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Complex Analysis and Operator Theory

Quantum Permutation Matrices

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Abstract

Quantum permutations arise in many aspects of modern "quantum mathematics". However, the aim of this article is to detach these objects from their context and to give a friendly introduction purely within operator theory. We define quantum permutation matrices as matrices whose entries are operators on Hilbert spaces; they obey certain assumptions generalizing classical permutation matrices. We give a number of examples and we list many open problems. We then put them back in their original context and give an overview of their use in several branches of mathematics, such as quantum groups, quantum information theory, graph theory and free probability theory.

Contents

1 Introduction
2 Quantum Permutation Matrices in Operator Theory
  2.1 Definition of Quantum Permutation Matrices
  2.2 Link to Classical Permutation Matrices
  2.3 Examples
  2.4 Quantum Permutation Matrices and Quantum Isomorphisms of Graphs
3 Open Problems
  3.1 Faithful Models

Dedicated to Jörg Eschmeier who sadly passed away in 2021.

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3.2 Hilbert Sudoku/SudoQ
3.3 Quantum Symmetries of Graphs
3.4 Quantum Sinkhorn Algorithm
3.5 Intermediate Quantum Permutations
3.6 More Examples, Constructions and Quantum Transposition Matrices
4 Quantum Permutation Matrices in Their Broader Context and Use
4.1 Quantum Mathematics
4.2 Quantum Groups and Quantum Permutation Groups
4.3 Quantum Symmetries of Graphs and Quantum Isomorphisms of Graphs
4.4 Quantum Information Theory
4.5 Free Probability Theory
References

1 Introduction

One of the most basic symmetry operations in mathematics, is given by permutations: Take N points $x_1, \dots, x_N$ listed in some order, and permute them. We may capture permutations in various ways in mathematics, and one way to do so is by using permutation matrices. Recall that a *permutation matrix* is an $N \times N$-matrix $\sigma \in M_N(\{0, 1\})$,

(a) whose entries σ_{ij} are either 0 or 1,
(b) such that each column and each row contains exactly one 1, all other entries being 0.

Here is an example of a 4×4 permutation matrix:

$$\sigma = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Such an $N \times N$ permutation matrix acts on $\mathbb{C}^N$ by permuting the canonical basis vectors $e_1, \dots, e_N \in \mathbb{C}^N$. So, in a way, we identify our N points $x_1, \dots, x_N$ from above with the basis vectors $e_1, \dots, e_N$ and permute them by letting the matrix σ act on them – we identify the N-elementary set $X = \{x_1, \dots, x_N\}$ with the N-dimensional space $\mathbb{C}^N$. Now, this is a very common theme in "quantum mathematics": We identify a classical space with another object having some more "functional analytic" properties—and this allows us to define and study "quantum versions" of this classical space. What exactly do we mean by this?

Let us postpone this discussion to Sect. 4.1 and let us define quantum permutation matrices right away. A *quantum permutation matrix* is an $N \times N$-matrix $u \in M_N(B(H))$, where H is some Hilbert space, such that

(a) all entries $u_{ij} \in B(H)$ are orthogonal projections (i.e. $u_{ij} = u_{ij}^* = u_{ij}^2$),
(b) and $\sum_{k=1}^N u_{ik} = \sum_{k=1}^N u_{kj} = 1$ for all $i, j = 1, \dots, N$.

Actually, an easy calculation (see Lemma 2.2) shows that the latter implies

(b') $u_{ik}u_{jk} = u_{ki}u_{kj} = 0$ for all $i, j, k = 1, \dots, N$ with $i \neq j$.

Here is an example of a 4×4-quantum permutation matrix with $H = \mathbb{C}^2$:

$$u = \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix}$$

Let us make a few observations: Firstly, the entries of any (classical) permutation matrix $\sigma \in M_N(\{0, 1\})$ satisfy the axioms of a quantum permutation matrix—a permutation matrix is a quantum permutation matrix, with $H = \mathbb{C}$ (and thus $B(\mathbb{C}) = \mathbb{C}$). Secondly, the entries $u_{ij} \in B(H)$ of a quantum permutation matrix do not need to commute, as the above example shows. Thirdly, the above matrix u is "quantum" indeed: While the above classical permutation matrix σ sends the first particle to the third one (meaning, it maps e_1 to e_3), the above quantum permutation matrix u sends the first particle a little bit to the third, and a little bit to the fourth – in the sense that the matrix $u \in M_4(M_2(\mathbb{C}))$ acting on $\mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2 \oplus \mathbb{C}^2$ sends the first copy of $\mathbb{C}^2$ to some part of the third copy (namely to $\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mathbb{C}^2 \subseteq \mathbb{C}^2$) and to some part of the fourth copy (namely to $\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mathbb{C}^2 \subseteq \mathbb{C}^2$). Quantum, isn't it?

However, the "quantum" aspect of the matrix will be neglected in the main part of the article:

- In Sect. 2, we will study quantum permutation matrices as such, in the realm of operator theory, and list a number of open problems.
- In Sect. 4, we will then explain how quantum permutation matrices fit into the broader context of quantum groups, quantum information, graph theory and free probability, and again list a number of open problems. And we give references and sketch the history of the field.

With the present article, we hope to provide a friendly access to quantum permutation matrices (also known as magic unitaries) for people interested in operator theory. In our functional analysis research seminar at Saarland University, we sometimes had a "friendly clash of cultures" between the groups of Jörg Eschmeier (together with Ernst Albrecht, Gerd Wittstock and Heinz König, and later also Michael Hartz's group) and the more operator algebraic ones by Roland Speicher and myself. However, a common ground was to consider operators on Hilbert spaces and to try to understand their properties. While Jörg was more interested in (tuples of) commuting operators, we were more interested in noncommuting operators—but in the end, we all dealt with operators on Hilbert spaces, to some extent. And maybe, Hardy spaces and their generalizations or complex analysis can be helpful when studying quantum permutation matrices at some point, in Jörg's spirit?

The quantum permutation matrices we are presenting here are interestingly diverse objects: They can be seen as an *array* of operators, containing certain tuples of commuting operators (within one row or one column, for instance), but also involving noncommutativity between other entries. It is good to think back to those happy times, when we had our interestingly diverse research seminar sessions at Saarland University, containing tuples from Jörg's group and tuples from Roland's and my groups, each group asking in the sense of:
– "But how about the Hardy space?"
– "And how about noncommutativity?"

2 Quantum Permutation Matrices in Operator Theory

In this section, we define and study quantum permutation matrices in the realm of operator theory and we list a number of open problems. See Sect. 4 for the historical origins.

2.1 Definition of Quantum Permutation Matrices

Let us define quantum permutation matrices first.

Definition 2.1 Let $N \in \mathbb{N}$ and let H be a (complex) Hilbert space. A *quantum permutation matrix* (also called *magic unitary*) is a matrix $u \in M_N(B(H))$ consisting of entries $u_{ij} \in B(H)$, $i, j = 1, \dots, N$, such that

(a) $u_{ij} = u_{ij}^* = u_{ij}^2$ for all $i, j = 1, \dots, N$ (i.e., the entries are projections)
(b) and $\sum_{k=1}^N u_{ik} = \sum_{k=1}^N u_{kj} = 1$ for all $i, j = 1, \dots, N$ with $i \neq j$.

$$u = \begin{pmatrix} u_{11} & u_{12} & \cdots & u_{1N} \\ u_{21} & u_{22} & \cdots & u_{2N} \\ & \vdots & & \\ u_{N1} & u_{2N} & \cdots & u_{NN} \end{pmatrix} \in M_N(B(H))$$

So, by definition, a quantum permutation matrix is an operator in $B(\bigoplus_{k=1}^N H)$. In some sense, it "permutes" the N copies of H, just as a classical permutation matrix permutes N copies of $\mathbb{C}$. If H is finite-dimensional of dimension d, we have $u \in M_N(M_d(\mathbb{C}))$, i.e. u is an $N \times N$-matrix whose entries are matrices themselves (of size $d \times d$). See Sect. 2.3 for examples. Let us derive further (well-known) relations amongst the entries of quantum permutation matrices.

Lemma 2.2 *For any quantum permutation matrix, we have*

$$u_{ik}u_{jk} = u_{ki}u_{kj} = 0$$

for all $i, j, k = 1, \dots, N$ *with* $i \neq j$.

Proof Projections summing up to 1 (or to simply to a projection) need to be mutually orthogonal. This follows from positivity. Indeed, if $p_1, \dots, p_N$ are projections auch that $\sum_k p_k = 1$, then for any $j = 1, \dots, N$,

$$\sum_{i=1, i\neq j}^{N} (p_i p_j)^*(p_i p_j) = \sum_{i=1}^{N} (p_i p_j)^*(p_i p_j) - p_j$$
$$= p_j \left(\sum_{i=1}^{N} p_i \right) p_j - p_j = p_j - p_j = 0.$$

Hence, the positive elements $(p_i p_j)^*(p_i p_j)$ with $i \neq j$ sum up to zero, which means that each of these summands needs to be zero; hence $p_i p_j = 0$ if $i \neq j$. □

Note that in case one wants to define quantum permutation matrices in general *-algebras, one should better add the relations from Lemma 2.2, as orthogonality is *not* implied in a general *-algebra [14, Rem. 4.10].

2.2 Link to Classical Permutation Matrices

Let us study the case $\dim H = 1$ in Definition 2.1. In that case, we obtain classical permutation matrices.

