Conway's Light on the Shadow of Mordell

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The celebrated Markov triples are positive integer solutions of the Markov equation

$$x^2 + y^2 + z^2 = 3xyz.$$

Andrey Andreyevich Markov (1856–1922) [6] showed that all such solutions can be found recursively from (1, 1, 1) using a natural action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on the solutions of this equation generated by the cyclic permutation of variables and the Vieta involution $(x, y, z) \mapsto (x, y, 3xy - z)$.

Markov introduced his triples in 1879 in relation to the Diophantine properties of the binary quadratic forms that can be interpreted as the description of the "most irrational" numbers. Since then, many surprising connections have been discovered between Markov triples and hyperbolic geometry, combinatorics of words, and, more recently, the theory of Frobenius manifolds, algebraic geometry, and cluster mutations (see [1, 2] and the references therein). This makes the Markovian theme one of the most fascinating stories in mathematics, which because of a still unproven uniqueness conjecture [1] is far from being finished.

Recently, Valentin Ovsienko [8], motivated by his attempts to understand possible super analogues of cluster algebras [9–11], introduced the so-called shadow version of Markov triples by considering the solutions of a certain version of the Markov equation over dual numbers $X = a + b\varepsilon$, $\varepsilon^2 = 0$.

The dual numbers were introduced in 1873 by William Clifford (1845–1879) and were used by Eduard Study (1862–1930) to describe the relative positions of two skew lines in space. They are the simplest case of the Grassmann numbers, which are used to describe the fermionic fields in modern quantum theory.

In fact, Ovsienko introduced the shadow versions of other sequences of integers, including Fibonacci numbers and (with Serge Tabachnikov) Somos-4 sequences [8, 11]. Recently, Andrew Hone [4] extended this study to a more general Somos-4 recurrence relation over dual numbers.

As always with a new development, a valid question is whether the shadow dynamics are interesting enough to deserve a detailed study. The aim of this note is to add few more arguments to those of Ovsienko and Hone in favor of a positive answer to this question.

Namely, we will study the shadow Mordell triples satisfying the Mordell equation

$$X^2 + Y^2 + Z^2 = 2XYZ + 1$$

with X, Y, Z dual integers. As was shown by Louis Mordell [7], this equation can be viewed as a "solvable" modification

This column is a place for those bits of contagious mathematics that travel from person to person in the community because they are so elegant, surprising, or appealing that one has an urge to pass them on. Contributions are most welcome. Submissions should be uploaded at the Intelligencer's website https://www.springer.com/journal/283 by clicking on "Submit manuscript" or sent directly to Sophie Morier-Genoud (sophie.morier-genoud@imj-prg.fr) or Valentin Ovsienko (valentin.ovsienko@univ-reims.fr).

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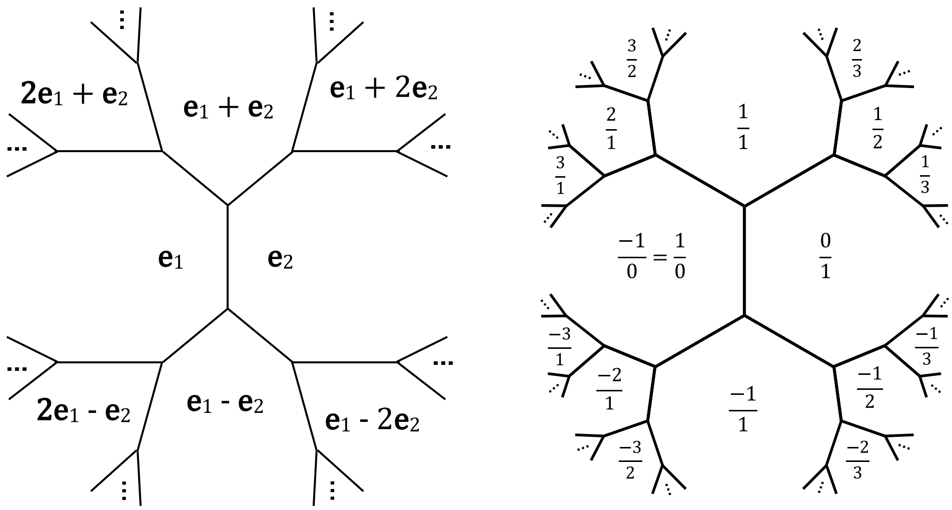


(a) (b)

Figure 1. (a) The superbase topograph and (b) the Farey tree.

of Markov's equation. This fact was used by Don Zagier [15] to study the growth of the Markov numbers.

Note that as in the case of Markov's equation, we have a natural action of the modular group $\mathrm{PSL}_2(\mathbb{Z})$ on the solutions of the Mordell equation. Remarkably, the shadow orbits of the solution (1, 1, 1), which is stationary for modular dynamics, are none other than the values of binary quadratic forms and thus can be visualized using the original Conway topograph [3].

