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ORIGINAL ARTICLE

Uncovering the Link Between the Theoretical and Probabilistic Models of the Global Production Function: A Copula Approach

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Abstract

We provide micro foundations for the global production function (GPF) based on the standard microeconomic model, and we develop a parallel probabilistic model with similar properties. The theoretical and probabilistic models of the GPF are integrated in the context of a technology choice problem. We construct a primitive, named the augmented transformation function, to obtain a GPF and its associated joint distribution that includes the output and labor-saving and capital-saving technological innovations. This type of primitive allows us not only to derive the theoretical GPF but also to consistently build a link between the substitutability micro-parameters (elasticities) and the probabilistic parameters (correlations). We find that the shape of the GPF is determined by all the relations among technological innovations and output and the way they are combined.

Keywords Global production function · Labor-saving and capital-saving technological innovations · Copula base models · Elasticity of substitution · Kendall's tau correlation coefficient

JEL Classification D24 · O12 · O33 · O4

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Introduction

Endogenous technology choice models obtain the shape of the GPF from two primitives: (1) the functional form of the local production function (LPF) and (2) the functional form of the technology frontier (TF), which is calculated as a level curve of a predefined joint distribution of labor- and capital-augmenting ideas (Growiec 2008a, b, 2013, 2018; Jones 2005, 2011). Then, the LPF is maximized subject to the TF to find the GPF, microfounding the shape of the GPF by assuming the probabilistic model that gives rise to the TF. Jones (2005) obtains a Cobb−Douglas GPF from a Leontief LPF and a TF viewed as a level curve of a joint distribution with independently Pareto-distributed ideas. He finds that the shape of the GPF is determined by the distribution of ideas rather than by the shape of the LPF. Growiec (2008a, b) shows that if ideas are independently Weibull-distributed (with the same shape parameter) or Pareto-distributed and dependent according to the Clayton copula, then the GPF can be of the CES class. He shows that the GPF is determined by the distribution of ideas and by the shape of the LPF. Growiec (2013) devises a specification of the R&D sector, microfounding the assumption of Weibull-distributed ideas by using extreme value theory. He finds that the Weibull distribution is an accurate approximation of the true distribution of ideas needed to obtain GPFs of the CES class. Growiec (2018) assumes homothetic primitives and finds that the TF and the GPF are dual objects, confirming that the GPF depends on the shapes of both the LPF and the TF unless one of them has the Cobb−Douglas form. He proves that if the idea distribution is homothetic and the copula is additively separable, then the TF is of the CES or Cobb−Douglas form, which translates into a requirement of Pareto or Weibull marginal idea distributions.

In general, the previous literature develops a new method to obtain the GPF and provides relevant insights into the determinants of its shape. However, the functional forms of the primitives impose implicit restrictions on the relations among output and factor-saving ideas and the way they are combined, which have a strong influence on the final GPF. In fact, such assumptions lead us to disregard some of the interdependencies among output and factor-augmenting ideas (Das 2021), to impose the mathematical structure of very specific extreme probability distributions on the TF and the GPF and to envisage the probabilistic model only as a primitive needed to obtain the GPF shape. We hold that these limitations can be avoided by microfounding the shape of the GPF based on the properties of the firm's production set, which is considered a primitive datum of the theory (Debreu 1959; Diewert 1973; Mas-Colel, 1995).

Thus, in this paper, we take a new look at the link between the theoretical and probabilistic models of the GPF in the context of the technology choice problem with given input levels. As opposed to the previous literature, we propose to obtain a microfounded theoretical GPF and link it with a probabilistic model by using a unique primitive: a neoclassical augmented transformation function (ATF), which is an equivalent way of describing the properties of the set of feasible production plans augmented by labor- and capital-saving technological innovations (ideas) (Debreu 1959; Diewert 1973; Acemoglu 2003). First, we use the ATF to calculate economically meaningful functional forms for the local production function (LPF) and the technology frontier (TF). Second, we maximize the LPF subject to the TF to obtain the GPF, microfounding the shape of the GPF by using an ATF whose shape is consistent with the assumptions (restrictions) imposed on the original augmented production set (Diewert 1973). In this context, we show that the resulting GPF can also be used to determine the corresponding set of production plans, and thus, we obtain a duality between our ATF, the set of feasible production plans, and the GPF (Diewert 1973). Finally, to integrate the theoretical model with the probabilistic model of the GPF, we propose to construct the proper joint probability distribution associated with our microfounded ATF by using Sklar's theorem.