Lemma 2.3 *Let $u \in M_N(B(H))$ be a quantum permutation matrix and let* $\dim H = 1$. *Then u is a (classical) permutation matrix.*

Proof If $\dim H = 1$, we have $H = \mathbb{C}$ and $B(H) = \mathbb{C}$. Thus, the entries u_{ij} of u are scalars. Now, since $u_{ij} = u_{ij}^*$, they are actually real, and as $u_{ij}^2 = u_{ij}$, we infer $u_{ij} \in \{0, 1\}$. In each row (and in each column), by Lemma 2.2, there is at most one nontrivial entry, and by (b) of Definition 2.1, there is exactly one. These are the defining properties of a permutation matrix. □

Also, if all entries u_{ij} commute, we are in the classical situation.[1]

Lemma 2.4 *Let $u \in M_N(B(H))$ be a quantum permutation matrix and let H be finite-dimensional of dimension $d \in \mathbb{N}$. Assume that all entries $u_{ij} \in B(H)$ commute, for all $i, j = 1, \dots, N$.*

Then, there are permutation matrices $\sigma_1, \dots, \sigma_d \in M_N(\mathbb{C})$, such that u is unitarily equivalent to

$$\begin{pmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_d \end{pmatrix} = \bigoplus_{t=1,\dots,d} \sigma_t \in \bigoplus_{t=1,\dots,d} M_N(\mathbb{C}) \subseteq M_{Nd}(\mathbb{C}) \cong M_N(M_d(\mathbb{C})).$$

[1] As an alternative to Lemma 2.4, one may show that for commuting entries u_{ij}, there are permutation matrices $\sigma_1, \dots, \sigma_d$ such that $u = \sum_{t=1}^{d} \sigma_t \otimes a_t \in M_N(\mathbb{C}) \otimes B(H)$ for $a_t = u_{1\sigma_t(1)} \dots u_{N\sigma_t(N)} \in B(H)$. [68].

Proof We identify $B(H)$ with $M_d(\mathbb{C})$.

Special case. For convenience, let us prove a special case first, assuming that the rank of each u_{ij} is at most one. Fix the first row $u_{11}, u_{12}, \ldots, u_{1N} \in M_d(\mathbb{C})$ of u. These are mutually orthogonal projections summing up to 1, by Definition 2.1 (and Lemma 2.2). We thus find a unitary $W \in M_d(\mathbb{C})$ such that each $w_{1j} := Wu_{1j}W^* \in M_d(\mathbb{C})$ is diagonal, for all $j = 1, \ldots, N$. We pick an orthonormal basis $e_1, \ldots, e_d$ of $\mathbb{C}^d$ according to this diagonalization and we denote by p_k, for $k = 1, \ldots, d$, the projection onto the one-dimensional space spanned by the basis vector e_k. Since the rank of each w_{1j} is at most one, we have

$$\{w_{1j} \mid j = 1, \ldots, N\} = \{p_k \mid k = 1, \ldots, d\}.$$

Now, by assumption, each element $w_{ij} := Wu_{ij}W^* \in M_d(\mathbb{C})$, $i, j = 1, \ldots, N$ commutes with all w_{1m}, $m = 1, \ldots, N$, i.e. it commutes with all p_k, $k = 1, \ldots, d$. Hence, each w_{ij} is a diagonal projection matrix of rank at most one and we have

$$\{w_{ij} \mid i, j = 1, \ldots, N\} = \{p_k \mid k = 1, \ldots, d\}.$$

Denote by $\sigma_t \in M_N(\{0, 1\})$, $t = 1, \ldots, d$ the matrix with

$$(\sigma_t)_{ij} := (w_{ij})_{tt},$$

i.e. σ_t consists in the t-th diagonal entries of all w_{ij}'s. Since the matrix w formed by the elements w_{ij}, $i, j = 1, \ldots, N$ is a quantum permutation matrix, we infer that $\sigma_1, \ldots, \sigma_d$ are permutation matrices.

General case. We now drop the assumption that the rank of each u_{ij} is at most one and we prove the general case. Again, we find a unitary $W_1 \in M_d(\mathbb{C})$ such that each $w_{1j}^{(1)} := W_1 u_{1j} W_1^* \in M_d(\mathbb{C})$ is diagonal, for all $j = 1, \ldots, N$. Define the quantum permutation matrix $w^{(1)}$ with entries $w_{ij}^{(1)} := W_1 u_{ij} W_1^*$. Since each $w_{ij}^{(1)}$ commutes with all elements from the first row of $w^{(1)}$, each $w_{ij}^{(1)}$ is block diagonal (but not necessarily diagonal, in contrast to the above special case) with respect to the blocks of the first row of $w^{(1)}$.

Now, consider the second row of $w^{(1)}$. Again, these are projections summing up to 1, and we find a unitary $W_2 \in M_d(\mathbb{C})$ such that each $w_{2j}^{(2)} := W_2 w_{2j}^{(1)} W_2^* \in M_d(\mathbb{C})$ is diagonal, for all $j = 1, \ldots, N$. The crucial point is, that we may choose W_2 to be block diagonal with respect to the blocks of the first row of $w^{(1)}$, since the second row of $w^{(1)}$ is block diagonal with respect to the first row of $w^{(1)}$. Thus, defining the quantum permutation matrix $w^{(2)}$ with entries $w_{ij}^{(2)} := W_2 w_{ij}^{(1)} W_2^*$, we note that both the first *and* the second row of $w^{(2)}$ are diagonal, since W_2 being block diagonal with respect to the first row did not change the first row.

Iterating, we may eventually diagonalize all elements of the matrix and we end up with a quantum permutation matrix $w = (w_{ij})$, which is unitarily equivalent to u and whose entries w_{ij} are all diagonal, i.e. each w_{ij} is a diagonal matrix with entries 0 or

1. Now, denote by $\sigma_t \in M_N(\{0,1\})$, $t = 1, \dots, d$ the matrix with

$$(\sigma_t)_{ij} := (w_{ij})_{tt},$$

which yields permutation matrices $\sigma_1, \dots, \sigma_d$, since w is a quantum permutation matrix. □

We observe, that Lemma 2.3 is a special case of Lemma 2.4 with $d = 1$. The next lemma shows, that if we want to go beyond the classical case, we need to choose $N \geq 4$.

Lemma 2.5 *Let $u \in M_N(B(H))$ be a quantum permutation matrix with $N \leq 3$. Then the entries of u all commute.*

Proof If $N = 1$ or $N = 2$, the situation is trivial (in the latter case: $u_{12} = 1 - u_{11}$, $u_{21} = 1 - u_{11}$, $u_{22} = u_{11}$). For $N = 3$, we copy the following proof from [53, Sect. 2.2]. Consider u_{ij} and u_{kl}. We want to show that they commute. They do, if $i = k$ or $j = l$, by Lemma 2.2, or since $u_{ij} = u_{kl}$, in case both equations hold, $i = k$ and $j = l$.

If now $i \neq k$ and $j \neq l$, there is some m with $m \neq j$ and $m \neq l$ such that $\{j, l, m\} = \{1, 2, 3\}$. Since $i \neq k$, we have by Lemma 2.2

$$u_{ij}u_{kj}u_{im} = 0, \qquad u_{ij}u_{km}u_{im} = 0, \qquad u_{ij}u_{kl}u_{il} = 0,$$

which yields, by Definition 2.1(b),

$$u_{ij}u_{kl}u_{im} = u_{ij}(u_{kj} + u_{kl} + u_{km})u_{im} = u_{ij}u_{im} = 0$$

and

$$u_{ij}u_{kl} = u_{ij}u_{kl}(u_{ij} + u_{il} + u_{im}) = u_{ij}u_{kl}u_{ij} = (u_{ij}u_{kl}u_{ij})^* = (u_{ij}u_{kl})^* = u_{kl}u_{ij}.$$

□

2.3 Examples

Let us now come to truly non-classical examples.

Example 2.6 Let $p, q \in B(H)$ be projections. The following is a (well-known) 4×4 quantum permutation matrix.

$$\begin{pmatrix} p & 1-p & 0 & 0 \\ 1-p & p & 0 & 0 \\ 0 & 0 & q & 1-q \\ 0 & 0 & 1-q & q \end{pmatrix}$$

Note that p and q need not commute. As a concrete example, let us rearrange the example from the introduction:

$$u = \begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} & \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} & \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} & \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \end{pmatrix}$$

For this matrix, the entries do not commute and the assumptions of Lemma 2.4 are violated. Indeed, this matrix cannot be written as a direct sum of classical permutation matrices (since the entries of a direct sum of permutation matrices all commute).

In order to have more examples, let us mention a construction by Woronowicz, see for instance [49, Def. 3.1]

Definition 2.7 Given two matrices $u, v \in M_N(B(H))$ their *Woronowicz tensor product* is defined as

$$u \ominus v := \sum_{i,j=1}^{N} E_{ij} \otimes \left(\sum_{k=1}^{N} u_{ik} \otimes v_{kj} \right).$$

It is easy to see that if u and v are quantum permutation matrices, so is $u \ominus v$.

