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Calculus of Variations

Size of the zero set of solutions of elliptic PDEs near the boundary of Lipschitz domains with small Lipschitz constant

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Abstract
Let $\Omega \subset \mathbb{R}^d$ be a C^1 domain or, more generally, a Lipschitz domain with small Lipschitz constant and $A(x)$ be a $d \times d$ uniformly elliptic, symmetric matrix with Lipschitz coefficients. Assume u is harmonic in Ω, or with greater generality u solves $\operatorname{div}(A(x)\nabla u) = 0$ in Ω, and u vanishes on $\Sigma = \partial\Omega \cap B$ for some ball B. We study the *dimension of the singular set* of u in Σ, in particular we show that there is a countable family of open balls $(B_i)_i$ such that $u|_{B_i \cap \Omega}$ does not change sign and $K \setminus \bigcup_i B_i$ has Minkowski dimension smaller than $d - 1 - \epsilon$ for any compact $K \subset \Sigma$. We also find upper bounds for the $(d-1)$-dimensional Hausdorff measure of the zero set of u in balls intersecting Σ in terms of the frequency. As a consequence, we prove a new *unique continuation principle at the boundary* for this class of functions and show that the *order of vanishing* at all points of Σ is bounded except for a set of Hausdorff dimension at most $d - 1 - \epsilon$.

Mathematics Subject Classification 31B05 · 31B20 · 35J25

1 Introduction

In this paper, we study the size of the zero set of solutions u of a certain class of elliptic PDEs (see Sect. 2.1) near the boundary of a Lipschitz domain. Assume $\Omega \subset \mathbb{R}^d$ is a Lipschitz domain with small Lipschitz constant and Σ is an open set of the boundary $\partial\Omega$ where u vanishes. We investigate the dimension of the set $\mathcal{S}'_\Sigma(u) = \{x \in \Sigma \mid u^{-1}(\{0\}) \cap B(x,r) \cap \Omega \neq \varnothing,\ \forall r > 0\}$, the *set where u changes sign in every neighborhood*.

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In a more regular setting, for example in the case Ω is a $C^{1,\text{Dini}}$ domain (see [2, 9, 19, 21, 22] for the definition) and u is harmonic, $\mathcal{S}'_\Sigma(u)$ coincides with the usual *singular set* at the boundary of u: $\mathcal{S}_\Sigma(u) = \{x \in \Sigma \mid |\nabla u(x)| = 0\}$ (see Proposition 1.9). Note that all $C^{1,\alpha}$ domains (for any $\alpha > 0$) are $C^{1,\text{Dini}}$ and all $C^{1,\text{Dini}}$ domains are C^1, but the converse is not true. Nonetheless, in the case where Ω is a Lipschitz domain, $\nabla u(x)$ (or $\partial_\nu u$) only exists in Σ in a weaker sense (see "Appendix A") as far as we know, which anticipates that we will not be able to find fine estimates of the size and dimension of $\mathcal{S}_\Sigma(u)$ (see Sect. 9). The situation is different for $\mathcal{S}'_\Sigma(u)$, for which we prove the following Minkowski dimension estimate:

Theorem 1.1 *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, and let $A(x)$ be a uniformly elliptic symmetric matrix with Lipschitz coefficients defined on $\overline{\Omega}$. Let B be a ball centered in $\partial\Omega$ and suppose that $\Sigma = B \cap \partial\Omega$ is a Lipschitz graph with slope $\tau < \tau_0$, where τ_0 is some positive constant depending only on d and the ellipticity of $A(x)$. Let $u \not\equiv 0$ be a solution of $\operatorname{div}(A(x)\nabla u(x)) = 0$ in Ω, continuous in $\overline{\Omega}$ that vanishes in Σ. Then there exists some small constant $\epsilon_1(d) > 0$ and a family of open balls $(B_i)_i$, $i \in \mathbb{N}$ centered on Σ such that*

(1) *$u|_{B_i\cap\Omega}$ is either strictly positive or negative, for all $i \in \mathbb{N}$,*
(2) *$K \setminus \bigcup_i B_i$ has Minkowski dimension at most $d - 1 - \epsilon_1$ for any compact $K \subset \Sigma$.*

Moreover, in the planar setting ($d = 2$), the set $K \setminus \bigcup_i B_i$ is finite for any compact $K \subset \Sigma$.

Recall that the upper Minkowski dimension of a set $E \subset \mathbb{R}^{d-1}$ can be defined as

$$\dim_{\overline{\mathcal{M}}} E = \limsup_{j\to\infty} \frac{\log(\#\{\text{dyadic cubes } Q \text{ of side length } 2^{-j} \text{ that satisfy } Q \cap E \neq \varnothing\})}{j \log 2}. \tag{1.1}$$

The previous result gives the following corollary:

Corollary 1.2 *Assume $\Omega \subset \mathbb{R}^d$, Σ, $A(x)$, $\tau < \tau_0$, and $u \not\equiv 0$ as in the statement of Theorem 1.1. Then, there exists a constant $\epsilon_1(d) > 0$ such that*

$$\dim_{\overline{\mathcal{M}}} \left(\mathcal{S}'_\Sigma(u) \cap K\right) \leq d - 1 - \epsilon_1$$

for any compact set $K \subset \Sigma$.

Remark 1.3 We remark that Theorem 1.1 and Corollary 1.2

- are new even in the harmonic case as the set $\mathcal{S}'_\Sigma(u)$ has not been well studied before (as far as I know),
- are valid for harmonic functions in Riemannian manifolds with Lipschitz boundary (with small Lipschitz constant depending on the metric),
- include the case when Ω is a C^1 domain, situation where not too much is known either,
- give Hausdorff dimension estimates for the set $\mathcal{S}'_\Sigma(u)$ by taking an exhaustion of Σ by compact sets. Note that Hausdorff dimension estimates are weaker than Minkowski dimension estimates but these were not known either.

Remark 1.4 Some of the results of the present paper suggest that the set $\mathcal{S}'_\Sigma(u)$ might be a natural substitute of the usual singular set $\mathcal{S}_\Sigma(u)$ in the case of Lipschitz domains (and rougher). These results are the fact that $\mathcal{S}_\Sigma(u) = \mathcal{S}'_\Sigma(u)$ in the case Ω is a $C^{1,\text{Dini}}$ domain (Proposition 1.9), the existence of an example of a Lipschitz domain where $\dim_{\mathcal{H}} \mathcal{S}_\Sigma(u) = d - 1$ and no better (see Sect. 9), and Corollary 1.2 showing that better dimension estimates are true for $\mathcal{S}'_\Sigma(u)$.

Moreover, we show an upper bound estimate on the size of the zero set of u on balls centered at Σ in terms of the *frequency function* $N(x, r)$ (see Definition 3.3):

Theorem 1.5 *Assume* Ω, Σ, $A(x)$, $\tau < \tau_0$, *and* $u \not\equiv 0$ *as in the statement of Theorem 1.1. Let* $x \in \Sigma$ *and* $0 < r < r_0$ *with* r_0 *depending on* $\operatorname{dist}(x, \partial\Omega \setminus \Sigma)$, *the Lipschitz constant* L_A *and the ellipticity* Λ_A *of* $A(x)$, *and* d. *There exists* $\tilde{x} \in \Omega$ *such that* $|x - \tilde{x}| \approx_{\Lambda_A} \operatorname{dist}(\tilde{x}, \partial\Omega) \approx_{\Lambda_A} r$ *and*

$$\mathcal{H}^{d-1}(\{u = 0\} \cap B(x, r) \cap \Omega) \leq C r^{d-1} (N(\tilde{x}, Sr) + 1)^{\alpha}$$

for some large S *depending on* L_A, Λ_A *and* d, *and some* $\alpha \geq 1$ *depending on* d.

The precise $\tilde{x}$ appearing in the statement of Theorem 1.5 is the center of a certain dyadic cube related to a Whitney cube decomposition of Ω but we have freedom in choosing it. For more details see Sect. 4.1 and Remark 6.5. This result is analogous to Theorem 2 in [28] for a more general class of functions but with a worse exponent. Further, we briefly discuss the application of this theorem to the study of the zero set of Dirichlet eigenfunctions of the operator $\operatorname{div}(A\nabla\cdot)$ in Ω in Sect. 6.6. See [28] for more background on this result and its applications in the harmonic case.

Let us give some historical background for the results of this paper. L. Bers asked the following question. Consider a harmonic function u in the upper half-plane $\mathbb{R}^d_+$, C^1 up to the boundary such that there exists $E \in \partial\mathbb{R}^d_+ = \mathbb{R}^{d-1}$ where $u = |\nabla u| = 0$ on E. Does $\text{measure}_{d-1}(E) > 0$ imply $u \equiv 0$? This question has positive answer in the plane, thanks to the subharmonicity of $\log |\nabla u|$. But in $\mathbb{R}^d_+$, $d \geq 3$, there are examples constructed by Bourgain and Wolff [6] which give a negative response in general.

A related conjecture by Lin [23] which is still open is the following:

Conjecture *Let* $\Omega \subset \mathbb{R}^d$ *be a Lipschitz domain and* $\Sigma = B \cap \partial\Omega$ *for some ball* B *centered in* $\partial\Omega$. *Let* u *be a harmonic function in* Ω *and continuous up to the boundary that vanishes in* Σ. *If the set where* $\partial_\nu u = 0$ *in* Σ *has positive surface measure, then* u *must be identically zero.*

This conjecture was proved in the $C^{1,1}$ case in [23], where it was also shown that $\mathcal{S}_\Sigma(u)$ is a $(d-2)$-dimensional set (see also [7] for more quantitative estimates). V. Adolfsson, L. Escauriaza and C. Kenig also gave a positive answer to the conjecture in [3] in the case Ω is a convex Lipschitz domain. Their work was followed by Kukavica and Nyström [19], and Adolfsson and Escauriaza [2] where the conjecture is solved in the case Ω is $C^{1,\text{Dini}}$. Moreover, in [2] it is also proved that $\mathcal{S}_\Sigma(u)$ has Hausdorff dimension at most $d-2$.

In the interior of the domain, the singular and critical sets of u have been extensively studied too. In particular, the recent results from Naber and Valtorta [30] are remarkable: they find $(d-2)$-dimensional Minkowski content bounds and prove they are $(d-2)$-rectifiable with the introduction of a new quantitative stratification. Some of their techniques have been adapted to study the Minkowski content of the singular set at the boundary in the convex Lipschitz case by McCurdy [29] and in the $C^{1,\text{Dini}}$ by Kenig and Zhao [21]. Unfortunately, the methods relying on pointwise monotonicity that have been commonly used to study this problem are no longer useful in general Lipschitz domains, and new ideas are needed in this case. The conjecture of F.-H. Lin saw no further progress until X. Tolsa proved the result for harmonic functions in Lipschitz domains with small Lipschitz constants in [33]. His proof uses the new powerful methods developed by A. Logunov and E. Malinnikova (see [24, 25, 27]) to study zero sets of Laplace eigenfunctions in compact Riemannian manifolds. These techniques are used on the boundary of the domain Ω with the trade-off of restricting its Lipschitz constant. This idea from [33] has also been successfully used by Logunov et al. [28] to study the zero set of Dirichlet Laplace eigenfunctions in Lipschitz domains with small Lipschitz constant. In [28], the authors also develop novel methods to control the zero set near the boundary that will be relevant in the present paper.

The main tool used in this paper is *Almgren's frequency function* (see Definition 3.3), a quantity that controls the doubling properties of the L^2 averages of u in spheres. This function was also used in most of the works mentioned in this introduction. Our proof of Theorem 1.1 requires two technical lemmas. One is Lemma 4.1, an adaptation of Lemma 3.1 - *Key lemma* from [33] and it controls the behavior of the frequency function at points in Ω near the boundary (see also Lemma 7 from [28]). The other is Lemma 5.1, inspired by Lemma 8 - *Second hyperplane lemma: cubes without zeros* from [28] and it controls the size of balls centered at the boundary that contain no zeros of u. Both results were originally proved only for harmonic functions, hence we first extend them for solutions of second order linear elliptic PDEs in divergence form. Afterwards, we combine both lemmas in a new combinatorial argument that controls the size of the zero set of u near the boundary. We remark that the extension of Lemma 8 from [28] to the elliptic case presents many difficulties and it might be interesting on its own.

Theorem 1.1 allows us to prove an analogous *unique continuation at the boundary* result to the one in [33] for more general elliptic PDEs in divergence form:

Corollary 1.6 *Assume Ω, Σ, $A(x)$, $\tau < \tau_0$, and $u \not\equiv 0$ as in the statement of Theorem 1.1. The set $\{x \in \Sigma \mid \partial_\nu u(x) = 0\}$ has $(d-1)$-Hausdorff measure 0.*

Observe that the assumption that u vanishes continuously in Σ implies that ∇u exists σ-a.e. as a non-tangential limit in Σ, and $\nabla u = (\partial_\nu u)\nu \in L^2_{\text{loc}}(\sigma)$. Here σ stands for the surface measure restricted to Σ and ν is the outer unit normal (see Remark 3.7 and "Appendix A").

The proof of Corollary 1.6 uses that the elliptic measure ω_A associated to the elliptic operator $\text{div}(A\nabla\cdot)$ is an $\mathcal{A}_\infty$ Muckenhoupt weight with respect to σ (see Definiton 7.1) and that, since $|u|$ is locally comparable to a Green function near most points of Σ (thanks to not changing sign in a neighborhood), $\partial_\nu u$ is comparable to $d\omega_A/d\sigma$.

Theorem 1.1 also has a second corollary that controls the *vanishing order of the zeros* in the set where u does not change sign nearby.

Definition 1.7 The *vanishing order of the zero at a point* $x \in \Sigma$ is defined as the supremum of the $\alpha > 0$ such that there exist $C_\alpha > 0$ finite and $r_0 > 0$ satisfying

$$\fint_{B(x,r)\cap\Omega} |u|\mathrm{d}y \leq C_\alpha r^\alpha, \quad 0 < r \leq r_0.$$

Corollary 1.8 *Assume Ω, Σ, $A(x)$, $\tau < \tau_0$, and $u \not\equiv 0$ as in the statement of Theorem 1.1. There exists some small constant $\epsilon_2 > 0$ depending on d, the Lipschitz constant τ of Σ, and the ellipticity Λ_A of the matrix $A(x)$ such that for all $x \in \Sigma$ outside a set of Hausdorff dimension $d-1-\epsilon_1$, the vanishing order of u at x is smaller than $1+\epsilon_2$. Moreover, for all $x \in \Sigma$, the vanishing order of u at x is greater than $1-\epsilon_2$.*

This corollary is proved by comparing u locally (in the neighborhoods where it does not change sign) with the Green function of a certain cone with angular opening related to the Lipschitz constant τ of Σ.

In Sect. 9 we provide an example showing that Corollary 1.6 cannot be improved in the sense of Hausdorff dimension estimates. This contrasts with the higher regularity case ($C^{1,\text{Dini}}$) where the set $\mathcal{S}_\Sigma(u)$ is known to be $(d-2)$-rectifiable and, a fortiori, has Hausdorff dimension at most $d-2$ (see [21]). Finally, in Sect. 10, we prove the following proposition relating $\mathcal{S}_\Sigma(u)$ and $\mathcal{S}'_\Sigma(u)$ in the smooth case.

Proposition 1.9 *Let $\Omega \subset \mathbb{R}^d$ be a $C^{1,\mathrm{Dini}}$ domain, B be a ball centered in $\partial\Omega$, and $\Sigma = B \cap \partial\Omega$. Let u be a harmonic function defined in Ω, continuous in $\overline{\Omega}$ that vanishes in Σ. Then $\mathcal{S}_\Sigma(u)$ coincides with $\mathcal{S}'_\Sigma(u)$.*

The proof of the proposition follows from a local expansion of u as the sum of a homogeneous harmonic polynomial and an error term of higher degree proved in [22].

1.1 Further questions

We present some open questions related to the previous results:

- Is it true that $\dim_{\overline{\mathcal{M}}} \mathcal{S}'_\Sigma(u) \cap K$ is at most $d-2$ for any compact $K \subset \Sigma$ ($\epsilon_1 \equiv 1$)? What about the (slightly) easier case where Ω is a C^1 domain?
- Is the set $\mathcal{S}'_\Sigma(u)$ $(d-2)$-rectifiable?
- Do these estimates hold for general Lipschitz domains? For the moment, even the conjecture of Lin is still open.
- Can there exist points with vanishing order ∞ in Σ in the Lipschitz case? Thanks to the results of [2, 19, 21, 22] we know this cannot happen in the $C^{1,\mathrm{Dini}}$ case.

1.2 Outline of the paper

In Sects. 2 and 3, we present some notation, tools and ideas that will be used throughout the paper (often without reference). The main aim of Sect. 4 is the proof of Lemma 4.1, although in Sect. 4.1 we construct a Whitney cube structure to Ω that will be used during the sections following after. Section 5 is devoted to Lemma 5.1. Both lemmas are then combined in a combinatorial argument in Sect. 6 to prove Theorems 1.1 and 1.5. In Sect. 6, we also briefly discuss the application of Theorem 1.5 to the study of the zero set of certain class of eigenfunctions. The rest of the paper is spent on the proofs of Corollaries 1.6, 1.8, the example of a Lipschitz domain and harmonic function u with “large" $\mathcal{S}_\Sigma(u)$, and the equality $\mathcal{S}_\Sigma(u) = \mathcal{S}'_\Sigma(u)$ in the $C^{1,\mathrm{Dini}}$ case. This last part does not require the Whitney cube structure or Lemmas 4.1 and 5.1. In “Appendix A”, we discuss the existence of the non-tangential limit of ∇u in Σ.

2 Lipschitz domains with small constant and some properties of elliptic PDEs

Notation: the letters $C, c, c', \tilde{c}$ are used to denote positive constants that depend on the dimension d and whose values may change on different proofs. The constants c_H and C_N retain their values. The notation $A \lesssim B$ is equivalent to $A \leq CB$, and $A \sim B$ is equivalent to $A \lesssim B \lesssim A$. Sometimes, we will also use the notation $A(x) = B(x) + O(x)$ to denote that $|A(x) - B(x)| \lesssim |x|$.

In the whole paper, we assume that Ω, Σ, and $u \not\equiv 0$ are as in Theorem 1.1. Moreover, we assume that Σ is a Lipschitz graph with Lipschitz constant τ with respect to the hyperplane $H_0 := \{x_d = 0\}$ and that locally Ω lies above Σ.

Remark 2.1 Note that a C^1 domain is a Lipschitz domain with local Lipschitz constant as small as we need. In particular, Theorems 1.1 and 1.5, and its corollaries are valid for C^1 domains.

Remark 2.2 The Lipschitz constant of Ω is invariant by rescalings. If we consider a more general (anisotropic) scaling given by multiplication by a positive definite symmetric matrix with ellipticity $\tilde{\Lambda}$, then the Lipschitz constant of Σ changes by a factor of at most $\tilde{\Lambda}^2$.

2.1 Divergence form elliptic PDEs with Lipschitz coefficients

The function u we study solves $\operatorname{div}(A(x)\nabla u(x)) = 0$ weakly in Ω where the matrix $A(x)$ satisfies that

- $A(x)$ is symmetric.
- There exists $\Lambda_A > 1$ such that $\Lambda_A^{-1}|y|^2 \le (A(x)y, y) \le \Lambda_A|y|^2$ for all $x \in \overline{\Omega}$, $y \in \mathbb{R}^d$ (*uniform ellipticity*).
- There exists a Lipschitz constant $L_A > 0$ such that $|A_{ij}(x) - A_{ij}(y)| \le L_A|x - y|$ (*Lipschitz coefficients*).

By standard elliptic PDE theory (see [13, 16] for example), we know that $u \in C^{1,\alpha}$ in any ball with closure inside Ω for any $0 < \alpha < 1$. Also $u \in W^{1,2}(\Omega)$ and $u \in W^{2,2}(\Omega')$ for any compactly embedded subdomain $\Omega' \subset\subset \Omega$.

We extend the function u by 0 outside of $\overline{\Omega}$ so that it is continuous through Σ. We also extend the matrix $A(x)$ in a way that it preserves the ellipticity Λ_A and Lipschitz L_A constants up to a constant factor (the particular extension we choose will not matter). In particular, note that the absolute value of the extended function $|u|$ is a subsolution in balls B such that $B \cap \partial\Omega \subset \Sigma$. This means that

$$\int_B (A\nabla|u|, \nabla\phi)\,\mathrm{d}x \le 0, \quad \forall\phi \in C_c^1(B).$$

2.2 Modifying the domain and $A(x)$

Remark 2.3 Throughout this paper, we will require at different points the constants $\max(\Lambda_A - 1, 1 - \Lambda_A^{-1})$ and L_A to be very small. We can obtain this by exploiting the fact that our considerations are local. Indeed, we can cover our initial domain Ω by a finite family of “small" domains and prove the results on the introduction on each one separately.

By “zooming" on a small domain, we decrease the Lipschitz constant L_A of the matrix. By rescaling the domain by multiplication with an adequate matrix, we can force $A(x) = I$ at a particular point x. This, in addition to the small Lipschitz constant L_A of the new matrix, implies small ellipticity Λ_A (we will show this below). Note, though, that this last operation changes the Lipschitz constant τ of Σ. For this reason, in the statement of Theorem 1.1, τ_0 depends on the ellipticity of $A(x)$. Intuitively, we use the Lipschitz regularity of $A(x)$ to exploit that, in small scales, $A(x)$ is very close to a constant matrix.

Let us show the effects of “zooming" and rescaling the domain when the domain is a ball. Suppose u solves weakly $\operatorname{div}(A\nabla u) = 0$ in a ball B_R for some $R < 1$, that is, for all $\phi \in C_c^1(B_R)$ we have

$$\int_{B_R} (A(x)\nabla u(x), \nabla\phi(x))\,\mathrm{d}x = 0.$$

If we “zoom" in on the origin by considering the function $\tilde{u}(y) = u(Ry)$ defined on B_1, we can show it satisfies

$$\int_{B_1} (\tilde{A}(y)\nabla\tilde{u}(y), \nabla\tilde{\phi}(y))\,\mathrm{d}y = 0, \quad \forall\tilde{\phi} \in C_c^1(B_1),$$

where $\tilde{A}(y) = A(Ry)$. Notice that $\tilde{A}(y)$ has improved Lipschitz constant $R \cdot L_A < L_A$.

Now suppose that $A(0) \neq I$ and we want to change the function, the domain and the equation so that "$A(0) = I$" for the solution of the new equation.

Consider $\tilde{u}(x) = u(\tilde{S}x)$ defined in $\tilde{S}^{-1}B_R$ where $\tilde{S}$ is the symmetric positive definite square root of $A(0)$ and B_R is a ball centered at the origin of radius $R > 0$. Then, for all $\tilde{\phi}(x) \in C_c^\infty(\tilde{S}^{-1}B_R)$, we have

$$\begin{aligned}\int_{\tilde{S}^{-1}B_R} (\tilde{S}^{-1}A(\tilde{S}x)\tilde{S}^{-1}\nabla\tilde{u}(x), \nabla\tilde{\phi}(x))\mathrm{d}x &= \int_{\tilde{S}^{-1}B_R} (A(\tilde{S}x)\tilde{S}^{-1}\nabla\tilde{u}(x), \tilde{S}^{-1}\nabla\tilde{\phi}(x))\mathrm{d}x \\ &= \int_{\tilde{S}^{-1}B_R} (A(\tilde{S}x)\nabla u(\tilde{S}x), \nabla\phi(\tilde{S}x))\mathrm{d}x \\ &= (\det\tilde{S})^{-1}\int_{B_R} (A(y)\nabla u(y), \nabla\phi(y))\mathrm{d}y \\ &= 0\end{aligned}$$

where $\phi(\tilde{S}x) = \tilde{\phi}(x)$. This implies that $\tilde{u}$ is a weak solution of the equation $\operatorname{div}(A_{\tilde{S}}(x)\nabla\tilde{u}) = 0$ in $\tilde{S}^{-1}B_R$ where $A_{\tilde{S}}(x) := \tilde{S}^{-1}A(\tilde{S}x)\tilde{S}^{-1}$.