By obtaining the theoretical and probabilistic models of the GPF from a unique ATF (convex, quasiconcave, continuous from above, nonincreasing in output, and nondecreasing in the levels of technological innovations and knowledge) that reflects the structure of the augmented production set (Diewert 1993), we add the following findings to the literature: (1) We calculate a general GPF with the following properties: continuous, quasiconcave, and nondecreasing in inputs and in the level of knowledge. More importantly, we show that the shape of the GPF and the elasticity of substitution are determined by all the relations among the technological innovations and output and the way they are combined, described by the firm's augmented production set rather than by specific functional forms of the LPF and the TF. (2) By mapping the theoretical ATF to a copula, the resulting probabilistic model also reflects the structure of the production set and the ATF (Diewert 1993). Thus, we can consistently relate the substitutability parameter set to the copula's dependence parameter set through statistical correlation measures such as Kendall's tau or Spearman's rho. Specifically, we link the microeconomic parameters (elasticities) to the probabilistic parameters (correlations) without imposing further restrictions on them, and (3) we illustrate our setup by assuming a primitive (ATF) with the CES property that allows us to find closed forms of all the well-known global production functions (linear, Leontief, Cobb–Douglas, and CES) and build a link between the substitutability microparameters and the statistical parameters (correlations).

This paper is organized as follows. "The Setup" establishes our approach to deliver global production functions based on a transformation function augmented by labor- and capital-saving technological innovations. We also show that the augmented production set, the ATF, and the GPF are dual objects. In addition, we show how our transformation function can be the basis of constructing a joint distribution that makes our setup empirically relevant. In "Illustration of How to Derive a Nested CES GPF and its Distribution Function Based on an Augmented Transformation Function", we apply our setup to obtain a novel nested CES GPF with two dependence parameters. "Conclusion" concludes and suggests potential directions of future research.

The Setup

Here, we construct our basic object, an augmented transformation function (ATF), which reflects the microeconomic structure of the set of feasible production plans available for firms in a competitive economy. This function is *augmented* because there are two types of technological innovations available for given levels of labor L and capital K in the domain of each production plan. The ATF reflects the relationship among the outputs and the factor-saving innovations, the *availability* of such innovations, and the *substitutability or complementarity between* them. We need to introduce this function because it is a primitive that allows us to theoretically derive the local production function (LPF) of a firm and its technology frontier (TF), from which we can derive all the textbook GPFs,[1]and it is naturally associated with a trivariate joint distribution function by using Sklar's theorem, which describes all the probabilistic relations among the technological innovations and output.

Theoretical Setup: A Transformation Function is Used to Obtain a Deterministic Microfounded GPF

We assume that there exists a set of feasible production plans, augmented by capital- and labor-saving technological innovations and a level of knowledge, from which an *augmented transformation function (ATF)* is realized. The level of knowledge determines the possible pairs of labor- and capital-saving technological innovations. We treat technological innovations as scarce goods that are substitutable or complementary like any other good in the economy. The higher the level of knowledge is, the more possible combinations of technological innovations. We show that the ATF embeds an LPF and a TF that are the basis for deriving a GPF.

Let $j \in J \subseteq \mathbb{R}$ index a discrete set of homogenous firms in a competitive economy. Let $i \in I \subseteq \mathbb{R}$ index a feasible production plan (i.e., a pair of labor-saving and capital-saving technological innovations a_i and b_i, respectively, an exogenous level of knowledge N_i, the input levels L_i and K_i, and the output level[2] $\widetilde{Y}_i$), which is available for the *j-th* firm at any given point in time $t \in T = [0, \tau]$. Note that the level of knowledge determines the feasible set of possible pairs of labor- and capital-saving technological innovations. In addition, a multiplicity of pairs (a_i, b_i) in this set can attain the same level of knowledge N_i.