Example 2.8 The following example is taken from [49, Ex. 3.12], where $p, p', q, q' \in B(H)$ may be any projections.

$$\begin{pmatrix} p \otimes p' & (1-p) \otimes q' & p \otimes (1-p') & (1-p) \otimes (1-q') \\ (1-p) \otimes p' & p \otimes q' & (1-p) \otimes (1-p') & p \otimes (1-q') \\ q \otimes (1-p') & (1-q) \otimes (1-q') & q \otimes p' & (1-q) \otimes q' \\ (1-q) \otimes (1-p') & q \otimes (1-q') & (1-q) \otimes p' & q \otimes q' \end{pmatrix}$$

In fact, this matrix arises as

$$\begin{pmatrix} p & 0 & 1-p & 0 \\ 1-p & 0 & p & 0 \\ 0 & q & 0 & 1-q \\ 0 & 1-q & 0 & q \end{pmatrix} \ominus \begin{pmatrix} p' & 0 & 1-p' & 0 \\ 1-p' & 0 & p' & 0 \\ 0 & q' & 0 & 1-q' \\ 0 & 1-q' & 0 & q' \end{pmatrix}$$

Example 2.9 And another one from [49, Ex. 3.13], again coming from taking the Woronowicz tensor product. Note that, as in the previous example, we may replace all p by p' in the second tensor leg, and by p'' in the third one, and the same for q.

$$\begin{pmatrix} \begin{matrix} p\otimes p\otimes p \\ +(1-p)\otimes q\otimes(1-p)\end{matrix} & \begin{matrix} p\otimes(1-p)\otimes q \\ +(1-p)\otimes(1-q)\otimes(1-q)\end{matrix} & \begin{matrix} p\otimes p\otimes(1-p) \\ +(1-p)\otimes q\otimes p\end{matrix} & \begin{matrix} p\otimes(1-p)\otimes(1-q) \\ +(1-p)\otimes(1-q)\otimes q\end{matrix} \\ \\ \begin{matrix} (1-p)\otimes p\otimes p \\ +p\otimes q\otimes(1-p)\end{matrix} & \begin{matrix} (1-p)\otimes(1-p)\otimes q \\ +p\otimes(1-q)\otimes(1-q)\end{matrix} & \begin{matrix} (1-p)\otimes p\otimes(1-p) \\ +p\otimes q\otimes p\end{matrix} & \begin{matrix} (1-p)\otimes(1-p)\otimes(1-q) \\ +p\otimes(1-q)\otimes q\end{matrix} \\ \\ \begin{matrix} q\otimes(1-p)\otimes p \\ +(1-q)\otimes(1-q)\otimes(1-p)\end{matrix} & \begin{matrix} q\otimes p\otimes q \\ +(1-q)\otimes q\otimes(1-q)\end{matrix} & \begin{matrix} q\otimes(1-p)\otimes(1-p) \\ +(1-q)\otimes(1-q)\otimes p\end{matrix} & \begin{matrix} q\otimes p\otimes(1-q) \\ +(1-q)\otimes q\otimes q\end{matrix} \\ \\ \begin{matrix} (1-q)\otimes(1-p)\otimes p \\ +q\otimes(1-q)\otimes(1-p)\end{matrix} & \begin{matrix} (1-q)\otimes p\otimes q \\ +q\otimes q\otimes(1-q)\end{matrix} & \begin{matrix} (1-q)\otimes(1-p)\otimes(1-p) \\ +q\otimes(1-q)\otimes p\end{matrix} & \begin{matrix} (1-q)\otimes p\otimes(1-q) \\ +q\otimes q\otimes q\end{matrix} \end{pmatrix}$$

Example 2.10 The following example is taken from [13, Sect. 2]. Let

$$g_1 := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad g_2 := \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \quad g_3 := \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad g_4 := \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

be the Pauli matrices. For a unitary matrix $x \in U_2 \subseteq M_2(\mathbb{C})$, we view the matrix $g_i x g_j \in M_2(\mathbb{C})$ as a vector in $\mathbb{C}^4$. Let $w_{ij}^{(x)} \in M_4(\mathbb{C})$ be the rank one projection onto the vector $g_i x g_j \in M_2(\mathbb{C}) \cong \mathbb{C}^4$. Then, $w_x = (w_{ij}^{(x)})$ is a quantum permutation matrix. Also, $x \mapsto w_x$, as a function in $C(U_2, M_4(\mathbb{C}))$, is a quantum permutation matrix.

Example 2.11 The following example is taken from [11, Def. 4.1] which is linked to quantum Latin squares. Let H be an N-dimensional Hilbert space and let $\xi_{ij} \in H$ be vectors, for $i, j = 1, \ldots, N$, such that

- for every $i = 1, \ldots, N$, the set $\{\xi_{ij} \mid j = 1, \ldots, N\}$ forms an orthonormal basis of H,
- and for every $j = 1, \ldots, N$, the set $\{\xi_{ij} \mid i = 1, \ldots, N\}$ forms an orthonormal basis of H.

Let $p_{ij} \in B(H)$ be the rank one projection onto the vector ξ_{ij}. Then, $p = (p_{ij})$ forms a quantum permutation matrix. In [11], amongst others the case is studied when $h \in M_N(\mathbb{C})$ is a complex Hadamard matrix: Denote its rows by $h_1, \ldots, h_N$. They may be viewed as invertible elements in the algebra $\mathbb{C}^N$ and considering the vectors $\xi_{ij} := h_i/h_j \in \mathbb{C}^N$ and their rank one projections p_{ij}, we may construct a quantum permutation matrix. See also [31].

2.4 Quantum Permutation Matrices and Quantum Isomorphisms of Graphs

There are quantum permutation matrices which generalize isomorphisms of graphs, see their use in Sect. 4.3. Given a finite simple graph $\Gamma = (V, E)$, we denote by $i \sim j$, if two vertices $i, j \in V$ are connected by an edge from E, and $i \nsim j$ otherwise. We specify $i \sim_1 j$ or $i \sim_2 j$ in case there are two graphs Γ_1 and Γ_2.

Definition 2.12 Let $N \in \mathbb{N}$ and let H be a Hilbert space. Let $\Gamma_1 = (V, E_1)$ and $\Gamma_2 = (V, E_2)$ be finite simple graphs, with $V = \{1, \dots, N\}$. A *quantum isomorphism matrix of* Γ_1 *and* Γ_2 is a matrix $u \in M_N(B(H))$ consisting of entries $u_{ij} \in B(H)$, $i, j = 1, \dots, N$, such that

(a) $u_{ij} = u_{ij}^* = u_{ij}^2$ for all $i, j = 1, \dots, N$,
(b) $\sum_{k=1}^N u_{ik} = \sum_{k=1}^N u_{kj} = 1$ for all $i, j = 1, \dots, N$ with $i \neq j$,
(c) and $u_{ij}u_{kl} = u_{kl}u_{ij} = 0$ if $i \sim_2 k$ and $j \not\sim_1 l$,
(d) as well as $u_{ij}u_{kl} = u_{kl}u_{ij} = 0$ if $i \not\sim_2 k$ and $j \sim_1 l$.

If $\Gamma = \Gamma_1 = \Gamma_2$, we say that u is a *quantum automorphism matrix of* Γ in that case.

Hence, a quantum isomorphism matrix is a quantum permutation matrix (by (a) and (b)). Another way of expressing relations (c) and (d) is to say, that $uA_1 = A_2u$ holds, where $A_k \in M_N(\{0, 1\})$ is the adjacency matrix of Γ_k, $k = 1, 2$, i.e. $(A_1)_{ij} = 1$, if $i \sim_1 j$ and $(A_1)_{ij} = 0$, if $i \not\sim_1 j$. Let us prove it.

Lemma 2.13 *Let u be a quantum permutation matrix. We have $uA_1 = A_2u$ if and only if relations (c) and (d) of Definition 2.12 hold.*

Proof Since

$$\sum_{s:\, s\sim_1 l} u_{is} = \sum_s u_{is}(A_1)_{sl} = (uA_1)_{il}$$

and

$$\sum_{t:\, i\sim_2 t} u_{tl} = \sum_t (A_2)_{it}u_{tl} = (A_2u)_{il},$$

we observe that $uA_1 = A_2u$ holds if and only if

$$\sum_{s:\, s\sim_1 l} u_{is} = \sum_{t:\, i\sim_2 t} u_{tl}.$$

Thus, assuming $uA_1 = A_2u$, we obtain in case $i \sim_2 k$ and $j \not\sim_1 l$

$$u_{ij}u_{kl} = \sum_{t:\, i\sim_2 t} u_{ij}(u_{tl}u_{kl}) = \sum_{s:\, s\sim_1 l} (u_{ij}u_{is})u_{kl} = 0.$$

Here, we used Lemma 2.2 to show that $u_{tl}u_{kl} = \delta_{tk}u_{kl}$ and $u_{ij}u_{is} = \delta_{js}u_{ij}$, i.e. $\sum_{t:\, i\sim_2 t} u_{tl}u_{kl} = u_{kl}$ (since k appears in the sum, because $i \sim_2 k$), whereas $\sum_{s:\, s\sim_1 l} u_{ij}u_{is} = 0$, since j does not appear in the sum. Likewise, we obtain in case $i \not\sim_2 k$ and $j \sim_1 l$

$$u_{ij}u_{kl} = \sum_{s:\, s\sim_1 l} (u_{ij}u_{is})u_{kl} = \sum_{t:\, i\sim_2 t} u_{ij}(u_{tl}u_{kl}) = 0.$$

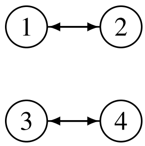
Conversely, assuming relations (c) and (d) of Definition 2.12, we infer

$$\sum_{s:\, s\sim_1 l} u_{is} = \sum_{s:\, s\sim_1 l} \sum_t u_{is}u_{tl} = \sum_{s:\, s\sim_1 l} \sum_{t:\, i\sim_2 t} u_{is}u_{tl} = \sum_s \sum_{t:\, i\sim_2 t} u_{is}u_{tl} = \sum_{t:\, i\sim_2 t} u_{tl}.$$

□

As in Sect. 2.2, we see that Definition 2.12 generalizes the classical situation: Given graphs Γ_1 and Γ_2, assume that $\sigma \in S_N$ is an isomorphism between them. We then have $\sigma A_1 \sigma^{-1} = A_2$, or equivalently $\sigma A_1 = A_2 \sigma$.