Properties of the new matrix $A_{\tilde{S}}(x)$:

- Observe that $A_{\tilde{S}}(0) = \tilde{S}^{-1}A(0)\tilde{S}^{-1} = I$.
- The coefficients of $A_{\tilde{S}}(x)$ are also Lipschitz with constant $L_{A_{\tilde{S}}}$ and the ratio between the Lipschitz constants of $A_{\tilde{S}}$ and A can be bounded above and below by positive constants depending only on d and Λ_A.
- $A_{\tilde{S}}(x)$ is uniformly elliptic with ellipticity constant bounded by $\min(\Lambda_A^2, L_{A_{\tilde{S}}}\operatorname{diam}(\tilde{S}^{-1}B_R)d)$. Indeed, the largest and smallest eigenvalues $\lambda_{\max}$ and $\lambda_{\min}$ of $A_{\tilde{S}}(z)$ at a point $z \in \tilde{S}^{-1}B_R$ satisfy

$$\begin{aligned}\max(\lambda_{\max} - 1, 1 - \lambda_{\min}) = \|A_{\tilde{S}}(z) - I\|_2 \le \|A_{\tilde{S}}(z) - I\|_F = \sqrt{\sum_{i,j} |a_{\tilde{S},i,j}(z) - \delta_{i,j}|^2} \\ \le L_{A_{\tilde{S}}}|z|d \le L_{A_{\tilde{S}}}\operatorname{diam}(\tilde{S}^{-1}B_R)d\end{aligned}$$

  where $\|\cdot\|_F$ is the Frobenius norm, $\|\cdot\|_2$ is the spectral norm, and $\delta_{i,j}$ is the Kronecker delta.

3 Frequency function for solutions of elliptic PDEs in divergence form

Let $x \in \Omega \cup \Sigma$ such that $A(x) = I$. For $r > 0$ such that $B_r(x) \cap \partial\Omega \subset \Sigma$, we denote $\mu_x(y) := (A(y)(y - x), y - x)/|y - x|^2$ and

$$H(x, r) := r^{1-d} \int_{\partial B_r(x)} \mu_x(z)|u(z)|^2 \mathrm{d}\sigma(z)$$

where $\mathrm{d}\sigma$ is the surface measure of the ball. Note that the quantity $H(x, r)$ is nonnegative. To simplify the notation we will assume from now on that x is the origin and we will write $H(r) := H(x, r)$ and $\mu(y) := \mu_x(y)$.

Remark 3.1 Observe that $\Lambda_A^{-1} \le \mu(y) \le \Lambda_A$ and, since A has Lipschitz coefficients, we have $|\mu(y) - 1| \lesssim L_A|y|$, $\|A(y) - I\|_2 \lesssim L_A|y|$, and $|\operatorname{tr}A(y) - d| \lesssim L_A|y|$.

We also denote

$$I(x,r) := r^{1-d}\int_{B_r(x)\cap\Omega}(A\nabla u,\nabla u)\mathrm{d}y.$$

Note that $I(x,r)$ is also nonnegative since A is positive definite. Moreover, $I(x,r)$ is finite thanks to Caccioppoli's inequality supposing $B_r(0)\cap\partial\Omega\subset\Sigma$. As before, we will write $I(r) := I(x,r)$.

Proposition 3.2 *Assume* $0\in\Omega\cup\Sigma$ *and* $A(0)=I$. *For* $r>0$ *such that* $B_r\cap\partial\Omega\subset\Sigma$, *the derivative of* $H(r)$ *is*

$$H'(r) = 2I(r) + O(L_A H(r)).$$

As a consequence there exists some constant $c>0$ *such that*

$$H'(r)\geq -cL_A H(r).$$

For the rest of this paper, we denote $c_H := cL_A$. In particular, $H(r)e^{c_H r}$ is a nondecreasing function of r. Note that in the harmonic case, $c_H=0$ and H is nondecreasing (which is also a corollary of the subharmonicity of $|u|^2$).

Note that the proof of Proposition 3.2 is quite simple in the case $B(0,r)\cap\partial\Omega=\varnothing$. But in our setting we require some extra considerations to apply the divergence theorem in balls touching the boundary since u only belongs in $W^{1,2}(B(0,r)\cap\Omega)$. To address this problem, we use "Appendix A" and follow the ideas of [33] in the harmonic case.

Proof First, using that $u\in W^{1,2}(B(0,r))$, we apply the divergence theorem to obtain

$$H(r) = r^{-d}\int_{B_r}\operatorname{div}(|u|^2A(x)x)\mathrm{d}x.$$

Using Remark 3.1, we expand the term inside the integral as

$$\begin{aligned}\operatorname{div}(|u|^2A(x)x) &= 2u(\nabla u, A(x)x) + |u|^2\underbrace{\operatorname{tr}(A(x))}_{d+O(L_A|x|)} + |u|^2\underbrace{\sum_{i,j}(\partial_i a_{ij})x_j}_{O(L_A|x|)}\\ &=: 2u(\nabla u, A(x)x) + d|u|^2\mu(x) + \underbrace{f(x)}_{O(L_A|x|)}|u|^2.\end{aligned}$$

We compute (formally) the derivative of H:

$$\begin{aligned}H'(r) &= -\mathrm{d}r^{-1}H(r) + r^{-d}\int_{\partial B_r}\big(2u(\nabla u, A(x)x) + d|u|^2\mu(x) + |u|^2 f(x)\big)\,\mathrm{d}\sigma(x)\\ &= r^{-d}\int_{\partial B_r}\big(2u(\nabla u, A(x)x) + |u|^2 f(x)\big)\,\mathrm{d}\sigma(x).\end{aligned}$$

Note that the only problematic term is $r^{-d}\int_{\partial B_r}2u(\nabla u, A(x)x)\mathrm{d}\sigma(x)$. Observe though, that $\int_{\partial B_r}|\nabla u|\mathrm{d}\sigma$ exists and is finite for almost every r since $\int_0^b\int_{\partial B_r}|\nabla u|\,\mathrm{d}\sigma\,\mathrm{d}r = \int_{B_r}|\nabla u|\mathrm{d}x$ is finite. Nonetheless, using the divergence theorem, we show that for all $[a,b]\subset$

$[0, \mathrm{dist}(0, \partial\Omega\setminus\Sigma))$:

$$\begin{aligned}\int_a^b H'(s)\mathrm{d}s &= \int_a^b s^{-d}\int_{\partial B_s}\left(2u(\nabla u, A(x)x) + |u|^2 f(x)\right)\mathrm{d}\sigma(x)\,\mathrm{d}s\\ &= \int_{\mathcal{A}(0,a,b)}\left(\frac{1}{|x|^d}2u(\nabla u, A(x)x) + \frac{1}{|x|^d}|u|^2 f(x)\right)\mathrm{d}x\\ &= \int_{\mathcal{A}(0,a,b)}\frac{1}{|x|^d}\,\mathrm{div}(|u|^2 A(x)x)\mathrm{d}x - \int_{\mathcal{A}(0,a,b)}\frac{\mathrm{d}}{|x|^d}|u|^2\mu(x)\,\mathrm{d}x\\ &= \int_{\mathcal{A}(0,a,b)}\mathrm{div}\left(|u|^2 A(x)\frac{x}{|x|^d}\right)\mathrm{d}x\\ &= H(b) - H(a).\end{aligned}$$

where $\mathcal{A}(0, a, b)$ is the annulus $B(0, b)\setminus B(0, a)$ for $0 < a < b$. Thus, we have the identity for $H'(r)$ for almost every r.

Now we want to prove that

$$r^{-d}\int_{\partial B_r} 2u(\nabla u, A(x)x)\,\mathrm{d}\sigma(x) = 2r^{1-d}\int_{B_r}(A\nabla u, \nabla u)\mathrm{d}x = 2I(r).$$

We denote $\Sigma_\epsilon = \Sigma + \epsilon e_d$ and $\Omega_\epsilon = \Omega + \epsilon e_d$ for small $\epsilon > 0$ where $e_d = (0, \ldots, 0, 1)$ and we have

$$r^{-d}\int_{\partial B_r\cap\Omega} 2u(\nabla u, A(x)x)\mathrm{d}\sigma(x) = \lim_{\epsilon\to 0} r^{-d}\int_{\partial(B_r\cap\Omega_\epsilon)} 2u(\nabla u, A(x)x)\mathrm{d}\sigma(x)$$

from the fact that u vanishes continuously in Σ and ∇u converges as a non-tangential limit in $L^2_{\mathrm{loc}}(\Sigma)$, as shown in “Appendix A”.

Now, since $u \in W^{2,2}(B_r \cap \Omega_\epsilon)$, we may use divergence’s theorem and that u solves $\mathrm{div}(A\nabla u) = 0$ to obtain

$$\begin{aligned}r^{-d}\int_{\partial(B_r\cap\Omega_\epsilon)} 2u(\nabla u, A(x)x)\mathrm{d}\sigma(x) &= r^{-d}\int_{B_r\cap\Omega_\epsilon} 2r\,\mathrm{div}(uA(x)\nabla u)\mathrm{d}x\\ &= r^{-d}\int_{B_r\cap\Omega_\epsilon} 2r(A(x)\nabla u, \nabla u)\mathrm{d}x.\end{aligned}$$

Finally, the limit as $\epsilon \to 0$ of the last term exists because $u \in W^{1,2}(B_r)$.

Summing up, we have

$$H'(r) = 2I(r) + r^{-d}\int_{\partial B_r}|u|^2 f(x)\mathrm{d}\sigma(x)$$

with the second term being $O(L_A H(r))$. □

Definition 3.3 Assume $x \in \Omega\cup\Sigma$ and $A(x) = I$. For $r > 0$ such that $B_r(x)\cap\partial\Omega \subset \Sigma$ we define the *Almgren’s frequency function* as

$$N(x, r) := r\frac{I(x, r)}{H(x, r)}.$$

We will assume, as before, that x is the origin and we will write $N(r) := N(x, r)$.

The following geometric lemma is essential to ensure good behavior for $N(x, r)$ in balls intersecting Σ.

Lemma 3.4 *Suppose that $x \in \Omega$ and $A(x) = I$. Assuming $B_r(x) \cap \partial\Omega \subset \Sigma$, r is small enough (depending on* $\operatorname{dist}(x, \Sigma)$ *and the Lipschitz constant L_A of $A(y)$), and the Lipschitz constant τ of the domain is also small enough (depending on* $\operatorname{dist}(x, \Sigma)/r$ *and $L_A r$), then*

$$(\nu(y), A(y)(y - x)) \geq 0$$

for $\mathcal{H}^{d-1}$-a.e. $y \in \Sigma \cap B_r(x)$ where ν is the outer unit normal to $\partial\Omega$. In particular, if x is the origin, we have

$$\int_{B_r \cap \partial\Omega} \mu(y)^{-1}(\nu, Ay)(A\nu, \nu)|\partial_\nu u|^2 \mathrm{d}\sigma(y) \geq 0.$$

Proof Without loss of generality, we assume x is the origin, denote $T = \operatorname{dist}(0, \Sigma)$ and suppose $T \leq r$. Otherwise, $\Sigma \cap B_r = \varnothing$.

First, note that $\|A(y) - I\|_2 \leq CL_A r$ for all $y \in B_r \cap \partial\Omega$ by the Lipschitz continuity of A. Using this, we obtain

$$\begin{aligned}(\nu(y), A(y)y) &= (\nu(y), y) + (\nu(y), (A - I)y) \\ &\geq (\nu(y), y) - |y|CL_A r \\ &= |y|\cos(\alpha) - |y|CL_A r\end{aligned}$$

where α is the angle between y and $\nu(y)$.

If the domain were flat ($\tau = 0$ and Σ were an open subset of a hyperplane) then all the points would have the same fixed normal vector $\tilde{\nu}$ and we would have $\cos\alpha \geq \frac{T}{r}$. Since the domain is not flat ($\tau \neq 0$), we have that the normal vector ν makes at most an angle of $\arctan(\tau)$ with $\tilde{\nu}$. Using this we can bound from above α by $\arccos\frac{T}{r} + \arctan\tau$.

Now, assuming r is small enough so that

$$\frac{T}{r} > CL_A r$$

and τ is also small enough so that

$$\cos\left(\arccos\frac{T}{r} + \arctan\tau\right) \geq CL_A r,$$

we obtain the desired inequality. □

Definition 3.5 We will say that the origin $0 \in \Omega \cup \Sigma$ (assuming $A(0) = I$) and a radius r are *admissible* if they satisfy the assumptions of the previous lemma. If $A(0) \neq I$, we will say that $0 \in \Omega \cup \Sigma$ and r are *admissible* if 0 and $\Lambda_A^{1/2} r$ are admissible for the transformed domain $\tilde{S}^{-1}\Omega$ and the matrix $A_{\tilde{S}}$ (see Remark 2.3). We can extend this definition to a point $x \neq 0$ by translating the domain.

Remark 3.6 (1) If $\operatorname{dist}(x, \Sigma) > r$, we have that $B_r \cap \partial\Omega = \varnothing$ and the integral is 0. In this case we also say that x and r are admissible.

(2) If x and r' are admissible, then x and r are admissible for all $0 \leq r < r'$.

(3) In the case $A(x) \neq I$, we can ensure admissibility if we impose

$$\cos\left(\arccos\left(\Lambda_A^{-1}\frac{\operatorname{dist}(x, \Sigma)}{r}\right) + \arctan(\Lambda_A \tau)\right) \geq CL_A\Lambda_A^{1/2} r.$$

The reason is that, in the transformed domain, the Lipschitz constant τ increases at most by a factor Λ_A and the distance from the point to Σ decreases at most by a factor $\Lambda_A^{1/2}$.

Remark 3.7 Since the outer unit normal vector ν is only defined σ-a.e. the derivative $\partial_\nu u$ may not exist everywhere. Its existence, the non-tangential convergence, and the fact that $\nabla u = (\nabla u, \nu)\nu$ in $L^2(\sigma)$ are proven in "Appendix A".

We prove an almost monotonicity property for $N(r)$ which was first observed in [14] in the interior of the domain.

Proposition 3.8 *Assume* 0 *and* r' *are admissible,* $A(0) = I$*, and the ellipticity constant* Λ_A *of* $A(x)$ *is smaller than* 2*. Then there exists* $C_N > 0$ *depending on* L_A *such that* $e^{rC_N} N(r)$ *is nondecreasing in the interval* $(0, r')$.

For the proof of this proposition we will only keep track of the relation between the constant C_N and the Lipschitz constant L_A. C_N also depends on the ellipticity constant Λ_A but, by assuming (for example) $\Lambda_A < 2$, we can omit it. We may do so thanks to Remark 2.3. Notice also that $C_N \equiv 0$ in the harmonic case.

Our proof is an adaptation of the one in [26] but with the inclusion of the case $B_r \cap \partial\Omega \neq \varnothing$. We need special care when $B_r \cap \partial\Omega \neq \varnothing$ as with the proof of Proposition 3.2. Again, we use "Appendix A" and follow the ideas of [33] in the harmonic case to circumvent these problems.

Proof Fix a compact interval $I \subset (0, r')$. We will show that the derivative of $N(r)$ is positive a.e. $r \in I$.

Since

$$I(r) = r^{1-d} \int_0^r \int_{\partial B_s} (A\nabla u, \nabla u)\, \mathrm{d}\sigma\, \mathrm{d}s$$

and r is bounded away from 0, we have that I is absolutely continuous. Also note that we are only considering the case $0 \in \Omega$ (since we ask for admissibility). For this reason, H is of class C^1 and bounded away from 0 and N is also absolutely continuous. The derivative is

$$N'(r) = \frac{(I(r) + rI'(r))H(r) - rI(r)H'(r)}{H(r)^2}.$$

Let's compute

$$I'(r) = (1-d)r^{-1}I(r) + r^{1-d} \underbrace{\int_{\partial B_r \cap \Omega} (A\nabla u, \nabla u)\mathrm{d}\sigma}_{\boxed{A}} .$$

The previous identity is true for a.e. $r \in I$.

Let $w(y) := \mu(y)^{-1}A(y)y$ be a vector field in Ω. Observe that $(w(y), y) = |y|^2$ and that $y/r = \nu$, the normal vector in ∂B_r. We can rewrite $\boxed{A}$ as

$$\begin{aligned} \boxed{A} &= \frac{1}{r} \int_{\partial B_r \cap \Omega} (A\nabla u, \nabla u)(\nu(y), w(y))\mathrm{d}\sigma(y) \\ &= \underbrace{\frac{1}{r} \int_{\partial(B_r \cap \Omega)} (A\nabla u, \nabla u)(\nu(y), w(y))\mathrm{d}\sigma}_{\boxed{B}} - \underbrace{\frac{1}{r} \int_{B_r \cap \partial\Omega} (A\nabla u, \nabla u)(\nu(y), w(y))\mathrm{d}\sigma}_{\boxed{C}} . \end{aligned}$$

To study $\boxed{C}$ we can use that $B_r \cap \partial\Omega \subset \Sigma$ and, thus, $\nabla u = (\nabla u, \nu)\nu$ on Σ (in the sense of non-tangential limits, see "Appendix A").

Then, we have that

$$\boxed{C} = \int_{B_r \cap \partial\Omega} (A\nu, \nu)|\partial_\nu u|^2(\nu(y), w(y))\mathrm{d}\sigma(y).$$

This term is a bit problematic and we will treat it later with the help of Lemma 3.4.

Let's use the divergence theorem on $\boxed{B}$ (we do this in $B_r \cap \Omega_\epsilon$ and then let $\epsilon \to 0$, as in the proof of Proposition 3.2), obtaining

$$\boxed{B} = \int_{B_r \cap \Omega} \mathrm{div}(w(x)(A\nabla u, \nabla u))\mathrm{d}x.$$

We have that

$$\begin{aligned}\mathrm{div}(w(x)(A\nabla u, \nabla u)) &= \mathrm{div}(w)(A\nabla u, \nabla u) + (w, \nabla(A\nabla u, \nabla u)) \\ &= \mathrm{div}(w)(A\nabla u, \nabla u) + 2(w, \mathrm{Hess}(u)(A\nabla u)) + (A_{D,w}\nabla u, \nabla u)\end{aligned} \tag{3.1}$$

where $A_{D,w} = \{\sum_k (\partial_k a_{ij}) w_k\}_{ij}$. Furthermore, $\mathrm{Hess}(u)$ is symmetric and

$$\mathrm{Hess}(u)(w) = \nabla(\nabla u, w) - (Dw)\nabla u.$$

Let's compute the integrals with all these terms in (3.1) one by one. First, we obtain

$$\int_{B_r \cap \Omega_\epsilon} (\mathrm{Hess}(u)w, A\nabla u)\mathrm{d}x = \int_{B_r \cap \Omega_\epsilon} ((\nabla(\nabla u, w), A\nabla u) - ((Dw)\nabla u, A\nabla u))\,\mathrm{d}x.$$

Using that u satisfies $\mathrm{div}(A\nabla u) = 0$ in Ω, we get

$$\begin{aligned}\int_{B_r \cap \Omega_\epsilon} (\mathrm{Hess}(u)w, A\nabla u)\mathrm{d}x &= \int_{B_r \cap \Omega_\epsilon} (\mathrm{div}((\nabla u, w)A\nabla u) - ((Dw)\nabla u, A\nabla u))\,\mathrm{d}x \\ &= \underbrace{\int_{\partial(B_r \cap \Omega_\epsilon)} (\nabla u, w)(A\nabla u, \nu)\mathrm{d}\sigma}_{\boxed{D}} - \underbrace{\int_{B_r \cap \Omega_\epsilon} ((Dw)\nabla u, A\nabla u)\mathrm{d}x}_{\boxed{E}}.\end{aligned}$$

We can rewrite the previous equation using the divergence theorem:

$$\boxed{D} = \int_{\partial B_r \cap \Omega_\epsilon} (\nabla u, w)(A\nabla u, \nu)\mathrm{d}\sigma + \int_{B_r \cap \partial\Omega_\epsilon} (\nabla u, w)(A\nabla u, \nu)\mathrm{d}\sigma$$

Using that $\nabla u|_{\Sigma_\epsilon}$ converges to $(\nabla u, \nu)\nu$ on Σ in $L^2_{\mathrm{loc}}(\sigma)$ as $\epsilon \to 0^+$ (see "Appendix A"), we get

$$\lim_{\epsilon \to 0} \boxed{D} = \int_{\partial B_r \cap \Omega} (\nabla u, w)(A\nabla u, \nu)\mathrm{d}\sigma + \underbrace{\int_{B_r \cap \partial\Omega} (\nu, w)(A\nu, \nu)|\partial_\nu u|^2\mathrm{d}\sigma}_{\boxed{C}}$$

where the last term on the right hand side coincides with $\boxed{C}$ defined previously.

Remember that $w(x) = \mu(x)^{-1}A(x)x$. Thus, we have the approximations $Dw = I + O(L_A|x|)$, and $\mathrm{div}(w) = d + O(L_A|x|)$ for some $a > 0$. We can use these estimates to show that

$$\lim_{\epsilon \to 0} \boxed{E} = r^{d-1}I(r) + O(L_A r^d I(r)).$$

One of the terms we missed in (3.1) behaves as

$$\int_{B_r\cap\Omega} \operatorname{div}(w)(A\nabla u, \nabla u)\mathrm{d}x = dr^{d-1}I(r) + O(L_A r^d I(r))$$

and the other as

$$\int_{B_r\cap\Omega} (A_{D,w}\nabla u, \nabla u)\mathrm{d}x \lesssim L_A \int_{B_r\cap\Omega} r|\nabla u|^2 = O(L_A r^d I(r)).$$

Summing everything up,

$$I'(r) = -r^{-1}I(r) + O(L_A I(r)) + 2r^{1-d}\int_{\partial B_r\cap\Omega} \mu(y)^{-1}(A\nabla u, \nu)^2\mathrm{d}\sigma + r^{-d}\boxed{C}$$

and we obtain

$$\begin{aligned} N'(r)N(r)^{-1} &= (rI'(r) + I(r))(rI(r))^{-1} - H'(r)(H(r))^{-1} \\ &= \left(2r^{1-d}\int_{\partial B_r\cap\Omega} \mu(y)^{-1}(A\nabla u, \nu)^2\mathrm{d}\sigma + r^{-d}\boxed{C}\right) I(r)^{-1} - 2r^{-1}N(r) + O(L_A) \\ &= \left(2r^{1-d}H(r)\int_{\partial B_r\cap\Omega} \mu(y)^{-1}(A\nabla u, \nu)^2\mathrm{d}\sigma + r^{-d}H(r)\boxed{C} - 2I(r)^2\right)(H(r)I(r))^{-1} \\ &\quad + O(L_A). \end{aligned}$$

Since $\boxed{C} \geq 0$ thanks to the fact that 0 and r' are admissible, and Lemma 3.4, we can use Cauchy–Schwarz inequality to show that the whole first term is positive. Indeed, note that from the proof of Proposition 3.2, we have

$$I(r) = r^{1-d}\int_{\partial B_r\cap\Omega} u(\nabla u, A(x)\nu)\,\mathrm{d}\sigma(x).$$

Thus, using Cauchy-Schwarz, we obtain

$$\begin{aligned} I(r)^2 &\leq r^{2-2d}\int_{\partial B_r\cap\Omega} \mu|u|^2\,\mathrm{d}\sigma \int_{\partial B_r\cap\Omega} \mu(y)^{-1}(A\nabla u, \nu)^2\mathrm{d}\sigma \\ &= r^{1-d}H(r)\int_{\partial B_r\cap\Omega} \mu(y)^{-1}(A\nabla u, \nu)^2\mathrm{d}\sigma. \end{aligned}$$

Therefore, there exists $C \geq 0$ such that

$$N'(r) \geq -CL_A N(r)$$

and we denote $C_N := CL_A$. □

3.1 Frequency function centered at arbitrary points

We have only considered $H(x, r)$ and $N(x, r)$ centered at points $x \in \Omega \cup \Sigma$ where $A(x) = I$. We can treat general points by making a change of variables such as the one in Sect. 2.2.

Assume $A(0) \neq I$. Let $\tilde{S}$ be the symmetric positive definite square root of $A(0)$, $\tilde{u}(x) = u(\tilde{S}x)$, and $A_{\tilde{S}} = \tilde{S}^{-1}A(\tilde{S}x)\tilde{S}^{-1}$. For the transformed equation $\operatorname{div}(A_{\tilde{S}}\nabla\tilde{u}) = 0$ with $A_{\tilde{S}}(0) = I$, we can compute

$$H(r) = r^{-1-d}\int_{\partial B_r} (A_{\tilde{S}}(x)x, x)|\tilde{u}(x)|^2\mathrm{d}\sigma(x).$$

After some computations and a change of variables, we can check that it is equal (in the original domain) to

$$H(r) = (\det \tilde{S})^{-1} r^{-d} \int_{\partial(\tilde{S}B_r)} |u(y)|^2 |A(0)^{-1} y| (A(y)\nu(y), \nu(y)) \mathrm{d}\sigma(y).$$

Remark 3.9 Note that $\det \tilde{S}$, $(A(y)\nu(y), \nu(y))$ and $r^{-1}|A(0)^{-1}y|$ can be upper and lower bounded by a constant depending only on the ellipticity constant Λ_A.

