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On the Stability of Solitons for the Maxwell-Lorentz Equations with Rotating Particle

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Abstract. We prove the stability of solitons of the Maxwell–Lorentz equations with extended charged rotating particle. The solitons are solutions which correspond to the uniform rotation of the particle. To prove the stability, we construct the Hamilton–Poisson representation of the Maxwell–Lorentz system. The construction relies on the Hamilton least action principle. The constructed structure is degenerate and admits a functional family of the Casimir invariants. This structure allows us to construct the Lyapunov function corresponding to a soliton. The function is a combination of the Hamiltonian with a suitable Casimir invariant. The function is conserved, and the soliton is its critical point. The key point of the proof is a lower bound for the Lyapunov function. This bound implies that the soliton is a strict local minimizer of the function. The bound holds if the effective moment of inertia of the particle in the Maxwell field is sufficiently large with respect to the "bar moment of inertia".

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Contents

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1. Introduction

The paper concerns the stability of solitons of the Maxwell–Lorentz equations with an extended charged rotating particle with fixed center of mass.The solitons are the solutions that correspond to the uniform rotation of the particle. We choose the units where the speed of light is $c = 1$. Then the Maxwell–Lorentz equations with the rotating particle read as [41,43]:

$$\left\{\begin{array}{ll} \dot E(x,t) = \operatorname{curl} B(x,t) - w(x,t)\rho(x), & \dot B(x,t) = -\operatorname{curl} E(x,t) \\ \operatorname{div} E(x,t) = \rho(x), & \operatorname{div} B(x,t) = 0 \\ I\dot\omega(t) = \big\langle x\wedge\big[E(x,t) + w(x,t)\wedge B(x,t)\big], \rho(x)\big\rangle & \end{array}\right| , \tag{1.1}$$

where $w(x,t)$ is the velocity field

$$w(x,t) := \omega(t)\wedge x. \tag{1.2}$$

We denote by $I > 0$ the moment of inertia of the particle, $\rho(x)$ is the charge distribution of the extended particle centered at the point $0 \in \mathbb{R}^3$, and the brackets $\langle , \rangle$ denote the inner product in the Hilbert space $L^2 := L^2(\mathbb{R}^3) \otimes \mathbb{R}^3$. Further, $\omega(t)$ is the angular velocity of the particle rotation: every fixed point $x(0) \in \mathbb{R}^3$ of the extended particle moves along the trajectory $x(t) = R(t)x(0)$, where $R(t) \in SO(3)$, and its velocity is

$$\dot x(t) = \dot R(t)x(0) = \dot R(t)R^{-1}(t)x(t) = \omega(t)\wedge x(t), \quad \omega(t)\wedge = \dot R(t)R^{-1}(t) \tag{1.3}$$

($\omega(t)$ is called the *angular velocity in the space*, see [4]).

The system (1.1) is a special case of the general Maxwell–Lorentz system considered in [41,43].

$$\left\{\begin{array}{ll} \dot E(x,t) = \operatorname{curl} B(x,t) - w(x,t)\rho(x-q(t)), & \dot B(x,t) = -\operatorname{curl} E(x,t) \\ \operatorname{div} E(x,t) = \rho(x-q(t)), & \operatorname{div} B(x,t) = 0 \\ m\ddot q(t) = \big\langle E(x,t) + w(x,t)\wedge B(x,t), \rho(x-q(t))\big\rangle & \\ I\dot\omega(t) = \big\langle (x-q(t))\wedge\big[E(x,t) + w(x,t)\wedge B(x,t)\big], \rho(x-q(t))\big\rangle & \end{array}\right| , \tag{1.4}$$

where $m > 0$ is the mass of the particle, $q(t)$ is its position, and $w(x,t) = \dot q(t) + x\wedge(x - q(t))$. The system (1.1) describes the solutions of (1.4) with $q(t) \equiv 0$. In particular, this identity holds for the solutions to (1.4) with $E(x,0)$ odd in x, $B(x,0)$ even in x, and $q(0) = 0$. However, we consider the system (1.1) as it stands, for all fields of finite energy.

The system (1.4) describes the *extended electron* coupled to the Maxwell field. This model was introduced by Abraham in 1903–1905 (see [1,2]). The model allows one to avoid the "ultraviolet divergence", that is, the infiniteness of the own energy and mass of electrons, in contrast to the case of point particles corresponding to $\rho(x) = \delta(x)$. Using this model, Abraham was the first to discover the mass-energy equivalence, thereby anticipating Einstein's theory of special relativity. This system also served as the classical Landé model of spin in Old Quantum Mechanics (1900–1924): see [44], and also Chapter 14 in [23] and Appendix A in [24]. Various approximations of the system (1.4) were introduced to explain the famous *radiation damping*: the *Lorentz–Dirac equation* (with *runaway solutions*) as introduced by Dirac in [9], and many other approximations, see Chapter 16 in [21]. The detailed account on the genesis and early investigations of the system by Dirac, Poincaré, Sommerfeld, and others can be found in Chapter 3 of [43].

The system (1.1) plays a crucial role in a rigorous analysis of radiation by moving particles, see [7,16,30–32,34,35,43]. Moreover, the mathematical analysis of the system is useful in connection to the related problems of nonrelativistic QED, see the survey by Spohn [43]. In particular, the similarity in the renormalization of mass was pointed out by Hiroshima and Spohn [13].

The system (1.1) is invariant under the group of rotations of the space $\mathbb{R}^3$. We will assume that the charge density $\rho(x)$ is spherically symmetric (2.1). In this case, the system (1.1) admits the Lagrangian structure [19]; see Remark 2.5. Below, we construct the corresponding Hamilton–Poisson structure. Moreover, as was discovered by Spohn [43], in the case (2.1) the system (1.1) admits solitons, that are solutions rotating with constant angular velocity. We calculate the effective moment of inertia of the solitons:

$$I_{\text{eff}} = I + \delta I, \qquad \delta I = \frac{2}{3}\int \frac{|\nabla\hat{\rho}(k)|^2}{k^2}dk = \frac{1}{6\pi}\int \frac{x\rho(x)\cdot y\rho(y)dxdy}{|x-y|}, \tag{1.5}$$

where $\hat{\rho}(k)$ is the Fourier transform of $\rho(x)$ (see Appendix B). Here I is the "bar moment of inertia" caused by the distribution of mass in the particle, while δI is the increment of the moment caused by the distribution of charge and its interaction with the Maxwell field.