The shadows of other Mordell triples can be written explicitly in terms of the solutions of Pell's equations, following Mordell [7]. We will discuss also the shadow growth along the paths in the Conway topograph, which can be described in terms of the Lyapunov function of the Euclid tree [12].

The Conway Topograph and the Modular Group

The values of binary quadratic forms on the integer lattice $\mathbb{Z}^2$ is one of the most classical objects of study in number theory, going back to Fermat, with later contributions from Euler, Gauss, and Jacobi. A famous question asks which integers n can be represented as the sum of two squares,

$$n = x^2 + y^2, \quad x, y \in \mathbb{Z},$$

and in how many ways. The answer is given by the Jacobi formula for the number $N(n)$ of such representations:

$$N(n) = 4(N_1(n) - N_3(n)),$$

where $N_1(n)$ and $N_3(n)$ are the numbers of the divisors of n with residues 1 and 3 modulo 4, respectively (see, e.g., [5]).

In the 1990s, this very classical story had a very interesting twist due to the intervention of John H. Conway. In his book *The Sensual (Quadratic) Form* [3], Conway described the following "topographic" way to "visualize" these values. He introduced the notions of the lax vector as a pair $(\pm v)$, $v \in \mathbb{Z}^2$, and the superbase of the integer lattice $\mathbb{Z}^2$ as a triple of lax vectors $(\pm e_1, \pm e_2, \pm e_3)$ such that (e_1, e_2) is a basis of the lattice and

$$e_1 + e_2 + e_3 = 0.$$

Every basis (e_1, e_2) can be included in exactly two superbases, namely $\pm(e_1, e_2, -e_1 - e_2)$ and $\pm(e_1, -e_2, -e_1 + e_2)$, so that the corresponding set of the superbases can be described using the binary tree embedded in the plane. The connected components of the complement to the tree (domains) are labeled by the primitive (that is, having coprime coordinates) lax vectors and the edges by the lax bases, while the superbases correspond to the vertices of the tree (see Figure 1a, where we have shown only one representative of the lax vectors). The projective version of the Conway superbase topograph is known also as the Farey tree, since it is related to Farey "addition"

$$\frac{a}{b} * \frac{c}{d} = \frac{a+c}{b+d}$$

(see Figure 1).

Let $Q(x, y) = ax^2 + hxy + by^2$ be a binary quadratic form (in Conway's notation) with $x, y \in \mathbb{Z}$; the coefficients a, b, h need not be integers. By taking the values of the form Q on the vectors of the superbase, we get what Conway called the topograph of Q, containing the values of Q on all primitive lattice vectors. In particular, if $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$, $\mathbf{e}_3 = -(1, 1)$, we have the values $Q(\mathbf{e}_1) = a$, $Q(\mathbf{e}_2) = b$, $Q(\mathbf{e}_3) = c := a + b + h$.

Conway's key idea[1] is that one can construct the topograph of Q *recursively*, starting from this triple and using

[1]Conway clearly believed that this is an important idea, claiming in the introduction to his lectures [3] that "the 'topograph' of the First Lecture is new."

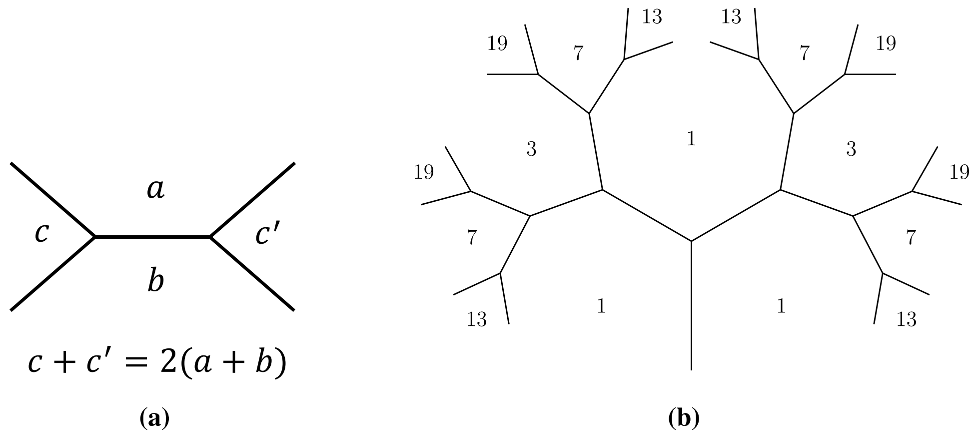


Figure 2. (a) The arithmetic progression rule; (b) the Conway topograph of the quadratic form $Q = x^2 + xy + y^2$.