In particular, by assuming Λ_A bounded, we have

$$H(r) \approx r^{1-d} \int_{\partial(\tilde{S}B_r)} |u(y)|^2 \mathrm{d}\sigma(y).$$

On the other hand, we have

$$I(r) = r^{1-d} \fint_{B_r} (A_{\tilde{S}}(x)\nabla\tilde{u}, \nabla\tilde{u})\mathrm{d}x$$

which, in the original domain, is equal to

$$I(r) = (\det \tilde{S})^{-1} r^{1-d} \int_{\tilde{S}B_r} (A(y)\nabla u(y), \nabla u(y))\mathrm{d}y \approx r^{1-d} \int_{\tilde{S}B_r} |\nabla u|^2 \mathrm{d}y.$$

This allows us to compute $N(x, r)$ for general points x. Beware that $A_{\tilde{S}}$ may have different Lipschitz and ellipticity constants but this is not a problem since the change can be controlled as discussed in Sect. 2.2.

3.2 Auxiliar lemmas on the behavior of *H*(*r*) and *N*(*r*)

First, we will present a lemma that controls the growth of $H(r)$ using $N(r)$.

Lemma 3.10 *Suppose* $0 \in \Omega \cup \Sigma$, $A(0) = I$, *and* $\alpha > 1$. *Then*

$$\int_\rho^{\alpha\rho} \left(2\frac{N(r)}{r} - c_H\right)\mathrm{d}r \le \log\left(\frac{H(\alpha\rho)}{H(\rho)}\right) \le \int_\rho^{\alpha\rho} \left(2\frac{N(r)}{r} + c_H\right)\mathrm{d}r.$$

Moreover, if 0 *and* $\alpha\rho$ *are admissible, we have*

$$\begin{aligned} &2N(\rho)(\log\alpha)e^{-C_N(\alpha-1)\rho} - c_H(\alpha-1)\rho \le \\ &\log\left(\frac{H(\alpha\rho)}{H(\rho)}\right) \le 2N(\alpha\rho)(\log\alpha)e^{C_N(\alpha-1)\rho} + c_H(\alpha-1)\rho. \end{aligned}$$

Proof Using Propositions 3.2 and 3.8 on the interval $[\rho, \alpha\rho]$, we can control

$$H'(r) \le 2I(r) + c_H H(r) = (2r^{-1}N(r) + c_H)H(r) \le (2r^{-1}N(\alpha\rho)e^{C_N(\alpha\rho - r)} + c_H)H(r).$$

Analogously,

$$H'(r) \ge 2I(r) - c_H H(r) \ge (2r^{-1}N(\rho)\mathrm{e}^{C_N(\rho-r)} - c_H)H(r).$$

Now we simply integrate H'/H in the interval $[\rho, \alpha\rho]$. □

The next lemma bounds L^2 norms in annuli by $H(r)$. We denote an annulus centered at x of outer radius r_2 and inner radius r_1 by $\mathcal{A}(x, r_1, r_2) := B(x, r_2)\backslash B(x, r_1)$.

Lemma 3.11 *Suppose* $0 \in \Omega \cup \Sigma$, $r, \delta \geq 0$, $B(0, r+\delta) \cap \partial\Omega \subset \Sigma$ *and* $A(0) = I$. *Then we have*

$$e^{c_H r} H(r) \frac{1}{d|B_1|} \leq \fint_{\mathcal{A}(0,r,r+\delta)} e^{c_H |x|} \mu(x) |u(x)|^2 dx \leq e^{c_H (r+\delta)} H(r+\delta) \frac{1}{d|B_1|}. \quad (3.2)$$

Proof Using polar coordinates, write

$$\int_{\mathcal{A}(0,r,r+\delta)} \mathrm{e}^{c_H |x|} \mu(x) |u(x)|^2 \mathrm{d}x = \int_r^{r+\delta} s^{d-1} \mathrm{e}^{c_H s} H(s) \mathrm{d}s,$$

and use that $\mathrm{e}^{c_H r} H(r)$ is nondecreasing (Proposition 3.2), and $|\mathcal{A}(0, r, r+\delta)| = |B_1|((r+\delta)^d - r^d)$. □

In a similar fashion, if $A(x_0) \neq I$ and we assume Λ_A bounded we obtain the following result.

Lemma 3.12 *Suppose* $x_0 \in \Omega \cup \Sigma$ *and* $B(x_0, \Lambda_A^{1/2} r) \cap \partial\Omega \subset \Sigma$. *Then we have*

$$\fint_{B(x_0,r)} |u|^2 \mathrm{d}x \lesssim \mathrm{e}^{c_H \Lambda_A^{1/2} r} H(x_0, \Lambda_A^{1/2} r).$$

Proof Make a change of variables so that $A(x_0) = I$. The ball $B(x_0, r)$ is sent to an ellipsoide contained in the ball $B(x_0, \Lambda_A^{1/2} r)$. Proceed as in the previous lemma by using that $\mu \approx 1$. □

The next lemma is a perturbation result for $H(z, r)$: it shows that we can bound $H(0, r)$ by $CH(z, r')$ if 0 and z are close compared to r. Moreover, it does not assume that $A(z) = I$.

Lemma 3.13 *Assume that* $0 \in \Omega \cup \Sigma$, $A(0) = I$, $\Lambda_A - 1$ *is small,* $z \in \overline{\Omega}$ *such that* $|z| \leq \gamma r$ *with* $\gamma \in (0, 1)$ *and* $B_{100r}(z) \cap \partial\Omega \subset \Sigma$. *Then for any* $\delta \in (0, 10)$, *we have*

$$H(0, r) \leq C(\gamma, \delta, d, r, L_A) H(z, \Lambda_A^{1/2} r(1 + \gamma + \delta))$$

for some constant $C > 0$ *depending on* $\gamma, \delta, d, r,$ *and* L_A.

We omit the dependence of C on Λ_A in this lemma.

Proof Let $\delta \in (0, 10)$, then using Lemma 3.11, we get

$$\mathrm{e}^{c_H r} H(0, r) \leq \frac{d|B_1|}{|\mathcal{A}(0, r, r(1+\delta))|} \int_{\mathcal{A}(0,r,r(1+\delta))} \mathrm{e}^{c_H |x|} \mu(x) |u(x)|^2 \mathrm{d}x.$$

Let $\tilde{S} := \sqrt{A(z)}$, $\lambda_{\min}$ be the minimum eigenvalue of $\tilde{S}$, $\lambda_{\max}$ be the maximum eigenvalue of $\tilde{S}$, $\mathcal{A}_0 := \mathcal{A}(0, r, r(1+\delta))$, and $\mathcal{A}_z = \mathcal{A}(0, \lambda_{\max}^{-1}(1-\gamma) r, \lambda_{\min}^{-1}(1+\delta+\gamma) r)$. These two annuli are defined so that $\mathcal{A}_0 \subset \{z\} + \tilde{S}\mathcal{A}_z$.

Moreover, we have the following estimates

$$\lambda_{\min} \geq \max(\Lambda_A^{-1/2}, 1 - O(L_A \gamma r))$$

and

$$\lambda_{\max} \leq \min(\Lambda_A^{1/2}, 1 + O(L_A \gamma r))$$

which will be useful in the proof of the next lemma.

Now we can bound

$$
\begin{aligned}
&\frac{|\mathcal{A}_0|}{d|B_1|}\mathrm{e}^{c_H r}H(0,r)\\
&\quad\le \int_{\mathcal{A}_0}\mathrm{e}^{c_H|x|}\mu(x)|u(x)|^2\mathrm{d}x \le (1+O(L_A\gamma r)+O(c_H r(1+\delta)))\int_{\mathcal{A}_0}|u|^2\mathrm{d}x\\
&\quad\le (1+O(L_A r(1+\delta)))\int_{\{z\}+\tilde{S}\mathcal{A}_z}|u|^2\mathrm{d}x
\end{aligned}
$$

where we have used that $\mathcal{A}_0\subset\{z\}+\tilde{S}\mathcal{A}_z$ and $\mathrm{e}^{c_H|x|}=1+O(L_A|x|)$.

We make the change of variables $x=\tilde{S}y$, $\mathrm{d}x=(\det\tilde{S})\mathrm{d}y=(1+O(L_A\gamma r))\,\mathrm{d}y$ and integrate in polar coordinates to get

$$
\begin{aligned}
\int_{\{z\}+\tilde{S}\mathcal{A}_z}|u|^2\mathrm{d}x &= (1+O(L_A\gamma r))\int_{\{z\}+\mathcal{A}_z}|u(\tilde{S}y)|^2\mathrm{d}y\\
&\le (1+O(L_A\gamma r))\int_{\lambda_{\max}^{-1}(1-\gamma)r}^{\lambda_{\min}^{-1}(1+\gamma+\delta)r}s^{d-1}H(z,s)\mathrm{d}s.
\end{aligned}
$$

Finally, we use that $\mathrm{e}^{c_H r}H(r)$ is increasing to see that

$$
\begin{aligned}
\int_{\lambda_{\max}^{-1}(1-\gamma)r}^{\lambda_{\min}^{-1}(1+\gamma+\delta)r}s^{d-1}H(z,s)\mathrm{d}s \le&(1+O(L_A r(1+\delta)))H(z,\lambda_{\min}^{-1}(1+\gamma+\delta)r)\\
&\frac{r^d[((\lambda_{\min}^{-1}(1+\delta))^d-(\lambda_{\max}^{-1}(1-\gamma))^d]}{d}.
\end{aligned}
$$

Summing up, we obtain

$$
\begin{aligned}
&H(0,r)\le (1+O(L_A r(1+\delta)))\\
&\quad\frac{[((\lambda_{\min}^{-1}(1+\gamma+\delta))^d-(\lambda_{\max}^{-1}(1-\gamma))^d]}{(1+\delta)^d-1}H(z,\lambda_{\min}^{-1}(1+\gamma+\delta)r).
\end{aligned}
$$

□

Finally, we prove a perturbation result for the frequency function N.

Lemma 3.14 *Let $r>0$ and $z\in\Omega$ with $|z|\le\gamma r$ for $\gamma>0$ small enough. Assume $L_A r$ is small enough, $0\in\Omega$, $A(0)=I$, the ellipticity Λ_A of $A(x)$ is small enough, and the point z and distance $4r$ are admissible. Then we have the following bound*

$$
N(0,r)\le O(\sqrt{L_A r})+O(\sqrt{\gamma})+N(z,4r)\left(1+O(\sqrt{L_A r})+O(\sqrt{\gamma})\right).
$$

If γ merely satisfies $0<\gamma<(\Lambda_A+1)^{-1}$, we obtain

$$
N(0,r)\le C+CN(z,4r)
$$

for some constant $C>0$.

Proof Let $\delta\in(0,1)$ to be chosen later. Using Lemma 3.11 we get the following upper and lower bounds

$$
\mathrm{e}^{c_H r}H(0,2r)\le\frac{\mathrm{d}|B_1|}{|\mathcal{A}(0,2r,r(2+\delta))|}\int_{\mathcal{A}(0,2r,r(2+\delta))}\mathrm{e}^{c_H|x|}\mu(x)|u(x)|^2\mathrm{d}x
$$

and

$$e^{c_H r} H(0,r) \geq \frac{d|B_1|}{|\mathcal{A}(0, r(1-\delta), r)|} \int_{\mathcal{A}(0,r(1-\delta),r)} e^{c_H|x|} \mu(x)|u(x)|^2 dx.$$

By Lemma 3.10, we have

$$2N(0,r)(\log 2)e^{-C_N r} - c_H r \leq \log \frac{H(0,2r)}{H(0,r)}.$$

We aim to upper bound this quantity by something of the form $\log(H(z,r_1)/H(z,r_2))$. To do this we will proceed as in Lemma 3.13.

Since $A(0) = I$, we have $\lambda_{\max}(A(z)) = 1 + O(L_A \gamma r)$ and $\lambda_{\min}(A(z)) = 1 - O(L_A \gamma r)$ (maximum and minimum eigenvalues of $A(z)$). Let $\tilde{S}_z$ be the positive definite symmetric square root of $A(z)$. From now on, we will denote $\lambda_{\max}(\tilde{S}_z) =: \lambda_{\max}$ and $\lambda_{\min}(\tilde{S}_z) =: \lambda_{\min}$, both depending on z.

We want to find an ellipsoidal annulus (with shape given by $\tilde{S}_z$) centered at z that contains

$$\mathcal{A}_0^2 := \mathcal{A}(0, 2r, r(2+\delta))$$

and another one that is contained in

$$\mathcal{A}_0^1 := \mathcal{A}(0, r(1-\delta), r).$$

If we take an annulus $\mathcal{A}(0, r_1, r_2)$ and deform it by $\tilde{S}_z$, we get an “ellipsoidal annulus" $\tilde{S}_z \mathcal{A}(0, r_1, r_2)$ such that

$$\mathcal{A}(0, \lambda_{\max} r_1, \lambda_{\min} r_2) \subset \tilde{S}_z \mathcal{A}(0, r_1, r_2) \subset \mathcal{A}(0, \lambda_{\min} r_1, \lambda_{\max} r_2).$$

Using this, we can choose the following annuli

$$\mathcal{A}_0^2 \subset \{z\} + \tilde{S}_z \mathcal{A}(0, \lambda_{\max}^{-1}(2-\gamma)r, \lambda_{\min}^{-1}(2+\delta+\gamma)r) =: \{z\} + \tilde{S}_z \mathcal{A}_z^2$$

and this other one (recall $|z| \leq \gamma r$)

$$\mathcal{A}_0^1 \supset \{z\} + \tilde{S}_z \mathcal{A}(0, \lambda_{\min}^{-1}(1-\delta+\gamma)r, \lambda_{\max}^{-1}(1-\gamma)r) =: \{z\} + \tilde{S}_z \mathcal{A}_z^1.$$

For this last annulus to be well defined, we need

$$\lambda_{\min}^{-1}(1-\delta+\gamma) < \lambda_{\max}^{-1}(1-\gamma) \iff \delta > (1+\gamma) - \left(\frac{\lambda_{\max}}{\lambda_{\min}}\right)^{-1}(1-\gamma).$$

We also require

$$\gamma \leq \frac{1}{\frac{\lambda_{\max}}{\lambda_{\min}} + 1} < \frac{1}{2}$$

so that δ can satisfy $\delta \leq 1$.

Now we can proceed in the exact same way as in Lemma 3.13 to get

$$H(0,2r) \leq (1 + O(L_A r(1+\delta))) \frac{|\mathcal{A}_z^2|}{|\mathcal{A}_0^2|} H(z, \lambda_{\min}^{-1}(2+\gamma+\delta)r).$$

In an analogous way, we can lower bound

$$H(0,r) \geq (1 + O(L_A r(1+\delta))) \frac{|\mathcal{A}_z^1|}{|\mathcal{A}_0^1|} H(z, \lambda_{\min}^{-1}(1-\delta+\gamma)r).$$

Putting both expressions together

$$\log\left(\frac{H(0,2r)}{H(0,r)}\right) \leq (1+O(L_A r(1+\delta))) + \log\left(\frac{|\mathcal{A}_z^2||\mathcal{A}_0^1|}{|\mathcal{A}_z^1||\mathcal{A}_0^2|}\right) + \log\left(\frac{H(z,\lambda_{\min}^{-1}(2+\gamma+\delta)r)}{H(z,\lambda_{\min}^{-1}(1-\delta+\gamma)r)}\right). \tag{3.3}$$

Using Lemma 3.10 again, we can upper bound $\log\frac{H(z,r_1)}{H(z,r_2)}$ in terms of N as

$$\log\left(\frac{H(z,\lambda_{\min}^{-1}(2+\gamma+\delta)r)}{H(z,\lambda_{\min}^{-1}(1-\delta+\gamma)r)}\right) \leq\ 2N(z,\lambda_{\min}^{-1}(2+\gamma+\delta)r)\log\left(\frac{2+\gamma+\delta}{1-\delta+\gamma}\right) (1+O(L_A r(1+\delta))) + O(L_A r(1+\delta)). \tag{3.4}$$

Now we need to choose δ. Remember that δ has to satisfy

$$1 > \delta > (1+\gamma) - \left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)(1-\gamma) = 2\gamma + O(L_A r)(1-\gamma). \tag{3.5}$$

We will choose δ equal to the geometric mean of the left hand side and the right hand side of inequality (3.5), that is

$$\delta = \sqrt{(1+\gamma) - \left(\frac{\lambda_{\min}}{\lambda_{\max}}\right)(1-\gamma)}.$$

On the other hand, γ has to satisfy

$$0 < \gamma < \frac{1}{\frac{\lambda_{\max}}{\lambda_{\min}}+1} = \frac{1}{2+O(L_A r)} = \frac{1}{2} - O(L_A r).$$

Notation: From now on, for this proof, we will write $1-\epsilon := \frac{\lambda_{\min}}{\lambda_{\max}}$. Thus $\epsilon = O(L_A r)$.

Notice that

$$\sqrt{\gamma+\epsilon} \leq \delta = \sqrt{2\gamma+\epsilon-\epsilon\gamma} \leq \sqrt{2\gamma+\epsilon} \leq O(\sqrt{\gamma}) + O(\sqrt{L_A r})$$

for ϵ small enough ($L_A r$ small enough). Now let's bound every term that has appeared before on Equations (3.3) and (3.4). First we bound

$$\log\left(\frac{2+\gamma+\delta}{1-\delta+\gamma}\right) = \log\left(\frac{2+\sqrt{(2-\epsilon)\gamma+\epsilon}+\gamma}{1-\sqrt{(2-\epsilon)\gamma+\epsilon}+\gamma}\right) \leq \log\left(\frac{2+\sqrt{(2-\epsilon)\gamma+\epsilon}}{1-\sqrt{(2-\epsilon)\gamma+\epsilon}}\right)$$
$$\leq \log\left(\frac{2+\sqrt{2\gamma+\epsilon}}{1-\sqrt{2\gamma+\epsilon}}\right) = \log(2) + O(\sqrt{\gamma+\epsilon}) \leq \log(2) + O(\sqrt{\gamma}) + O(\sqrt{L_A r}).$$

To bound

$$\log\left(\frac{|\mathcal{A}_0^1||\mathcal{A}_z^2|}{|\mathcal{A}_0^2||\mathcal{A}_z^1|}\right) = \log\left(\frac{(1-(1-\delta)^d)((\lambda_{\min}^{-1}(2+\delta+\gamma))^d - (\lambda_{\max}^{-1}(2-\gamma))^d)}{((2+\delta)^d-2^d)((\lambda_{\max}^{-1}(1-\gamma))^d - (\lambda_{\min}^{-1}(1-\delta+\gamma))^d)}\right)$$

we separate it in two terms,

$$\log\boxed{A} = \log\left(\frac{1-(1-\delta)^d}{((1-\epsilon)(1-\gamma))^d - (1-\delta+\gamma)^d}\right)$$

and

$$\log \boxed{B} = \log \left(\frac{(2+\delta+\gamma)^d - ((1-\epsilon)(2-\gamma))^d}{(2+\delta)^d - 2^d} \right).$$

Let's write the first order expansion of the terms in δ:

- $(1-\delta)^d = 1 - d\delta + O(\delta^2)$
- $(1-\epsilon)^d = 1 - O(L_A r)$
- $(1-\gamma)^d \geq 1 - O(\delta^2)$
- $(1-\delta+\gamma)^d = (1-\delta)^d + O(\gamma) = 1 - d\delta + O(\delta^2)$

Now we can use this in the first term

$$\begin{aligned}
\boxed{A} &= \frac{d\delta + O(\delta^2)}{(1-O(L_A r))(1-O(\gamma)) - (1 - d\delta + O(\delta^2))} \\
&\leq \frac{d\delta + O(\delta^2)}{(1-O(L_A r))(1-O(\delta^2)) - (1 - d\delta + O(\delta^2))} \\
&= \frac{d\delta + O(\delta^2)}{d\delta - O(L_A r) - O(\delta^2)} \\
&= 1 + O(L_A r/\delta) + O(\delta) \\
&\leq 1 + O(\sqrt{L_A r}) + O(\sqrt{\gamma}).
\end{aligned}$$

As for the other term, we proceed similarly

$$\begin{aligned}
\boxed{B} &= \frac{(2+\delta)^d + O(\gamma) - (1-O(L_A r))(2^d - O(\gamma))}{d2^{d-1}\delta + O(\delta^2)} \\
&= \frac{d2^{d-1}\delta + O(\delta^2) + O(\gamma) + O(L_A r)}{d2^{d-1}\delta + O(\delta^2)} \\
&= 1 + O(\delta) + O(\gamma/\delta) + O(L_A r/\delta) \\
&\leq 1 + O(\sqrt{L_A r}) + O(\sqrt{\gamma}).
\end{aligned}$$

Using all the bounds obtained in the last paragraphs together with Eqs. (3.3) and (3.4), we get

$$N(0,r) \leq O(\sqrt{L_A r}) + O(\sqrt{\gamma}) + \tilde{N}(z, \lambda_{\min}^{-1}(2+\gamma+\delta)r)\left(1 + O(\sqrt{L_A r}) + O(\sqrt{\gamma})\right)$$

for $L_A r$ and γ small enough.

Finally, we can bound

$$\lambda_{\min}^{-1}(2+\gamma+\delta)r \leq 4r$$

and use that $e^{C_N r} N(r)$ is increasing to get a simpler expression.

Remark 3.15 To prove the second part of the lemma (γ not necessarily small), we just need to choose δ as the arithmetic mean of the left hand side and the right hand side of inequality (3.5). The rest of the proof is straightforward.

□

4 Behavior of the frequency function on cubes near the boundary

The aim of this section is to prove the first technical lemma concerning the behavior of N near the boundary. For an analogous proof in the harmonic case, see Section 3 of [33] or Sections 4.1 and 4.2 of [28].

4.1 Whitney cube structure on Ω

We will consider the same Whitney cube structure in Ω as [33].

Let H_0 be the horizontal hyperplane through the origin, and B_0 be a ball centered in Σ such that $MB_0 \cap \partial\Omega \subset \Sigma$ for some very large M. We also assume that $MB_0 \cap \partial\Omega$ is a Lipschitz graph with slope τ small enough with respect to H_0.

We consider the following Whitney decomposition of Ω: a family $\mathcal{W}$ of dyadic cubes in $\mathbb{R}^d$ with disjoint interiors and constants $W > 20$ and $D_0 \geq 1$ such that

(1) $\bigcup_{Q\in\mathcal{W}} Q = \Omega$,
(2) $10Q \subset \Omega, \forall Q \in \mathcal{W}$,
(3) $WQ \cap \partial\Omega \neq \varnothing, \forall Q \in \mathcal{W}$,
(4) there are at most D_0 cubes $Q' \in \mathcal{W}$ such that $10Q \cap 10Q' \neq \varnothing, \forall Q \in \mathcal{W}$. Further, for such cubes Q' we have $\frac{1}{2}\ell(Q') \leq \ell(Q) \leq 2\ell(Q')$.

We will denote by $\ell(Q)$ the side length of Q and by x_Q the center of the cube Q. From these properties it is clear that $\operatorname{dist}(Q, \partial\Omega) \approx \ell(Q)$. Also we consider the cubes small enough so that $\operatorname{diam}(Q) < \frac{1}{20}\operatorname{dist}(Q, \partial\Omega)$.

Now we will introduce a “tree" structure of parents, children and generations to this Whitney cube decomposition.

Let Π denote the orthogonal projection on H_0 and choose $R_0 \in \mathcal{W}$ such that $R_0 \subset \frac{M}{2}B_0$. It will be the root of the tree and we define $\mathcal{D}^0_{\mathcal{W}}(R_0) = \{R_0\}$ (that is the set of cubes of generation 0 of the rooted tree). To characterize the generations $\mathcal{D}^k_{\mathcal{W}}(R_0)$ for $k \geq 1$, we define first

$$J(R_0) = \{\Pi(Q) : Q \in \mathcal{W} \text{ such that } \Pi(Q) \subset \Pi(R_0) \text{ and } Q \text{ is below } R_0\}.$$

We have that $J(R_0)$ is a family of $d-1$ dimensional dyadic cubes in H_0, all of them contained in $\Pi(R_0)$. Let $J_k(R_0) \subset J(R_0)$ be the subfamily of $(d-1)$-dimensional dyadic cubes in H_0 with side length equal to $2^{-k}\ell(R_0)$. To each $Q' \in J_k(R_0)$ we assign some $Q \in \mathcal{W}$ such that $\Pi(Q) = Q'$ and such that Q is below R_0 (notice that there may be more than one choice for Q but the choice is irrelevant), see [33, Lemma B.2], and we write $s(Q') = Q$. Then we define

$$\mathcal{D}^k_{\mathcal{W}}(R_0) := \{s(Q') : Q' \in J_k(R_0)\}$$

and

$$\mathcal{D}_{\mathcal{W}}(R_0) = \bigcup_{k\geq 0} \mathcal{D}^k_{\mathcal{W}}(R_0).$$

Finally, for each $R \in \mathcal{D}^k_{\mathcal{W}}(R_0)$ and $j \geq 1$, we denote

$$\mathcal{D}^j_{\mathcal{W}}(R) = \left\{Q \in \mathcal{D}^{k+j}_{\mathcal{W}}(R_0) : \Pi(Q) \subset \Pi(R)\right\}.$$

By the properties of the Whitney cubes, we can observe that

$$Q \in \mathcal{D}_{\mathcal{W}}(R_0) \Rightarrow \operatorname{dist}(Q, \Sigma) = \operatorname{dist}(Q, \partial\Omega) \approx \ell(Q).$$

Further, for any $Q \in \mathcal{W}$, we denote its center by x_Q, its associated cylinder by

$$\mathcal{C}(Q) := \Pi^{-1}(\Pi(Q)),$$

and the $(d-1)$-dimensional Lebesgue measure on the hyperplane H_0 by m_{d-1}. In Appendix B of [33] one can find more details about the construction of this Whitney cube structure and its projections.