An *augmented production plan* $(a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i)$ is feasible if the combination of technological innovations $\left(a_i, b_i\right)$ available for the level of knowledge N_i and the levels of inputs L_i and K_i can produce the level of output $\widetilde{Y}_i$.

Assumption 2.1.1 Let $(a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i)$ represent a single production plan augmented by a_i, b_i, and N_i, and let $B_j \subseteq \mathbb{R}^6$ represent the set of augmented feasible production plans for the *j-th* firm at time t, where $\widetilde{Y}_i$ is the output and the available

[1] In this paper, we analyze a technology choice problem, which is different from the classic production problem of choosing the inputs and the output level by maximizing profits.

[2] The level of production $\widetilde{Y}_i$ can be attained by different combinations of innovations for a given level of knowledge and inputs.

technology pair of innovations (a_i, b_i) is associated with the levels of labor and capital, L_i and K_i, respectively. Let B_j be a nonempty, closed, and convex subset of the nonnegative orthant in $\mathbb{R}^6$ that satisfies the following properties:

a. If $(a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i) \in B_j$ and $(a_l, L_l, b_l, K_l, N_l) \geq (a_i, L_i, b_i, K_i, N_i)$, then $(a_l, L_l, b_l, K_l, \widetilde{Y}_i, N_l) \in B_j$,[3]
b. $0 = (0, 0, 0,0, 0,0) \in B_j$, and if for an augmented production plan $(a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i) \in B_j$, $a_i = 0$, $L_i = 0$, $b_i = 0$, $K_i = 0$ or $N_i = 0$, then $\widetilde{Y}_i = 0$ (possibility of inaction and no free lunch),
c. If $(a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i) \in B_j$ and $0 \leq \widetilde{Y_\zeta} \leq \widetilde{Y}_i$, then $(a_i, L_i, b_i, K_i, \widetilde{Y_\zeta}, N_i) \in B_j$, and
d. For each vector$(a_i, L_i, b_i, K_i, N_i) \geq 0$,, the set $\left\{ \widetilde{Y}_i : (a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i) \in B_j \right\}$ of feasible outputs is bounded from above.

Assumption 2.1.1 not only states the *substitutability or complementarity* between L_i and K_i and a_i and b_i, respectively, but also describes the relationship of such inputs and the level of the available factor-saving technological innovations and output. Properties (i) and (iii) indicate that if we can produce a certain quantity of output with a given amount of inputs and technological innovations, then higher quantities can produce at least the same level of output. Property (ii) indicates that firms can decide not to produce and not incur costs for production and that all inputs and technological innovations are necessary for production. Property (iv) indicates that with finite levels of inputs and technological innovations, we can produce only finite amounts of output.

Note that we might use convex analysis to define and solve the firm's technology choice problem, but it could be unmanageable. To have a functional representation of B_j, we define an *augmented transformation function* that indicates if a feasible augmented production plan $(a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i)$ of a is technologically efficient for a profit-maximizing firm.

Definition 2.1.1 Let B_j satisfy Assumption 2.1.1; then, $T : \mathbb{R}^6_+ \to \mathbb{R}$ defined on the vector $\boldsymbol{z_i} = (a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i)$ as

$$T(a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i) := \begin{cases} \max\limits_{\widetilde{Y}_i} \left\{ \widetilde{Y}_i : \boldsymbol{z_i} \in B_j \right\} - \widetilde{Y}_i; & if\ \boldsymbol{z_i} \in B_j \\ -1 & if\ \boldsymbol{z_i} \notin B_j \end{cases}$$

[3] We say that vector $\boldsymbol{x} = (x_1, x_2, \dots, x_n)$ is either greater than or equal to $\boldsymbol{y} = (y_1, y_2, \dots, y_n)$ if $x_i \geq y_i$ for all $i = 1,2, \dots, n$.

is an augmented transformation function (ATF). Note that $(a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i) \geq 0$[4] and $T(a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i) = 0$ if and only if $\widetilde{Y}_i = \max_{\widetilde{Y}_i} \left\{ \widetilde{Y}_i : (a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i) \in B_j \right\}$.