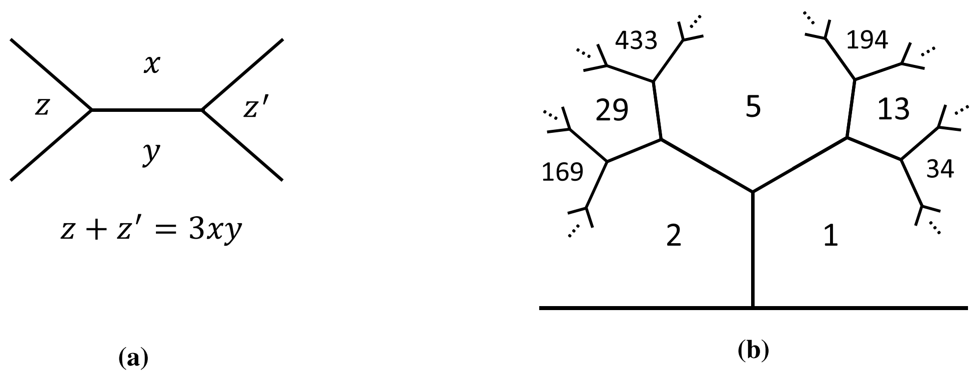


Figure 3. (a) Vieta involution and (b) Conway topograph of Markov triples.

the following property of any quadratic form, which Conway called the arithmetic progression rule (known also in geometry as the parallelogram law):

$$Q(\mathbf{u}+\mathbf{v}) + Q(\mathbf{u}-\mathbf{v}) = 2(Q(\mathbf{u}) + Q(\mathbf{v})), \quad \mathbf{u}, \mathbf{v} \in \mathbb{R}^2. \quad (1)$$

In fact, the quadratic forms (in any dimension) can be characterized as the continuous functions Q satisfying relation (1). In particular, for $Q(x, y) = x^2 + xy + y^2$, $(x, y) \in \mathbb{Z}^2$, describing the square lengths of vectors in a regular hexagonal lattice, we have the Conway topograph shown in Figure 2.

As Conway nicely explained, the modular group $PSL_2(\mathbb{Z})$ can be viewed as the natural symmetry group of his topograph. Indeed, it is well known that $PSL(2, \mathbb{Z}) = \mathbb{Z}_2 * \mathbb{Z}_3$ is generated by

$$U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad V = \begin{pmatrix} 0 & -1 \\ 1 & 1 \end{pmatrix},$$

where U acts on the topograph combinatorially by rotation by π about the edge center, and V acts by rotation by $2\pi/3$ about the vertex. Thus, within Klein's Erlangen program, Conway's topograph can be viewed as a discrete version of the hyperbolic plane with its isometry group $PSL_2(\mathbb{R})$ replaced by $PSL_2(\mathbb{Z})$.

This means that Conway's topograph can be used to describe any $PSL_2(\mathbb{Z})$ dynamics. In particular, we have a natural action of $PSL_2(\mathbb{Z})$ on the solutions of Markov's equation $x^2 + y^2 + z^2 = 3xyz$, generated by the Vieta involution and cyclic permutation of the variables. As Markov showed in [6], all integer solutions of Markov's equation form one orbit of $PSL_2(\mathbb{Z})$ acting on $(1, 1, 1)$. A part of the corresponding Conway topograph of the Markov triples is shown in Figure 3 with the local Vieta rule generating them.

Shadow Markov and Mordell triples

Let

$$\mathbb{D}(\mathbb{Z}) = \{a + b\varepsilon,\ a, b \in \mathbb{Z},\ \varepsilon^2 = 0\}$$

be the commutative ring of dual integers. By analogy with the Gaussian integers $a + bi$, $i^2 = -1$, $a, b \in \mathbb{Z}$, we will call them Clifford integers. The invertible elements (units) in $\mathbb{D}(\mathbb{Z})$ have the form $\pm 1 + b\varepsilon$ for $b \in \mathbb{Z}$.

Ovsienko [8] proposed the shadow version of Markov triples as the solutions over Clifford integers of the following version of the Markov equation:

$$X^2 + Y^2 + Z^2 = (3 - 2\varepsilon)XYZ, \quad X, Y, Z \in \mathbb{D}(\mathbb{Z}), \quad (2)$$

as the $PSL_2(\mathbb{Z})$-orbit of the initial triple of units $X = 1$, $Y = Z = 1 + \varepsilon$ (for the motivation for this choice, see [8]).