The main result of the present paper is the stability of the rotating solitons of the system (1.1) under suitable condition on their effective moments of inertia. Our basic assumption is as follows:

$$I_{\text{eff}} \gg I. \tag{1.6}$$

This condition holds if the charge of the particle is sufficiently large.

To prove the stability, we construct the Hamilton–Poisson representation for the system (1.1), explicitly calculating the *structural operator*, which is the integral kernel of the Poisson bracket. The calculation relies on the Hamilton least action principle and the Lie–Poincaré calculus [3,15] (see Appendix A), which is based on the ideas of Lie and Poincaré [37,42].

The Hamilton–Poisson structure is degenerate and admits a functional family of Casimir invariants. Hence, the theory of orbital stability [10] is not formally applicable in our case. We construct a Lyapunov function as a combination of the Hamiltonian with a suitable Casimir invariant (such a strategy is known as the “energy–Casimir method” [12,14,40]). The Lyapunov function is conserved, and the soliton is its critical point. The key point of the proof is a lower bound for the Lyapunov function under the condition (1.6). This bound implies that the soliton is a strict local minimizer of the Lyapunov function.

Let us comment on related results.

The pioneering work of Arnold [3] opened a novel chapter in the theory of stability in hydrodynamics treating the Euler equations as a Hamiltonian system with a symmetry group. This theory was developed by Marsden, Weinstein, Holm, and others, for the Maxwell–Vlasov system, equations of magnetohydrodynamics, and others, see [5,15,38,39] for surveys and references. The theory relies on the reduction of the systems by the action of the corresponding symmetry groups. This theory develops the ideas of Lie and Poincaré, which were introduced in the context of finite-dimensional dynamical systems, [37,42]. In the present paper, we develop this stability theory for the system (1.1).

The Hamilton principle and conservation laws for the system (1.1) were established first by Nodvik [41], who used the Euler angles representation. The coordinate-free proof of the Hamilton principle was given in [19] on the basis of the technique of [6,42]: the Poincaré equations, and expansions over right-invariant vector fields on the Lie group $SO(3)$. This technique was developed in [20], where the general theory of invariants was constructed for the Poincaré equations on manifolds and applied to the construction of invariants for the system (1.1). The coordinate-free proof of the conservation laws for the system (1.1) was given by Kiessling [22]. A Hamilton structure for the system (1.1) was constructed in [8] in the Euler angles for sufficiently smooth solutions. However, in contrast to our result, the Legendre transformation in [8] is not invertible. The proofs in [22,41] and [8,19,20] rely on the assumptions that all the differentiations and integrations by parts are correct. In [28], we give a new coordinate-free proof of the Hamilton principle using the Lie–Poincaré calculus. Now all calculations in [28] are rigorously justified, and the corresponding Legendre transformation is invertible.

In [7], Bambusi and Galgani proved the existence and orbital stability of solitons with velocities $|v| < 1$ for the Maxwell–Lorentz system (1.1) without spinning (the first three lines of (1.1) with $\omega(t) \equiv 0$). The proof relies on the transition to a comoving frame and a lower bound for the reduced Hamiltonian. The global convergence to solitons for the same system was proved in [16]. In [31], the global attraction to stationary states was established for a similar system with a relativistic particle in presence of an external confining potential. In [30,32] the results of [16,31] were proved for a similar systems with a scalar field instead of the Maxwell field. The results [16,30–32,34] provide the first rigorous proof of the radiation damping in classical electrodynamics. All these results were obtained under the *Wiener-type condition* on the charge density ρ. For the corresponding surveys, see [25] and [26].

The global convergence to rotating solitons was established in [17] for solutions of the system (1.1) with $q(t) \equiv 0$ in the case of sufficiently small charge density

ρ. This result was strengthened in [34] under a considerably weaker Wiener-type condition.

The adiabatic effective dynamics of solitons was proved in [29] for a relativistic particle in the scalar field. In [35] this result was extended to the Maxwell–Lorentz equations without spinning.

In [18], the asymptotic completeness was proved for scattering of solutions to the Maxwell–Lorentz equations without spinning. The result refers to solutions which are close to a solitary manifold.

In [27], we have proved the orbital stability of moving and rotating solitons for the 2D analog of the system (1.4) using the reduction by conservation of the corresponding linear and angular momenta. In the present paper, we extend this result to the rotating solitons of the 3D system (1.1). This extension required several essential modification of our approach [27].

2. The Lagrangian Structure

In this section we state the well-posedness and the Hamilton least action principle for the system (1.1). Denote the Sobolev spaces $H^s = H^s(\mathbb{R}^3) \otimes \mathbb{R}^3$ with $s \in \mathbb{R}$, and $\dot{H}^1 = \dot{H}^1(\mathbb{R}^3) \otimes \mathbb{R}^3$. All the derivatives are understood in the sense of distributions. We assume that the charge density $\rho(x)$ is smooth and spherically-invariant, i.e.,

$$\rho \in C_0^\infty(\mathbb{R}^2), \qquad \rho(x) = \rho_1(|x|); \qquad \rho(x) = 0 \;\text{ for } |x| \geq R_\rho. \tag{2.1}$$

2.1. The Maxwell Potentials

For the proof of the stability of the solitons, we need the Hamilton form of the system (1.1). This is why we should rewrite the system in the Maxwell potentials $A(x,t) = (A_1(x,t), A_2(x,t), A_3(x,t))$ and $\Phi(x)$ (see [21]):

$$B(x,t) = \operatorname{curl} A(x,t), \qquad E(x,t) = -\dot{A}(x,t) - \nabla\Phi(x,t). \tag{2.2}$$

We choose the Coulomb gauge

$$\operatorname{div} A(x,t) = 0. \tag{2.3}$$

Note that (2.1) implies

$$\operatorname{div}[\omega\wedge x\rho(x)] = 0, \qquad x \in \mathbb{R}^3.$$

Now the first two lines of (1.1) are equivalent to the system

$$\left\{\begin{array}{l} -\ddot{A}(x,t) = -\Delta A - \omega\wedge x\rho(x) \\ -\Delta\Phi(x) = \rho(x) \end{array}\right| . \tag{2.4}$$