4.2 Lemma on the behavior of the frequency in the Whitney tree

Now we can present the first main lemma required in the proofs of Theorems 1.1 and 1.5. This lemma controls probabilistically the behavior of the frequency function in the tree of Whitney cubes defined in the last section. See [33, Lemma 3.1] for a version of this lemma for harmonic functions. Our proof is very similar. The reader only needs to consider that the properties of the frequency function for elliptic PDEs are slightly worse than those of the frequency function for harmonic functions, and that $A(x)$ is a perturbation of the identity matrix. Also this Lemma should be compared with the (interior) Hyperplane lemma of [24] and [28, Lemma 7]. Note that, in what follows, we refer to the frequency function N of a solution of $\operatorname{div}(A\nabla u) = 0$ in Ω as in the statement of Theorem 1.1.

Lemma 4.1 *Let $N_0 > 1$ be big enough. There exists some absolute constant $\delta_0 > 0$ such that for all $S \gg 1$ big enough the following holds, assuming also the Lipschitz constant τ of Σ is small enough. Let R be a cube in $\mathcal{D}_{\mathcal{W}}(R_0)$ with $\ell(R)$ small enough depending on S and L_A that satisfies $N(x_R, S\ell(R)) \geq N_0$. Then, there exists some positive integer $K = K(S)$ big enough such that if we let*

$$\mathcal{G}_K(R) = \left\{ Q \in \mathcal{D}^K_{\mathcal{W}}(R) : N(x_Q, S\ell(Q)) \leq \frac{1}{2} N(x_R, S\ell(R)) \right\}$$

then:

(1) $m_{d-1}\left(\bigcup_{Q \in \mathcal{G}_K(R)} \Pi(Q)\right) \geq \delta_0 m_{d-1}(\Pi(R))$,

(2) *for $Q \in \mathcal{D}^K_{\mathcal{W}}(R)$, it holds*

$$N(x_Q, S\ell(Q)) \leq (1 + CS^{-1/2}) N(x_R, S\ell(R)).$$

Note that (2) does not require $N(x_R, S\ell(R)) \geq N_0$.

It is important that δ_0 does not depend on S. Other constants such as M, K, and the upper bound on τ do depend on S. Finally, N_0 only depends on the dimension d.

Remark 4.2 Fix $S, T > 0$ and $R \in \mathcal{D}_{\mathcal{W}}(R_0)$ with $\ell(R)$ small enough depending on L_A and T. If $x \in \Omega$ satisfies

$$\operatorname{dist}(x, R) \leq T\ell(R) \quad \text{and} \quad \operatorname{dist}(x, \partial\Omega) \geq T^{-1}\ell(R),$$

then x and $S\ell(R)$ are admissible, assuming $M \gg T$, $M \gg S$, and that τ is small enough. The proof is analogous to the proof of [33, Remark 3.3].

An important tool for the proof of the Lemma 4.1 is the following quantitative Cauchy uniqueness theorem.

Theorem 4.3 (Quantitative Cauchy uniqueness) *Let u be a solution of* $\operatorname{div}(A(x)\nabla u(x)) = 0$ *in the half ball*

$$B_+ = \left\{ y = (y', y'') \in \mathbb{R}^{d-1} \times \mathbb{R} \mid \left|y'\right|^2 + |y''|^2 < 1,\ y'' > 0 \right\}$$

and suppose u is C^1 smooth up to the boundary and $A(x)$ is as discussed in Sect. 2.1. Let

$$\Gamma := \{x \in \mathbb{R}^d \mid x_d = 0,\ |x| \leq 3/4\}$$

Suppose that $\|u\|_{L^2(B^+)} \leq 1$ and $\|u\|_{W^{1,\infty}(\Gamma)} \leq \epsilon$ for some $\epsilon \in (0, 1)$. Then

$$\sup_{B((0,\dots,0,1/2),1/4)} |u| \leq C\epsilon^{\alpha} \quad \text{and} \quad \sup_{\frac{1}{4}B_+} |u| \leq C'\epsilon^{\alpha'}$$

where C, C', α, and α' are positive constants depending only on the ellipticity and the Lipschitz constant of the matrix $A(x)$ and the dimension d.

This result is proved in great generality in [5, Theorem 1.7]. It will also be useful for the proof of Lemma 5.1. Before starting the proof of Lemma 4.1, we note that we will require both $\Lambda_A - 1$ and L_A very small in what follows (see again Remark 2.3 to see why we can do so).

Proof of Lemma 4.1 Let $S \gg 1$ and then choose $R \in \mathcal{D}_{\mathcal{W}}(R_0)$ with $\ell(R)$ small enough depending on S and L_A. For some $j \gg 1$ independent of S that will be fixed below, consider the hyperplane L parallel to H_0 (and above H_0) such that

$$\text{dist}(L, \Sigma \cap \mathcal{C}(R)) = 2^{-j}\ell(R).$$

From now on, we will denote by J the family of cubes from $\mathcal{W}$ that intersect $L \cap \mathcal{C}(\frac{1}{2}R)$. By our construction of the Whitney cubes, we have $\ell(Q) \approx 2^{-j}\ell(R)$ and $\Pi(Q) \subset \Pi(R)$ for all $Q \in J$. Notice that if τ is small enough (depeding on j), then

$$\text{dist}(x, \Sigma \cap \mathcal{C}(10R)) \approx 2^{-j}\ell(R) \quad \text{for } x \in L \cap \mathcal{C}(10R).$$

Denote by $\text{Adm}(2WQ)$ the set of points $x \in \Omega \cap 2WQ$ such that the interval $(0, \text{diam}(25WQ))$ is admissible for x. We assume that the Lipschitz constant of the domain τ is small enough so that $3Q \subset \text{Adm}(2WQ)$ (using Remark 4.2). Then by Lemma 3.14, for Q small enough

$$\sup_{x \in \text{Adm}(2WQ)} N(x, \text{diam}(5WQ)) \leq C_0 N(x_Q, \text{diam}(20WQ)) + C_0 \tag{4.1}$$

where C_0 is an absolute constant. Note that $|x - x_Q| \leq \text{diam}(WQ) < \frac{1}{4}\,\text{diam}(5WQ)$ since $x \in 2WQ$, hence it satisfies the conditions required by Lemma 3.14 (we use that the ellipticity constant is small enough).

Claim *There exists some $Q \in J$ such that*

$$N(x_Q, \text{diam}(20WQ)) \leq \frac{N(x_R, S\ell(R))}{4C_0} \tag{4.2}$$

if j is big enough (but independent of S) and we assume that τ_0 is small enough depending on j, and also N_0 is big enough.

Proof of the claim From now on, we denote $N = N(x_R, S\ell(R))$. Our aim is to prove the claim using Theorem 4.3 in a small half-ball centered at z_R, the projection of x_R onto the hyperplane L. Set

$$B_+ := \{x \in B(z_R, \ell(R)/4) \mid x_d > (z_R)_d\} \subset \Omega.$$

Also, let $\tilde{z}_R := z_R + (0, \dots, 0, \ell(R)/8) \in B_+$. Note that, rescaling B_+, $\tilde{z}_R$ corresponds to the point $(0, \dots, 0, 1/2)$ in the statement of Theorem 4.3.

We aim for a contradiction, so we assume that $N(x_Q, \mathrm{diam}(20WQ)) > N/(4C_0)$ for all $Q \in J$, where $N = N(x_R, S\ell(R))$. For each $Q \in J$, we have

$$\sup_{2Q} |u|^2 \lesssim \fint_{B(x_Q,\mathrm{diam}(3Q))} |u|^2 \mathrm{d}x$$

by standard properties of solutions of elliptic PDEs. Then, by Lemma 3.12, we obtain

$$\fint_{B(x_Q,\mathrm{diam}(3Q))} |u|^2 \mathrm{d}x \lesssim H(x_Q, \mathrm{diam}(20WQ)) \mathrm{e}^{c_H \mathrm{diam}(20WQ)}$$

where c_H is the constant from Lemma 3.10. Note that $\mathrm{e}^{c_H \mathrm{diam}(20WQ)} = 1 + O(L_A \ell(Q))$, omiting the dependence on Λ_A (which we may assume is very close to 1). Using Lemma 3.10 we obtain the following bound using the frequency function:

$$\begin{aligned} &H(x_Q, \mathrm{diam}(20WQ)) \\ &\quad \le H(x_Q, \ell(R)) \left(\frac{\mathrm{diam}(20WQ)}{\ell(R)} \right)^{2N(x_Q,\mathrm{diam}(20WQ))(1+O(L_A\ell(Q)))} (1 + O(L_A \ell(Q))). \end{aligned}$$

Note that for the previous step we need τ small enough depending on j. At other points of the proof we will require τ small enough but without further reference.

Now we estimate $H(x_Q, \ell(R))$ as follows

$$\begin{aligned} H(x_Q, \ell(R)) &\approx \ell(R)^{1-d} \int_{\partial B(\ell(R))} |u(x_Q + A(x_Q)^{1/2} y)|^2 \mathrm{d}\sigma(y) \\ &\lesssim \frac{(1 + O(L_A \ell(R)))}{|\mathcal{A}(x_Q, \ell(R), 2\ell(R))|} \int_{\mathcal{A}(0,\ell(R),2\ell(R))} |u(x_Q + A(x_Q)^{1/2} y)|^2 \mathrm{e}^{c_H |y|} \mathrm{d}y \\ &\lesssim \frac{(1 + O(L_A \ell(R)))}{|B(\tilde{z}_R, C_1 \ell(R))|} \int_{B(0,C_1\ell(R))} |u(\tilde{z}_R + A(\tilde{z}_R)^{1/2} y)|^2 \mathrm{e}^{c_H |y|} \mathrm{d}y \end{aligned}$$

where we have used Remark 3.9, and that for some fixed $C_1 > 0$ we have that $A(x_Q)^{1/2} \mathcal{A}(x_Q, \ell(R), 2\ell(R)) \subset \mathcal{A}(\tilde{z}_R)^{1/2} B(\tilde{z}_R, C_1 \ell(R))$. Finally, using Lemma 3.11 we can bound

$$H(x_Q, \ell(R)) \lesssim (1 + O(L_A \ell(R))) H(\tilde{z}_R, C_1 \ell(R)).$$

Moreover, using again Lemma 3.10, we can further bound as follows

$$\begin{aligned} H(\tilde{z}_R, C_1 \ell(R)) &\le H(\tilde{z}_R, \ell(R)/16)(16C_1)^{2N(\tilde{z}_R,C_1\ell(R))(1+O(L_A\ell(R)))} (1 + O(L_A \ell(R))) \\ &\lesssim \frac{(1 + O(L_A \ell(R)))}{|B(\tilde{z}_R, \ell(R)/8)|} \int_{B(0,\ell(R)/8)} \mathrm{e}^{c_H |y|} |u(\tilde{z}_R + A(\tilde{z}_R)^{1/2} y)|^2 \mathrm{d}y \\ &\quad (16C_1)^{2N(\tilde{z}_R,C_1\ell(R))(1+O(L_A\ell(R)))} \end{aligned}$$

and recalling that $B(\tilde{z}_R, \ell(R)/8) \subset B_+$ even after considering the reescaling by $A(\tilde{z}_R)^{1/2}$, we obtain

$$H(\tilde{z}_R, C_1 \ell(R)) \lesssim \frac{(1 + O(L_A \ell(R)))}{|B_+|} \int_{B_+} |u|^2 \mathrm{d}y \cdot (16C_1)^{2\tilde{N}(\tilde{z}_R,C_1\ell(R))(1+O(L_A\ell(R)))}.$$

Now, using that $L_A \ell(R)$ is very small, we can bound all the terms $O(L_A \ell(R))$ by 1, for example. Thus, summing up all the computations we have done, we can write

$$\sup_{2Q} |u|^2 \lesssim \left(\frac{\mathrm{diam}(20WQ)}{\ell(R)} \right)^{N(x_Q,\mathrm{diam}(20WQ))} (16C_1)^{4N(\tilde{z}_R,C_1\ell(R))} \fint_{B_+} |u|^2 \mathrm{d}y. \quad (4.3)$$

By Lemma 3.14, we have

$$N(\tilde{z}_R, C_1\ell(R)) \leq CN(x_R, S\ell(R)) + C \leq C'N \tag{4.4}$$

for a suitable constant $C > 2C_1$ if S is large enough, and N_0 too.

Recalling that we assumed for all $Q \in J$ that $N(x_Q, \operatorname{diam}(20WQ)) > \frac{N}{4C_0}$, we get

$$\begin{aligned}\sup_{2Q} |u|^2 &\lesssim (16C_1)^{C'N} \left(\frac{\operatorname{diam}(20WQ)}{\ell(R)}\right)^{N/4C_0} \fint_{B_+} |u|^2 \mathrm{d}y \\ &= 2^{-jC''N + C'''N} \fint_{B_+} |u|^2 \mathrm{d}y\end{aligned}$$

for some positive constants C'' and C'''. We have used that $\operatorname{diam}(20WQ) < \ell(R)$ by choosing j large enough.

Using interior estimates for solutions of elliptic PDEs

$$\sup_{\frac{3}{2}Q} |\nabla u|^2 \lesssim \frac{\sup_{2Q} |u|^2}{\ell(Q)^2} \lesssim \frac{2^{2j}}{\ell(R)^2} 2^{-jC''N + C'''N} \fint_{B_+} |u|^2 \mathrm{d}y.$$

From the last two estimates we deduce that if j is big enough (depending on the absolute constants C'' and C''') and N_0 (and thus also N) is big enough too, then there exists some $c' > 0$ such that

$$\sup_{\frac{3}{2}Q} \left(|u|^2 + \ell(R)^2 |\nabla u|^2\right) \lesssim 2^{-jc'N} \fint_{B_+} |u|^2 \mathrm{d}y.$$

Since the cubes $\frac{3}{2}Q$ with $Q \in J$ cover the flat part of the boundary of B_+, we can apply a rescaled version of Theorem 4.3 to B_+ to get

$$\sup_{B(\tilde{z}_R, \ell(R)/16)} |u|^2 \lesssim 2^{-jc'N\alpha} \fint_{B_+} |u|^2 \mathrm{d}y \tag{4.5}$$

for some $\alpha > 0$. Observe that we can lower bound

$$\sup_{B(\tilde{z}_R, \ell(R)/16)} |u|^2 \gtrsim H(\tilde{z}_R, \ell(R)/16)$$

and upper bound

$$2^{-jc'N\alpha} \fint_{B_+} |u|^2 \mathrm{d}y \lesssim 2^{-jc'N\alpha} H(\tilde{z}_R, \ell(R)).$$

Using these bounds in (4.5) we obtain

$$H(\tilde{z}_R, \ell(R)/16) \lesssim 2^{-jc'N\alpha} H(\tilde{z}_R, \ell(R)).$$

By Lemma 3.10, this implies

$$2N(\tilde{z}_R, \ell(R))(\log 16)(1 + O(L_A\ell(R))) + O(L_A\ell(R)) \geq \log\left(\frac{H(\tilde{z}_R, \ell(R))}{H(\tilde{z}_R, \ell(R)/16)}\right) \gtrsim c' jN\alpha$$

for some fixed $c' > 0$. But for j big enough this contradicts the fact that $N(\tilde{z}_R, \ell(R)) \lesssim N$ by (4.4). Observe though that j "big enough" does not depend on the election of S. □

Now we may introduce the set $\mathcal{G}_K(R)$. Fix $Q_0 \in J$ such that (4.2) holds for Q_0. Notice that, by (4.1)

$$\sup_{x\in \mathrm{Adm}(2WQ_0)} N(x, \mathrm{diam}(5WQ_0)) \le C_0 N(x_{Q_0}, \mathrm{diam}(20WQ_0)) + C_0 \le \frac{N}{2}\cdot\left(\frac{10}{11}\right)^2 \tag{4.6}$$

since $N \ge N_0$ and we assume N_0 big enough. The precise value of the constant $\left(\frac{10}{11}\right)^2$ is not important. Finally, we can define

$$\mathcal{G}_K(R) = \{Q \in \mathcal{D}_{\mathcal{W}}^{j+k}(R) : \Pi(Q) \subset \Pi(Q_0)\}$$

with $k = \lceil \log_2 S\rceil$. Thus, we have $\mathcal{G}_K(R) \subset \mathcal{D}_{\mathcal{W}}^K(R)$ with $K = j + k$ and it holds $\ell(Q) = 2^{-k}\ell(Q_0)$ for every $Q \in \mathcal{G}_K(R)$.

The property (1) follows from (4.6). Indeed if $P \in \mathcal{G}_K(R)$, then taking into account that $x_P \in \mathrm{Adm}(2WQ_0)$ for τ small enough (depending on S) and using Proposition 3.8 we get

$$N(x_P, S\ell(P)) \le \frac{11}{10} N(x_P, \ell(Q_0)) \le \left(\frac{11}{10}\right)^2 N(x_P, W\ell(Q_0)) \le \frac{N}{2}$$

where we have bounded the terms $1 + O(L_A\ell(Q_0))$ by $11/10$. Notice also that

$$m_{d-1}\left(\bigcup_{Q\in\mathcal{G}_K(R)} \Pi(Q)\right) = l\,(Q_0)^{d-1} \approx \left(2^{-j}\ell(R)\right)^{d-1}$$

and recall that j is independent of S. So (1) holds with $\delta_0 \approx 2^{-j(d-1)}$.

The property (2) is a consequence of Lemma 3.14. Indeed for any $P \in \mathcal{D}_{\mathcal{W}}^K(R)$, since $|x_P - x_R| \lesssim \ell(R)$, taking $\gamma \approx S^{-1}$ in the Lemma 3.14, we deduce

$$\begin{aligned} N(x_P, S\ell(P)) &\le (1 + O(L_A S\ell(R)))N(x_P, S\ell(R)/3) \\ &\le (1 + O(L_A S\ell(R)))\left[O(\sqrt{L_A S\ell(R)}) + O(\sqrt{S^{-1/2}})\right] \\ &\quad + (1 + O(L_A S\ell(R)))N(x_R, S\ell(R))\left[1 + O(\sqrt{L_A S\ell(R)}) + O(\sqrt{S^{-1/2}})\right]. \end{aligned}$$

Assuming that $\ell(R) \lesssim S^{-2}$ and N_0 large enough, we obtain

$$N(x_P, S\ell(P)) \le (1 + CS^{-1/2})N(x_R, S\ell(R))$$

for certain constant C. □

5 Balls without zeros near the boundary

In this section we will prove the second main lemma concerning the behavior of N near the boundary. This lemma shows that if we have a ball near the boundary with bounded frequency, then we can find a smaller ball centered at the boundary where u does not change sign. The following lemma should be compared with [28, Lemma 8] that treats the harmonic case. Note that, in what follows, we refer to the frequency function $N(x, r)$ of a solution of $\mathrm{div}(A\nabla u) = 0$ in Ω as in the statement of Theorem 1.1. Moreover, we consider Ω with the Whitney structure defined in Sect. 4.1.

Lemma 5.1 *For any $N > 0$ and $S \gg 1$ large enough there exist positive constants $\tau_0(N, S)$ and $\rho(N, S)$ such that the following statement holds. Suppose the Lipschitz constant τ of Σ is smaller than τ_0 and Q is a cube in $\mathcal{D}_{\mathcal{W}}(R)$ such that $N(x_Q, S\ell(Q)) \leq N$. Then there exists a ball B centered in $\Sigma \cap \mathcal{C}(Q)$ with radius $\rho\ell(Q)$ such that u does not vanish in $B \cap \Omega$.*

First, we will prove a "toy" version of this lemma on the half ball B_+ for harmonic functions. The following lemma is essentially [28, Lemma 9] but formulated using the frequency function instead of the doubling index (a closely related quantity used in [28]).

Lemma 5.2 *Let B be the unit ball in $\mathbb{R}^d$ and let B_+ be the half ball,*

$$B_+ = \left\{y = \left(y', y''\right) \in \mathbb{R}^{d-1} \times \mathbb{R} \mid |y'|^2 + |y''|^2 < 1, y'' > 0\right\}.$$

Let u be a function harmonic in B_+ such that $u \in C(\overline{B_+})$, $u = 0$ on $\Gamma := \partial B_+ \cap \left\{y'' = 0\right\}$, and

$$\sup_{\frac{1}{4}B_+} |u| = 1.$$

For any $N > 0$ and $0 < r_0 < 1/16$, there exist $\rho = \rho(N, r_0) > 0$ and $c_0 = c_0(N, r_0) > 0$ such that if $N(0, 1/2) \leq N$, then there is $x' \in \mathbb{R}^{d-1}$ with $\left|x'\right| < r_0$ such that

$$|u(y)| \geq c_0 y'', \quad \text{for any} \quad y = (y', y'') \in B((x', 0), \rho) \cap B_+.$$

In particular, u does not vanish in $B\left(\left(x', 0\right), \rho\right) \cap B_+$.

The notation $\frac{1}{n}B_+$ used in the previous statement stands for $\{y \in \mathbb{R}^d \mid ny \in B_+\}$.

Proof Let B_- be the reflection of the half-ball B_+ with respect to $\Gamma = \{y'' = 0\} \cap \partial B_+$. Since u vanishes on Γ, u can be extended to a harmonic function in B by the Schwarz reflection principle. We also denote this extension by u.

Using Cauchy estimates we can uniformly bound every partial derivative of u inside $B(0, 1/8)$ obtaining

$$\sup_{x \in B(0,1/8)} |\nabla u(x)| \lesssim \sup_{B(0,1/4)} |u| = 1.$$

Let $\delta := \max_{\substack{x' \in \mathbb{R}^{d-1},\\ |x'| \leq r_0}} |\nabla u(x', 0)|$. Then, Theorem 4.3 applied to $r_0 B_+$ implies that

$$\sup_{B(0,r_0/4)} |u| \leq C\delta^\gamma$$

for some positive C and $\gamma \in (0, 1)$. Then $⨍_{\partial B(0,r_0/4)} u^2 \mathrm{d}\sigma \leq C\delta^{2\gamma}$ and by subharmonicity $⨍_{\partial B(0,1/2)} u^2 \mathrm{d}\sigma \geq c \sup_{B(0,1/4)} u^2 = c$.

By the monotonicity of N in the harmonic case, we also have

$$\log_{2/r_0} \frac{⨍_{\partial B(0,1/2)} u^2 \mathrm{d}\sigma}{⨍_{\partial B(0,r_0/4)} u^2 \mathrm{d}\sigma} = \log_{2/r_0} \frac{H(0, 1/2)}{H(0, r_0/4)} \leq N(0, 1/2) \leq N.$$

We can conclude that

$$\log_{2/r_0} \frac{c}{C\delta^{2\gamma}} \leq N \implies \delta \geq c' \left(\frac{2}{r_0}\right)^{-\frac{N}{2\gamma}}.$$

Now, choose $x'_* \in \mathbb{R}^{d-1}$, $|x'_*| \leq r_0$ such that $|\nabla u(x'_*, 0)| = \delta$. Clearly, at this point, $|\nabla u(x'_*, 0)| = |\partial_d u(x'_*, 0)|$ (the derivative in the direction normal to Γ). Without loss of generality, assume that $\partial_d u(x'_*, 0) = \delta$. Observe that the second derivatives of u are uniformly bounded in $B(0, 1/8)$ using Cauchy estimates (as we have done before). Thus, we have

$$\partial_d u(y) > \delta/2 \quad \text{when } \operatorname{dist}(y, (x'_*, 0)) < \rho = \min\{c_0\delta, r_0\}$$

where c_0 only depends on the bound on the second derivatives, and thus is an absolute constant. Using this, we finally get

$$u(y) \geq \frac{\delta}{2} y'' \geq c'' \left(\frac{2}{r_0}\right)^{-\frac{N}{2\gamma}} y'',$$

for $y = \left(y', y''\right) \in B\left(\left(x'_*, 0\right), \rho\right) \cap B_+$. □

Now we will prove an elliptic extension of the previous lemma. Unfortunately, the proof presents some complications since we do not have an adequate substitute to Schwarz's reflection principle. To overcome this, we assume that our function u is a solution of $\operatorname{div}(A(x)\nabla u) = 0$ where $A(x)$ is a small (Lipschitz) perturbation of the identity matrix, and we show that there is a harmonic function v very close to u in C^1 norm for which the previous lemma holds. Then, we obtain that there is a smaller ball where u does not vanish either.