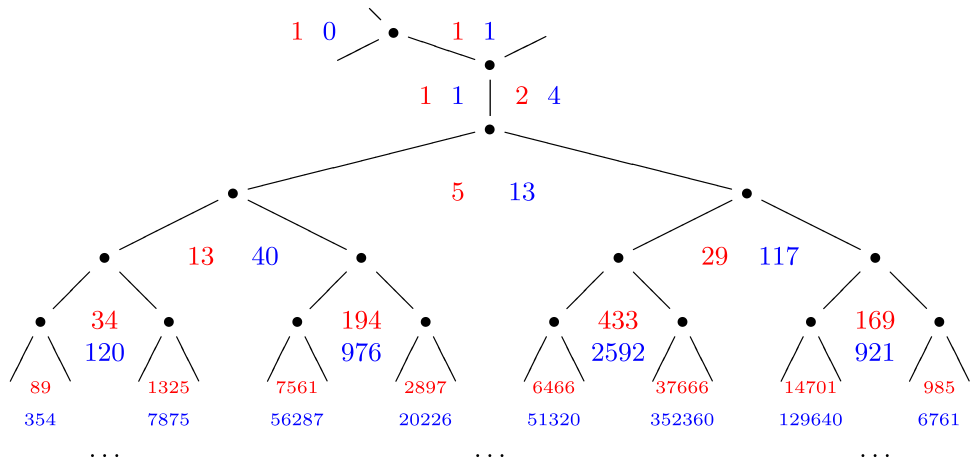


Figure 4. The shadow Markov tree (from [8]).

Ovsienko observed that the shadow companion of the very left (Fibonacci) branch of the Markov tree

$$1, 4, 13, 40, 120, 354, 1031, 2972, 8495, \ldots$$

is the known sequence A238846, which is the convolution of two bisections of the Fibonacci sequence, and he asked for a possible meaning of the other shadows (see Figure 4). This is a very interesting question, which is still largely open.

We will discuss the simpler Diophantine equation

$$x^2 + y^2 + z^2 = 2xyz + 1\,, \tag{3}$$

which was studied by Mordell [7] and can be viewed as a "solvable" modification of Markov's equation.[2] Recently, it was realized that it is closely related to the two-valued formal group in complex K-theory studied by Buchstaber and Novikov (see [2]).

Mordell showed that this equation has integer solutions of two types:

$$\begin{aligned} &\mathrm{I}: && x = 1,\ y = z, \\ &\mathrm{II}: && 2x = \xi^a + \eta^a,\ 2y = \xi^b + \eta^b,\ 2z = \xi^c + \eta^c, \end{aligned}$$

where $a, b, c \in \mathbb{Z}$ satisfy $a + b = c$, and $\xi = p + q\sqrt{d}$, $\eta = p - q\sqrt{d}$, and p, q are solutions of Pell's equation

$$p^2 - dq^2 = 1\,,$$

where $d \in \mathbb{N}$ is not a perfect square. It is well known that all positive integer solutions of this classical equation (mistakenly ascribed to Pell by Euler) can be given by

$$p + q\sqrt{d} = \left(p_0 + q_0\sqrt{d}\right)^k, \quad k \in \mathbb{N},$$

where (p_0, q_0) is the minimal positive (fundamental) solution, which can be found from the continued fraction expansion of $\sqrt{d}$ (see, e.g., [5]). One can also use Conway's topograph to describe the solutions of Pell's equation, as is nicely explained by Weissman in [14].

In particular, when $d = 2$, the fundamental solution of Pell's equation $p^2 - 2q^2 = 1$ is $p_0 = 3$, $q_0 = 2$, which leads to the Mordell triples of type II with $\xi = 3 + 2\sqrt{2}$, $\eta = 3 - 2\sqrt{2}$. As in the Markov case, we have the action of $\mathrm{PSL}_2(\mathbb{Z})$ generated by the Vieta involution

$$(x, y, z) \mapsto (x, y, 2xy - z)$$

and by cyclic permutations of the variables. However, in contrast to the Markov case, we see that $\mathrm{PSL}_2(\mathbb{Z})$ has infinitely many orbits here. In particular, we have a special type-I orbit consisting of the single point $(1, 1, 1)$ fixed under the action of $\mathrm{PSL}_2(\mathbb{Z})$.

Let us look at the shadows $X = 1 + a\varepsilon$, $Y = 1 + b\varepsilon$, $Z = 1 + c\varepsilon$ of this special orbit. The corresponding orbit on the Conway topograph satisfies the Vieta rule $Z + Z' = 2XY$, giving

$$c + c' = 2(a + b)\,,$$

which is none other than Conway's arithmetic progression rule! Thus we have the following nice surprise.