Here the second equation can be solved explicitly:

$$\Phi(x,t) = \Phi(x) = \frac{1}{4\pi}\int \frac{\rho(y)}{|x-y|}dy. \tag{2.5}$$

In the Fourier transform,

$$\hat{\Phi}(k) = \frac{\hat{\rho}(k)}{k^2}. \tag{2.6}$$

Hence,

$$\Phi(\cdot) \in \dot{H}^1(\mathbb{R}^3). \tag{2.7}$$

Definition 2.1. The Hilbert space $F^0 := \{A \in L^2 : \operatorname{div} A(x) \equiv 0\}$, and $\dot{F}^1$ is the completion of $F^0 \cap [C_0^\infty(\mathbb{R}^3) \otimes \mathbb{R}^3]$ w.r.t. the norm

$$\|\psi\|^2_{\dot{F}^1} = \int |\nabla\psi(x)|^2 dx < \infty. \tag{2.8}$$

The first equation of (2.4) is equivalent to the wave equation

$$\ddot{A} = \Delta A + \omega\wedge x\rho(x), \tag{2.9}$$

Now the system (1.1) becomes

$$\left\{\begin{array}{l} \ddot{A} = \Delta A + \omega\wedge x\rho(x) \\ I\dot{\omega}(t) = \big\langle x\wedge[-\dot{A}(x,t) + (\omega\wedge x)\wedge\operatorname{curl} A(x,t)], \rho(x)\big\rangle \end{array}\right| . \tag{2.10}$$

where the brackets $\langle , \rangle$ denote the inner product in the real Hilbert space L^2. In the last equation, we have canceled the term involving $\Phi(x)$ because

$$\big\langle x\wedge\nabla\Phi(x), \rho(x)\big\rangle = \big\langle i\nabla\wedge ik\hat{\Phi}(k), \hat{\rho}(k)\big\rangle = 0. \tag{2.11}$$

This follows from the rotation-invariance (2.1) since

$$\nabla\wedge k = (\partial_{\varphi_1}, \partial_{\varphi_2}, \partial_{\varphi_3}), \tag{2.12}$$

where φ_j is the angle of rotation about the axis k_j.

2.2. Well-Posedness

Here we state the well-posedness for the system (2.10). Denote the Hilbert spaces

$$\mathbb{Y} := \dot{F}^1 \oplus F^0 \oplus \mathbb{R}^3, \qquad \mathbb{V} = F^0 \oplus H^{-1} \oplus \mathbb{R}^3. \tag{2.13}$$

Proposition 2.2. *i) For any initial state $Y(0) = (A(x,0), \dot{A}(x,0), \omega(0)) \in \mathbb{Y}$, the system (2.10) admits a unique solution*

$$Y(t) = (A(x,t), \dot{A}(x,t), \omega(t)) \in C(\mathbb{R}, \mathbb{Y}) \cap C^1(\mathbb{R}, \mathbb{V}). \tag{2.14}$$

ii) The map $W(t) : Y(0) \mapsto Y(t)$ is continuous in $\mathbb{Y}$ for every $t \in \mathbb{R}$.

iii) The energy is conserved:

$$E(t) := \frac{1}{2}\int [\dot{A}^2(x,t) + (\operatorname{curl} A(x,t))^2]dx + \frac{1}{2}I\omega^2(t) = \text{const}, \tag{2.15}$$

iv) Let

$$\left\{\begin{array}{ll} A(x,0) \in C^3(\mathbb{R}^3)\otimes\mathbb{R}^3, \quad \dot{A}(x,0) \in C^2(\mathbb{R}^3)\otimes\mathbb{R}^3 & \\ A(x,0) = 0, \ \ \dot{A}(x,0) + \nabla\Phi(x) = 0, & |x| > R \end{array}\right| \tag{2.16}$$

for some $R > 0$. Then

$$\left\{\begin{array}{l} A(x,t) \in C^2(\mathbb{R}^3 \times \mathbb{R})\otimes\mathbb{R}^3, \\ |\partial_x^\alpha A(x,t)| + |\partial_x^\alpha \dot{A}(x,t)| \le C_\alpha(1+|x|)^{-2-|\alpha|}, \quad |x| > \overline{R}(t), \quad \forall\alpha \end{array}\right| , \tag{2.17}$$

where $\overline{R}(t) = \max(R, R_\rho) + |t| + 1$.

We sketch the proof. All details can be found in [28] and [31]. It suffices to consider the case of smooth initial functions $A(x,0), \dot{A}(x,0)$ with compact supports. The elimination of the fields reduces the system (2.10) to the system of nonlinear integral equation for $\omega(t)$. The existence and uniqueness of the solution $\omega(t)$ for small $|t|$ follows by application of the contraction mapping principle as in [31]. The corresponding fields $A(\cdot,t), \dot{A}(\cdot,t)$ are smooth and have compact supports, hence, the energy conservation (2.15) follows by standard integration by parts. Therefore, the solution $Y(t)$ can be extended to all $t \in \mathbb{R}$.

The continuity of the map $W(t)$, as constructed on the dense subspace of $\mathbb{Y}$, follows from the continuity of the map $\mathbb{Y} \to C^2(0,t;\mathbb{R}^3)$ defined as $Y(0) \mapsto \omega(\cdot)|_{[0,t]}$. Now for general initial state $Y(0) \in \mathbb{Y}$, the existence of solutions and the energy conservation (2.15) follow by suitable approximations of $Y(0)$.

To prove ii), note that $\omega(\cdot) \in C^2(\mathbb{R}) \otimes \mathbb{R}^3$ by (2.14) and the last two equations of (2.10). This fact and the first line of (2.16) imply the first line of (2.17) due to the Kirchhoff integral representation of solutions of the first equation from (2.4). Further, applying curl to both sides of the first equation in (2.2), we obtain by (2.3) that

$$\Delta A(x,t) = -\operatorname{curl} B(x,t).$$

The second line of (2.16) implies that

$$E(x,0) = B(x,0) = 0, \qquad |x| > R. \tag{2.18}$$

Hence,

$$E(x,t) = B(x,t) = 0, \qquad |x| > \overline{R}(t)$$

by the integral representation (A.4) from [31]. Hence,

$$A(x,t) = \int_{|y|<\overline{R}(t)} \frac{\operatorname{curl} B(y,t)dy}{4\pi|x-y|} = \operatorname{curl} \int_{|y|<\overline{R}(t)} \frac{B(y,t)dy}{4\pi|x-y|}. \tag{2.19}$$

Now the second line of (2.17) follows.

