_1| \geq \frac{1}{4}|\overline{x}_1 - \overline{x}_j|, \quad \text{if } |y - \overline{x}_1| \leq \frac{1}{4}|\overline{x}_1 - \overline{x}_j|,$$

and

$$|y - \overline{x}_j| \geq |y - \overline{x}_1| \geq \frac{1}{4}|\overline{x}_1 - \overline{x}_j|, \quad \text{if } |y - \overline{x}_1| \geq \frac{1}{4}|\overline{x}_1 - \overline{x}_j|,$$

$$|y - \underline{x}_i| \geq \frac{1}{4}|\overline{x}_1 - \underline{x}_1| \geq C\Big(\frac{r}{k}\Big).$$

Then

$$\begin{aligned}
&\Big(\sum_{j=2}^{k} U_{\overline{x}_j,\Lambda} + \sum_{j=1}^{k} U_{\underline{x}_j,\Lambda}\Big) \\
&\quad\leq \frac{C}{\big(1+|y-\overline{x}_1|\big)^{N-2-\alpha}}\Big[\sum_{j=2}^{k}\frac{1}{\big(1+|y-\overline{x}_j|\big)^{\alpha}} + \frac{1}{\big(1+|y-\underline{x}_1|\big)^{\alpha}}\Big] \\
&\quad\leq \frac{C}{\big(1+|y-\overline{x}_1|\big)^{N-2-\alpha}}\Big[\sum_{j=2}^{k}\frac{1}{|\overline{x}_1-\overline{x}_j|^{\alpha}} + \frac{1}{|\overline{x}_1-\underline{x}_1|^{\alpha}}\Big] \\
&\quad\leq \frac{C}{\big(1+|y-\overline{x}_1|\big)^{N-2-\alpha}}\Big(\frac{k}{\mathbf{r}}\Big)^{\alpha}.
\end{aligned}$$

□

Lemma B.2. *Under the condition $(r, h, \Lambda) \in \mathscr{S}_k$, for $y \in \Omega_1^+$ we have*

$$\Big(\sum_{j=2}^{k} \overline{\mathbb{Z}}_{2j} + \sum_{j=1}^{k} \underline{\mathbb{Z}}_{2j}\Big) \leq C\Big(\frac{k}{\mathbf{r}}\Big)^{\alpha}\frac{\mathbf{r}}{\big(1+|y-\overline{x}_1|\big)^{N-1-\alpha}},$$

with $\alpha = (1, N-1)$.

Proof. The proof of Lemma B.2 is similar to Lemma B.1. We omit the details for concise. □

For each fixed i and j, $i \neq j$, we consider the following function

$$g_{ij}(y) = \frac{1}{(1+|y-x_j|)^{\gamma_1}}\frac{1}{(1+|y-x_i|)^{\gamma_2}},$$

where $\gamma_1 \geq 1$ and $\gamma_2 \geq 1$ are two constants.

Lemma B.3. (Lemma B.1, [31]) *For any constants $0 < \upsilon \leq \min\{\gamma_1, \gamma_2\}$, there is a constant $C > 0$, such that*

$$g_{ij}(y) \leq \frac{C}{|x_i - x_j|^{\upsilon}}\Big(\frac{1}{(1+|y-x_i|)^{\gamma_1+\gamma_2-\upsilon}} + \frac{1}{(1+|y-x_j|)^{\gamma_1+\gamma_2-\upsilon}}\Big).$$

Lemma B.4. (Lemma B.2, [31]) *For any constant $0 < \beta < N-2$, there is a constant $C > 0$, such that*

$$\int_{\mathbb{R}^N}\frac{1}{|y-z|^{N-2}}\frac{1}{(1+|z|)^{2+\beta}}\mathrm{d}z \leq \frac{C}{(1+|y|)^{\beta}}.$$

Lemma B.5. *Suppose that $N \geq 5$ and $\tau \in (0,2), y = (y_1, \ldots, y_N)$. Then there is a small $\sigma > 0$, such that when $y_3 \geq 0$,*

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} W_{r,h,\Lambda}^{\frac{4}{N-2}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-\overline{x}_j|)^{\frac{N-2}{2}+\tau}} \,\mathrm{d}z \leq C \sum_{j=1}^{k} \frac{1}{(1+|y-\overline{x}_j|)^{\frac{N-2}{2}+\tau+\sigma}},$$

and when $y_3 \leq 0$,

$$\int_{\mathbb{R}^N} \frac{1}{|y-z|^{N-2}} W_{r,h,\Lambda}^{\frac{4}{N-2}}(z) \sum_{j=1}^{k} \frac{1}{(1+|z-\underline{x}_j|)^{\frac{N-2}{2}+\tau}} \,\mathrm{d}z \leq C \sum_{j=1}^{k} \frac{1}{(1+|y-\underline{x}_j|)^{\frac{N-2}{2}+\tau+\sigma}}.$$

Proof. The proof of Lemma B.5 is similar to Lemma B.3 in [31]. Here we omit it. □

Lemma B.6. *Suppose that $N \geq 5$ and m satisfies (1.6). We have*

$$\mathbf{r} \max\left\{ \frac{1}{k^{(\frac{m}{N-2-m})(N+2-2\frac{N-2-m}{N-2}-2\epsilon_1)}}, \frac{1}{k^{(\frac{N-2}{N-2-m})\min\{2m,m+3\}}} \right\} \leq \frac{C}{k^{\left(\frac{m(N-2)}{N-2-m}+\frac{(N-3)}{N-1}+\sigma\right)}}, \tag{B.1}$$

provided with σ, ϵ_1 small enough.

Proof. It's easy to show that

$$\frac{\mathbf{r}}{k^{(\frac{N-2}{N-2-m})\min\{2m,m+3\}}} \leq \frac{C}{k^{\left(\frac{m(N-2)}{N-2-m}+\frac{(N-3)}{N-1}+\sigma\right)}},$$

for $m \geq 2$. In order to get (B.1), we just need to show

$$\frac{\mathbf{r}}{k^{(\frac{m}{N-2-m})(N+2-2\frac{N-2-m}{N-2}-2\epsilon_1)}} = \frac{k^{\frac{N-2}{N-2-m}}}{k^{(\frac{m}{N-2-m})(N+2-2\frac{N-2-m}{N-2}-2\epsilon_1)}} \leq \frac{C}{k^{\left(\frac{m(N-2)}{N-2-m}+\frac{(N-3)}{N-1}+\sigma\right)}}, \tag{B.2}$$

for some σ, ϵ_1 small. The problem to show (B.2) can be reduced to show that $6 + \frac{(N-3)}{N-1} < 3(\frac{N-2}{N-2-m}) + 2\frac{N-2-m}{N-2}$, for m satisfying (1.6). This inequality follows by simple computations. This fact concludes the proof. □

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Received: 12 January 2023.
Accepted: 29 January 2023.