Theorem 1. *Shadow* $\mathrm{PSL}_2(\mathbb{Z})$*-orbits of the special Mordell triple* $(1, 1, 1)$ *can be naturally visualized by Conway topographs of the values of binary quadratic forms.*

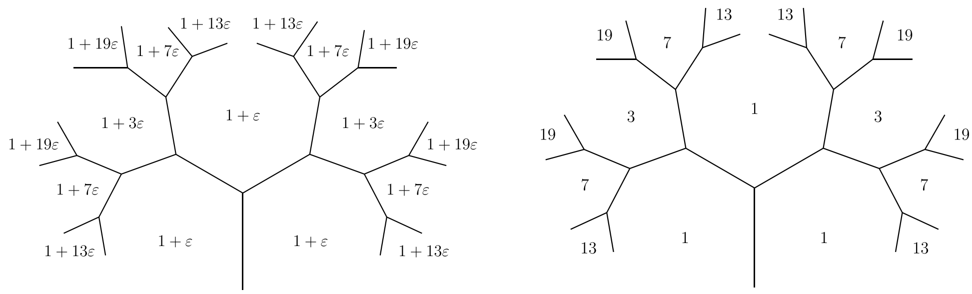
Note that this is true for any coefficients of the binary quadratic forms and any Mordell triples, not necessarily integers (Figure 5).

Before considering the shadows of other Mordell triples, let us look at the underlying geometry. Geometrically, the Mordell equation (3) determines a particular affine realization of the classical Cayley's nodal cubic surface with the maximal number (which is 4) of conical singularities at the points $(1, 1, 1), (1, -1, -1), (-1, 1, -1), (-1, -1, 1)$ (see the real version in Figure 6).

The following observation, going back to Mordell, allows one to linearize the Mordell equation.

Theorem 2. [7] *The real Mordell triples* (x, y, z) *with* $1 \le x, y \le z$ *can be parametrized as*

[2]It is very interesting to read Mordell's comments in [7, p. 505] about what solvability could mean for Diophantine equations.



(a) **(b)**

Figure 5. (a) Shadow of the special orbit (1, 1, 1); (b) the Conway topograph of $Q = x^2 + xy + y^2$.

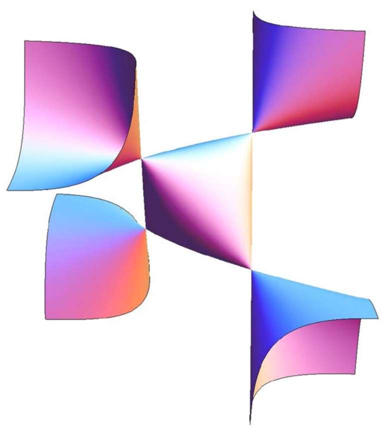


Figure 6. A real Cayley–Mordell surface.

$$x = \cosh u, \;\; y = \cosh v, \;\; z = \cosh w \tag{4}$$

with real $u, v, w \geq 0$ satisfying the Euclid relation $w = u + v$.

Now we can use the fact that any analytic relation can be extended to the dual numbers using the formula

$$f(a + b\varepsilon) = f(a) + bf'(a)\varepsilon\,,$$

since $\varepsilon^2 = 0$. This leads us to the following shadow Mordell triples:

$$\begin{aligned} X &= \cosh u + \alpha\varepsilon \sinh u, \\ Y &= \cosh v + \beta\varepsilon \sinh v, \\ Z &= \cosh w + \gamma\varepsilon \sinh w, \end{aligned} \tag{5}$$

where the parameters satisfy the Euclid relation

$$w = u + v, \quad \gamma = \alpha + \beta. \tag{6}$$

Theorem 3. *The formulas (5), (6) describe all solutions of the Mordell equation*

$$X^2 + Y^2 + Z^2 = 2XYZ + 1 \tag{7}$$

over real dual numbers having real parts $1 \leq x,\ y \leq z$, with the exception of the special triple $x = y = z = 1$, for which the shadows were described above.

First, note that $X = x + \tilde{x}\varepsilon$, $Y = y + \tilde{y}\varepsilon$, $Z = z + \tilde{z}\varepsilon$ satisfy (7) iff

$$x^2 + y^2 + z^2 = 2xyz + 1$$

and

$$(x - yz)\tilde{x} + (y - xz)\tilde{y} + (z - xy)\tilde{z} = 0. \tag{8}$$

Let $X = \cosh u + \tilde{x}\varepsilon$, $Y = \cosh v + \tilde{y}\varepsilon$, $Z = \cosh w + \tilde{z}\varepsilon$ be a shadow of the triple (4) with $w = u + v$. Then using the addition formula for cosh, we can rewrite (8) as

$$\sinh v \sinh w \tilde{x} + \sinh u \sinh w \tilde{y} - \sinh u \sinh v \tilde{z} = 0.$$

Dividing this equation by $\sinh u \sinh v \sinh w$, assumed to be nonzero, we see that the parameters $\alpha = \tilde{x}/\sinh u$, $\beta = \tilde{y}/\sinh v$, $\gamma = \tilde{z}/\sinh w$ satisfy the relation $\gamma = \alpha + \beta$. This means that if none of x, y, z is 1, then the corresponding shadows indeed have the form (5), (6). For the Mordell triples $(1, y, y)$ with $y \neq 1$, the shadows have the form $(1, y + \tilde{y}\varepsilon, y + \tilde{z}\varepsilon)$ with arbitrary $\tilde{y}, \tilde{z}$, which agrees with (5), (6), since α could be arbitrary because $\sinh u = 0$. The case of the special triple $(1, 1, 1)$ is the only exceptional one here, but it was discussed above.