Lemma 5.3 *Let u be a solution of $\operatorname{div}(A\nabla u) = 0$ in B_+ such that $u \in C(\overline{B_+})$, $u = 0$ on $\Gamma := \partial B_+ \cap \left\{y'' = 0\right\}$, and*

$$\sup_{\frac{1}{4}B_+} |u| = 1$$

where $\frac{1}{4}B_+ = \left(\frac{1}{4}B\right)_+$. For any $N > 0$ and $0 < r_0 < 1/32$, there exist $\rho = \rho(N, r_0) > 0$ and $c_0 = c_0(N, r_0) > 0$ such that if $N(0, 1/2) \leq N$, then there is $x' \in \mathbb{R}^{d-1}$ with $\left|x'\right| < r_0$ such that

$$|u(y)| \geq c_0 y'', \quad \textit{for any} \quad y = \left(y', y''\right) \in B\left(\left(x', 0\right), \rho\right) \cap B_+$$

assuming that L_A and $\Lambda_A - 1$ are small enough depending on N and r_0 (where L_A and Λ_A are the Lipschitz and ellipticity constants of $A(x)$, respectively). In particular, u does not have zeros in $B\left(\left(x', 0\right), \rho\right) \cap B_+$.

Proof Let v be the harmonic extension of $u|_{\partial(\frac{1}{2}B_+)}$ defined in $\frac{1}{2}B_+$. We intend to use Lemma 5.2 to find a ball B such that $|v(y', y'')| \gtrsim y''$ for $(y', y'') \in B \cap \Omega$. Afterwards, we will see that if L_A and $\Lambda_A - 1$ are small enough, then the difference $v - u$ is arbitrarily small in $W^{1,\infty}(B \cap \Omega)$ which will prove the lemma (for a smaller concentric ball).

First, we bound the frequency (associated to Δ) of v as follows

$$N^v(0, 1/2) = \frac{1}{2} \frac{\int_{\frac{1}{2}B_+} |\nabla v|^2 \mathrm{d}x}{\int_{\partial \frac{1}{2}B_+} v^2 \mathrm{d}\sigma} \lesssim \frac{\int_{\frac{1}{2}B_+} |\nabla u|^2 \mathrm{d}x}{\int_{\partial \frac{1}{2}B_+} u^2 \mathrm{d}\sigma}$$

using that v is the minimizer of the Dirichlet energy for the boundary condition $u|_{\partial(B_+/2)}$. We also have that

$$\frac{1}{2} \frac{\int_{\frac{1}{2}B_+} |\nabla u|^2 \mathrm{d}x}{\int_{\partial \frac{1}{2}B_+} u^2 \mathrm{d}\sigma} \approx \frac{1}{2} \frac{\int_{\frac{1}{2}B_+} (A(x)\nabla u, \nabla u) \mathrm{d}x}{\int_{\partial \frac{1}{2}B_+} \mu u^2 \mathrm{d}\sigma} = N^u(0, 1/2),$$

obtaining an upper bound for $N^v(0,1/2)$. Analogously, we may also obtain a lower bound for $N^v(0,1/2)$ in terms of $N^u(0,1/2)$ using that u minimizes a weighted Dirichlet energy.

Consider the function $h = v - u$ defined in $\frac{1}{2}B_+$. Note that h is a solution of

$$\begin{cases} \operatorname{div}(A\nabla h) = \operatorname{div}((A-I)\nabla v), & \text{in } \frac{1}{2}B_+, \\ h = 0, & \text{on } \partial \frac{1}{2}B_+. \end{cases}$$

We aim to bound $h(x)$ and $\nabla h(x)$ in $\frac{1}{16}B_+$ in terms of N, L_A, and Λ_A. Using the Green's function $G_A(x,y)$ for the elliptic operator $\operatorname{div}(A(x)\nabla\cdot)$ in $\frac{1}{2}B_+$, we can represent $h(x)$ as

$$h(x) = -\int_{\frac{1}{2}B_+} (\nabla_y G_A(x,y), (A-I)(y)\nabla v(y))\mathrm{d}y$$

for $x \in B_+/16$. We split this integral in two parts and take absolute values:

$$\begin{aligned} |h(x)| \le &\int_{\frac{1}{8}B_+} |\nabla_y G_A(x,y)||(A-I)(y)||\nabla v(y)|\mathrm{d}y \\ &+ \int_{\frac{1}{2}B_+ \setminus \frac{1}{8}B_+} |\nabla_y G_A(x,y)||(A-I)(y)||\nabla v(y)|\mathrm{d}y. \end{aligned}$$

In both integrals we bound $|A-I|$ by $\Lambda_A - 1$. Also, in the first integral, we are going to bound $|\nabla v(y)| \lesssim 2^{cN}$ for some $c > 0$ using Cauchy estimates. To this end, we consider v extended to $\frac{1}{2}B$ using Schwarz reflection principle, and then, for $y \in \frac{1}{8}B$, we obtain

$$|\nabla v(y)|^2 \lesssim \int_{\partial \frac{1}{4}B} v^2 \mathrm{d}\sigma \approx H^v(0,1/4) \le H^v(0,1/2) \approx H^u(0,1/2) \lesssim H^u(0,1/4)\, 2^{cN} \lesssim 1 \cdot 2^{cN}$$

where we have also used $H^u(0,1/4) \lesssim \sup_{\frac{1}{4}B_+} |u|^2 = 1$. Remember that $H^v(x,r) = r^{1-d} \int_{\partial B_r(x)} |u(z)|^2 \mathrm{d}\sigma(z)$, as v is harmonic.

Then, using that $\nabla_y G_A(x,\cdot)$ has weak $L^{\frac{d}{d-1}}$ norm bounded by a constant depending only on Λ_A and d (see [17, estimate (1.6)] together with the symmetry of A), we get

$$\int_{\frac{1}{8}B_+} |\nabla_y G_A(x,y)|\mathrm{d}y \le C(d, \Lambda_A).$$

Thus, we can bound the first integral by

$$\int_{\frac{1}{8}B_+} |\nabla_y G_A(x,y)||(A-I)(y)||\nabla v(y)|\mathrm{d}y \lesssim C'(d,\Lambda_A) 2^{cN}.$$

For the other integral, we use Cauchy–Schwarz to obtain

$$\int_{\frac{1}{2}B_+ \setminus \frac{1}{8}B_+} |\nabla_y G_A(x,y)||\nabla v(y)|\mathrm{d}y \le \underbrace{\left(\int_{\frac{1}{2}B_+ \setminus \frac{1}{8}B_+} |\nabla_y G_A(x,y)|^2 \mathrm{d}y\right)^{1/2}}_{\boxed{A}} \underbrace{\left(\int_{\frac{1}{2}B_+} |\nabla v(y)|^2 \mathrm{d}y\right)^{1/2}}_{\boxed{B}}.$$

Invoking [17, Theorem 3.3], we have that $|\nabla_y G_A(x,y)| \le C|x-y|^{1-d} \lesssim 1$ for $x \in \frac{1}{16}B_+$ and $y \in \frac{1}{2}B_+ \setminus \frac{1}{8}B_+$, which allows us to bound $\boxed{A}$. We estimate $\boxed{B}$ as follows:

$$\int_{\frac{1}{2}B_+} |\nabla v(y)|^2 \mathrm{d}y = 2N^v(0,1/2) \int_{\partial \frac{1}{2}B_+} v^2 \mathrm{d}\sigma \lesssim 2N 2^{cN},$$

where we have used that $H^v(0, 1/2) \lesssim 2^{cN}$, as shown before.

Summing up all the previous estimates, we have obtained $|h(x)| \lesssim C(N, \Lambda_A)(\Lambda_A - 1)$ for $x \in \frac{1}{16}B_+$. Now, [16, Corollary 8.36] gives us

$$\|h\|_{C^{1,\alpha}(\frac{1}{32}B_+)} \lesssim \|h\|_{L^\infty(\frac{1}{16}B_+)} + \|(A-I)\nabla v\|_{C^{0,\alpha}(\frac{1}{16}B_+)}$$

for some $\alpha \in (0, 1)$. Using the product rule for derivatives, we get

$$\|(A-I)\nabla v\|_{C^{0,\alpha}(\frac{1}{16}B_+)} \lesssim \|(A-I)\nabla v\|_{C^{0,1}(\frac{1}{16}B_+)} \lesssim L_A\|\nabla v\|_{L^\infty(\frac{1}{16}B_+)} + (\Lambda_A - 1)\|D^2 v\|_{L^\infty(\frac{1}{16}B_+)}.$$

Finally, using interior Cauchy estimates to estimate $\|D^2 v\|_{L^\infty(\frac{1}{16}B_+)}$ and $\|\nabla v\|_{L^\infty(\frac{1}{16}B_+)}$ in terms of N, we get

$$|\nabla h(x)| \leq C(N)\left(L_A + (\Lambda_A - 1)\right), \quad x \in \frac{1}{32}B_+.$$

To end the proof, we apply Lemma 5.2 to v to get a ball where

$$|v(y)| \geq c_0 y'', \quad \text{for any} \quad y = \left(y', y''\right) \in B\left(\left(x', 0\right), \rho\right) \cap B_+$$

and we make L_A and $\Lambda_A - 1$ small so that $\partial_d h(x) \leq c_0/2$ in $\frac{1}{32}B_+$. This implies $|h(y', y'')| \leq \frac{c_0}{2}y''$ and, since $|u| \geq |v| - |h|$, it finishes the proof. □

We need to state a last lemma before proving Lemma 5.1.

Lemma 5.4 *Let $\Sigma \subset \mathbb{R}^d$ be a Lipschitz graph with respect to the hyperplane H_0 with Lipschitz constant $\tau < 1/2$, assume that $0 \in \Sigma$, and let u be a solution of $\operatorname{div}(A\nabla u) = 0$ in the domain $\Omega = \{x + re_d \,|\, x \in \Sigma \cap B(0,1), r \in (0,1)\}$. Then, there exist positive constants M and δ depending on the Lipschitz constant L_A and the ellipticity constant Λ_A of $A(x)$ such that the following statement holds. If $u \geq -1$ in $\Omega \cap B(0,1)$, $u \equiv 0$ on Σ and $u \geq M$ on $\{x + re_d \,|\, x \in \Sigma \cap B(0,1), r \in (\delta, 1)\} \cap B(0,1)$, then $u \geq 0$ in $\Omega \cap B(0, 1/2)$.*

The proof of this lemma is Step 1 of the proof of [11, Theorem 1.1]. For a simpler proof in the harmonic case (in Lipschitz domains with small Lipschitz constant), see Appendix B in [13].

In the following proof, we will assume again that L_A and $\Lambda_A - 1$ are very small (see Remark 2.3).

Proof of Lemma 5.1 Let $S \gg 1$ and Q be a cube of our Whitney cube structure such that x_Q (the center of Q) and $S\ell(Q)$ are admissible and $N(x_Q, S\ell(Q)) \leq N$. Note that, to attain this, we need the Lipschitz constant of the domain τ small enough depending on S. Further, we assume that $S\ell(Q) = 8$ by rescaling the domain and the Whitney cube structure. This rescaling changes the Lipschitz constant L_A of the matrix $A(x)$ corresponding to the elliptic operator. But if the cube Q is small enough, the rescaling improves it, that is, makes L_A smaller.

Let $\tilde{x}$ be the projection (in the direction e_d) of x on Σ. Then, if S is big enough, we have

$$\begin{aligned}\log\frac{H(\tilde{x}, S\ell(Q)/2^k)}{H(\tilde{x}, S\ell(Q)/2^{k+1})} &\leq \log c + \log\frac{H(x_Q, S\ell(Q)/2^{k-1})}{H(x_Q, S\ell(Q)/2^{k+2})} \\ &\leq \log c + Nc', \ \forall k \in \{1, \ldots, 5\}. \end{aligned} \tag{5.1}$$

by Lemmas 3.10 and 3.13 (since we do not assume $A(x_Q) = I$ or $A(\tilde{x}) = I$, we use first Remark 3.9). We need S large enough so that $B(\tilde{x}, \Lambda_A S\ell(Q)/2^k) \subset B(x_Q, S\ell(Q)/2^{k-1})$

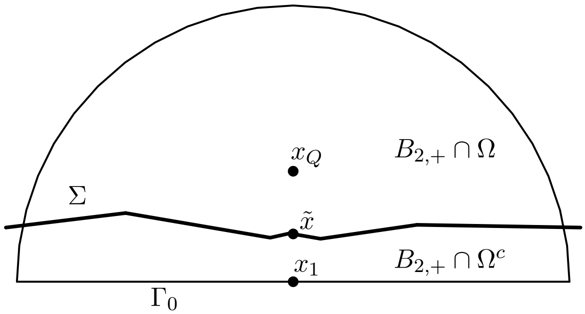

Fig. 1 The domain Ω, the Lipschitz graph Σ, and the half ball $B_{2,+}$

and $B(\tilde{x}, S\ell(Q)/2^{k+1}) \supset B(x_Q, \Lambda_A S\ell(Q)/2^{k+2})$ (we are using that Λ_A is very close to 1) for $k = 1, \dots, 5$. From now on, S is fixed.

We also fix the following normalization for u:

$$\sup_{B(\tilde{x},3)\cap\Omega} |u| = 1 \tag{5.2}$$

(remember that $S\ell(Q) = 8$).

Let $x_1 = \tilde{x} - 3\tau e_d$. Observe that

$$\Gamma_0 = \{x = (x', x'') \in B(x_1, 2) : x'' = x_1''\}$$

doesn't intersect $\overline{\Omega}$. Also let $B_1 := B(x_1, 1)$, $B_2 := B(x_1, 2)$ and $B_{k,+}$ the upper half of B_k, $k = 1, 2$ (the half of B_k that intersects Ω), see Fig. 1.

Let g_0 be the solution of $\operatorname{div}(A\nabla g_0) = 0$ on $B_{2,+}$ such that $g_0 \equiv 1$ on $\partial B_{2,+}\setminus\Gamma_0$ and $g_0 \equiv 0$ on Γ_0. By the maximum principle $g_0 \geq 0$ on $B_{2,+}$ and $g_0 \geq |u|$ on $\Omega \cap B_2 \subset B_{2,+}$ because of the normalization (5.2) of u. Notice that, moreover, we have the bound

$$g_0(x) \leq C_1(x'' - x_1'')$$

for $x \in B_{1,+}$ which gives us a bound for $|u|$ in $\Omega \cap B_1$. In the case $A \equiv I$, this follows from reflection and interior Cauchy estimates for ∇u. In the general case, we may use that $g_0(x)$ is comparable to the Green function $G_A(x, y)$ of the domain $B_{2,+}$ (with pole $y = (0, \dots, 0, 1.5)$ for example) by the boundary Harnack inequality inside $B_{1,+}$. By [17, Theorem 3.3], since $B_{2,+}$ satisfies an exterior sphere condition, we have that $G(x, y) \lesssim \operatorname{dist}(x, \partial B_{2,+})|x - y|^{1-d} \approx \operatorname{dist}(x, \partial B_{2,+})$. Further, for $x \in B_{1,+}$ we have that $\operatorname{dist}(x, \partial B_{2,+}) = x'' - x_1''$ which gives us the desired bound.

Let g be the solution of $\operatorname{div}(A\nabla g) = 0$ in $B_{1,+}$ with Dirichlet boundary conditions $g \equiv u$ on $\partial B_{1,+} \cap \Omega$ and $g \equiv 0$ on $\partial B_{1,+}\setminus\Omega$. Since $|u| \leq g_0$ in $B_{2,+} \cap \Omega$, we have $|g| \leq g_0 \leq C_1(x'' - x_1'')$. Also,

$$|g - u| = \begin{cases} 0, & \text{on } \Omega \cap \partial B_{1,+}, \\ |g| \leq 4C_1\tau, & \text{on } \partial\Omega \cap B_{1,+}, \end{cases} \tag{5.3}$$

because of the bound of $|g| \leq g_0 \leq C_1(x'' - x_1'')$.

By (5.1), Remark 3.9, Lemma 3.12, Lemma 3.11, Proposition 3.2, and (5.2), we have

$$\int_{\partial B(\tilde{x},\frac{1}{8})\cap\Omega} |u|^2 \mathrm{d}\sigma \gtrsim \mathrm{e}^{-c'N} \int_{\partial B(\tilde{x},4)\cap\Omega} |u|^2 \mathrm{d}\sigma \approx \mathrm{e}^{-c'N} H(\tilde{x},4) \gtrsim \mathrm{e}^{-c'N} \int_{B(\tilde{x},4)} |u|^2 \mathrm{d}y$$
$$\geq \mathrm{e}^{-c'N} \int_{B(x_{\sup},1)} |u|^2 \mathrm{d}y \gtrsim \mathrm{e}^{-c'N} |u(x_{\sup})|^2 = \mathrm{e}^{-c'N}$$

where $x_{\sup}$ is a point in $\partial B(\tilde{x},3)$ where $|u(x_{\sup})| = 1$. Assuming that τ is small enough, we have $B(\tilde{x},\frac{1}{8}) \cap \Omega \subset \frac{1}{4}B_{1,+}$. Using (5.3) and Lemma 3.13, we get

$$\left(\int_{\partial \frac{1}{4}B_{1,+}} g^2 \mathrm{d}\sigma\right)^{1/2} \geq \left(\int_{\partial \frac{1}{4}B_{1,+}} u^2 \mathrm{d}\sigma\right)^{1/2} - C_2\tau$$
$$\geq \left(\tilde{c}\int_{\partial B\left(\tilde{x},\frac{1}{8}\right)} u^2 \mathrm{d}\sigma\right)^{1/2} - C_2\tau.$$

If we assume that τ (depending on N) is small enough, we conclude that

$$\|g\|_{L^2\left(\partial\left(\frac{1}{4}B_{1,+}\right)\right)} \geq c_1 \mathrm{e}^{-c'N/2}. \tag{5.4}$$

Now let's estimate the doubling of $H^g(x_1,r)$ for g by using (5.2) and (5.4):

$$\log \frac{\int_{\partial \frac{1}{2}B_{1,+}} g^2 d\sigma}{\int_{\partial \frac{1}{4}B_{1,+}} g^2 d\sigma} \leq \log \frac{\sigma(\partial \frac{1}{2}B_1)}{c_1 \mathrm{e}^{-c'N/2}} = \log c'' + c'N/2 \leq C(N+1).$$

This gives us an upper bound for $N^g(x_1, 1/4)$, the frequency for g at x_1. Also note that (5.4) implies

$$\sup_{\frac{1}{4}B_{1,+}} |g| \geq c\mathrm{e}^{-c'N/2}.$$

Then, by Lemma 5.3, there exists $x_* \in \Gamma_0 \cap B_{1/S}$, $c_2 = c_2(C(N+1), S) > 0$, and $\rho = \rho(C(N+1), S)$ such that

$$|g(x)| \geq c_2\left(x'' - x_1''\right) \text{ for } x = \left(x', x''\right) \in B\left(x_*, \rho\right) \cap B_{1,+}.$$

We may assume that $g > 0$ in $B(x_*,\rho) \cap B_{1,+}$, otherwise we consider $-u$ and $-g$ in place of u and g. From (5.3), we obtain

$$u(x) \geq g(x) - 4C_1\tau \geq c_2\left(x'' - x_1''\right) - 4C_1\tau \quad \text{in} \quad B\left(x_*,\rho\right) \cap \Omega. \tag{5.5}$$

Note that ρ does not depend on τ and for τ small enough we have $B\left(x_*, \frac{\rho}{4}\right) \cap \partial\Omega \neq \varnothing$.

Our goal is to show that $u > 0$ on $B(x_*, \frac{\rho}{2}) \cap \Omega$ and we will use Lemma 5.4 to this end (from now on, constants δ and M come from the statement of Lemma 5.4). Restrict τ to be small enough so that $u > 0$ in $B(x_*,\rho) \cap \Omega \cap \{(x',x'')|x'' > x_*'' + \delta\rho\}$, which we can do thanks to (5.5). Now, we multiply the function u by a very large constant K so that $Ku \geq M$ in the same set $B(x_*,\rho) \cap \Omega \cap \{(x',x'')|x'' > x_*'' + \delta\rho\}$. Finally, we make τ even smaller so that $Ku \geq -1$ in $B(x_*,\rho) \cap \Omega$, thanks again to (5.5). Now, by a rescaled version of Lemma 5.4 to $B(x_*,\rho) \cap \Omega$, the function Ku restricted to $B(x_*,\rho/2) \cap \Omega$ is positive, which implies that u is positive too and ends the proof. □

6 Proof of Theorems 1.1 and 1.5

We will combine Lemmas 4.1 and 5.1 to prove Theorems 1.1 and 1.5.

Note that, in the present section, we refer to the frequency function N of a solution of $\operatorname{div}(A\nabla u) = 0$ in Ω as in the statement of Theorem 1.1. Without further mention, we will use that the constants L_A and $\Lambda_A - 1$ from the matrix $A(x)$ are very small thanks to Remark 2.3. Moreover, we consider Ω with the Whitney structure defined in Sect. 4.1.

First, we give a corollary to Lemma 5.1 in a language closer to that of Lemma 4.1.

Corollary 6.1 *For any $N > 0$ and $S \gg 1$ large enough there exist positive constants $\tau_0(N, S)$ and $K(N, S)$ such that the following statement holds. Suppose $\Omega \subset \mathbb{R}^d$ has Lipschitz constant $\tau < \tau_0$ and Q is a cube in $\mathcal{D}_{\mathcal{W}}(R)$ such that $N(x_Q, S\ell(Q)) \leq N$. Then, for all $\tilde{K} \geq K$, there exist cubes $Q''_1, \dots, Q''_{2^{(d-1)(\tilde{K}-K)}}$ such that, for all j,*

1. *the center of Q''_j lies in Σ,*
2. *$u|_{Q''_j \cap \Omega}$ does not have zeros,*
3. *there exists $Q'_j \in \mathcal{D}^{\tilde{K}}_{\mathcal{W}}(Q)$ such that Q''_j is a vertical translation of Q'_j.*

In particular, there exists $\delta_0(N, S) > 0$ such that

$$m_{d-1}\left(\bigcup_i \Pi(Q'_i)\right) \geq \delta_0 m_{d-1}(\Pi(Q)).$$

These cubes Q''_j are the cubes that are contained in the ball B given by Lemma 5.1 and are vertical translation of cubes in $\mathcal{D}^{\tilde{K}}_{\mathcal{W}}(Q)$. From now on, given a cube $Q \in \mathcal{D}_{\mathcal{W}}(R)$, we denote by $t(Q)$ the unique cube Q' such that its center lies on Σ and Q' is a vertical translation of Q.

Next, we present a modified frequency function for which we prove good behavior as a consequence of Lemmas 4.1 and 5.1.

6.1 Modified frequency function

Let R be a cube in $\mathcal{W}$ such that it satisfies the conditions of Lemma 4.1 with $S = S_1 \gg 1$ large enough so that $CS_1^{-1/2}$ (also from the statement of Lemma 4.1) is small enough (we will specify the precise relation later). The use of this lemma gives us constants $K_1 := K(S_1)$ and $\delta_1 := \delta_0$ (that does not depend on S_1).

Consider also Corollary 6.1 with fixed $N = 2N_0 + 1$ (where N_0 is the constant of Lemma 4.1) and $S = S_2$ large enough but smaller than S_1 (note that both constants are independent). The use of this corollary gives us constants $K_2 := K(N, S_2)$ and $\delta_2 := \delta_0(N, S_2)$. In particular, we may assume that K_2 is smaller or equal than K_1 and that $CS_1^{-1/2}$ is small enough depending on $\min(\delta_1, \delta_2)$.

We remark that the use of Lemma 4.1 and Corollary 6.1 with constants S_1 and S_2 respectively requires that the domain has small enough Lipschitz constant τ. For the rest of this section, we will denote $\epsilon := CS_1^{-1/2}$, $K := K_1$, and $\delta_0 = \min(\delta_1, \delta_2)$ (in particular, δ_0 and S_1 are independent constants).