Remark 2.3. This proposition allows us to justify all the operations involving classical derivatives of solutions: calculation of variational derivatives, the proof of conservation laws, etc. Namely, it suffices to justify the operations for C^2-solutions with initial data satisfying (2.16), and conclude the results for general initial data by the continuity of the map $W(t)$ in the space $\mathbb{Y}$.

2.3. The Hamilton Least Action Principle

According to [19], under the assumption (2.1), sufficiently smooth trajectories $X(t) = (A(t), R(t))$ which correspond to solutions of the system (2.10) with $\omega(t)\wedge = \dot{R}(t)R^{-1}(t)$, satisfy the Hamilton least action principle, i.e., they are stationary points of the Lagrangian action: for any $a, b \in \mathbb{R}$,

$$\delta S_{ab}(X) = 0 \qquad \text{if} \quad \delta X(a) = \delta X(b) = 0, \tag{2.20}$$

where the action is

$$S_{ab} := \int_a^b L(X(t), \dot{X}(t))dt. \tag{2.21}$$

The Lagrangian L is well known (see [36, (28.6)]: due to spherical symmetry (2.1), we have

$$L(A, \dot{A}, R, \dot{R}) = \frac{1}{2}\int (E^2 - B^2)dx + \frac{I\omega^2}{2} + \langle j(x), A(x)\rangle. \tag{2.22}$$

Here $\omega\wedge = \dot{R}R^{-1}$, E and B are expressed in terms of A and Φ according to (2.2) (with the potential Φ given by (2.5)), and the current density $j(x)$ is expressed in accordance with (1.2):

$$j(x) := \omega\wedge x\rho(x). \tag{2.23}$$

The proof of the Hamilton principle (2.20) in [19] relies on the variational Poincaré equations on the symmetry group $SO(3)$, [6, 42]. The proofs in [19] assume that all the derivatives and integrals exist and all partial integrations are correct. In [28], we give a novel proof, relying on the Lie–Poincaré calculus [3, 15], and justify all the calculations. We sketch some calculations from [28] which are used in further analysis.

The Euler–Lagrange Equations. The least action principle (2.20) can be written as

$$\frac{\delta S_{ab}}{\delta A} = 0, \qquad \frac{\delta S_{ab}}{\delta R} = 0, \tag{2.24}$$

where

$$\delta A(a) = \delta A(b) = 0, \qquad \delta R(a) = \delta R(b) = 0. \tag{2.25}$$

The first equation of (2.24) can be represented in the Euler–Lagrange form, since the field A varies in the corresponding linear space:

$$\frac{d}{dt}D_{\dot{A}}L = D_A L. \tag{2.26}$$

The calculations in [19, 28] show that this equation is equivalent to the first equation of the system (2.10). The last equation of (2.24) is also formally equivalent to the Euler–Lagrange equation

$$\frac{d}{dt}D_{\dot{R}}L = D_R L. \tag{2.27}$$

However, it is not easy to express this equation in the variables A and R, since R varies in the rotation group $SO(3)$, and the calculations require suitable local coordinates on the Lie group $SO(3)$ [15, Theorem 4.1].

The Euler–Poincaré Equation. In this section, we obtain an *Euler–Poincaré* form of equation (2.27) using the Lie–Poincaré calculus [3, 15]. First, we note that the Lagrangian (2.22) depends on R and $\dot{R}$ only through the angular velocity ω, which can be identified with the skew-symmetric matrix

$$\hat{\omega}(t) = \dot{R}(t)R^{-1}(t) \in so(3) \simeq \mathbb{R}^3.$$

Here we denote

$$\hat{\omega} := \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \qquad \omega = (\omega_1, \omega_2, \omega_3) \in \mathbb{R}^3. \tag{2.28}$$

Now

$$L(A,\dot A,R,\dot R)=l(A,\dot A,\omega), \tag{2.29}$$

where l is the *reduced Lagrangian*. Using (2.2) and the Coulomb gauge (2.3), we can rewrite the reduced Lagrangian as

$$l(A,\dot A,\omega)=\frac{1}{2}\int(\dot A^2-(\operatorname{curl}A)^2)dx+\frac{I\omega^2}{2}+\big\langle \omega\wedge x\rho(x),A(x)\big\rangle \tag{2.30}$$

up to an additive constant depending on the function Φ. The reduced Lagrangian l is well defined and Fréchet differentiable on the phase space $\mathbb{Y}$ introduced in (2.13).

Now the last equation of (2.24) can be written as

$$\int_a^b\left\langle\frac{\delta l}{\delta\omega}(t),\delta\omega(t)\right\rangle dt=0, \tag{2.31}$$

where the brackets mean the pairing of elements of $so(3)^*$ with $so(3)$. However, we cannot conclude that $\frac{\delta l}{\delta\omega}(t)=0$ since the variation $\delta\omega(t)$ is not an arbitrary function with values in the Lie algebra $so(3)$. On the other hand, the Lie–Poincaré technique [3, 15] allows us to show that

$$\delta\omega(t)=\dot\Sigma(t)+\Sigma(t)\wedge\omega(t),\qquad \hat\Sigma(t):=(\delta R(t))R^{-1}(t), \tag{2.32}$$

where we used the notation (2.28). We will prove (2.32) in Appendix A. Now (2.31) implies

$$\int_a^b\left\langle\frac{\delta l}{\delta\omega}(t),\dot\Sigma(t)+\Sigma(t)\wedge\omega(t)\right\rangle dt=0, \tag{2.33}$$

It is crucially important that

$$\delta\Sigma(a)=\delta\Sigma(b)=0, \tag{2.34}$$

since $\delta R(a)=\delta R(b)=0$ according to (2.20) and (2.32). Hence, integrating by parts, we obtain

$$\int_a^b\left\langle-\frac{d}{dt}\frac{\delta l}{\delta\omega}(t)+\omega(t)\wedge\frac{\delta l}{\delta\omega}(t),\Sigma(t)\right\rangle dt=0. \tag{2.35}$$