Thus, the shadows of the Mordell triples (5) on the Conway topograph are determined by the pairs of Euclid's triples (6). In particular, we have a natural choice when these triples are proportional:

$$\alpha = \mu u, \;\; \beta = \mu v, \;\; \gamma = \mu w, \quad u \pm v \pm w = 0\,,$$

leading to the following shadows, which we will call principal:

$$\begin{aligned} X &= \cosh(u + \alpha\varepsilon) = \cosh u + \mu u\varepsilon \sinh u\,, \\ Y &= \cosh(v + \beta\varepsilon) = \cosh v + \mu v\varepsilon \sinh v\,, \\ Z &= \cosh(w + \gamma\varepsilon) = \cosh w + \mu w\varepsilon \sinh w\,. \end{aligned}$$

Let us return now to the integer Mordell triples of type II,

$$x = \frac{1}{2}(\xi^a + \eta^a), \;\; y = \frac{1}{2}\left(\xi^b + \eta^b\right), \;\; z = \frac{1}{2}(\xi^c + \eta^c), \tag{9}$$

where the integers a, b, c satisfy $a + b = c$ and $\xi = p + q\sqrt{d}$, $\eta = p - q\sqrt{d}$, with p, q a solution of Pell's equation $p^2 - dq^2 = 1$.

Theorem 4. *The following formulas provide an explicit form of the principal integer shadows of Mordell triples (9):*

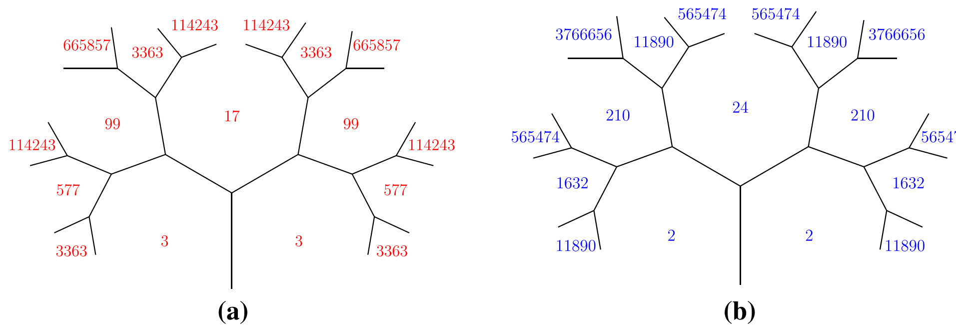


Figure 7. (a) A Mordell tree; (b) its shadow.

$$\begin{aligned}\tilde{x} &= \frac{1}{2\sqrt{d}} ma(\xi^a - \eta^a)\,,\\ \tilde{y} &= \frac{1}{2\sqrt{d}} mb(\xi^b - \eta^b)\,,\\ \tilde{z} &= \frac{1}{2\sqrt{d}} mc(\xi^c - \eta^c)\,,\end{aligned} \tag{10}$$

where $m, a, b, c \in \mathbb{Z}$, $a + b = c$.

Indeed, since $\xi\eta = (p + q\sqrt{d})(p - q\sqrt{d}) = p^2 - dq^2 = 1$, the equality $\cosh u = \frac{1}{2}(\xi^a + \eta^a)$ implies that $u = \pm a \log \xi$ and $\sinh u = \pm\frac{1}{2}(\xi^a - \eta^a)$. Now we have only to choose the values of the parameter μ in the formula for principal shadows to make them integers, which leads to formula (10). In particular, for Pell's equation $p^2 - 2q^2 = 1$ with $p = 3$, $q = 2$, $m = 1$, we have the Conway topographs of the corresponding Mordell triples and their shadows shown in Figure 7.

Note that in contrast to Markov's equation, where we have only the local Vieta rule on the topograph to generate the solutions, in the Mordell case we have also the explicit formula (10) for the Mordell triples and their shadows in terms of the corresponding Euclid triples (cf. Mordell's comments in [7]).

Shadow Growth on the Conway Topograph

As we can see in Figure 7, in the Conway topograph the shadows grow faster than the Mordell triples. Let us look at their growth in more detail.

Consider the corresponding Euclid tree of triples $a, b, c \in \mathbb{Z}$ with $a + b = c$ parametrizing the Mordell triples (9). Note that $2x(a) := \xi^a + \eta^a = 2x(-a)$, so we can restrict ourselves to positive a, b, c (see Figure 8). Traversing this tree can be viewed as an application of the classical Euclidean algorithm for finding the greatest common divisor of a and b (assumed here to be 1).