Define

$$N(Q) := N(x_Q, S_1\ell(Q))$$

for any $Q \in \mathcal{D}_{\mathcal{W}}^{jK}(R)$ for $j \geq 0$. Notice that $1 + 2N(Q) \geq N(x_Q, S_2\ell(Q))$ thanks to Proposition 3.8 (we assume x_Q and $S_1\ell(Q)$ are admissible for all children Q of R in the Whitney tree structure, thanks to τ being small enough).

We define the modified frequency function $N'(Q)$ for $Q \in \mathcal{D}_{\mathcal{W}}^{jK}(R)$, $j \geq 0$, inductively. For $j = 0$, we define $N'(R) = \max(N(R), N_0/2)$. Assume we have $N'(P)$ defined for all cubes $P \in \mathcal{D}_{\mathcal{W}}^{iK}(R)$ for $0 \leq i < j$. Fix $\widehat{Q} \in \mathcal{D}_{\mathcal{W}}^{(j-1)K}(R)$ and consider its vertical translation $t(\widehat{Q})$ centered on Σ. Then:

(a) if u restricted to $t(\widehat{Q}) \cap \Omega$ has no zeros, define $N'(Q) = N'(\widehat{Q})/2$ for $\lceil \delta_0 \cdot \mathrm{card}\{\mathcal{D}_{\mathcal{W}}^{K}(\widehat{Q})\}\rceil$ of the cubes Q in $\mathcal{D}_{\mathcal{W}}^{K}(\widehat{Q})$ and $N'(Q) = (1+\epsilon)N'(\widehat{Q})$ for the rest of cubes in $\mathcal{D}_{\mathcal{W}}^{K}(\widehat{Q})$ (the particular choice is irrelevant),

(b) if u restricted to $t(\widehat{Q}) \cap \Omega$ has zeros, choose $Q \in \mathcal{D}_{\mathcal{W}}^{K}(\widehat{Q})$, and

   1. if its vertical translation $t(Q)$ satisfies that u restricted to $t(Q) \cap \Omega$ has no zeros, define $N'(Q) = N'(\widehat{Q})/2$,
   2. else define $N'(Q) = \max(N(Q), N_0/2)$.

Note that if a cube Q satisfies that u restricted to $t(Q) \cap \Omega$ has no zeros, then all its descendants in the Whitney cube structure will satisfy the same property and (a) applies to them. Alternatively, if a cube Q satisfies that u restricted to $t(Q) \cap \Omega$ changes sign, then all its predecessors in the Whitney cube structure will satisfy the same and (b2) applies to them.

Now, a combination of Lemma 4.1 and Corollary 6.1 yields the following behavior for $N'(Q)$ for $Q \in \mathcal{D}_{\mathcal{W}}^{jK}(R)$, $j \geq 0$. Consider a cube $\widehat{Q}$ and its vertical translation $t(\widehat{Q})$. Then:

- If u restricted to $t(\widehat{Q}) \cap \Omega$ has zeros and $N(\widehat{Q}) \geq N_0$, then Lemma 4.1 tells us that at least $\lceil \delta_0 \cdot \mathrm{card}\{\mathcal{D}_{\mathcal{W}}^{K}(\widehat{Q})\}\rceil$ cubes in $\mathcal{D}_{\mathcal{W}}^{K}(\widehat{Q})$ satisfy $N(Q) \leq N(\widehat{Q})/2$. Moreover, in this case, $N'(Q) = \max(N(Q), N_0/2) \leq N(\widehat{Q})/2 = N'(\widehat{Q})/2$ where we have used that $N(\widehat{Q}) > N_0$ and that (b) applies. For the rest of the cubes Q in $\mathcal{D}_{\mathcal{W}}^{K}(\widehat{Q})$, we have $N'(Q) = \max(N(Q), N_0/2) \leq (1+\epsilon)N(\widehat{Q}) = (1+\epsilon)N'(\widehat{Q})$ where we have used again that $N(\widehat{Q}) > N_0$ and (b) applies.
- If u restricted to $t(\widehat{Q}) \cap \Omega$ has zeros and $N(\widehat{Q}) < N_0$, then Corollary 6.1 enters in play and it tells us that at least $\lceil \delta_0 \cdot \mathrm{card}\{\mathcal{D}_{\mathcal{W}}^{K}(\widehat{Q})\}\rceil$ cubes Q in $\mathcal{D}_{\mathcal{W}}^{K}(\widehat{Q})$ satisfy that $t(Q) \cap \Omega$ does not contain zeros of u. For these cubes, $N'(Q) = N'(\widehat{Q})/2 = \max(N(\widehat{Q}), N_0/2)/2 < N_0/2$. For the rest of the cubes Q in $\mathcal{D}_{\mathcal{W}}^{K}(\widehat{Q})$, we have $N'(Q) = \max(N(Q), N_0/2) \leq \max((1+\epsilon)N(\widehat{Q}), N_0/2) \leq (1+\epsilon)N'(\widehat{Q})$.
- If u restricted to $t(\widehat{Q}) \cap \Omega$ has no zeros, then we have defined $N'(Q) = N'(\widehat{Q})/2$ for $\lceil \delta_0 \cdot \mathrm{card}\{\mathcal{D}_{\mathcal{W}}^{K}(\widehat{Q})\}\rceil$ cubes Q in $\mathcal{D}_{\mathcal{W}}^{K}(\widehat{Q})$. We have defined $N'(Q) = (1+\epsilon)N'(\widehat{Q})$ for the rest of cubes Q in $\mathcal{D}_{\mathcal{W}}^{K}(\widehat{Q})$.

Summing up, for $\widehat{Q} \in \mathcal{D}_{\mathcal{W}}(R)$ and random $Q \in \mathcal{D}_{\mathcal{W}}^{K}(\widehat{Q})$, we have

$$N'(Q) \leq \begin{cases} N'(\widehat{Q})/2, & \text{with probability at least } \delta_0, \\ N'(\widehat{Q})(1+\epsilon), & \text{with probability at most } 1-\delta_0. \end{cases}$$

This is similar to the behavior of the frequency function N given by Lemma 4.1 but without the restriction $N'(\widehat{Q}) \geq N_0$.

Let's summarize the dependence of the constants that have appeared in this section (omitting its dependence on the dimension d and on L_A and Λ_A by assuming we are in a setting like the one described in Remark 2.3). On one hand, the constants K_2 and δ_2 given by Corollary 6.1 are absolute since they depend on S_2 and $N = 2N_0 + 1$ which also are absolute constants. We do require τ small enough to use Corollary 6.1. On the other hand, we have

K_1 depending on S_1, $\epsilon = CS_1^{-1/2}$, and an absolute constant δ_1 given by Lemma 4.1. To use Lemma 4.1, we require τ small enough depending on S_1. For the arguments that follow, we need ϵ small enough depending on $\delta_0 = \min(\delta_1, \delta_2)$. Though, since both constants (ϵ and δ_0) are independent, we can choose S_1 large enough and, thus, we require τ small enough (depending on S_1) to use Lemma 4.1.

6.2 Proof of Theorem 1.1

The idea behind the proof is that Lemma 4.1 allows us to use that most cubes in any generation of the Whitney tree satisfy $N(x_Q, S\ell(Q)) \leq N_0$. Then, we can apply Lemma 5.1 to these cubes, thus covering most of Σ by balls where u does not change sign.

Proof of Theorem 1.1 We will prove the result for the projection of a single cube $\Pi(R)$. Afterwards, we can cover any compact in Σ by a finite union of such cubes which leaves stable the Minkowski dimension estimate.

For $x \in \Pi(R)$, we denote by $Q_j(x)$ the unique cube $Q_j \in \mathcal{D}_{\mathcal{W}}^{jK}(R)$ such that $x \in \Pi(Q_j)$ for some integer K large enough that will be fixed later. We will say that $Q_j(x)$ is a *good* cube if $N'(Q_j) \leq \frac{1}{2} N'(Q_{j-1})$ and that it is *bad* otherwise.

Remark 6.2 Note that, with the previous definitions, $N'(Q) < N_0/2$ implies that for all $x \in t(Q) \cap \Sigma$ there is a neighborhood where u does not vanish in Ω. Thus we only need to study the Minkowski dimension of the set of points $x \in \Pi(R)$ such that they are not in $\Pi(Q)$ for some $Q \in \mathcal{D}_{\mathcal{W}}(R)$ with $N'(Q) < N_0/2$. Also, notice that the map $\Pi : \Sigma \cap \Pi^{-1}(\Pi(R)) \mapsto \Pi(R)$ is biLipschitz and thus it preserves Minkowski dimensions.

We define the *goodness frequency* of a point $x \in \Pi(R)$ as

$$F_j(x) = \frac{1}{j} \#\{\text{good cubes in } Q_1(x), \ldots, Q_j(x)\}$$

for $j \in \mathbb{N}$. We define $\overline{F}(x) = \limsup_{n \to \infty} F_n(x)$. Let $\alpha(\delta_0) > 0$ be such that

$$\frac{\delta_0}{1 - \delta_0} \frac{1 - \alpha}{\alpha} = 3$$

(in particular, $\alpha < \delta_0$) and $\epsilon_0(\alpha) > 0$ such that

$$\alpha = \frac{\log(1 + \epsilon_0)}{\log(1 + \epsilon_0) + \log 2}.$$

Note that for any $0 < \epsilon < \epsilon_0$, we have

$$\left(\frac{1}{2}\right)^{\alpha} (1 + \epsilon)^{1-\alpha} < 1. \tag{6.1}$$

For all $j > 1$, define

$$\mu_j = \frac{1}{j} \log_2 \left(\frac{2N'(R)}{N_0}\right) \geq 0.$$

Claim *For all $j > 1$, the following holds*

$$F_j(x) \geq \alpha + \mu_j \implies N'(Q_j(x)) < \frac{N_0}{2}.$$

Proof of claim We have that

$$N'(Q_j(x)) \leq \left(\frac{1}{2}\right)^{j(\alpha+\mu_j)} (1+\epsilon)^{j(1-\alpha-\mu_j)} N'(R) < \left(\frac{1}{2}\right)^{j\mu_j} (1+\epsilon)^{-j\mu_j} N'(R) < \frac{N_0}{2}$$

by (6.1), and the definition of $F_j(x)$ and μ_j. □

Now, thanks to the previous claim and Remark 6.2, we can reduce the problem to studying the Minkowski dimension of the set of points

$$E = \{x \in \Pi(R) \mid F_j(x) < \alpha + \mu_j, \ \forall j \in \mathbb{N}\}.$$

If we consider a random sequence of cubes $(Q_j)_j$ with $Q_0 = R$ and $Q_j \in \mathcal{D}^K_{\mathcal{W}}(Q_{j-1})$, and let $x \in \bigcap_{j\geq 0} \Pi(Q_j)$, the probability that $F_j(x) \leq \beta_j$ for $j \in \mathbb{N}$ and $\beta_j \in (0, 1)$ is bounded above by

$$\sum_{i=0}^{\lfloor j\beta_j \rfloor} \binom{j}{i} \delta_0^i (1-\delta_0)^{j-i}.$$

Note that choosing randomly such a sequence is equivalent to choosing a random $x \in \Pi(R)$ uniformly. In what follows we will assume that β_j satisfy $2 < \frac{\delta_0}{1-\delta_0}\frac{1-\beta_j}{\beta_j} < 4$ for all $j > 0$, in particular $\beta < \delta_0$. Let's find an upper bound for the previous quantity for very large j:

$$\boxed{A}_j := \sum_{i=0}^{\lfloor j\beta_j \rfloor} \binom{j}{i} (1-\delta_0)^{j-i} \delta_0^i = (1-\delta_0)^j \sum_{i=0}^{\lfloor j\beta_j \rfloor} \binom{j}{i} \left(\frac{\delta_0}{1-\delta_0}\right)^i.$$

Observe that for $\beta_j < 1/2$ we have

$$\binom{j}{k-1} < \frac{\beta_j}{1-\beta_j}\binom{j}{k}, \quad \text{for } 0 < k \leq \lfloor \beta_j j \rfloor.$$

This is because

$$\frac{\binom{j}{k-1}}{\binom{j}{k}} = \frac{k!(j-k)!}{(k-1)!(j-k+1)!} = \frac{k}{j-k+1} < \frac{\beta_j}{1-\beta_j}.$$

Iterating this inequality, we obtain

$$\binom{j}{i} < \left(\frac{\beta_j}{1-\beta_j}\right)^{\lfloor j\beta_j \rfloor - i} \binom{j}{\lfloor j\beta_j \rfloor}, \quad \text{for } i < \lfloor j\beta_j \rfloor.$$

Using this observation we can bound $\boxed{A}_j$ by

$$(1-\delta_0)^j \sum_{i=0}^{\lfloor j\beta_j \rfloor} \binom{j}{i} \left(\frac{\delta_0}{1-\delta_0}\right)^i \leq (1-\delta_0)^j \left(\frac{\beta_j}{1-\beta_j}\right)^{\lfloor j\beta_j \rfloor} \binom{j}{\lfloor j\beta_j \rfloor} \sum_{i=0}^{\lfloor j\beta_j \rfloor} \left(\frac{\delta_0}{1-\delta_0}\frac{1-\beta_j}{\beta_j}\right)^i.$$

We use Stirling's formula to approximate

$$\binom{j}{\lfloor j\beta_j \rfloor} \approx \frac{\sqrt{2\pi j}\left(\frac{j}{e}\right)^j}{\sqrt{2\pi\beta_j j}\left(\frac{\beta_j j}{e}\right)^{\beta_j j}\sqrt{2\pi(1-\beta_j)j}\left(\frac{(1-\beta_j)j}{e}\right)^{(1-\beta_j)j}}$$

$$= \frac{1}{\sqrt{2\pi\beta_j(1-\beta_j)j}}\left(\frac{1}{\beta_j^{\beta_j}(1-\beta_j)^{1-\beta_j}}\right)^j.$$

We also estimate

$$\left(\frac{\beta_j}{1-\beta_j}\right)^{\lfloor j\beta_j \rfloor} \approx \left(\frac{\beta_j}{1-\beta_j}\right)^{j\beta_j} \quad \text{and} \quad \sum_{i=0}^{\lfloor j\beta_j \rfloor} \left(\frac{\delta_0}{1-\delta_0}\frac{1-\beta_j}{\beta_j}\right)^i \approx \left(\frac{\delta_0}{1-\delta_0}\frac{1-\beta_j}{\beta_j}\right)^{j\beta_j}.$$

Observe that the comparability constants in the previous approximations depend on the upper and lower bounds of β_j but not on j for j large enough depending on δ_0.

Summing everything up, we obtain

$$\boxed{A}_j \lesssim \frac{2}{\sqrt{2\pi\beta_j(1-\beta_j)j}} \underbrace{\left(\frac{(1-\delta_0)^{1-\beta_j}\delta_0{}^{\beta_j}}{\beta_j^{\beta_j}(1-\beta_j)^{1-\beta_j}}\right)^j}_{z(\beta_j)^j},$$

for j large enough. Observe that $z(\beta_j) < 1$ for $\beta_j < \delta_0$.

Now, we will choose a suitable covering of the set E by projections of cubes in $\mathcal{D}_{\mathcal{W}}^{jK}(R)$. First, pick $\epsilon < \epsilon_0$ (equivalently S_1 large enough, recall the discussion in Sect. 6.1). Observe that by choosing ϵ we also fix K. For $j \geq 1$, set

$$E_j := \{x \in \Pi(R) \mid F_j(x) \leq \alpha + \mu_j\},$$

so that $E = \bigcap_j E_j$. Let's upper bound the (K-adic) Minkowski dimension of E by finding a certain cover of E_j by projections of cubes in $\mathcal{D}_{\mathcal{W}}^{jK}(R)$ (note that there are $M = 2^{(d-1)K}$ cubes in $\mathcal{D}_{\mathcal{W}}^{K}(R)$).

Using the previous asymptotics (setting $\beta_j = \alpha + \mu_j$), we can cover E_j (for j large enough) with

$$CM^j \frac{2}{\sqrt{2\pi(\alpha+\mu_j)(1-\alpha-\mu_j)j}} z(\alpha+\mu_j)^j$$

projections of cubes in $\mathcal{D}_{\mathcal{W}}^{jK}(R)$ and each of those cubes has side length $M^{-j/(d-1)}$.

Now we are ready to upper bound the Minkowski dimension of the set E. We will use the following definition of upper Minkowski dimension

$$\dim_{\overline{\mathcal{M}}} E = \limsup_{j\to\infty} \frac{\log \#(\{K-\text{adic cubes } Q \text{ of side length } K^{-j} \text{ that satisfy } Q \cap E \neq \varnothing\})}{j \log K}.$$

which is equivalent to the dyadic one in (1.1). By covering a single set $E_j \supset E$ and making $j \to \infty$, we obtain the following upper Minkowski dimension estimate

$$\begin{aligned}\dim_{\overline{\mathcal{M}}}(E) &\leq \lim_{j\to\infty} \frac{j(\ln M + \ln z(\alpha+\mu_j)) + \ln\left(\frac{2}{\sqrt{2\pi(\alpha+\mu_j)(1-\alpha-\mu_j)j}}\right)}{(d-1)^{-1} j \ln M} \\ &= (d-1)\frac{\ln M + \ln z(\alpha)}{\ln M} < d-1\end{aligned}$$

since $z(\alpha) < 1$. ◻

6.3 Planar case of Theorem 1.1

In the planar case, Theorem 1.1 asserts that we can cover any compact $K \subset \Sigma$ by balls where u does not change sign inside apart from a finite set of points. Moreover, this is valid for general Lipschitz domains (and domains with worse boundary regularity).

Proof of Theorem 1.1 in the planar case Without loss of generality, suppose that Ω is simply connected and bounded.

By [4, Theorem 16.1.4], there exists a K-quasiconformal map $\phi : \mathbb{C} \to \mathbb{C}$ such that $u = w \circ \phi$ with w harmonic in $\phi(\Omega)$. Since ϕ is biHölder continuous it is enough to prove the desired result for harmonic functions in Jordan domains.

Now, consider a conformal mapping between $\phi(\Omega)$ and the disk $\mathbb{D}$ which extends to a homeomorphism up the boundary, and let v be the induced harmonic function in the disk.

Denote by $\tilde{\Sigma}$ the open set in $\partial\mathbb{D}$ where v vanishes and by $\tilde{E}$ the set of points in $\tilde{\Sigma}$ such that for every neighborhood v changes sign. Then $\tilde{E}$ must be a discrete set. This is because it coincides with the zero set of ∇v which is holomorphic (we can locally extend v to $\mathbb{D}^c$ by reflection using the Kelvin transform near $\tilde{\Sigma}$). Thus, since it is a discrete set, it is countable and finite inside any compact. Note that all maps we have considered are homeomorphisms, thus the set where v changes sign in every neighborhood is transported to another countable discrete set in $\Sigma \subset \overline{\Omega}$. □

6.4 Estimates on the measure of nodal sets in the interior of the domain

Theorem 6.1 in [24] estimates the $(d-1)$-dimensional Hausdorff measure of the nodal set in a cube in terms of its *doubling index*, which is a quantity intimately related to the frequency function used in this paper. We present the following reformulation of Logunov's theorem avoiding the use of doubling indices.

Theorem 6.3 *There exist positive constants r, R, C depending on the Lipschitz constant L_A of $A(x)$, ellipticity constant Λ_A of $A(x)$, and dimension d such that the following statement holds. Let u be a solution of* $\operatorname{div}(A\nabla u) = 0$ *on $B(0, R) \subset \mathbb{R}^d$. Then, for any cube $Q \subset B(0, r)$, we have*

$$\mathcal{H}^{d-1}(\{u = 0\} \cap Q) \leq C\ell(Q)^{d-1}(N(x_Q, 16\,\mathrm{diam}(Q)) + 1)^{\alpha}$$

for certain $\alpha = \alpha(d) > 0$ and where x_Q is the center of Q.

Remark 6.4 If we assume $N(x_Q, 16\,\mathrm{diam}(Q)) > N_0$ for some N_0 positive, we can rewrite the previous theorem as

$$\mathcal{H}^{d-1}(\{u = 0\} \cap Q) \leq C\ell(Q)^{d-1}(N(x_Q, 16\,\mathrm{diam}(Q)))^{\alpha}$$

where now C depends also on N_0 and α.

6.5 Proof of Theorem 1.5

To prove Theorem 1.5 we will follow the ideas of [28] but exchanging the use of the Donnelly–Fefferman estimate for the size of nodal sets (see [10]) by Logunov's estimate (Theorem 6.3). This gives rise to a worse estimate (polynomial in the frequency) than the one of [28] (which is linear in the frequency but only valid for harmonic functions).

Proof of Theorem 1.5 Let Q be a small enough cube of the Whitney structure (so that we can use the modified frequency function defined in Sect. 6.1). Again, we will use the notation $t(Q)$ for the unique cube Q' with center on Σ such that Q' is a vertical translation of Q.

Claim *The following equation holds*

$$\mathcal{H}^{d-1}(\{u=0\}\cap t(Q)\cap K) \le C_0 N'(Q)^{\alpha}\ell(Q)^{d-1} \tag{6.2}$$

for any K compact inside Ω with C_0 and α independent of K.

Proof of claim Note that Eq. (6.2) holds for all cubes Q small enough, since $t(Q)\cap K=\varnothing$ if Q is small enough. This is because $\operatorname{dist}(K,\Sigma)>0$. We will proceed to prove this estimate by induction going from small cubes to large cubes.

Assume it holds for all small cubes with $\ell(Q) < s$. Now choose a larger cube P in the Whitney cube structure with $\ell(P) < 2^K\ell(Q)$ (with K given by Lemma 4.1 as discussed in Sect. 6.1). Given such a cube P, we can cover $t(P)\cap\Omega$ with small cubes $t(Q_i)$ (intersecting the boundary) where $Q_i\in\mathcal{D}^K_{\mathcal{W}}(P)$ and with small cubes Q'_i far from the boundary (small enough so that we can apply Theorem 6.3 on them).

Using that the Lipschitz constant of $\partial\Omega$ is small, we can bound the number of small cubes Q'_i necessary. Moreover, using Lemma 3.14, we can bound $N(x_{Q'}, 16\operatorname{diam}(Q')) < 2N(P)+1$. Note that $N(P)\le N'(P)$ in the case that $t(P)\cap\Omega\cap\{u=0\}\neq\varnothing$. This allows us to bound the size of the nodal set on $\bigcup Q'_i\cap\Omega$ using Theorem 6.3 by

$$\sum_{Q'_i}\mathcal{H}^{d-1}(\{u=0\}\cap Q'_i)\le C_1N'(P)\ell(P)^{d-1}.$$

We still need to bound the size of the nodal set in the boundary cubes $t(Q_i)$, which satisfy $N'(Q_i)\le(1+\epsilon)N'(P)$. Moreover, we know that at least for $\lceil\delta_0\operatorname{card}\{\mathcal{D}^K_{\mathcal{W}}(P)\}\rceil$ cubes Q_i we have $N'(Q_i)\le\frac{N'(P)}{2}$. Now we can use the induction hypothesis

$$\begin{aligned}\mathcal{H}^{d-1}(\{u=0\}\cap K\cap\bigcup_{Q_i}t(Q_i)) &\le \sum_{N'(Q_i)>N'(P)/2}\mathcal{H}^{d-1}(\{u=0\}\cap K\cap t(Q_i))\\ &\quad+\sum_{N'(Q_i)\le N'(P)/2}\mathcal{H}^{d-1}(\{u=0\}\cap K\cap t(Q_i))\\ &\le \sum_{N'(Q_i)>N'(P)/2}C_0N'(Q_i)^{\alpha}\ell(Q_i)^{d-1}\\ &\quad+\sum_{N'(Q_i)\le N'(P)/2}C_0N'(Q_i)^{\alpha}\ell(Q_i)^{d-1}\\ &\le C_0N'(P)^{\alpha}\ell(P)^{d-1}\left((1+\epsilon)^{\alpha}(1-\delta_0)+\frac{\delta_0}{2^{\alpha}}\right).\end{aligned}$$

We choose ϵ small enough (by increasing S_1) so that

$$(1+\epsilon)^{\alpha}(1-\delta_0)+\frac{\delta_0}{2^{\alpha}}<1$$

noting that δ_0 does not depend on S_1 or ϵ. Finally, we choose C_0 large enough so that it absorbs all terms, that is,

$$C_0\left((1+\epsilon)^{\alpha}(1-\delta_0)+\frac{\delta_0}{2^{\alpha}}\right)+C_1<C_0.$$

Note that the previous estimates do not depend on the compact K chosen and we prove the claim. □

To treat a general (small) ball B centered at Σ, we first cover it by a comparable cube Q centered at Σ. Now, we choose a translate and dilation of the dyadic cube structure of $\mathbb{R}^d$ such that $Q = t(P)$ for some P in the Whitney cube structure and we apply the previous claim. □

Remark 6.5 In the statement of Theorem 1.5, the point $\tilde{x}$ that appears is the center of a particular Whitney cube appearing in the proof. Nonetheless, Lemma 3.14 (and that the Lipschitz constant τ of the boundary is small to preserve admissibility) gives us a lot of freedom to choose $\tilde{x}$.