Here $\Sigma(t)$ is an arbitrary function with values in $so(3)\simeq\mathbb{R}^3$ according to (2.32). Hence, we get the corresponding Euler–Poincaré equation

$$-\frac{d}{dt}\frac{\delta l}{\delta\omega}(t)+\omega(t)\wedge\frac{\delta l}{\delta\omega}(t)=0. \tag{2.36}$$

In other words,

$$\dot\pi=\omega\wedge\pi,\qquad \pi(t):=\frac{\delta l}{\delta\omega}. \tag{2.37}$$

Differentiating (2.30), we find the angular momentum of the particle in the Maxwell field,

$$\pi=I\omega+\langle x\wedge A(x),\rho(x)\rangle\,. \tag{2.38}$$

Remark 2.4. Substituting into (2.36), we get

$$I\dot{\omega}(t) + \Big\langle x\wedge\dot{A}(x,t), \rho(x)\Big\rangle = \omega(t)\wedge\langle x\wedge A(x,t), \rho(x)\rangle\,. \tag{2.39}$$

The formulas (6.16)–(6.18) in [19] demonstrate that (2.39) is equivalent to the last equation of (2.10).

The Euler–Lagrange–Poincaré equations. As we have shown above, the system (2.10) is equivalent to the system of the equations (2.26) and (2.37):

$$\dot{\Pi} = D_A l, \qquad \dot{\pi} = \omega\wedge\pi, \tag{2.40}$$

where $\Pi := D_{\dot{A}} l$ is the conjugate momentum. Differentiating (2.30), we obtain

$$\Pi = \dot{A}, \qquad D_A l = \Delta A + \omega\wedge x\rho(x). \tag{2.41}$$

Thus, the system (2.40) reads as

$$\left\{\begin{array}{ll} \dot{A} = \Pi, & \dot{\Pi} = \Delta A + \omega\wedge x\rho(x) \\ \\ \dot{\pi} = \omega\wedge\pi & \end{array}\right|. \tag{2.42}$$

Remark 2.5. The proof of the equivalence of (2.27) with the last equation of (2.24) substantially relies on the spherical symmetry of the *charge distribution* (2.1); see [19] and Remark 2.5 in [28]. We suppose that such role of the requirement (2.1) is related to the fact that the Lagrangian (2.22) and the last equation of (1.1) correspond to spherically symmetric *mass distribution*. In the case of non-symmetric mass distribution we must substitute the scalar I by the 3×3 inertia matrix. In this case the reduction (2.29) is impossible since we must keep the rotation R in the Lagrangian. The corresponding Hamiltonian structure is an open question, and the stability theory as well.

3. The Hamilton–Poisson Representation

In this section, we show that the system (2.42) admits a representation in the Hamiltonian form

$$\dot{Y} = J(Y)DH(Y), \qquad Y = (A, \Pi, \pi), \tag{3.1}$$

where $H(Y)$ is a Hamiltonian, and $J(Y)$ is a skew-symmetric *structural operator*:

$$\big(J(Y)Y_1, Y_2\big) = -\big(Y_1, J(Y)Y_2\big), \qquad Y, Y_1, Y_2 \in \mathbb{Y}, \tag{3.2}$$

where the brackets $\big(,\big)$ denote the inner product in the Hilbert space

$$\mathbb{Y}^0 := F^0 \oplus F^0 \oplus \mathbb{R}^3.$$

3.1. The Legendre Transformation and the Hamiltonian

The conserved *energy functional* is defined as the Legendre transformation of the reduced Lagrangian [4,15]:

$$E(A,\dot{A},\omega)=\langle\Pi,\dot{A}\rangle+\pi\cdot\omega-l. \tag{3.3}$$

The Hamiltonian is this functional expressed as a function of (A,Π,π). By (2.38) and (2.41), we have

$$\Pi=\dot{A},\qquad \pi=I\omega+\langle x\wedge A(x),\rho(x)\rangle. \tag{3.4}$$

Hence, the map $(A,\dot{A},\omega)\mapsto(A,\Pi,\pi)$ is a diffeomorphism of the phase space $\mathbb{Y}$ defined in (2.13). Substituting (3.4) in (3.3), we obtain the Hamiltonian

$$\begin{aligned}H(A,\Pi,\pi)&=\langle\Pi,\Pi\rangle+[I\omega+\langle x\wedge A(x),\rho(x)\rangle]\cdot\omega\\&\quad-\frac{1}{2}\langle\Pi,\Pi\rangle+\frac{1}{2}\langle\operatorname{curl}A,\operatorname{curl}A\rangle-\frac{1}{2}I\omega^2-\langle(\omega\wedge x\rho(x),A(x)\rangle\\&=\frac{1}{2}\langle\Pi,\Pi\rangle+\frac{1}{2}\langle\operatorname{curl}A,\operatorname{curl}A\rangle+\frac{1}{2}I\omega^2+[\langle x\wedge A(x),\rho(x)\rangle]\cdot\omega\\&\quad-\langle\omega\wedge x\rho(x),A(x)\rangle=\frac{1}{2}\langle\Pi,\Pi\rangle+\frac{1}{2}\langle\operatorname{curl}A,\operatorname{curl}A\rangle+\frac{1}{2}I\omega^2\\&=\frac{1}{2}\int[\Pi^2+(\operatorname{curl}A)^2]dx+\frac{1}{2I}[\pi-\langle x\wedge A(x),\rho(x)\rangle]^2,\end{aligned} \tag{3.5}$$

which coincides with (2.15). The Hamiltonian H is well defined and Fréchet differentiable on the Hilbert phase space $\mathbb{Y}$ introduced in (2.13).