Theorem 5. *On the Conway topograph, the growth of the Mordell numbers and their principal shadows (10) are related asymptotically for large a by*

$$\tilde{x}(a) \sim Cax(a), \quad C = m/\sqrt{d},$$

where a is the corresponding entry in the Euclid tree. In other words, asymptotically, we have the shadow growth

$$\tilde{x} \sim cx \ln x, \quad c = m/\sqrt{d} \ln \xi.$$

Indeed, we know that $\xi = p + q\sqrt{d} > 1$, $\eta = p - q\sqrt{d} < 1$, since $\xi\eta = p^2 - dq^2 = 1$. This implies that for large positive a, we asymptotically have $2x(a) = \xi^a + \eta^a \sim \xi^a$, and thus in the Conway topograph the shadows grow as

$$2\tilde{x}(a) = ma(\xi^a - \eta^a)/\sqrt{d} \sim ma\xi^a/\sqrt{d} \sim Cax(a).$$

Here we have used the standard notation $f(a) \sim g(a)$ when $\lim_{a\to\infty} f(a)/g(a) = 1$.

Let us look now more closely at the growth along the paths on the Conway topograph. Using a Farey tree, we can label the directed infinite paths by $\xi \in \mathbb{R}P^1$, where ξ is the limit of Farey fractions along the path (see Figure 9).

Following [12], we define the Lyapunov function $\Lambda(\xi)$ describing the growth of the Euclid triples

$$(a_n(\xi), b_n(\xi), c_n(\xi))$$

along the path $\gamma(\xi)$:

$$\Lambda(\xi) := \limsup_{n\to\infty} \frac{\ln |a_n(\xi)|}{n} \tag{11}$$

(here $a_n(\xi)$ can be replaced by $b_n(\xi)$ or $c_n(\xi)$).

Theorem 6. ([12]) *The function $\Lambda(\xi)$ has the following properties*:

- $\Lambda(\xi)$ is defined for all $\xi \in \mathbb{R}P^1$ and is $\mathrm{GL}_2(\mathbb{Z})$-invariant:
$$\Lambda\left(\frac{a\xi + b}{c\xi + d}\right) = \Lambda(\xi), \quad \xi \in \mathbb{R}P^1, \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z})\,.$$
- $\Lambda(\xi)$ vanishes almost everywhere, but the Hausdorff dimension of its support is 1.
- The Lyapunov spectrum $\mathrm{Spec}_E := \{\Lambda(\xi),\ \xi \in \mathbb{R}P^1\}$ of the Euclid tree is
$$\mathrm{Spec}_E = [0, \ln \varphi],$$
where $\varphi = \left(1 + \sqrt{5}\right)/2$ is the golden ratio.

Using this function, we can describe the growth of both Mordell triples and their shadows.

The special orbit (1, 1, 1) is stationary for the modular dynamics, so it has zero growth, but as we have seen, the shadows are the values of binary quadratic forms. Their growth in the Conway topograph was studied by Spalding and the author in [13], which leads to the following result.

Consider the growth in the Conway topograph of the $PSL_2(\mathbb{Z})$-orbit of $X = 1 + a\varepsilon$, $Y = 1 + b\varepsilon$, $Z = 1 + c\varepsilon$. Let

$$Q(x, y) = ax^2 + (c - a - b)xy + by^2$$

be the corresponding binary quadratic form and

$$D(a, b, c) = a^2 + b^2 + c^2 - 2ab - 2ac - 2bc$$

its discriminant, which we assume for simplicity to be nonzero. When $D > 0$, the quadratic form is indefinite; otherwise, it is either positive or negative definite.[3] As a corollary of the results of [13], we have the following result.

Theorem 7. *When $D(a, b, c) < 0$, the growth of the shadows of the special Mordell triple (1, 1, 1) along the path $\gamma(\xi)$ satisfies*

$$\limsup_{n\to\infty} \frac{1}{n} \ln |a_n(\xi)| = 2\Lambda(\xi)\,.$$

When $D(a, b, c) > 0$, the same is true except for $\xi = \alpha, \bar{\alpha}$, which are the roots of the quadratic equation $Q(\xi, 1) = 0$ defining the ends of the corresponding Conway river, where the growth is zero.

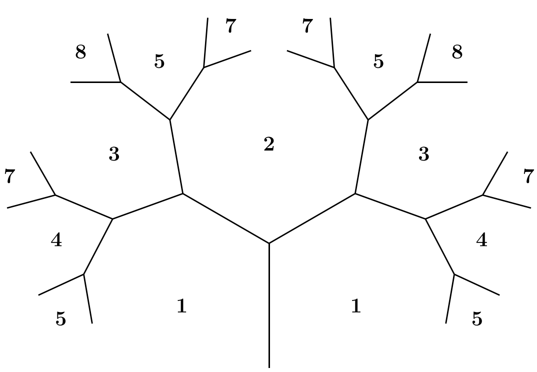


Figure 8. The Euclid tree.