Example 2.14 Let Γ be the undirected graph on $V = \{1, 2, 3, 4\}$ given by:



The matrices from Example 2.6 are quantum automorphism matrices of this graph.

3 Open Problems

There are many open problems related with quantum permutation matrices. Let us state these problems first, as problems in operator theory, and let us shift the background information to the next section.

3.1 Faithful Models

We don't have a (finite-dimensional[2]) faithful model of quantum permutation matrices, for $N \geq 5$. The task is to find a quantum permutation matrix u with $u_{ij} \in B(H)$ on some finite-dimensional Hilbert space H, such that for any polynomial p in the N^2 entries and for any other quantum permutation matrix v, if $p(u) = p(u_{11}, u_{12}, \ldots, u_{NN}) = 0$, then also $p(v) = 0$. In other words, if a polynomial relation holds for u, then it holds for any other v, too.

Problem 3.1 *Find a faithful model (for polynomial relations[3]) of quantum permutation matrices of size N.*

There are models for $N = 4$: The map $x \mapsto w_x$ of Example 2.10 produces a faithful model, see [13]. See also [7, 11, 25, 27] for more on (not necessarily faithful) models, in particular [27, Conj. 5.7] for a conjecture on inner faithfulness of some

[2] We will see in Sect. 4.2 that we may define a universal C^*-algebra $A_S(N)$ generated by the entries of a quantum permutation matrix. By the noncommutative Gelfand-Naimark Theorem, we may represent this C^*-algebra faithfully on some Hilbert space – but it might be an infinite-dimensional one.

[3] More generally, we are interested in faithful, finite-dimensional *-representations of $A_S(N)$.

matrix model for quantum permutation matrices of any size $N \geq 4$. See also [6] for more on the case $N = 4$.

As a more concrete problem, consider the matrix of Example 2.6. It is not a faithful model. Indeed, the polynomial $p(u) = u_{13}$ vanishes for the matrix in Example 2.6, but not for the matrix in Example 2.8. Also the matrix from the latter example is not a faithful model. Indeed, consider the polynomial $p(u) = u_{11}u_{23}$. It vanishes for the matrix in Example 2.8 (as $p(u) = (p \otimes p')((1-p) \otimes (1-p')) = 0$ for that matrix), but not for the matrix in Example 2.9, as

$$\begin{aligned} p(u) &= \big(p \otimes p \otimes p + (1-p) \otimes q \otimes (1-p)\big)\big((1-p) \otimes p \otimes (1-p) + p \otimes q \otimes p\big) \\ &= p \otimes pq \otimes p + (1-p) \otimes qp \otimes (1-p) \\ &\neq 0 \end{aligned}$$

for suitable choices of the projections. Let us rephrase the questions from [49, Question 4.10, 4.11].

Problem 3.2 *Is the matrix in Example 2.9 a faithful model of size $N = 4$?*

$$\begin{pmatrix} \begin{matrix} p \otimes p \otimes p \\ +(1-p) \otimes q \otimes (1-p) \end{matrix} & \begin{matrix} p \otimes (1-p) \otimes q \\ +(1-p) \otimes (1-q) \otimes (1-q) \end{matrix} & \begin{matrix} p \otimes p \otimes (1-p) \\ +(1-p) \otimes q \otimes p \end{matrix} & \begin{matrix} p \otimes (1-p) \otimes (1-q) \\ +(1-p) \otimes (1-q) \otimes q \end{matrix} \\ \begin{matrix} (1-p) \otimes p \otimes p \\ +p \otimes q \otimes (1-p) \end{matrix} & \begin{matrix} (1-p) \otimes (1-p) \otimes q \\ +p \otimes (1-q) \otimes (1-q) \end{matrix} & \begin{matrix} (1-p) \otimes p \otimes (1-p) \\ +p \otimes q \otimes p \end{matrix} & \begin{matrix} (1-p) \otimes (1-p) \otimes (1-q) \\ +p \otimes (1-q) \otimes q \end{matrix} \\ \begin{matrix} q \otimes (1-p) \otimes p \\ +(1-q) \otimes (1-q) \otimes (1-p) \end{matrix} & \begin{matrix} q \otimes p \otimes q \\ +(1-q) \otimes q \otimes (1-q) \end{matrix} & \begin{matrix} q \otimes (1-p) \otimes (1-p) \\ +(1-q) \otimes (1-q) \otimes p \end{matrix} & \begin{matrix} q \otimes p \otimes (1-q) \\ +(1-q) \otimes q \otimes q \end{matrix} \\ \begin{matrix} (1-q) \otimes (1-p) \otimes p \\ +q \otimes (1-q) \otimes (1-p) \end{matrix} & \begin{matrix} (1-q) \otimes p \otimes q \\ +q \otimes q \otimes (1-q) \end{matrix} & \begin{matrix} (1-q) \otimes (1-p) \otimes (1-p) \\ +q \otimes (1-q) \otimes p \end{matrix} & \begin{matrix} (1-q) \otimes p \otimes (1-q) \\ +q \otimes q \otimes q \end{matrix} \end{pmatrix}$$

There are hints [39] that this is indeed the case! However, there is no proof yet.

3.2 Hilbert Sudoku/SudoQ

An interesting problem is the following. Given a rectangular array of projections $(u_{ij})_{i=1,\ldots,m;\, j=1,\ldots,N}$, $m < N$ such that $\sum_{k=1}^{N} u_{ik} = 1$ for all $i = 1, \ldots, m$ and such that $u_{ij}u_{kj} = 0$ for all $i \neq k$ and all $j = 1, \ldots, N$.

Problem 3.3 *Can we fill up a given rectangular matrix to a quantum permutation matrix? More precisely, can we find further projections $(u_{ij})_{i=m+1,\ldots,N;\, j=1,\ldots,N}$ such that $u = (u_{ij})_{i,j=1,\ldots,N}$ is a quantum permutation matrix?*

Let us give a concrete example. Suppose the following array is given:

$$\begin{matrix} u_{11} & u_{12} & u_{13} & u_{14} \\ u_{21} & u_{22} & u_{23} & u_{24} \end{matrix}$$

All u_{ij} are projections, we have $u_{11}+u_{12}+u_{13}+u_{14} = 1$ and $u_{21}+u_{22}+u_{23}+u_{24} = 1$ as well as $u_{1j}u_{2j} = 0$ for $j = 1, 2, 3, 4$.

Problem 3.4 *Can you always fill up this array to a* 4×4 *quantum permutation matrix?*

Or can you give a counter example of eight projections such that the above array may not be filled up? The answer is unknown. (And it is yes, if all projections commute, see Lemma 2.4.) See [32, Sect. 5] for the source of this problem and see [31] for the links to Hadamard matrices and Example 2.11. As a side remark, such a rectangular matrix needs to be filled up to a quadratic matrix – we may not fill it up to a rectangular matrix[4]

As a starting point for Problem 3.4, you might want to investigate the following question.

Problem 3.5 *Given two projections* $p, q \in B(H)$*, classify how they can be decomposed into projections* $p = p_1 + p_2$ *and* $q = q_1 + q_2$ *such that* $p_1 \perp q_1$ *and* $p_2 \perp q_2$*.*

This might be useful firstly, when investigating how the given rectangular array of Problem 3.4 may look like in general, with $p_1 = u_{11}$, $p_2 = u_{21}$, $q_1 = u_{12}$ and $q_2 = u_{22}$; secondly given such a rectangular array, you must find a decomposition of $p := 1 - (u_{11} + u_{21})$ and $q := 1 - (u_{12} + u_{22})$ into $p = u_{31} + u_{41}$ and $q = u_{32} + u_{42}$ on the way to solve Problem 3.4; and thirdly, this shall also help when trying to find counterexamples. Recall also Halmos's investigation of two projections in generic position [46].

In fact, you may even think of more complicated situations: Think of a quantum permutation matrix with "empty spots". Can you always fill it up to a complete quantum permutation matrix? You might want to call this game "Hilbert Sudoku" or "SudoQ" [62].

Problem 3.6 *Is there a Hilbert Sudoku consisting in entries* u_{ij} *which all commute, such that no "classical" solution exists (i.e. no completion to a quantum permutation matrix such that all entries commute) but a nonclassical one (some of the additional* u_{ij} *do not commute)?*

See also [62, Ex. 4.3, 4.4, Conj. 4.2, Conj. on page 3] for more on the Hilbert Sudoku game.