3.2. The Structural Operator

Note that

$$D_\pi H=\frac{1}{I}[\pi-\langle x\wedge A(x),\rho(x)\rangle]=\omega, \tag{3.6}$$

by (2.38), and hence, the last equation of the system (2.42) can be written as

$$\dot{\pi}=-\pi\wedge D_\pi H. \tag{3.7}$$

Now it is easy to check that the system (2.42) can be written as

$$\left\{\begin{array}{ll}\dot{A}=D_\Pi H, & \dot{\Pi}=-D_A H\\ \dot{\pi}=-\pi\wedge D_\pi H & \end{array}\right| . \tag{3.8}$$

Obviously, this system admits the representation (3.1) with the skew-symmetric structural operator

$$J(Y)=\begin{pmatrix}0&1&0\\-1&0&0\\0&0&-\pi\wedge\end{pmatrix},\qquad Y=(A,\Pi,\pi). \tag{3.9}$$

Note that the operator $J(Y)$ is not invertible for all $Y=(A,\Pi,\pi)$, and

$$\dim\operatorname{Ker}J(Y)=\left\{\begin{array}{l}1,\ \pi\neq 0\\ 3,\ \pi=0\end{array}\right| . \tag{3.10}$$

Remark 3.1. In the case

$$H = \frac{1}{2}\Big(\frac{\pi_1^2}{I_1} + \frac{\pi_2^2}{I_2} + \frac{\pi_3^2}{I_3}\Big),$$

equation (3.7) coincides with the Euler system describing rotations of the free rigid body with a fixed center of mass [15,33].

3.3. The Energy Conservation

For initial data satisfying (2.16), the conservation of energy (2.15) follows directly from (3.1) since the structural operator $J(Y)$ is skew-symmetric:

$$\frac{d}{dt}H(Y(t)) = \big(DH, \dot{Y}\big) = \big(DH, J(Y)DH\big) = 0, \tag{3.11}$$

where all the expressions are well-defined and the identities hold from (2.17). For arbitrary initial data from $\mathbb{Y}$, the conservation follows from the continuity of the map $W(t)$ in $\mathbb{Y}$ (Proposition 2.2, ii)).

3.4. Casimir Invariants

The system (3.8) admits the functional family of invariants

$$C(Y) = f(|\pi|), \qquad Y = (A, \Pi, \pi), \qquad f \in C^1(\mathbb{R}). \tag{3.12}$$

Such invariants are known as*Casimir invariants*. Their presence is due to the fact that the matrix $J(Y)$ is not invertible. It suffices to prove the conservation of $C(Y)$ for $\pi \neq 0$. In this case $D_\pi C = \frac{\pi}{|\pi|}f'(|\pi|)$, and so

$$\begin{aligned}\partial_t C(Y(t)) &= \big(DC(Y(t)), \dot{Y}(t)\big) = \big(DC(Y(t)), J(Y(t))DH(Y(t))\big)\\ &= -\big(J^*(Y(t))DC(Y(t)), DH(Y(t))\big) = 0\end{aligned} \tag{3.13}$$

since the structural operator $J(Y)$ is skew-symmetric by (3.9), and

$$J^*(Y)DC(Y) = J^*(Y)\begin{pmatrix}0\\0\\\frac{\pi}{|\pi|}f'(|\pi|)\end{pmatrix} = \begin{pmatrix}0\\0\\-\pi\wedge\frac{\pi}{|\pi|}f'(|\pi|)\end{pmatrix} = \begin{pmatrix}0\\0\\0\end{pmatrix}. \tag{3.14}$$

4. The Solitons

The solitons of the system (1.1) are stationary solutions

$$X_\omega = (E_\omega(x), B_\omega(x), \omega), \tag{4.1}$$

where $E_\omega, B_\omega \in L^2$ for solutions with finite energy (2.15). The solitons correspond to stationary solutions

$$S_\omega := (A_\omega(x), \Pi_\omega(x), \pi_\omega) \tag{4.2}$$

of the system (2.42):

$$\left\{\begin{array}{ll}0 = \Pi_\omega, & 0 = \Delta A_\omega + \omega\wedge x\rho(x)\\ 0 = \omega\wedge\pi_\omega & \end{array}\right|. \tag{4.3}$$

The field A_ω can be calculated from the second equation of this system. This equation is easy to solve in the case $|v|<1$: in the Fourier transform

$$\hat{A}_\omega(k)=\frac{-i\omega\wedge\nabla\hat{\rho}(k)}{k^2}. \tag{4.4}$$

Now (2.1) implies that

$$A_\omega\in\dot{F}^1,\qquad \omega\in\mathbb{R}^3. \tag{4.5}$$

The component Π_ω valishes by first equation of (4.3), and so,

$$S_\omega:=(A_\omega(x),0,\pi_\omega),\qquad \pi_\omega=I\omega+\langle x\wedge A_\omega(x),\rho(x)\rangle. \tag{4.6}$$

Now the Maxwell fields (4.1) are expressed by (2.2) and (2.5):

$$E_\omega(x)=-\nabla\Phi(x),\qquad B_\omega(x)=\operatorname{curl}A_\omega(x). \tag{4.7}$$

We still need to check the last equation of the system (4.3), which is equivalent to the relation

$$\pi_\omega\|\omega. \tag{4.8}$$

This relation holds by the next lemma, which is proved in Appendix B.

Lemma 4.1. *For solitons* (4.1),

$$\pi_\omega=I_{\rm eff}\,\omega, \tag{4.9}$$

where $I_{\rm eff}$ *is given by the formula* (1.5).

5. The Stability of Solitons

To prove the stability of a soliton S_ω, we construct the corresponding Lyapunov function $\Lambda_\omega(Y)$, which is an invariant of the system (3.8). The soliton S_ω has to be a strict local minimizer for Λ_ω. In particular, the soliton must be a critical point:

$$D\Lambda_\omega(S_\omega)=0. \tag{5.1}$$

We will see that the Hamiltonian $H(Y)$ does not satisfy this identity. The *energy-Casimir method* [12,14,40] consists in correcting the Hamiltonian by a suitable Casimir invariant (3.12):

$$\Lambda_\omega(Y)=H(Y)+f_\omega(|\pi|),\qquad Y=(A,\Pi,\pi). \tag{5.2}$$

Obviously, such function is an invariant for the system (3.8). Let us show that the identity (5.1) can be satisfied with a suitable choice of the function f_ω. Indeed, the condition (5.1) is equivalent to

$$D_A\Lambda_\omega(S_\omega)=0,\qquad D_\Pi\Lambda_\omega(S_\omega)=0,\qquad D_\pi\Lambda_\omega(S_\omega)=0. \tag{5.3}$$