Recall that the Conway river [3] is the infinite path separating the positive and negative values of the indefinite binary quadratic forms in the Conway topograph (see Figure 10, and for more details, [13]).

For the type-II Mordell triples, we can describe the relative growth of the shadows along the path $\gamma(\xi)$. Let $x_n(\xi)$ and $\tilde{x}_n(\xi)$ be the Mordell numbers (9) and their shadows (10) along the path $\gamma(\xi)$.

Theorem 8. *The relative growth of the principal shadows $\tilde{x}_n(\xi)$ of the type-II Mordell numbers $x_n(\xi)$ along the path $\gamma(\xi)$ can be described by*

$$\limsup_{n\to\infty} \frac{1}{n} \ln \frac{\tilde{x}_n(\xi)}{x_n(\xi)} = \Lambda(\xi)\,.$$

Indeed, from the explicit formulas (9), (10), it follows that

$$\limsup_{n\to\infty} \frac{1}{n} \ln \frac{\tilde{x}_n(\xi)}{x_n(\xi)} = \limsup_{n\to\infty} \frac{\ln a_n(\xi)}{n} = \Lambda(\xi)\,.$$

Note that the condition on the shadows being principal can be removed here. For the generic shadows of the orbit (1, 1, 1), this limit is $2\Lambda(\xi)$. I believe that similar results about shadow growth hold also for Ovsienko's shadow Markov numbers, but that is still to be established.

Another natural research direction is to study similar questions for the elliptic version of the Mordell equation

$$X^2 + Y^2 + Z^2 = 1 + 2XYZ - k^2\left(1 - X^2\right)\left(1 - Y^2\right)\left(1 - Z^2\right), \tag{12}$$

corresponding to the addition formula for the Jacobi elliptic function $cn(u, k)$ (see [2]). As explained in [2], these equations have significant meaning in topology: the Mordell equation is closely related to the formal group law in complex K-theory, while its elliptic version is related to the elliptic cohomology.

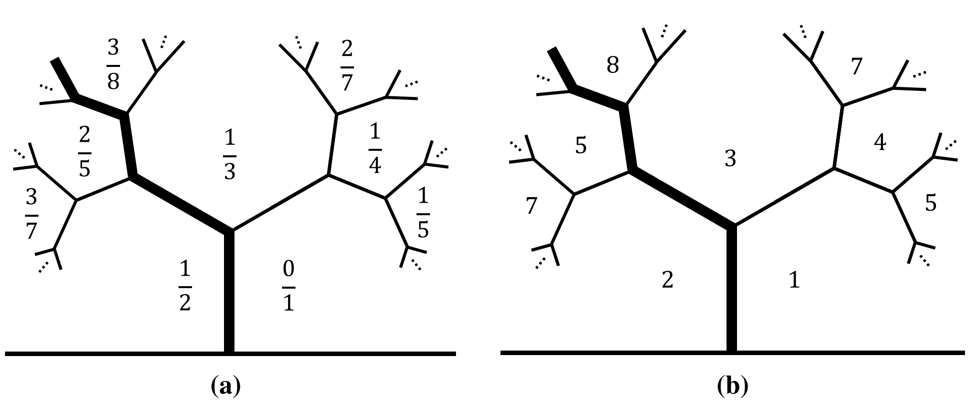


Figure 9. (a) Farey and (b) Euclid trees with the "golden" Fibonacci path.

[3]In the corresponding real projective plane, the equation $D(a, b, c) = 0$ defines a conic, with the disk $D < 0$ the celebrated Cayley–Klein model of the hyperbolic plane.

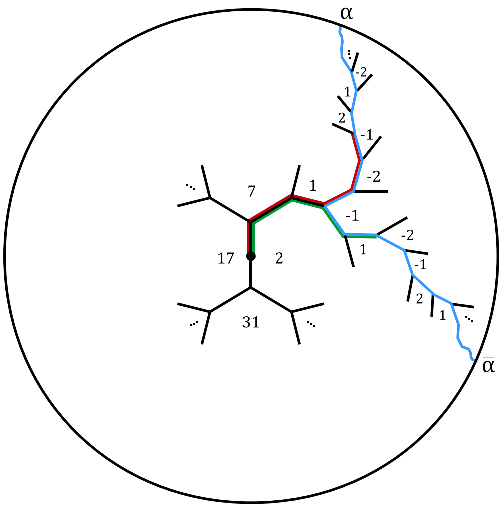


Figure 10. Conway's river for $Q = 17x^2 - 12xy + 2y^2$.

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