The Hilbert sudoku problem (in its rectangular form) is linked to the following classical situation. Suppose, we have two finite, simple graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ with $m = |V_1| \leq |V_2| = N$. Assume we have an injective graph

[4] There cannot be a "rectangular quantum permutation matrix": Given a matrix $(u_{ij})_{i=1,\dots,m;\, j=1,\dots,N}$ of projections u_{ij} such that $\sum_{k=1}^{N} u_{ik} = 1$ and $\sum_{k=1}^{m} u_{kj} = 1$, we have

$$m \cdot 1 = \sum_{i=1}^{m} \sum_{j=1}^{N} u_{ij} = \sum_{j=1}^{N} \sum_{i=1}^{m} u_{ij} = N \cdot 1,$$

so $m = N$ holds. Acknowledgements to Alexander Mang for this short and sweet argument. See also [1, after Lemma 4.1].

homomorphism $\varphi : \Gamma_1 \to \Gamma_2$, i.e. if $i \sim_1 j$, then $\varphi(i) \sim_2 \varphi(j)$. We can write this homomorphism as an array $(\sigma_{ij})_{i=1,\dots,m;\,j=1,\dots,N}$ with $\sigma_{ij} = 1$, if $\varphi(i) = j$ and zero otherwise. Since φ is defined on all of V_1, we have $\sum_{k=1}^{N} \sigma_{ik} = 1$ for all $i = 1, \dots, m$. Since φ is injective, we have $u_{ij}u_{kj} = 0$ for all $i \neq k$ and all j. And since Γ_1 and the subgraph $\varphi(\Gamma_1)$ of Γ_2 are isomorphic, it is easy to fill up the graph Γ_1 to a larger graph Γ_1' which is then isomorphic to Γ_2. So, we may rephrase the question of filling up rectangular arrays $(u_{ij})_{i=1,\dots,m;\,j=1,\dots,N}$ to quantum permutation matrices: Suppose that the u_{ij} in that rectangular array also satisfy the relations (c) and (d) of Definition 2.12 for two graphs Γ_1 and Γ_2. In that case, we have a graph Γ_1 which we may view as a subgraph of Γ_2 to some extent (it is quantum isomorphic to a subgraph of Γ_2).

Problem 3.7 *Can we complete the graph Γ_1 to a graph Γ_1' which is quantum isomorphic to Γ_2?*

This is unclear!

3.3 Quantum Symmetries of Graphs

There is a whole community interested in the following question: Given a finite, simple graph Γ, is there a quantum automorphism matrix of Γ (in the sense of Definition 2.12) whose entries do not all commute (i.e. there are some i, j, k, l such that $u_{ij}u_{kl} \neq u_{kl}u_{ij}$)? In that case, we say that the graph has quantum symmetries. On the other hand, if all quantum automorphism matrices of Γ have commuting entries, we say that Γ has no quantum symmetries.

As an example, take the complete graph (no self-edges, no multiple edges, undirected) on four vertices. It does have quantum symmetries. Indeed, any 4×4 quantum permutation matrix is a quantum automorphism of the complete graph (note that the relations (c) and (d) of Definition 2.12 are redundant for the complete graph), and the matrix of Example 2.6 has noncommuting entries. The same holds true for the graph in Example 2.14, it has quantum symmetries. Adding one edge to that graph however, we obtain a graph without quantum symmetries. See also [36, 73] for more on quantum symmetries and these examples of simple graphs on a small number of vertices.

For quite a few simple graphs, the question of the existence of quantum symmetries is settled. See also Sect. 4.3. However, there are also many open questions in this regard. For instance, how about the Johnson graph $J(6, 3)$, the Tutte 12-cage or the graph constructed from the linear constraint system on $K_{3,3}$ – do these graphs have quantum symmetries? [71, Ch. 8.2]

Problem 3.8 *Can you find a quantum permutation matrix with noncommuting entries which is a quantum automorphism matrix for one of these graphs?*

Another interesting question in that context:

Problem 3.9 *Can you find an asymmetric graph Γ (i.e.a graph whose automorphism group is trivial) and a quantum automorphism matrix of Γ such that one of its entries u_{ij} with $i \neq j$ is nonzero?*

We have no clue whether such a graph exists, but it would be a very exciting example. See Sect. 4.3. In fact, it seems that certain automorphism groups are "quantum excluding" [26]. Generalizing the above question, we may ask:

Problem 3.10 *Can you find a graph* Γ *whose automorphism group is either the trivial one* $\{e\}$*, a cyclic group* $\mathbb{Z}_k$ *or the symmetric group* S_3 *(also for the alternating groups* A_n*, in particular for* A_5*, it is open) and a quantum automorphism matrix of* Γ *such that one of its entries* u_{ij} *with* $i \neq j$ *is nonzero?*

And yet another question:

Problem 3.11 *Can you find two simple connected asymmetric graphs* Γ_1 *and* Γ_2 *which are non-isomorphic, but for which a quantum isomorphism matrix as in Definition 2.12 exists?*

Actually, solving Problem 3.11 in the affirmative solves Problem 3.9 in the affirmative by taking Γ as the disjoint union of Γ_1 and Γ_2. We know that there are such graphs as in Problem 3.11, if we drop the asymmetry assumption: There are non-isomorphic graphs which are quantum isomorphic [53, Sect. 4.4]. However, the smallest such example we know has 24 vertices.

Problem 3.12 *Can you find two graphs* Γ_1 *and* Γ_2*, each having less than 24 vertices, which are non-isomorphic, but for which a quantum isomorphism matrix as in Definition 2.12 exists? What is the smallest such example?*

It is known to experts that the smallest example must have at least 16 vertices. [68]

3.4 Quantum Sinkhorn Algorithm

Sinkhorn's algorithm, in a nutshell, is a procedure to construct bistochastic matrices: Take an arbitrary orthogonal matrix $A \in M_N(\mathbb{R})$ and normalize the rows such that their entries sum up to one, for each row. Then do the same for the columns. Oh no, you just destroyed the row sums – their sum might now differ from one! Nevermind, simply normalize again the rows, then the columns, then the rows etc. Eventually, magically, this will converge to a bistochastic matrix, i.e. to an orthogonal matrix $B \in M_N(\mathbb{R})$ such that $\sum_k B_{ik} = \sum_k B_{kj} = 1$ for all i and j.

In [27], a Sinkhorn type algorithm for quantum permutation matrices has been presented. Pick N^2 rank one projections and put them in an $N \times N$ matrix. Then normalize alternatingly the rows and the columns. In [64], we refined this algorithm: We may also insert a graph Γ in this algorithm and adjust the normalization procedure so that the iterations tend to respect the graph better and better. When restricting our algorithm to a friendly subclass of vertex-transitive graphs (to so called quasi Cayley graphs), we obtain very good results: The algorithm predicts with high accuracy whether or not the given graph has quantum symmetries. However, unlike in the case of the classical Sinkhorn algorithm, we are unable to *prove* convergence (althought we *observe* it).

Problem 3.13 *Does this quantum Sinkhorn algorithm converge?*

This is related to many other such questions in operator theory regarding the stability property: Given a sequence of operators which approximately almost satisfy a certain condition – can we find a limit object, which actually precisely satisfies this condition? We don't know for the quantum Sinkhorn algorithm.

3.5 Intermediate Quantum Permutations

The next open problem we want to present here has been puzzling the quantum group community for quite a while. We rephrase it in terms of operator theory, which looks a bit cumbersome, since we avoid the language of quantum groups and C^*-algebras.

Problem 3.14 *Given $N \geq 6$. Find (or disprove the existence of)*

(a) a polynomial p in the N^2 entries of a $N \times N$ matrix,
(b) and two $N \times N$ quantum permutation matrices u and v each with some noncommuting entries (at least two entries shall not commute),

such that

(c) $p(\sigma) = 0$ for all permutation matrices $\sigma \in M_N(\{0, 1\})$,
(d) $p(u) = 0$ (i.e. the polynomial relation vanishes on the entries of u),
(e) $p(v) \neq 0$ (it does not vanish on the entries of v),
(f) and whenever we take any quantum permutation matrix w with $p(w) = 0$, then also $p(w') = 0$, where $w'_{ij} := \sum_{k=1}^{N} w_{ik} \otimes w_{kj}$.

In that case, you just proved the existence of a famous intermediate quantum permutation group, see Sect. 4.2.

3.6 More Examples, Constructions and Quantum Transposition Matrices

Finally, we need more examples of quantum permutation matrices.

Problem 3.15 *Is there a good machine for constructing quantum permutation matrices?*

Is there a nice subclass of quantum permutation matrices, which may be studied separately? Possibly on some nice Hilbert spaces, like functional Hilbert spaces, Hardy spaces, Bergman spaces, Fock spaces? See [7, 13, 27, 49] for some models of quantum permutation matrices.

Actually, recall that Examples 2.8 and 2.9 come from the Woronowicz tensor product. In fact, the class of quantum permutation matrices is closed under taking the Woronowicz tensor product, conjugation with a diagonal unitary $W \oplus \ldots \oplus W$ (as in Lemma 2.4), or other operations, see for instance [11, Def. 3.6]. Now, the theory of classical permutation matrices allows for a nice generating set: Transpositions – every permutation matrix may be written as a product of transpositions. We do not have an analog for quantum permutation matrices.

Problem 3.16 *Are there "quantum transposition matrices" such that every quantum permutation matrix can be constructed from a tuple of quantum transposition matrices? Are there natural building blocks of quantum permutation matrices?*

4 Quantum Permutation Matrices in Their Broader Context and Use

In this final section, we briefly sketch the context of the above objects and problems and we give references for further reading. We try to be very selective here in order to keep it short and simple. In particular, we cannot reflect the whole variety of the existing research on these topics.

4.1 Quantum Mathematics

The notion "quantum mathematics" is not well-established yet, but it is used from time to time to subsume various areas of mathematics losely related to quantum physics. A common theme is noncommutativity, i.e. we consider algebraic structures where we might have $xy \neq yx$ for some elements x and y. In the 1930s and 1940s, the theories of von Neumann algebras [58, 59] and of C^*-algebras [35, 42] have been founded, as a starting point of "noncommutative analysis" or "quantum analysis".