6.6 ($d-1$)-dimensional Hausdorff measure of Dirichlet eigenfunctions

Theorem 1.5 allows us to study the zero set of solutions of the Dirichlet eigenvalue problem

$$\begin{cases} \operatorname{div}(A\nabla u_\lambda) = -\lambda u_\lambda, & \text{in } \Omega, \\ u_\lambda = 0, & \text{on } \partial\Omega. \end{cases}$$

In fact, one can show that

$$\mathcal{H}^{d-1}(\{u_\lambda = 0\}) \le C(\Omega, \Lambda_A, L_A)\lambda^{\frac{\alpha}{2}} \tag{6.3}$$

for bounded domains Ω with local Lipschitz constant small enough for some $\alpha = \alpha(d) > 1$. For a detailed account of the proof in the harmonic case (and a sharper result), see Section 6 in [28]. We will only briefly sketch the main ideas behind its proof. Also note that this problem is intimately related to *Yau's conjecture on nodal sets of Laplace eigenfunctions in manifolds* (see [10, 24, 25, 27]).

The first step consists in passing from eigenfunctions to solutions of $\operatorname{div}(A\nabla u) = 0$. Consider the function $u(x,t) = u_\lambda(x)e^{\sqrt{\lambda}t}$ in the cylinder domain $\Omega \times \mathbb{R} \subset \mathbb{R}^{d+1}$. Let

$$\tilde{A}(x) = \begin{pmatrix} A(x) & 0 \\ 0 & 1 \end{pmatrix},$$

then we have that u solves

$$\operatorname{div}(\tilde{A}\nabla u) = \underbrace{\operatorname{div}(A\nabla u_\lambda)}_{-\lambda u_\lambda(x)} e^{\sqrt{\lambda}t} + \lambda u_\lambda(x)e^{\sqrt{\lambda}t} = 0.$$

Clearly, we have $\{u(x,t) = 0\} = \{u_\lambda(x) = 0\} \times \mathbb{R}$. Thus, we can restrict us to the study of the nodal set of $u(x,t)$.

The next necessary step is the *Donnelly–Fefferman frequency estimate* [10]. For small balls B contained in the domain, it is shown in [10, 26] that $N^u(x_B, r(B)) \le C(\Omega, \Lambda_A, L_A)\sqrt{\lambda}$. The previous result is also true for balls intersecting the boundary (see Lemma 10 in [28] for a proof in the harmonic case).

Finally, using the previous estimate together with Theorems 1.5 and 6.3, one obtains the result in (6.3).

7 Proof of Corollary 1.6

Theorem 1.1 tells us that we can decompose Σ in its intersection with a countable family of balls $(B_i)_i$ and a set of Hausdorff dimension smaller than $d-1$ by taking an exhaustion of Σ by compacts. Thanks to countable additivity, we only need to prove that

$$\mathcal{H}^{d-1}(\{x \in \Sigma \mid \partial_\nu u(x) = 0\} \cap B) = 0$$

for any ball $B \in (B_i)_i$ given by the decomposition of Theorem 1.1.

Before starting the proof, we define the concept of $\mathcal{A}_\infty$ weight.

Definition 7.1 We say that a measure $\omega \in \mathcal{A}_\infty(\sigma)$ if there exist $0 < \gamma_1, \gamma_2 < 1$ such that for all balls B and subsets $E \subset B$, $\sigma(E) \leq \gamma_1 \sigma(B)$ implies $\omega(E) \leq \gamma_2 \omega(B)$.

Proof of Corollary 1.6 Consider B centered on Σ such that $u|_{B\cap\Omega}$ does not change sign. Without loss of generality, we assume that u is positive.

By Dahlberg's theorem [8], harmonic measure for the domain $B \cap \Omega$ is an $\mathcal{A}_\infty$ weight with respect to surface measure. By [12], since the matrix $A(x)$ is uniformly elliptic and has Lipschitz coefficients, its associated elliptic measure ω_A is another $\mathcal{A}_\infty$ weight. In particular, it is well known that this implies that the density $\frac{d\omega_A}{\mathrm{d}\sigma}$ can only vanish in a set of zero surface measure.

On the other hand, the density of elliptic measure is comparable with $(A\nabla g, \nu)$ at the boundary (where g is the Green function with pole outside $2B$). By the boundary Harnack inequality (see [11] for example), since u is positive in $B \cap \Omega$, we have that $A\nabla u$ on $\Sigma \cap B$ is comparable to $A\nabla g$. This finishes the proof. □

Remark 7.2 There is also a different approach to proving Corollary 1.6, which is the one adopted in [33] for harmonic functions. The ingredients are Lemma 4.1, [2, Lemma 0.2] (which is also valid for the type of PDEs we consider, see the paragraph below the proof of [2, Lemma 2.2], also [33, Lemma 4.3]), and a modification of [33, Lemma 4.1]. In particular, the tools of Sects. 5 and 6 are not indispensable for this result.

8 Proof of Corollary 1.8

Definition 8.1 We define a non-truncated cone $\tilde{C}_\tau$ with aperture $\tau \in \mathbb{R}$ and vertex at 0 as

$$\tilde{C}_\tau(0) = \{(y', y'') \in \mathbb{R}^{d-1} \times \mathbb{R} \mid y'' > \tau |y'|\}.$$

Notice that when $\tau = 0$, then $\tilde{C}_0$ is a half space and when $\tau < 0$, $\tilde{C}_\tau$ is a non-convex cone. A truncated cone $C_{\tau,s}$ is a non-truncated cone $\tilde{C}_\tau$ intersected with the ball $B(0, s)$.

First we will prove the result for harmonic functions.

Proof of Corollary 1.8 in the harmonic case Let $x \in B \cap \Sigma$ for some ball B where u does not vanish given by Theorem 1.1. Assume without loss of generality that $u|_{B\cap\Omega} > 0$.

Since Σ is Lipschitz with Lipschitz constant τ, we can find a small truncated cone $C_{\tau,s}$ of aperture τ with vertex at x and contained in Ω.

Let g be the Green function (for the Laplacian) with pole at infinity of the non-truncated cone $\tilde{C}_\tau$. Since $u|_{C_{\tau,s}}$ is nonnegative up to the boundary, we can lower bound it by some adequate multiple of g (that is $u \gtrsim g$ in $C_{\tau,s}$) by boundary Harnack inequality. Notice that g is of the form $g(x) = g_r(|x|) g_\theta(\frac{x}{|x|})$ where g_θ is the first Dirichlet eigenfunction of the Laplace-Beltrami operator Δ in the domain $\tilde{C}_\tau \cap \partial B(0,1)$ with eigenvalue λ_τ and $g_r(|x|) = |x|^{-\frac{d}{2}+1+\frac{\sqrt{(d-2)^2-4\lambda_\tau}}{2}}$ (see [1, Theorem 1.1]). Thus g_r gives an upper bound on the order of vanishing at the origin. Also notice that $\lambda_0 = 1-d$ (when $\tilde{C}_0$ is a half space) and thus $g_r(|x|) = |x|$. Since the Dirichlet eigenvalues λ_τ of $\tilde{C}_\tau \cap \partial B(0,1)$ for the Laplace-Beltrami operator vary continuously with the aperture τ of $\tilde{C}_\tau$, we can ensure that the order of vanishing of g is close enough to 1 by making τ small enough. The continuity of the variation of the eigenvalues with the aperture can be easily shown using the Rayleigh quotient.

For the lower bound on the order of the vanishing at the origin, consider a cone $\tilde{C}_{-\tau}$ of aperture $-\tau$ (concave cone) and its Green function g with pole at the infinity. Now $g|_{\Omega\cap\tilde{C}} \gtrsim u$ and we can follow the same argument. □

Next, we deal with the elliptic case, but first, we need several lemmas.

Lemma 8.2 *Let u be a solution of* $\operatorname{div}(A\nabla u) = 0$ *in Ω and $x_0 \in \Omega \cup \Sigma$. Then the vanishing order α of u at x_0 satisfies*

$$\alpha(x_0) = \lim_{r\downarrow 0} \log_2 \frac{\sqrt{H(x_0,2r)}}{\sqrt{H(x_0,r)}}$$

when the limit exists.

Proof The proof has two parts. First, we see that

$$\fint_{B(x_0,r)} |u|\,\mathrm{d}x \le C_\alpha r^\alpha \iff \sqrt{H(x_0,r)} \le C'_\alpha r^\alpha.$$

Notice that for r small enough, we have

$$\fint_{B(x_0,\Lambda_A^{-1}r)} |u|\mathrm{d}y \le \left(\fint_{B(x_0,\Lambda_A^{-1}r)} |u(y)|^2\mathrm{d}y\right)^{1/2} \lesssim \sqrt{H(x_0,r)}$$

thanks to Cauchy–Schwarz and Lemma 3.12. On the other hand, if we let $x_* \in \overline{B(x_0,r)}$ be such that $|u(x_*)| = \sup_{B(x_0,r)} |u|$, then

$$\sqrt{H(x_0,r)} \lesssim \sup_{B(x_0,r)} |u| \lesssim \fint_{B(x_*,\Lambda_A^{-1}r)} |u|\mathrm{d}y \lesssim \fint_{B(x_0,2\Lambda_A^{-1}r)} |u|\mathrm{d}y$$

using standard estimates for solutions of elliptic PDEs.

The second part consists in showing that

$$\lim_{r\downarrow 0} \log_2 \frac{\sqrt{H(x_0,2r)}}{\sqrt{H(x_0,r)}} = \alpha$$

implies

$$\sqrt{H(x_0,r)} \le C_\gamma r^\gamma, \quad 0<\gamma<\alpha, \quad \text{and} \quad \lim_{r\to 0} \frac{\sqrt{H(x_0,r)}}{r^\gamma} = \infty, \quad \gamma > \alpha.$$

Fix $\gamma < \alpha$. Then there exists some $k_\gamma > 0$ such that for all $k > k_\gamma$, we have

$$\log_2 \frac{\sqrt{H(x_0,2^{-k+1})}}{\sqrt{H(x_0,2^{-k})}} > \gamma.$$

Then, for all n large enough, we have

$$\sqrt{H(x_0, 2^{-n})} = \sqrt{H(x_0, 1)} \prod_{j=1}^{n} \frac{\sqrt{H(x_0, 2^{-j})}}{\sqrt{H(x_0, 2^{-j+1})}} < C2^{-\gamma(n-k_\gamma)} = C' \left(2^{-n}\right)^{\gamma}.$$

Now, fix $\gamma > \alpha$. As before,there exists some $k_\gamma > 0$ such that for all $k > k_\gamma$, we have

$$\log_2 \frac{\sqrt{H(x_0, 2^{-k+1})}}{\sqrt{H(x_0, 2^{-k})}} < \gamma - \delta.$$

for some small $\delta > 0$. Finally, for all n large enough, we obtain

$$\sqrt{H(x_0, 2^{-n})} = \sqrt{H(x_0, 1)} \prod_{j=1}^{n} \frac{\sqrt{H(x_0, 2^{-j})}}{\sqrt{H(x_0, 2^{-j+1})}} > C2^{-(\gamma+\delta)(n-k_\gamma)} > C' \left(2^{-n}\right)^{\gamma} 2^{\delta n}.$$

Since $\lim_{n\to\infty} 2^{\delta n} = \infty$, this finishes the proof. □

Lemma 8.3 *Let u be a positive solution to* $\operatorname{div}(A\nabla u) = 0$ *in a cone $C_{\tau,s}$ such that $u = 0$ on $\partial C_{\tau,s} \cap B(0, s)$. Assume also that $A(0) = I$. Then, its frequency $N(0, r)$ is bounded above for* $0 < r < \frac{s}{2}$.

Proof Remember that

$$N(0, r) = r \frac{\int_{B(0,r)} (A\nabla u, \nabla u)\mathrm{d}y}{\int_{\partial B(0,r)} \mu u^2 \mathrm{d}\sigma} \approx r \frac{\int_{B(0,r)} |\nabla u|^2 \mathrm{d}y}{\int_{\partial B(0,r)} u^2 \mathrm{d}\sigma}.$$

Using that u extended by 0 outside of $C_{\tau,s}$ is a subsolution, by Caccioppoli's inequality we have

$$\int_{B(0,r)} |\nabla u|^2 \mathrm{d}x \lesssim \frac{1}{r^2} \int_{B(0,2r)} u^2 \mathrm{d}x.$$

By Remark 3 (page 953) in [2], we obtain the doubling property

$$\int_{B(0,4r)} u^2 \,\mathrm{d}x \lesssim \int_{B(0,\Lambda_A^{-1} r)} u^2 \,\mathrm{d}x.$$

uniformly for all r small enough. The remark is stated for convex domains, but the condition needed is that the domain is star-shaped with respect to 0. In the case of cones, this is trivially true. Using Lemma 3.12, we obtain that

$$\fint_{B(0,\Lambda_A^{-1} r)} u^2 \mathrm{d}x \lesssim \fint_{\partial B(0,r)} u^2 \mathrm{d}\sigma.$$

Summing up, we get

$$N(0, r) \approx r \frac{\int_{B(0,r)} |\nabla u|^2 \mathrm{d}x}{\int_{\partial B(0,r)} u^2 \mathrm{d}x} \lesssim 1$$

for all r small enough. □

The following lemma shows that the blow-up of positive solutions in cones converges to the Green function for the Laplacian in the domain $\tilde{C}_\tau$ with pole at ∞ (see the proof of Corollary 1.8 in the harmonic case).

Lemma 8.4 *Let u be a positive solution of* $\operatorname{div}(A\nabla u) = 0$ *in a truncated cone $C_{\tau,s}$ that vanishes on $\partial C_{\tau,s} \cap B(0,s)$ and assume that $A(0) = I$. Consider any sequence of radii $r_k \downarrow 0$ (such that $r_1 < |p|$). Let*

$$u_k(x) := \frac{u(r_k x)}{\left(\int_{\partial B_1} |u(r_k y)|^2 d\sigma(y)\right)^{1/2}}.$$

Then u_k converges in $W^{1,2}_{loc}(\tilde{C}_\tau)$ and in $C^{1,\alpha}_{loc}\left(\overline{\tilde{C}_\tau}\setminus\{0\}\right)$ to a multiple of the Green function g with pole at ∞ for the Laplacian in $\tilde{C}_\tau$ for some exponent $0 < \alpha < 1$. In particular, u and g have the same vanishing order at 0.

Proof Without loss of generality assume that $s > 2$ and $r_1 = 1$. Clearly, $u \in W^{1,2}(C_{\tau,1})$ and $u_k \in W^{1,2}(C_{\tau,1/r_k})$ for all $k > 0$. Also notice that $N^u(0, r_k) = N^{u_k}(0, 1)$ where N^{u_k} is the frequency for the new PDE. Moreover, each u_k satisfies $1 \approx H^{u_k}(0, 1)$ and $\|\nabla u_k\|^2_{L^2(C_{\tau,1})} \approx N^{u_k}(0, 1)$. Observe that we can also bound $\|\nabla u_k\|_{L^2(C_{\tau,1/r_j})}$ and $H^{u_k}(0, r_j)$ for any $j \leq k$ (with bound depending on r_j).

Thanks to Lemma 8.3, we have that the sequence $(u_k)_k$ is bounded in Sobolev norm in any bounded set. Moreover, by boundary Schauder estimates (see Lemma 6.18 in [16]), $u_k \in C^{1,\alpha}(K)$ for any compact $K \subset \overline{\tilde{C}_\tau}\setminus\{0\}$ (for k large enough depending on K). By Arzelà-Ascoli, there is a subsequence $(u_{k_j})_j$ that converges in norm C^1 to $\tilde{u}$ in $\overline{C_{\tau,1}}\setminus B_{1/8}$.

Moreover, we can also assume that $\tilde{u}$ is a weak limit in $W^{1,2}_{\text{loc}}(\tilde{C}_\tau)$. Thus, $\tilde{u}$ is harmonic in $\tilde{C}_\tau$. To see this, fix any compact $K \subset\subset \tilde{C}_\tau$. Then, for all $\phi \in C^1_C(K)$, we have

$$\int_K (A(r_k x)\nabla u_k, \nabla\phi)\mathrm{d}x = 0, \quad \forall k > 0.$$

On the other hand, since $\tilde{u}$ is the weak limit of u_k, we have

$$\begin{aligned}\int_K (\nabla\tilde{u}, \nabla\phi)\mathrm{d}x &= \lim_{k\to\infty} \int_K (\nabla u_k, \nabla\phi)\mathrm{d}x \\ &= \lim_{k\to\infty} \underbrace{\int_K (A(r_k x)\nabla u_k, \nabla\phi)\mathrm{d}x}_{\boxed{A}} - \underbrace{\int_K ((A(r_k x) - I)\nabla u_k, \nabla\phi)\mathrm{d}x}_{\boxed{B}}.\end{aligned}$$

The term $\boxed{A}$ is zero by the definition of weak solution and we can estimate $\boxed{B}$ as

$$\begin{aligned}\int_K ((A(r_k x) - I)\nabla u_k, \nabla\phi)\mathrm{d}x &\lesssim L_A r_k(\operatorname{dist}(0, K) \\ &+ \operatorname{diam}(K))\|\nabla u_k\|_{L^2(K)}\|\nabla\phi\|_{L^2(K)} \to_{k\to\infty} 0.\end{aligned}$$

Summing up, we get that $\tilde{u}$ is harmonic in K.

Claim *The only functions which are positive and harmonic in $\tilde{C}_\tau$ are the multiples of the Green function g for the Laplacian in $\tilde{C}_\tau$ with pole at ∞.*

For the proof of the claim see [18, Lemma 3.7].

Since we have C^1 convergence away from the pole and $\int_{\partial B_1} u_k^2\,\mathrm{d}\sigma = 1$ for all $k > 0$, the multiple of the Green function in the claim is fixed. Thus, the convergence does not depend on the sequence r_k chosen and is not up to subsequences. For this reason, we assume without loss of generality that $r_k = 2^{-k+1}$.

We will see now that the limit

$$\lim_{r \downarrow 0} \log_2 \sqrt{H^u(x, 2r)} - \log_2 \sqrt{H^u(x, r)}$$

exists and by Lemma 8.2 this coincides with the order of vanishing at zero.

If $r \in (1/2^k, 1/2^{k-1})$, then

$$\log_2 \sqrt{H^u(0, 2r)} - \log_2 \sqrt{H^u(0, r)} = \log_2 \frac{\sqrt{H^{u_{k-1}}(0, 2^{k-1}r)}}{\sqrt{H^{u_{k-1}}(0, 2^{k-2}r)}}.$$

Since $r \in (1/2^k, 1/2^{k-1})$, this happens in the region where u_k converges to $\tilde{u}$ in C^1 norm. Thus $H^{u_k}(0, r) \to_{k \to \infty} H^{\tilde{u}}(0, r)$ for $1/8 < r < 1$. By Lemma 8.2 this implies that both u and $\tilde{u}$ have the same vanishing order. □

The general elliptic case is a direct consequence of the previous lemma following the same proof as in the harmonic case.

9 Example with $\dim_{\mathcal{H}}(\{x \in \Sigma \mid \lim_{r \downarrow 0} \omega(B(x, r)) r^{1-d} = 0\}) > d - 1 - \epsilon$

The following example by Xavier Tolsa shows that in a Lipschitz domain Ω (even with small Lipschitz constant) there is no hope for a (non-trivial) Hausdorff dimension bound of the set of points of Σ where $\partial_\nu u(x) = 0$.

Remark 9.1 We are interested in the normal derivative, but it may not exist at every point $x \in \Sigma$. Nonetheless, since we are in a Lipschitz domain Ω for every point $x \in \Sigma$ we can consider a non-tangential cone $C_\tau(x)$ contained in the domain. Thus, we can consider a non-tangential approach to the normal derivative. That is, for $x \in \Sigma$, we define

$$\nabla_{nt} u(x) = \limsup_{\substack{y \to x \\ y \in C_\tau(x)}} \frac{|u(y) - u(x)|}{|y - x|}.$$

We will show that the set $\{x \in \Sigma \mid \nabla_{nt} u(x) = 0\}$ can be very large (in terms of Hausdorff dimension).

The following lemma shows that non-tangential derivatives for positive harmonic functions are closely related with the density of harmonic measure.

Lemma 9.2 *Assume u is positive and harmonic in Ω and vanishes in Σ. Then, for $x \in \Sigma$, $\nabla_{nt} u(x) = 0$ if and only if $\lim_{r \downarrow 0} \frac{\omega(B(x,r))}{r^{d-1}} = 0$ where ω is the harmonic measure for the Laplacian in Ω.*

Proof of Remark 9.1 Fix a point $p \in \Omega$ far from Σ and consider the Green function g with pole at p. Then, in a neighborhood of Σ, the boundary Harnack inequality (see [11]) implies that the non-tangential derivative of u is 0 if and only if the non-tangential derivative of g is 0 wherever one of the two exists (if one exists and is 0 the other exists too by boundary Harnack inequality). This does not depend on the pole p chosen.

Now, the non-tangential derivative of g is 0 if and only if $\lim_{r \downarrow 0} \frac{\omega(B(x,r))}{r^{d-1}} = 0$. This is a consequence of [8, Lemma 1] which shows

$$r^{d-2} g(x + Cre_d) \approx \omega^p(B(x, r))$$

for some C depending on the dimension and the Lipschitz constant of the domain. □

Now we can start setting up an appropriate domain. Consider, for $\lambda \in (0,1)$, the λ-Cantor set defined as

$$\mathcal{C}_\lambda = \bigcap_{n=1}^{\infty} E_n^\lambda$$

where $E_0^\lambda = [0,1]$ and $E_k^\lambda = \frac{1-\lambda}{2} E_{k-1}^\lambda \cup (\frac{1+\lambda}{2} + \frac{1-\lambda}{2} E_{k-1}^\lambda)$ for $k \geq 1$. We will refer to the intervals in $[0,1] \backslash E_k^\lambda$ by *gaps*.

Remark 9.3 The set $\mathcal{C}_\lambda$ has Hausdorff dimension between 0 and 1 depending on λ, but by making λ small enough we can obtain a set with dimension arbitrarily close to 1. From now, on we will denote it by $s = s(\lambda) = \dim_{\mathcal{H}} \mathcal{C}_\lambda$.

Remark 9.4 If $\lambda = 1/(2k+1)$ for some $k \in \mathbb{N}$, then the set $\mathcal{C}_\lambda$ coincides with the set of real numbers in $[0,1]$ such that its decimal expansion in basis $2k+1$ does not contain the digit k. Instead of choosing λ, we will choose k large enough so that the Hausdorff dimension s of $\mathcal{C}_\lambda$ is as close as we want to 1.

Theorem 9.5 *Let* $\lambda = 1/(2k+1)$ *for some* $k \in \mathbb{N}$, *and* $s = s(\lambda) = \dim_{\mathcal{H}}(\mathcal{C}_\lambda)$. *Then,* $\mathcal{H}^s$-*a.e.* $x \in \mathcal{C}_\lambda$ *satisfies that each possible digit* $\{0, 1, \ldots, k-1, k+1, \ldots, 2k\}$ *in its decimal representation in basis* $2k+1$ *appears with the same asymptotic frequency* $1/(2k)$.

The proof of the theorem is analogous to the proof of Borel's normal number theorem. Also, we will say that the points $x \in \mathcal{C}_\lambda$ that satisfy this are $\mathcal{C}_\lambda$-*normal*.

Fix $a \in (0,1)$ and let

$$\Omega_\lambda := B(0,2) \cap \bigcup_{p \in \mathcal{C}_\lambda} \{C_a(p)\} \subset \mathbb{R}^2,$$

where $C_a(p)$ are vertical open cones of aperture a with vertex at p. Thus, Ω_λ is the union of all the open cones with aperture a centered at a point of the Cantor set $\mathcal{C}_\lambda$. The domain Ω_λ clearly has Lipschitz boundary and its Lipschitz constant can be made arbitrarily small as it depends only on the aperture of the cones a.

Remark 9.6 Let the interval (ξ_1, ξ_2) be a gap in $[0,1] \backslash E_j^\lambda$. Then $\partial (C_a(\xi_1) \cup C_a(\xi_2)) \cap \{(x,y) \in \mathbb{R}^2 \,|\, x \in (\xi_1, \xi_2)\} \subset \partial \Omega_\lambda$. In other words, the boundary of $\partial\Omega_\lambda$ looks like a cone (pointing upwards) where the Cantor set $\mathcal{C}_\lambda$ has gaps.