The first and second identities hold with any choice of f_ω due to the first two equations of (4.3). On the other hand, (3.6) implies that

$$D_\pi H(S_\omega)=\omega. \tag{5.4}$$

Hence, the last identity of (5.3) holds for $f_\omega = 0$ in the case $\omega = 0$. Otherwise, the identity holds if

$$\omega + \frac{\pi_\omega}{|\pi_\omega|} f'_\omega(|\pi_\omega|) = 0. \tag{5.5}$$

where $f_\omega \in C^1(\mathbb{R})$, and $\pi_\omega \neq 0$ for $\omega \neq 0$ by (4.9). The relation (4.9) gives

$$\omega = \lambda \pi_\omega, \qquad \lambda = 1/I_{\rm eff} > 0. \tag{5.6}$$

We choose the function

$$f_\omega(r) = -\lambda |\pi_\omega| r = -|\omega| r, \tag{5.7}$$

for which (5.5) obviously holds. Now (5.2) becomes

$$\Lambda_\omega(Y) = H(Y) - |\omega||\pi|. \tag{5.8}$$

For the proof of stability of the soliton, we need the following lower bound. Denote $\nu = I_{\rm eff}/I$.

Proposition 5.1. *i) Let the condition (2.1) hold. Then there exists a constant $\nu_* > 0$, independent of $\omega \in \mathbb{R}^3$, such that, for $\nu > \nu_*$, the following lower bound holds with a $\varkappa > 0$:*

$$\delta\Lambda_\omega := \Lambda_\omega(S_\omega + \delta Y) - \Lambda_\omega(S_\omega) \geq \varkappa \|\delta Y\|^2_{\mathbb{Y}} \tag{5.9}$$

for $\delta Y \in \mathbb{Y}$ with sufficiently small norm $\|\delta Y\|_{\mathbb{Y}}$.

ii) In the case $\omega = 0$, the above bound holds for any $\nu > 0$.

Proof. Denote $\delta Y = (\alpha, \beta, \gamma)$, so $A = A_\omega + \alpha$, $\Pi = \Pi_\omega + \beta = \beta$, $\pi = \pi_\omega + \gamma$. Note that

$$\text{div}\ \alpha(x) = \text{div}\ \beta(x) = 0. \tag{5.10}$$

We have

$$\begin{aligned}\delta\Lambda_\omega = &\frac{1}{2}\int [|\beta|^2 + |\mathrm{curl}\,(A_\omega + \alpha)|^2]\,dx - \frac{1}{2}\int |\mathrm{curl}\,A_\omega|^2\,dx \\ &+ \frac{1}{2I}[\pi_\omega + \gamma - \langle x\wedge(A_\omega(x) + \alpha(x)), \rho(x)\rangle \\ &- \frac{1}{2I}[\pi_\omega - \langle x\wedge A_\omega(x), \rho(x)\rangle]^2 + f_\omega(|\pi_\omega + \gamma|) - f_\omega(|\pi_\omega|).\end{aligned} \tag{5.11}$$

After rearrangements, we obtain

$$\begin{aligned}\delta\Lambda_\omega = &\frac{1}{2}\int (|\beta|^2 + |\mathrm{curl}\,\alpha|^2)\,dx + \int \mathrm{curl}\,A_\omega(x)\cdot \mathrm{curl}\,\alpha(x)\,dx \\ &+ \frac{1}{2I}[(M_\omega + \delta M)^2 - M_\omega^2] \\ &+ f_\omega(|\pi_\omega + \gamma|) - f_\omega(|\pi_\omega|),\end{aligned} \tag{5.12}$$

where

$$M_\omega := \pi_\omega - \langle x\wedge A_\omega(x), \rho(x)\rangle = I\omega, \qquad \delta M := \gamma - \langle x\wedge\alpha(x), \rho(x)\rangle. \tag{5.13}$$

Using the second equation of (4.3) and (4.3), and taking into account (5.10), we obtain

$$\int \operatorname{curl} A_\omega(x)\cdot\operatorname{curl}\alpha(x)\,dx = -\big\langle \Delta A_\omega(x),\alpha(x)\big\rangle$$
$$= \big\langle x\wedge\omega\rho(x),\alpha(x)\big\rangle = -\omega\cdot\delta M+\omega\cdot\gamma. \tag{5.14}$$

Substituting into (5.12), we get

$$\delta\Lambda_\omega = \frac12\int(|\beta|^2+|\nabla\alpha|^2)\,dx+\frac{1}{2I}[(M_\omega+\delta M)^2-M_\omega^2]-\omega\cdot\delta M$$
$$+\omega\cdot\gamma+f_\omega(|\pi_\omega+\gamma|)-f_\omega(|\pi_\omega|). \tag{5.15}$$

Remark 5.2. The presence of the term $\omega\cdot\gamma$ corresponds to (5.4).

Now the bound below (5.9) follows from (5.15) by the following arguments.
I. The first line of (5.15) reads as

$$J_1 := \frac12\int(|\beta|^2+|\nabla\alpha|^2)\,dx+\frac{I}{2}(\delta M/I)^2$$
$$= \frac12\int(|\beta|^2+|\nabla\alpha|^2)\,dx+\frac{1}{2I}[\gamma-\big\langle x\wedge\alpha(x),\rho(x)\big\rangle]^2. \tag{5.16}$$

Lemma 5.3. *Let the conditions of Proposition 5.1 hold. Then, for small $\varkappa>0$*

$$\int|\nabla\alpha|^2\,dx+[\gamma-\big\langle x\wedge\alpha(x),\rho(x)\big\rangle]^2\ge\varkappa(\|\nabla\alpha\|_{L^2}^2+\gamma^2), \tag{5.17}$$

Proof. Denote $\mu:=\gamma-\big\langle x\wedge\alpha(x),\rho(x)\big\rangle$. Then the bound (5.17) can be rewritten equivalently as

$$\|\nabla\alpha\|_{L^2}^2+[\mu+\langle x\wedge\alpha(x),\rho(x)\rangle]^2\le C\left(\int|\nabla\alpha(x)|^2\,dx+\mu^2\right). \tag{5.18}$$

It remains to note that

$$|\big\langle x\wedge\alpha(x),\rho(x)\big\rangle|\le C(\rho)\|\alpha\|_{L^6}\le C_1(\rho)\|\nabla\alpha\|_{L^2} \tag{5.19}$$

by the Sobolev embedding theorem. □

This lemma and (5.16) imply the bound

$$J_1\ge\varkappa_1(I)\|\delta Y\|_{\mathbb{Y}}, \tag{5.20}$$

where $\varkappa_1(I)>0$.