4.1.1 Gelfand (Naimark) Philosophy

The famous Gelfand-Naimark Theorem from 1943 [42] (see also [24, Thm. II.2.2.4]) states that a unital C^*-algebra is commutative if and only if it is isomorphic to an algebra of continuous functions on a compact Hausdorff space. This is even a functorial/categorial relation and we may thus identify commutative C^*-algebras with compact spaces – and view noncommutative C^*-algebras as analogues of "noncommutative" compact spaces, in some sense. This Gelfand duality might look a bit strange when one sees it for the first time, but it is just the beginning of a whole philosophy of "quantum" versions of classical theories.

4.1.2 Extensions of the Gelfand Philosophy

While the theory of C^*-algebras can be viewed, to some extent, as noncommutative (or "quantum") topology [41], the theory of von Neumann algebras is viewed as noncommutative measure theory [2, Sect. 3], [24, Ch. III]; there is Connes's noncommutative (differential) geometry [33], Voiculescu's free probability [57, 63, 77, 78] (as a counter part to probability theory; you might also consider the community of "quantum probability" here), and there is Woronowicz's theory of compact quantum groups [65, 75, 81, 83]. They are all building on this Gelfand philosophy: commutative algebras correspond to the classical situation, while noncommutative ones correspond to their quantum counterpart. To some extent, you may also add quantum information theory (in analogy to information theory) [61, 80] and Taylor's free analysis/noncommutative function theory (in analogy to complex analysis) [52, 74] to this family – although the latter ones rely little on Gelfand duality. However, all these theories found their pioneers in the 1980s (besides von Neumann algebras: in the 1930s and C^*-algebras: in the 1940s) and more and more interdependences have been revealed in the past few decades.

So this is why it might be the time to subsume all these theories under the name "quantum mathematics" and to view it as a deeply intervowen branch of modern mathematics. See also [82] for a very brief introduction and overview, or [38].

4.2 Quantum Groups and Quantum Permutation Groups

In the context of "quantum mathematics", the role of symmetries is played by quantum groups rather than by groups.

4.2.1 Quantum Groups

In the 1980s, Woronowicz [83, 84] defined compact quantum groups and he showed that this class naturally generalizes the class of compact groups. See also the books [65, 75]. Let us present the slightly easier definition of a compact matrix quantum group [83]. See also the introductory notes [81].

Definition 4.1 Let $N \in \mathbb{N}$. A *compact matrix quantum group* is a pair $G = (A, u)$ with $u = (u_{ij})_{i,j=1,\dots,N}$ such that

(a) A is a unital C^*-algebra which is generated by the elements $u_{ij} \in A$, with $i, j = 1, \dots, N$,
(b) $u = (u_{ij})$ and $\bar{u} = (u_{ij}^*)$ are invertible matrices in $M_N(A)$,
(c) the map $\Delta : A \to A \otimes_{\min} A$ given by $\Delta(u_{ij}) = \sum_{k=1}^{N} u_{ik} \otimes u_{kj}$ is a *-homomorphism.

Woronowicz proved a Gelfand-Naimark type theorem for compact matrix quantum groups (A, u): If the C^*-algebra A is commutative, then A is isomorphic to the algebra of continuous functions $C(G)$ on some compact matrix group $G \subseteq GL_N(\mathbb{C})$, the generators u_{ij} are then the evaluation maps of the matrix entries, and Δ arises from matrix multiplication (hence, from the group operation of G). See [75, Prop. 5.1.3] for the more general statement on compact quantum groups. Let us mention, that there also other (strongly related) notions of quantum groups, mostly in a purely algebraic setting rather than in Woronowicz's analytic one, see for instance [50]. See [75, Sect. 5.4] for some links.

4.2.2 Quantum Permutation Groups

In the 1990s, Sh. Wang [79] defined S_N^+, a quantum version of the symmetric group S_N. It is given by the universal C^*-algebra

$$A_S(N) := C^*(u_{ij}, i, j = 1, \dots, N \mid u_{ij} = u_{ij}^* = u_{ij}^2, \sum_k u_{ik} = \sum_k u_{kj} = 1).$$

So, it is the universal C^*-algebra generated by the entries of a "universal quantum permutation matrix". If we add commutativity of all generators to this C^*-algebra, we obtain a commutative C^*-algebra, which is – by Gelfand-Naimark's Theorem – isomorphic to the algebra of continuous functions on some compact space. Which space? S_N, the symmetric group! So, in Woronowicz's theory (which also takes the group structure on S_N into account, in Gelfand duality), S_N^+ is a reasonable quantum counterpart of S_N.

4.2.3 Quantum Permutation Matrices as Representations of $A_S(N)$; Further Reading

Now, we immediately see that any quantum permutation matrix, as defined in Definition 2.1, gives rise to a *-representation of $A_S(N)$, so Sect. 2 basically boils down to the representation theory of this C^*-algebra. This is also the origin of quantum permutation matrices, better known under the name *magic unitaries*. They can also be viewed as generalized latin squares. It is impossible to list the whole literature on S_N^+, but here is a small collection: [6, 10–12, 15, 20, 22, 23, 28, 30, 37, 47, 79]. See in particular the survey [9]. See [45, 72] for a discussion on possible definitions of S_∞^+. See also [54] for another point of view on quantum permutations.

4.2.4 Operator Algebras Associated to S_N^+

Note that the C^*-algebras $A_S(N)$—or more generally, the C^*-algebras A associated to compact matrix quantum groups $G = (A, u)$ – lead to very interesting and deeply studied examples of operator algebras. In fact, any compact quantum group possesses a distinguished state, the Haar state, see for instance [75, Thm. 5.1.6, Def. 5.1.9]. So, by GNS construction, we obtain a reduced version of any quantum group, and also a von Neumann algebra associated to it. Now, these objects are at the same time very intricate, and yet tractable due to their extra structure, with striking connections to operator algebras associated with classical discrete groups. See for instance [81, Sect. 7.3, Sect. 7.4.6] for a short and incomplete (and also slightly outdated) overview on these operator algebraic aspects. See also the literature mentioned in the previous subsubsection.

4.2.5 "Easy" Quantum Groups

The representation theory of S_N^+ is quite combinatorial using partitions of sets, similar to Brauer diagrams and Schur-Weyl duality. See the work on "easy" quantum groups for this larger class of "combinatorial quantum groups" containing S_N^+; here is some excerpt: [17, 18, 29, 40, 43, 60, 67, 76] Here is an introduction to the field: [81].

4.2.6 Intermediate Quantum Permutation Groups

For the famous question on the existence of intermediate quantum permutation groups mentioned in Sect. 3.5, let us state it here more properly:

Problem 4.2 *Is there some $N \in \mathbb{N}$ and a quantum group G such that $S_N \subsetneq G \subsetneq S_N^+$?*

The answer is no for $N \leq 5$, see [4, 15], and it is unknown for $N \geq 6$. Providing a polynomial p and a solution to the question in Sect. 3.5 would produce such an intermediate quantum group G whose associated C^*-algebra is the quotient of $C(S_N^+)$ by the relations $p = 0$.

4.2.7 S_N^+ as a Symmetry Object

By the way, just like S_N is the symmetry object of N points within the category of groups (i.e. it is the maximal group acting on N points), S_N^+ is the symmetry object of N points within the theory of quantum groups: We may define actions of quantum groups on N points (after identifying N points with $\mathbb{C}^N$ in the sense of Gelfand duality), and we observe that S_N^+ is the maximal object acting on it [79, 81]. As S_N^+ contains S_N, due to a natural definition of what containment means here, we have more ways of quantum permuting points than just permuting them. The quantum world has a richer notion of symmetry!

4.3 Quantum Symmetries of Graphs and Quantum Isomorphisms of Graphs

Let us comment on further use of quantum permutation matrices as symmetry objects.

4.3.1 Quantum Automorphism Groups of Graphs

In 2005, Banica defined the quantum automorphism group of a finite graph [3, 21]. It is given by the quotient of the above C^*-algebra $A_S(N)$ by the relations (c) and (d) of Definition 2.12. It naturally generalizes the automorphism group of a simple finite graph $\Gamma = (\{1, \ldots, N\}, E)$, which in turn is given by

$$\mathrm{Aut}(\Gamma) = \{\sigma \in S_N \mid \sigma A = A\sigma\},$$

where $A \in M_N(\{0, 1\})$ is the adjacency matrix of Γ. The quantum automorphism group of Γ contains the automorphism group, and we say that Γ has quantum symmetries in case this is a strict containment; this definition is consistent with the one given in Sect. 3.3.

Again, the quantum world has a richer notion of symmetry—for a graph having quantum symmetries, we have more ways of quantum permuting its vertices than just permuting them. The literature on quantum symmetries is growing rapidly these days, and here is a short collection: [3, 5, 8, 21, 44, 69–71, 73]. Let us also mention some Erdös-Renyi type results in this context: [53, Thm. 3.15] and [34, 48].

4.3.2 Quantum Isomorphism of Graphs

Closely linked is the concept of quantum isomorphism of two graphs. We say that two graphs are *quantum isomorphic*, if a quantum isomorphism (in the sense of Definition 2.12) between them exists. Surprisingly, there are graphs, which are non-isomorphic but quantum isomorphic, see [53, Sect. 4.4]. This is striking—the quantum isomorphism class of a graph is larger than its isomorphism class, in general!