Note that, if $\lambda = 1/(2k+1)$, the gaps correspond to subintervals of $[0,1]$ where all the numbers have the digit k in the position j of the decimal expansion in base $2k+1$.

Now we will prove that every $\mathcal{C}_\lambda$-normal point $(x,0) \in \mathcal{C}_\lambda \subset \mathbb{R} \cap \overline{\Omega_\lambda}$ satisfies that the density of harmonic measure at $(x,0)$ is 0.

Proposition 9.7 *Let* Ω_λ *be the Lipschitz domain defined above. Let* $(x,0)$ *be a* $\mathcal{C}_\lambda$-*normal point in* $\mathcal{C}_\lambda$. *Then* $\lim_{r \downarrow 0} \frac{\omega(B(x,r))}{r^{d-1}} = 0$ *in* $(x,0)$ *where* ω *is the harmonic measure of* Ω_λ *with pole in* $(x,1)$.

Proof Theorem 6.2. in [15, page 149] (called $\mathrm{d}x/\theta(x)$ estimate) tells us

$$\omega(B((x,0),r)) \leq \frac{8}{\pi} \exp\left(-\pi \int_r^1 \frac{\mathrm{d}s}{s\Theta(s)}\right) \tag{9.1}$$

for r small enough where $s\Theta(s)$ is the length of the connected arc in $\Omega \cap \partial B((x, 0), s)$ that separates $(x, 0)$ and the pole $(x, 1)$. We want to prove that

$$\limsup_{r\to 0} \frac{\omega(B((x, 0), r))}{r} = 0$$

which is a consequence of

$$\limsup_{r\to 0} -\pi \int_r^1 \left(\frac{1}{\Theta(s)} - \frac{1}{\pi}\right) \frac{\mathrm{d}s}{s} \to -\infty$$

by inequality (9.1).

Since the domain is contained in the upper half plane and $x \in \mathbb{R}$ we have that $0 < \Theta(s) \leq \pi$. This implies that we can change the lim sup by a lim. Also we can rewrite the integral as

$$-\pi \lim_{n\to\infty} \sum_{j=1}^{n} \int_{(2k+1)^{-j}}^{(2k+1)^{1-j}} \left(\frac{1}{\Theta(s)} - \frac{1}{\pi}\right) \frac{\mathrm{d}s}{s}.$$

Now we will use that x is $\mathcal{C}_\lambda$-normal. This implies that, asymptotically, for a fixed proportion of digits in the decimal expansion of x in basis $2k + 1$ the digit is $k - 1$ or $k + 1$.

If the j-th digit in the decimal expansion of x in basis $2k+1$ is either $k-1$ or $k+1$, then for $s \in [2.1(2k + 1)^{-j}, (2k + 1)^{1-j}]$ we have that $\Theta(s) < \pi - \epsilon$ for some $\epsilon > 0$ depending only on the aperture of the cones a. This is because the arc of circumference centered at $(x, 0)$ with radius s meets with a cone $C_a(p)$ corresponding to a gap in $[0, 1]\backslash E_j^\lambda$ (see Remark 9.6) where p is the truncation to j digits of x in basis $2k + 1$. Thus, $\Theta(s) < \pi - \epsilon$. Since x is $\mathcal{C}_\lambda$-normal, this happens for positive proportion of digits, the integral tends to $-\infty$, and $\lim_{r\downarrow 0} \frac{\omega(B(x,r))}{r^{d-1}} = 0$. □

As a consequence of Proposition 9.7 and the previous claim, we can construct a domain where $\lim_{r\downarrow 0} \frac{\omega(B(x,r))}{r^{d-1}} = 0$ in a set of dimension s and s can be made as close to 1 as desired. This implies that the non-tangential derivative of a harmonic function (the Green function of the domain) can be 0 in a set of Hausdorff dimension arbitrarily close to 1. Moreover, we can obtain a domain where this set has Hausdorff dimension 1 by concatenating domains of the form Ω_λ considered before.

10 $\mathcal{S}_\Sigma(u) = \mathcal{S}'_\Sigma(u)$ in the $C^{1,\mathrm{Dini}}$ case

Recall that for a harmonic function u in a Lipschitz domain that vanishes in a relatively open subset of the boundary $\Sigma \subset \partial\Omega$, we define

$$\mathcal{S}_\Sigma(u) = \{x \in \Sigma \mid |\nabla u(x)| = 0\}$$

and

$$\mathcal{S}'_\Sigma(u) = \{x \in \Sigma \mid u^{-1}(\{0\}) \cap B(x, r) \cap \Omega \neq \varnothing, \ \forall r > 0\}.$$

Remark 10.1 Note that $\mathcal{S}_\Sigma(u)$ is well defined in the $C^{1,\mathrm{Dini}}$ case since $u \in C^1(\Omega \cup \Sigma)$ thanks to the results of [9].

The proof of Proposition 1.9 follows from a local expansion of u as the sum of a homogeneous harmonic polynomial and an error term of higher degree (see [22, Theorem 1.1]).

Proof of Proposition 1.9 By [22, Theorem 1.1], for every $x \in \Sigma$ there exists a positive radius $R = R(x)$ and a positive integer $N = N(x)$ such that

$$u(y) = P_N(y - x) + \psi(y - x), \quad \text{in } B_R(x) \cap \Omega$$

where P_N is a non-trivial homogeneous harmonic polynomial of degree N and the error term ψ satisfies

$$\lim_{y\to 0} |\psi(y)||y|^{-N} = 0 \quad \text{and} \quad \lim_{y\to 0} |\nabla\psi(y)||y|^{-N+1} = 0.$$

We will show that $N(x) = 1$ implies $x \notin \mathcal{S}_\Sigma(u)$ and $x \notin \mathcal{S}'_\Sigma(u)$ and that $N(x) > 1$ implies $x \in \mathcal{S}_\Sigma(u)$ and $x \in \mathcal{S}'_\Sigma(u)$.

Case $N = 1$: We have that $\nabla u(x) = \nabla P_1(0) + \nabla\psi(0) = \nabla P_1(0) \neq 0$ since P_1 is a non-trivial linear function, thus $x \notin \mathcal{S}_\Sigma(u)$. Also, we know that $u(y) = 0$ for $y \in \partial\Omega \cap B_R(x)$. Suppose there exist $(z_n)_n \in \Omega \cap B_R(x)$ tending to x such that $u(z_n) = 0$. Then, by Rolle's theorem, we get a contradiction since the derivative of u in the direction $\nabla P_1/|\nabla P_1|$ does not vanish in a neighborhood of x (because $\lim_{y\to 0} |\nabla\psi(y)| = 0$) but u vanishes on $(z_n)_n$ and on $\partial\Omega \cap B_R(x)$. Thus $x \notin \mathcal{S}'_\Sigma(u)$.

Case $N > 1$: Clearly, $\nabla P_N(0) = 0$ if $N > 1$ which implies that $x \in \mathcal{S}_\Sigma(u)$. Assume that $x \notin \mathcal{S}'_\Sigma(u)$ and, without loss of generality, u is positive near x. Then, by the generalized Hopf principle of [31], we get that $\partial_\nu u(x) < 0$ contradicting $x \in \mathcal{S}_\Sigma(u)$. □

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Appendix A. Existence of non-tangential limits for ∇u

We will closely follow the ideas of Appendix A of [33] in order to prove the L^2 convergence of the non-tangential limits of ∇u in Σ.

Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, B a ball centered on $\partial\Omega$, $\Sigma = B \cap \partial\Omega$, and σ denote the surface measure on Σ. Without loss of generality, we assume Ω is locally above Σ in the direction of $e_d = (0, \dots, 0, 1)$. For σ-a.e. $x \in \Sigma$, the outer unit normal vector $\nu(x)$ is well defined. For a parameter $a \in (0, 1)$ and $x \in \Sigma$, we consider the inner cone and outer cone

$$X_a^+(x) = \{y \in \mathbb{R}^d | (x - y, \nu(x)) > a|y - x|\}, \quad X_a^-(x) = \{y \in \mathbb{R}^d | - (x - y, \nu(x)) > a|y - x|\},$$

respectively. For a function f defined on $\mathbb{R}^d \backslash \Sigma$, we define the non-tangential limits

$$f_{+,a}(x) = \lim_{X_a^+(x) \ni y \to x} f(y), \quad f_{-,a}(x) = \lim_{X_a^-(x) \ni y \to x} f(y),$$

when they exist.

We prove the following theorem about the convergence of the non-tangential limits of the gradient of the solution of an elliptic PDE (of the type of Sect. 2.1).

Theorem A.1 *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain, B be an open ball centered in $\partial\Omega$, and $\Sigma = B \cap \partial\Omega$ be a Lipschitz graph such that $\Omega \cap B$ is above Σ. Let u be a solution of $\operatorname{div}(A\nabla u) = 0$ in Ω with A as in Sect. 2.1. Assume that u is continuous up to the boundary and that it vanishes continuously on Σ. Then, for all $a \in (0, 1)$ large enough, $(\nabla u)_{+,a}$ exists σ-a.e. and belongs to $L^2_{\text{loc}}(\sigma)$. Further, $(\nabla u)_{+,a}$ has vanishing tangential component. That is, $(\nabla u)_{+,a} = (\nabla u, \nu)\nu$. Further,*

$$\lim_{\varepsilon\to 0} \nabla u\,(\cdot + \varepsilon e_n) \to (\nabla u, \nu)\,\nu \quad \text{in } L^2_{\text{loc}}(\sigma).$$

Proof We extend u by 0 out of Ω and denote $u^+ = \max(u, 0)$, $u^- = \max(-u, 0)$. Both u^+ and u^- are continuous and subsolutions (that is $-\operatorname{div}(A\nabla u^\pm) \le 0$ in B).

Claim *In the sense of distributions,* $\operatorname{div}(A\nabla u) = \operatorname{div}(A\nabla u^+) - \operatorname{div}(A\nabla u^-)$ *restricted to B is a signed Radon measure supported on Σ.*

Proof of the claim To see this, take arbitrary $\phi \in C_c^\infty(B)$. For $0 < \epsilon \ll r(B)$, denote $\Sigma_\epsilon = \Sigma + \epsilon e_d$ and $\Omega_\epsilon = \Omega + \epsilon e_d$, where $e_d = (0, \ldots, 0, 1)$. Let ψ_δ be a bump function in an δ-neighborhood of Σ such that $\|\nabla\psi_\delta\|_\infty \lesssim \delta^{-1}$ and $\|\nabla^2\psi_\delta\|_\infty \lesssim \delta^{-2}$. Then, writing $\phi = \phi\psi_\delta + \phi(1 - \psi_\delta)$ for $\phi \in C_c^\infty(B)$, we get

$$\left|\int_{B\cap\Omega} A\nabla u\nabla\phi \mathrm{d}x\right| \le \underbrace{\left|\int_{B\cap\Omega} A\nabla u\nabla(\phi\psi_\delta)\mathrm{d}x\right|}_{\boxed{A}} + \underbrace{\left|\int_{B\cap\Omega} A\nabla u\nabla(\phi(1-\psi_\delta))\mathrm{d}x\right|}_{\boxed{B}}.$$

We directly obtain $\boxed{B} = 0$ since u is a weak solution of the PDE in Ω. We have, because $u \in W^{1,2}(B)$ and the divergence theorem, that

$$\begin{aligned}\int_{B\cap\Omega} A\nabla u\nabla(\phi\psi_\delta)\mathrm{d}x &= \lim_{\epsilon\to 0}\int_{B\cap\Omega_\epsilon} A\nabla u\nabla(\phi\psi_\delta)\mathrm{d}x\\ &= \lim_{\epsilon\to 0}\underbrace{\int_{\partial\Omega_\epsilon\cap B} u(A\nabla(\phi\psi_\delta), \nu)\mathrm{d}\sigma}_{\boxed{C}} - \underbrace{\int_{\Omega_\epsilon\cap B} u\operatorname{div}(A\nabla(\phi\psi_\delta))\mathrm{d}x}_{\boxed{D}}\end{aligned}$$

The term $\boxed{C}$ converges uniformly to 0 as $\epsilon \to 0$ as u is 0 in Σ. On the other hand,

$$\begin{aligned}|\operatorname{div}(A\nabla(\phi\psi_\delta))| &\le |\operatorname{div}(A\nabla\psi_\delta)\phi| + |\operatorname{div}(A\nabla\phi)\psi_\delta| + |2(A\nabla\phi, \nabla\psi_\delta)|\\ &\lesssim \frac{1}{\delta^2}\|\phi\|_{L^\infty(\operatorname{supp}\psi_\delta)} + \|\nabla^2\phi\|_\infty + \frac{1}{\delta}\|\nabla\phi\|_\infty\end{aligned}$$

with constants depending on the Lipschitz and ellipticity constants of A.

We can bound $\boxed{D}$ as

$$\begin{aligned}&\left|\int_{\Omega\cap B} u\operatorname{div}(A\nabla(\phi\psi_\delta))\mathrm{d}x\right|\\ &\quad\lesssim \left(\frac{1}{\delta^2}\|\phi\|_{L^\infty(\operatorname{supp}\psi_\delta)} + \|\nabla^2\phi\|_\infty + \frac{1}{\delta}\|\nabla\phi\|_\infty\right)\int_{\operatorname{supp}\psi_\delta\cap\Omega}|u|\mathrm{d}x. \qquad \text{(A.1)}\end{aligned}$$

Using the boundary Harnack inequality (see [11], for example), we can bound

$$\int_{\text{supp } \psi_\delta \cap \Omega} |u| \mathrm{d}x \lesssim \int_{\text{supp } \psi_\delta \cap \Omega} g \, \mathrm{d}x \lesssim \delta^2$$

where g is the Green function with fixed pole far from B. To prove the second inequality, we cover supp ψ_δ by a finite family of cubes Q_i with side length $\ell(Q) \approx \delta$. We can do this with approximately $\left(\frac{r(B)}{\delta}\right)^{d-1}$ cubes. By standard estimates for elliptic measure, we have

$$g(x) \lesssim \frac{\omega(4Q)}{\ell(Q)^{n-2}} \approx \omega(4Q)\delta^{2-d} \quad \text{for all } x \in Q \cap \Omega.$$

where ω is the elliptic measure associated to $\text{div}(A\nabla\cdot)$ for Ω with respect to a fixed pole $p \in \Omega \backslash B$.

Finally, we have

$$\int_{\text{supp } \psi_\delta \cap \Omega} g \, \mathrm{d}x \leq \sum_Q \int_{Q \cap \Omega} g \, \mathrm{d}x \lesssim \sum_Q \omega(4Q)\delta^{2-d} \int_Q \mathrm{d}x \approx \delta^2 \omega(\Sigma)$$

where we have used the doubling properties of elliptic measure.

Summing up, we take the limit as $\delta \to 0$ in Eq. (A.1) to obtain

$$((\text{div}(A\nabla u)), \phi) \leq C\|\phi\|_{L^\infty(\Sigma)}$$

where we have used that the supp $\psi_\delta \to \Sigma$ as $\delta \to 0$, Thus $\text{div}(A\nabla u)|_B$ is a Radon measure supported on Σ. □

Moreover, this Radon measure is absolutely continuous with respect to elliptic measure.

Claim *In the sense of distributions, we have*

$$\text{div}(A\nabla u)|_B = \rho\omega|_\Sigma,$$

where $\rho \in L^\infty_{loc}(\Sigma)$ *and* ω *is the elliptic measure associated to* $\text{div}(A\nabla\cdot)$ *for* Ω *with respect to a fixed pole* $p \in \Omega \backslash B$.

We can assume that B is small enough so that $\Omega \backslash 2B \neq \varnothing$ and $p \in \Omega \backslash 2B$.

Proof of the claim To prove the claim, let B' be an open ball concentric with B such that $\overline{B'} \subset B$. We will show that that there exists some constant C depending on B' and p such that for any compact set $K \subset \Sigma \cap B'$, it holds

$$(\text{div}(A\nabla u), \chi_K) \leq C\omega(K). \tag{A.2}$$

By duality, this implies the claim.

Given $\epsilon \in \left(0, \frac{1}{2}\text{dist}(K, \mathbb{R}^d \backslash B')\right)$, let $\{Q_i\}_{i\in I}$ be a lattice of cubes covering $\mathbb{R}^d$ such that each Q_i has diameter $\epsilon/2$. Let $\{\phi_i\}_{i\in I}$ be a partition of unity of $\mathbb{R}^n$, so that each ϕ_i is supported in $2Q_i$ and satisfies $\|\nabla^j \phi_i\|_\infty \lesssim \ell(Q_i)^{-j}$, for $j = 0, 1, 2$. Then, we have

$$(\text{div}(A\nabla u), \chi_K) = (\text{div}(A\nabla u), \sum_{i\in I'} \phi_i) - (\text{div}(A\nabla u), \sum_{i\in I'} \phi_i - \chi_K),$$

where I' is the collection of indices $i \in I$ that satisfy $2Q_i \cap K \neq \varnothing$. By the regularity properties of Radon measures, we obtain

$$|(\text{div}(A\nabla u), \sum_{i\in I'} \phi_i - \chi_K)| \leq \left(|\text{div}(A\nabla u)|, \chi_{U_\epsilon(K)\backslash K}\right) \to 0 \quad \text{as } \epsilon \to 0,$$

where $U_\epsilon(K)$ is the ϵ-neighborhood of K. For the other term, we have

$$|(\operatorname{div}(A\nabla u), \sum_{i\in I'} \phi_i)| \leq \sum_{i\in I'} |(u, \operatorname{div}(A\nabla\phi_i))| \lesssim \sum_{i\in I'} \frac{1}{\ell(Q_i)^2} \int_{2Q_i} |u| \mathrm{d}m$$

using that A is elliptic and Lipschitz. Since $|u|$ is a continuous subsolution in B that vanishes in $B\setminus\Omega$, by the boundary Harnack inequality, we have again $|u(x)| \lesssim g(x)$ for all $x \in B' \cap \Omega$, where g is the Green function of Ω for $\operatorname{div}(A\nabla\cdot)$ with pole in $\Omega\setminus B$. The constant C depends on u, p, Λ_A (the ellipticity constant of A), and B', but not on K. Thus, proceeding as in the proof of the previous claim, we obtain

$$|(\operatorname{div}(A\nabla u), \sum_{i\in I'} \phi_i)| \lesssim \sum_{i\in I'} \frac{1}{\ell(Q_i)^2} \int_{2Q_i} g(x)\mathrm{d}x \approx \sum_{i\in I'} \omega(4Q_i) = \omega(U_{4\epsilon(K)}).$$

Letting $\epsilon \to 0$, we have $\omega(U_{4\epsilon}(K)) \to \omega(K)$ from which Eq. (A.2) follows. □

By the solvability of the Dirichlet problem with L^2 data in Lipschitz domains for divergence form elliptic equations with Hölder coeficients (see Remark 1.4 in [20]), we have that ω is a B_2 weight with respect to the surface measure σ. In particular, the density function $\frac{\mathrm{d}\omega}{\mathrm{d}\sigma}$ belongs to $L^2_{loc}(\sigma)$. Therefore, in the sense of distributions,

$$\operatorname{div}(A\nabla u)|_B = h\sigma, \quad \text{for some } h \in L^2_{loc}(\sigma).$$

Next, we will show that $(\nabla u)_{+,a}$ exists σ-a.e. and moreover $(\nabla u)_{+,a} = (\nabla u, \nu)\nu \in L^2_{loc}(\sigma)$. Consider an arbitrary open ball $\tilde{B}$ centered in Σ such that $4\tilde{B} \subset B$. Let ϕ be a C^∞ function which equals 1 on $2\tilde{B}$ is supported on $3\tilde{B}$, and let $v = \phi u$. Observe that

$$\begin{aligned} v(x) &= \int \mathcal{E}(x, y) \operatorname{div}(A\nabla v)(y) \\ &= \int \mathcal{E}(x, y)\phi(y) \operatorname{div}(A\nabla u)(y) + \int \mathcal{E}(x, y)u(y) \operatorname{div}(A\nabla\phi)(y) \\ &\quad + 2\int \mathcal{E}(x, y)(A\nabla u(y), \nabla\phi(y)), \end{aligned} \tag{A.3}$$

where $\mathcal{E}$ is the fundamental solution of $\operatorname{div}(A\nabla\cdot)$ (we consider a Lipschitz, elliptic extension of A defined in $\mathbb{R}^d$). Note that $\nabla u \in L^2_{\mathrm{loc}}(B)$, by Caccioppoli's inequality.

For a finite Borel measure η, let $T\eta$ be the gradient of the single layer potential of η (in the case $A \equiv I$ it coincides with the $(d-1)$-dimensional Riesz transform of η). That is,

$$T\eta(x) = \int \nabla_1 \mathcal{E}(x, y)\mathrm{d}\eta(y),$$

in the sense of truncations. From identity (A.3), we obtain for $x \notin \Sigma$,

$$\nabla v(x) = c_d\big(T(\phi h\sigma|_\Sigma)(x) + T(u \operatorname{div}(A\nabla\phi)m)(x) + 2T((A\nabla u, \nabla\phi)m\big)(x)\big)$$

(where m is the Lebesgue measure in $\mathbb{R}^d$). Observe that $T(u \operatorname{div}(A\nabla\phi)m)(x)$ and $T((A\nabla u, \nabla\phi)m\big)(x)$ are continuous functions in $\tilde{B}$. On the other hand, the non-tangential limit $T(\phi h\sigma|_\Sigma)_{\pm,a}(x)$ exists for σ-a.e. $x \in \partial\Omega$, by the jump formulas for the gradients of single layer potentials transforms (see [20, Theorem 4.4] or [32, Theorem 1] for the constant coefficients case in rectifiable sets, for example). From the fact that $\nabla v = \nabla u$ in $\tilde{B}$, it follows that $(\nabla u)_{\pm,a}(x)$ exists for σ-a.e. $x \in \Sigma \cap \tilde{B}$. By the $L^2(\sigma)$ boundedness of T on Lipschitz graphs (see [20, Theorem 3.1]), we deduce that $(\nabla u)_{\pm,a} \in L^2(\sigma|_{\Sigma\cap\tilde{B}})$.

Since $u \equiv 0$ in Ω^c, we have $(\nabla u)_{-,a} = 0$ in $\Sigma \cap \tilde{B}$. As the tangential component of $T(\phi h\sigma|_\Sigma)(x)$ is continuous across $\partial\Omega$ for σ-a.e. $x \in \partial\Omega$, again by the jump formulas for T ([20, Theorem 4.4]), we deduce that the tangential component of $(\nabla u)_{+,a}$ coincides with the tangential component of $(\nabla u)_{-,a}$ σ-a.e. in $\Sigma \cap \tilde{B}$, and thus $(\nabla u)_{+,a} = (\nabla u, \nu)\nu$ in $\Sigma \cap \tilde{B}$.

We will now prove that $(\mathrm{div}(A\nabla u))|_B = -(A\nabla u, \nu)\sigma|_\Sigma$ in the sense of distributions. Let ψ be a function in $C_c^\infty(\tilde{B})$. Again, for $0 < \epsilon \ll r(\tilde{B})$, consider $\Sigma_\epsilon = \Sigma + \epsilon e_d$ and $\Omega_\epsilon = \Omega + \epsilon e_d$, where $e_d = (0, \dots, 0, 1)$. Then, we have

$$\begin{aligned} \langle \mathrm{div}(A\nabla u), \psi \rangle &= \int_{\tilde{B}\cap\Omega_\epsilon} u \,\mathrm{div}(A\nabla\psi)\mathrm{d}m = \lim_{\epsilon\to 0} \int_{\tilde{B}\cap\Omega_\epsilon} u \,\mathrm{div}(A\nabla\psi)\mathrm{d}m \\ &= \lim_{\epsilon\to 0} \int_{\tilde{B}\cap\partial\Omega_\epsilon} u(A\nabla\psi, \nu)\mathrm{d}\sigma - \lim_{\epsilon\to 0} \int_{\tilde{B}\cap\partial\Omega_\epsilon} \psi(A\nabla u, \nu)\mathrm{d}\sigma \\ &= 0 - \int_\Sigma \psi(A\nabla u, \nu)\mathrm{d}\sigma. \end{aligned} \tag{A.4}$$

The last identity follows from the uniform convergence of u to 0 as $\Sigma_\epsilon \to \Sigma$ and that $\nabla u(\cdot + \epsilon e_n)$ converges to $(\nabla u)_{+,a}$ in $L^2(\sigma|_{\Sigma\cap\tilde{B}})$ (this is proven by arguments analogous to the ones above for the σ-a.e. existence of the limit $(\nabla u)_{+,a}(x)$ in Σ). From (A.4), we deduce that $\mathrm{div}(A\nabla u) = -(A\nabla u, \nu)\sigma$ in $\tilde{B}$, and thus also in B. □

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