II. The second line of (5.15) for $\omega\ne0$ (and $\pi_\omega\ne0$) reads as

$$\omega\cdot\gamma+f_\omega(|\pi_\omega+\gamma|)-f_\omega(|\pi_\omega|)=\omega\cdot\gamma-\lambda|\pi_\omega|(|\pi_\omega+\gamma|)-|\pi_\omega|)$$
$$=-\frac{\lambda}{2}\Big[\gamma^2-\Big(\frac{\pi_\omega}{|\pi_\omega|}\cdot\gamma\Big)^2\Big]+O(|\gamma|^3)\ge-\frac{\lambda}{2}\gamma^2+O(|\gamma|^3),\quad|\gamma|\to0. \tag{5.21}$$

Note that this expression can be negative for small $|\gamma|$ since $\lambda > 0$ by (5.6). The expression follows from the Taylor expansion

$$\begin{aligned} |\pi_\omega + \gamma| &= \sqrt{\pi_\omega^2 + 2\pi_\omega \cdot \gamma + \gamma^2} \\ &= |\pi_\omega| + \frac{\pi_\omega \cdot \gamma}{|\pi_\omega|} + \frac{1}{2|\pi_\omega|}\Big[\gamma^2 - \Big(\frac{\pi_\omega}{|\pi_\omega|} \cdot \gamma\Big)^2\Big] + O(|\gamma|^3). \end{aligned} \tag{5.22}$$

In the case $\omega = 0$, the bound (5.20) immediately implies (5.9). It remains to consider the case $\omega \neq 0$. Then the second line of (5.15) is estimated below by (5.21), where $\lambda \to 0$ as $I_{\text{eff}} \to \infty$, in accordance with (5.6). Hence, the bound below (5.9) follows from (5.20) for sufficiently large $\nu(v)$. □

Remark 5.4. The arguments above demonstrate that the case $\omega = 0$ formally corresponds to $\lambda = 0$, i.e., to the limit $I_{\text{eff}} \to \infty$. This correspondence can be clarified by the reformulation of the condition (1.6) as follows:

$$\lambda = \frac{1}{I_{\text{eff}}} = \frac{|\omega|}{|\pi_\omega|} \ll 1, \tag{5.23}$$

which automatically includes the case $\omega = 0$ and $I_{\text{eff}} = \infty$.

Proposition 5.1 implies the following theorem which is the main result of this paper.

Theorem 5.5. *Let the condition (2.1) hold. Then*

i) there exists a constant $\nu_* > 0$, independent of $\omega \in \mathbb{R}^3$ such that, for $\nu > \nu_*$, the solitons S_ω with all $\omega \in \mathbb{R}^3$ are stable;
ii) In the case $\omega = 0$, the soliton S_0 is stable for any $\nu > 0$.

It is easy to reformulate this theorem in terms of the Maxwell fields (4.1). Namely, the stability of the soliton S_ω in the Hilbert phase space $\mathbb{Y}$ is equivalent to the stability of the soliton X_ω in the Hilbert phase space

$$\mathbb{X} := L^2 \oplus L^2 \oplus \mathbb{R}^3. \tag{5.24}$$

Appendix A. Lie–Poincaré Calculus

The formula (2.32) follows, e.g., by method [15, p. 219]. Namely, denote $r(t) = \delta R(t)$, so $\hat{\Sigma} = rR^{-1}$. Differentiating $\hat{\omega} = \dot{R}R^{-1}$, we obtain

$$\delta\hat{\omega} = \dot{r}R^{-1} - \dot{R}R^{-1}rR^{-1} = \dot{r}R^{-1} - \hat{\omega}\hat{\Sigma}. \tag{A.1}$$

Similarly,

$$\partial_t\hat{\Sigma} = \dot{r}R^{-1} - rR^{-1}\dot{R}R^{-1}. \tag{A.2}$$

Hence,

$$\dot{r}R^{-1} = \partial_t\hat{\Sigma} + rR^{-1}\dot{R}R^{-1} = \partial_t\hat{\Sigma} + \hat{\Sigma}\hat{\omega}. \tag{A.3}$$

Finally, substituting the last two expressions into (A.1), we get

$$\delta\hat{\omega} = \partial_t\hat{\Sigma} + \hat{\Sigma}\hat{\omega} - \hat{\omega}\hat{\Sigma} = \partial_t\hat{\Sigma} + [\hat{\Sigma}, \hat{\omega}]. \tag{A.4}$$

Appendix B. Effective Moment of Inertia of Solitons

Here we prove Lemma 4.1. Substituting (4.4) into (2.38), we obtain

$$\begin{aligned}\pi_\omega &= I\omega + \left\langle (-i\nabla)\wedge\frac{(-i\omega\wedge\nabla)\hat{\rho}(k)}{k^2}, \hat{\rho}(k)\right\rangle = I\omega - \left\langle \frac{\omega\wedge\nabla\hat{\rho}(k)}{k^2}\wedge\nabla\hat{\rho}(k)\right\rangle \\ &= I\omega - \left\langle \frac{\omega\wedge\frac{k}{|k|}}{k^2}\wedge\frac{k}{|k|}|\nabla_{|k|}\hat{\rho}(k)|^2\right\rangle.\end{aligned} \tag{B.5}$$

We may assume that $\omega = (|\omega|, 0, 0)$. Then

$$\omega\wedge k = (0, -|\omega|k_3, |\omega|k_2), \quad (\omega\wedge k)\wedge k = (-|\omega|k_3^2 - |\omega|k_2^2, |\omega|k_1k_2, |\omega|k_1k_3). \tag{B.6}$$

Substituting into (B.5), and using the antisymmetry in k_2 and k_3, we obtain

$$\pi_\omega = I_{\text{eff}}\omega, \qquad I_{\text{eff}} = I + \left\langle \frac{k_3^2 + k_2^2}{|k|^4}|\nabla_{|k|}\hat{\rho}(k)|^2\right\rangle = I + \frac{2}{3}\left\langle \frac{|\nabla\hat{\rho}(k)|^2}{k^2}\right\rangle. \tag{B.7}$$

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