This is also very interesting in the context of graph homomorphism counts. In the 1960s, Lovasz showed, that two graphs Γ_1 and Γ_2 are isomorphic, if and only if for all graphs Γ', the graph homomorphism counts from Γ' to Γ_1 and from Γ' to Γ_2 coincide. Do we really need all graphs Γ'? May we restrict to graph homomorphism counts for a smaller class, say planar graphs Γ' for instance? In 2019, Mancinska and Roberson proved a Quantum Lovasz Theorem [56] (or rather [55] for the full version): Two graphs Γ_1 and Γ_2 are quantum isomorphic, if and only if for all planar graphs Γ', the graph homomorphism counts from Γ' to Γ_1 and from Γ' to Γ_2 coincide. As there are graphs, which are quantum isomorphic but not isomorphic, we may not restrict to planar graphs in Lovasz's Theorem.

The result by Mancinska and Roberson is exciting in many ways: Not only is it a strong theorem in graph theory—it has been achieved by means from quantum group theory and quantum information, revealing a nice interplay between these three fields.

4.4 Quantum Information Theory

Let us elaborate more on the link to quantum information theory.

4.4.1 Graph Isomorphism Game

Consider the nonlocal game, as described in [53, 55]: Given two graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ with $|V_1| = |V_2| = N$, a referee passes a vertex $x_A \in X$ to Alice and a vertex $x_B \in X$ to Bob; here, $X := V_1 \sqcup V_2$ is the disjoint union of V_1 and V_2. Alice replies with a vertex $y_A \in X$ which is not from the same graph as x_A, and likewise Bob replies with $y_B \in X$ different from the graph where y_A is from. The players win the game, if

(1) the set $\{x_A, x_B, y_A, y_B\}$ has four elements, two of them being from V_1 (let us call them j and l) and two of them from V_2 (calling them i and k) and we have $j \sim_1 l$ if and only if $i \sim_2 k$,
(2) or the set $\{x_A, x_B, y_A, y_B\}$ has two elements, one from V_1 and one from V_2.

4.4.2 Classical Winning Strategy

There is a perfect strategy (i.e. a strategy with which they may always win regardless of the referee's input), if and only if Γ_1 and Γ_2 are isomorphic. Indeed, in that case let $\varphi : V_1 \to V_2$ be an isomorphism of Γ_1 and Γ_2 and instruct Alice to reply with $\varphi(x_A) \in V_2$ in case $x_A \in V_1$ and with $\varphi^{-1}(x_A) \in V_1$ in case $x_A \in V_2$; likewise for Bob.

4.4.3 Quantum Winning Strategy

Now, may Alice and Bob increase their chances to win when applying a quantum strategy? Technically, they are now performing quantum measurements, on a shared entangled state. And the answer is: Yes, they can! If the given graphs Γ_1 and Γ_2 are non-isomorphic but quantum isomorphic, there is no classical winning strategy, but there is a perfect quantum strategy. The strategy comes from a quantum isomorphism matrix, as in Definition 2.12, of course.

So, the existence of perfect quantum strategies for this game in quantum information theory is linked with quantum permutation matrices (and hence also with quantum groups), and also with quantum isomorphisms of graphs and a quantum version of Lovasz's Theorem from graph theory. Beautiful, isn't it? See also [14, 66] for more on such links.

4.5 Free Probability Theory

Let us mention another use of quantum permutation matrices, or rather of the quantum permutation group S_N^+.

4.5.1 Classical de Finetti Theorem

In probability theory, De Finetti's Theorem may be stated as follows: Given a sequence $(x_n)_{n\in\mathbb{N}}$ of real random variables, this sequence is iid (independent, identically distributed) over the tail algebra if and only if it is exchangeable (i.e. its distribution is invariant under the action of the symmetric groups S_N, $N \in \mathbb{N}$ on finite tuples of the sequence).

4.5.2 Free de Finetti Theorem

Now, in free probability theory [57, 63, 77, 78], there is a notion of free independence, a kind of a noncommutative counterpart of classical independence, for noncommuting random variables. And just as S_N is the distributional symmetry object for classical (conditional) independence, S_N^+ is the distributional symmetry object for (conditional) free independence—Köstler and Speicher's De Finetti Theorem [51] states: Given a sequence $(x_n)_{n\in\mathbb{N}}$ of selfadjoint noncommutative random variables, this sequence is freely independent and identically distributed over the tail algebra if and only if it is quantum exchangeable (i.e. its distribution is invariant under the action of the quantum permutation groups S_N^+, $N \in \mathbb{N}$ on finite tuples of the sequence).

4.5.3 More on de Finetti Theorems and Other Stochastic Aspects

This is another instance of the interplay between various fields of "quantum mathematics": Just as groups provide the correct symmetries for probability theory, the correct symmetries for free probability are provided by quantum groups. See for instance [19], [16, Sect. 1.1] for more on such de Finetti theorems, or [18] for other stochastic aspects of S_N^+.

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References

1. Atserias, A., Mančinska, L., Roberson, D.E., Šámal, R., Severini, S., Varvitsiotis, A.: Quantum and non-signalling graph isomorphisms. J. Combin. Theory Ser. B **136**, 289–328 (2019)
2. Anantharaman, C. and Popa, S.: An introduction to II_1 factors. (draft)
3. Banica, T.: Quantum automorphism groups of homogeneous graphs. J. Funct. Anal. **224**(2), 243–280 (2005)
4. Banica, T.: Homogeneous quantum groups and their easiness level. Kyoto J. Math. **61**(1), 171–205 (2021)
5. Banica, T., Bichon, J.: Quantum automorphism groups of vertex-transitive graphs of order ≤ 11. J. Algebraic Combin. **26**(1), 83–105 (2007)
6. Banica, T., Bichon, J.: Quantum groups acting on 4 points. J. Reine Angew. Math. **626**, 75–114 (2009)
7. Banica, T., Bichon, J.: Random walk questions for linear quantum groups. Int. Math. Res. Not. IMRN **24**, 13406–13436 (2015)
8. Banica, T., Bichon, J., Chenevier, G.: Graphs having no quantum symmetry. Ann. Inst. Fourier (Grenoble) **57**(3), 955–971 (2007)
9. Banica, T., Bichon, J., Collins, B.: Quantum permutation groups: a survey. In: Noncommutative harmonic analysis with applications to probability, volume 78 of Banach Center Publ., pp. 13–34. Polish Acad. Sci. Inst. Math., Warsaw, (2007)
10. Banica, T., Bichon, J., Natale, S.: Finite quantum groups and quantum permutation groups. Adv. Math. **229**(6), 3320–3338 (2012)
11. Banica, T., Bichon, J., Schlenker, J.-M.: Representations of quantum permutation algebras. J. Funct. Anal. **257**(9), 2864–2910 (2009)
12. Banica, T., Collins, B.: Integration over quantum permutation groups. J. Funct. Anal. **242**(2), 641–657 (2007)
13. Banica, T., Collins, B.: Integration over the Pauli quantum group. J. Geom. Phys. **58**(8), 942–961 (2008)
14. Brannan, M., Chirvasitu, A., Eifler, K., Harris, S., Paulsen, V., Xiaoyu, S., Wasilewski, M.: Bigalois extensions and the graph isomorphism game. Comm. Math. Phys. **375**(3), 1777–1809 (2020)
15. Brannan, M., Chirvasitu, A., Freslon, A.: Topological generation and matrix models for quantum reflection groups. Adv. Math. **363**, 106982 (2020)
16. Baraquin, I., Cébron, G., Franz, U., Maaßen, L., Weber, M.: De finetti theorems for the unitary dual group. arxiv:2203.05852, (2022)
17. Banica, T., Curran, S., Speicher, R.: Classification results for easy quantum groups. Pac. J. Math. **247**(1), 1–26 (2010)
18. Banica, T., Curran, S., Speicher, R.: Stochastic aspects of easy quantum groups. Probab. Theory Relat. Fields **149**(3–4), 435–462 (2011)
19. Banica, T., Curran, S., Speicher, R.: De Finetti theorems for easy quantum groups. Ann. Probab. **40**(1), 401–435 (2012)
20. Bichon, J., Franz, U., Gerhold, M.: Homological properties of quantum permutation algebras. N. Y. J. Math. **23**, 1671–1695 (2017)

21. Bichon, J.: Quantum automorphism groups of finite graphs. Proc. Am. Math. Soc. **131**(3), 665–673 (2003)
22. Bichon, J.: Free wreath product by the quantum permutation group. Algebr. Represent. Theory **7**(4), 343–362 (2004)
23. Bichon, J.: Algebraic quantum permutation groups. Asian-Eur. J. Math. **1**(1), 1–13 (2008)
24. Blackadar, B.: Operator algebras, volume 122 of Encyclopaedia of Mathematical Sciences. Springer-Verlag, Berlin,: Theory of C^*-algebras and von Neumann algebras. Operator Algebras and Non-commutative Geometry, III (2006)
25. Banica, T., Moroianu, S.: On the structure of quantum permutation groups. Proc. Am. Math. Soc. **135**(1), 21–29 (2007)
26. Banica, T., McCarthy, J.P.: The frucht property in the quantum group setting. arxiv:2106.04999, (2021)
27. Banica, T., Nechita, I.: Flat matrix models for quantum permutation groups. Adv. Appl. Math. **83**, 24–46 (2017)
28. Brannan, M.: Reduced operator algebras of trace-perserving quantum automorphism groups. Doc. Math. **18**, 1349–1402 (2013)
29. Banica, T., Speicher, R.: Liberation of orthogonal Lie groups. Adv. Math. **222**(4), 1461–1501 (2009)
30. Banica, T., Skalski, A.: Two-parameter families of quantum symmetry groups. J. Funct. Anal. **260**(11), 3252–3282 (2011)
31. Banica, T., Skalski, A.: The quantum algebra of partial Hadamard matrices. Linear Algebra Appl. **469**, 364–380 (2015)
32. Banica, T., Skalski, A., Sołtan, P.: Noncommutative homogeneous spaces: the matrix case. J. Geom. Phys. **62**(6), 1451–1466 (2012)
33. Connes, A.: Noncommutative geometry. Academic Press Inc, San Diego, CA (1994)
34. Chirvasitu, A., Wasilewski, M.: Random quantum graphs. Trans. Am. Math. Soc. **375**(5), 3061–3087 (2022)
35. Doran, R.S. (editor): C^*-algebras: 1943–1993, volume 167 of Contemporary Mathematics. American Mathematical Society, Providence, RI, 1994. A fifty year celebration
36. Eder, C., Levandovskyy, V., Schanz, J., Schmidt, S., Steenpass, A., Weber, M.: Existence of quantum symmetries for graphs on up to seven vertices: a computer based approach. arXiv:1906.12097, pp. 15 pages + appendix, (2019)
37. Franz, U., Kula, A., Skalski, A.: Lévy processes on quantum permutation groups. In: Noncommutative analysis, operator theory and applications, volume 252 of Oper. Theory Adv. Appl., pp. 193–259. Birkhäuser/Springer, [Cham], (2016)
38. Franz, U., Skalski, A.: Noncommutative mathematics for quantum systems. Cambridge-IISc Series. Cambridge University Press, Delhi (2016)
39. Faroß, N., Weber, M.: Towards a concrete model for the quantum permutation group on 4 points. in preparation
40. Freslon, A., Weber, M.: On the representation theory of partition (easy) quantum groups. J. Reine Angew. Math. **720**, 155–197 (2016)
41. Gracia-Bondía, J.M., Várilly, J.C., Figueroa, H.: Elements of noncommutative geometry. Birkhäuser Advanced Texts: Basler Lehrbücher. [Birkhäuser Advanced Texts: Basel Textbooks]. Birkhäuser Boston, Inc., Boston, MA, (2001)
42. Gelfand, I., Neumark, M.: On the imbedding of normed rings into the ring of operators in Hilbert space. In C^*-algebras: 1943–1993 (San Antonio, TX, 1993), volume 167 of Contemp. Math., pages 2–19. Amer. Math. Soc., Providence, RI, 1994. Corrected reprint of the 1943 original [MR **5**, 147]
43. Gromada, D.: Classification of globally colorized categories of partitions. Infin. Dimens. Anal. Quant. Probab. Relat. Top. **21**(4), 1850029 (2018)
44. Gromada, D.: Quantum symmetries of cayley graphs of abelian groups. arxiv:2106.08787, (2021)
45. Goswami, D., Skalski, A.: On two possible constructions of the quantum semigroup of all quantum permutations of an infinite countable set. In: Operator algebras and quantum groups, volume 98 of Banach Center Publ., pages 199–214. Polish Acad. Sci. Inst. Math., Warsaw, (2012)
46. Halmos, P.R.: Two subspaces. Trans. Am. Math. Soc. **144**, 381–389 (1969)
47. Józiak, P.: Quantum increasing sequences generate quantum permutation groups. Glasg. Math. J. **62**(3), 631–639 (2020)
48. Junk, L., Schmidt, S., Weber, M.: Almost all trees have quantum symmetry. Arch. Math. (Basel) **115**(4), 367–378 (2020)
49. Jung, S., Weber, M.: Models of quantum permutations. J. Funct. Anal. **279**(2), 108516 (2020)

50. Kassel, C.: Quantum groups. Graduate Texts in Mathematics, vol. 155. Springer-Verlag, New York (1995)
51. Köstler, C., Speicher, R.: A noncommutative de Finetti theorem: invariance under quantum permutations is equivalent to freeness with amalgamation. Comm. Math. Phys. **291**(2), 473–490 (2009)
52. Kaliuzhnyi-Verbovetskyi, D.S., Vinnikov, V.: Foundations of free noncommutative function theory. Mathematical Surveys and Monographs, vol. 199. American Mathematical Society, Providence, RI (2014)
53. Lupini, M., Mančinska, L., Roberson, D.E.: Nonlocal games and quantum permutation groups. J. Funct. Anal. **279**(5), 108592 (2020)
54. McCarthy, J.P.: A state-space approach to quantum permutations. arxiv:2104.02817, (2021)
55. Mančinska, L., Roberson, D.E.: Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs. arXiv:1910.06958, pp. 55 (2019)
56. Mančinska, L., Roberson, D.E.: Quantum isomorphism is equivalent to equality of homomorphism counts from planar graphs. In: 2020 IEEE 61st Annual Symposium on Foundations of Computer Science, pp. 661–672. IEEE Computer Soc., Los Alamitos, CA, [2020] (2020)
57. Mingo, J.A., Speicher, R.: Free probability and random matrices, volume 35 of Fields Institute Monographs. Springer, New York; Fields Institute for Research in Mathematical Sciences, Toronto, ON, (2017)
58. Murray, F.J., Von Neumann, J.: On rings of operators. Ann. Math. **37**(1), 116–229 (1936)
59. Murray, F.J., von Neumann, J.: On rings of operators. IV. Ann. Math. **2**(44), 716–808 (1943)
60. Mang, A., Weber, M.: Categories of two-colored pair partitions Part II: categories indexed by semigroups. J. Combin. Theory Ser. A **180**, 105409 (2021)
61. Michael, A.: Nielsen and Isaac L. Chuang. Quantum computation and quantum information. Cambridge University Press, Cambridge (2000)
62. Nechita, I., Pillet, J.: SUDOQ–a quantum variant of the popular game using quantum coherence. Quant. Inf. Comput. **21**(9–10), 781–799 (2021)
63. Nica, A., Speicher, R.: Lectures on the combinatorics of free probability. London Mathematical Society Lecture Note Series, vol. 335. Cambridge University Press, Cambridge (2006)
64. Nechita, I., Schmidt, S., Weber, M.: Sinkhorn algorithm for quantum permutation groups. Exp. Math., pp. 1–13 (2021). https://doi.org/10.1080/10586458.2021.1926005
65. Neshveyev, S., Tuset, L.: Compact quantum groups and their representation categories. Cours Spécialisés [Specialized Courses], vol. 20. Société Mathématique de France, Paris (2013)
66. Roberson, D.E., Schmidt, S.: Quantum symmetry vs nonlocal symmetry. arxiv:2012.13328, (2020)
67. Raum, S., Weber, M.: The full classification of orthogonal easy quantum groups. Comm. Math. Phys. **341**(3), 751–779 (2016)
68. Schmidt, S.: Private communication
69. Schmidt, S.: The Petersen graph has no quantum symmetry. Bull. Lond. Math. Soc. **50**(3), 395–400 (2018)
70. Schmidt, S.: On the quantum symmetry of distance-transitive graphs. Adv. Math. **368**, 107150 (2020)
71. Schmidt, S.: Quantum automorphism groups of finite graphs. PhD thesis (2020)
72. Skalski, A., Sołtan, P.M.: Projective limits of quantum symmetry groups and the doubling construction for Hopf algebras. Infin. Dimens. Anal. Quant. Probab. Relat. Top. **17**(2), 1450012 (2014)
73. Schmidt, S., Weber, M.: Quantum symmetries of graph C^*-algebras. Can. Math. Bull. **61**(4), 848–864 (2018)
74. Taylor, J.L.: Functions of several noncommuting variables. Bull. Am. Math. Soc. **79**, 1–34 (1973)
75. Timmermann, T.: An invitation to quantum groups and duality. EMS Textbooks in Mathematics. European Mathematical Society (EMS), Zürich, 2008. From Hopf algebras to multiplicative unitaries and beyond
76. Tarrago, P., Weber, M.: Unitary easy quantum groups: the free case and the group case. Int. Math. Res. Not. IMRN **18**, 5710–5750 (2017)
77. Voiculescu, D.: Lectures on free probability theory. In: Lectures on probability theory and statistics (Saint-Flour, 1998), volume 1738 of Lecture Notes in Math., pp. 279–349. Springer, Berlin, (2000)
78. Voiculescu, D.-V., Stammeier, N., Weber, M. (editors): Free probability and operator algebras. Münster Lectures in Mathematics. European Mathematical Society (EMS), Zürich, 2016. Lecture notes from the masterclass held in Münster, Sept 2–6 (2013)
79. Wang, S.: Quantum symmetry groups of finite spaces. Comm. Math. Phys. **195**(1), 195–211 (1998)
80. Watrous, J.: The Theory of Quantum Information. Cambridge University Press (2018)

81. Weber, M.: Introduction to compact (matrix) quantum groups and Banica-Speicher (easy) quantum groups. Proc. Indian Acad. Sci. Math. Sci. **127**(5), 881–933 (2017)
82. Weber, M.: Quantum symmetry. Snapshots of modern mathematics from Oberwolfach, p. 5:16 (2020)
83. Woronowicz, S.L.: Compact matrix pseudogroups. Comm. Math. Phys. **111**(4), 613–665 (1987)
84. Woronowicz, S.L.: Compact quantum groups. In Symétries quantiques (Les Houches, 1995), pages 845–884. North-Holland, Amsterdam (1998)