Then, an augmented transformation function inherits the mathematical properties of the set B_j, such as the shape of its level surfaces and its quasiconcavity and continuity. To analyze the firm's technology choice problem, we are interested in choosing the optimal levels of feasible technological innovations a_i and b_i that allow the firm to obtain the maximum amount of production for given input levels.

Proposition 2.1.1 *If the transformation function $T(a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i)$ satisfies Definition 2.1.1, then it is monotone, concave, quasiconcave, continuous from above, nonincreasing in $\widetilde{Y}_i$, and nondecreasing in a_i, L_i, b_i, K_i, and N_i.*

Proof See Appendix A.1.

Definition 2.1.2 Given the augmented transformation function (ATF) $T(a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i)$, we can define the following set:

$$B_j^{'} := \left\{ z_i = \left(a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i \right) : T(z_i) \geq 0; z_i \geq 0 \right\}$$

Proposition 2.1.2 *If $T(a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i)$ satisfies the conditions given in Proposition 2.1.1, the set $B_j^{'}$, defined in 2.1.2, satisfies the conditions of a proper set of augmented feasible production plans.*

Proof The proof follows directly from Definition 2.1.2.

Then, we have proven that there exists a duality between the set of augmented feasible production plans, B_j, and the augmented transformation function, $T(a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i)$, because we can define each object from the other one, maintaining their properties.

Now, we suppose that L and K are fixed for a given time t and a representative firm. As a result, subscripts are not provided for these variables in the remainder of the derivations. Thus, we can write the *transformation function* as $T\left(a_i, b_i, \widetilde{Y}_i, N_i\right) := T\left(a_i, L, b_i, K, \widetilde{Y}_i, N_i\right)$. Figure 1 shows an augmented transformation function for a given level of knowledge N.

Then, Assumption 2.1.1 and Definition 2.1.1 imply that a representative firm will choose to produce at the maximum of $\left\{ \widetilde{Y}_i : T\left(a_i, b_i, \widetilde{Y}_i, N_i\right) \geq 0 \right\}$ for each pair $(a_i, b_i) \geq 0$ at given levels of L and$K$. Then, an LPF can be obtained from the transformation function. Note that the level of knowledge N at time t is a given

[4] The expression $\boldsymbol{x} = \left(x_1, x_2, \ldots, x_n\right) \geq \left(y_1, y_2, \ldots, y_n\right) = \boldsymbol{y}$ means that $x_i \geq y_i$ for all $i = 1, 2, \ldots, n$.

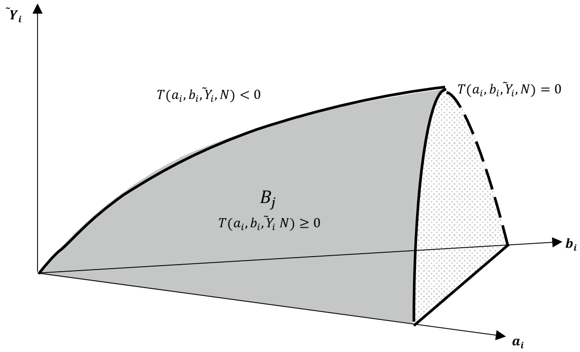

Fig. 1 Surface level of the set of augmented production plans B_j and the augmented transformation function $T\left(a_i, b_i, \widetilde{Y}_i, N\right)$ for given levels of L, K, and N

exogenous variable because it is unrelated to the level of production in the short run. Consequently, the LPF can be considered a function that does not depend on the level of knowledge$N$. In addition, the maximum of T always exists due to Weierstrass's theorem, as it is a *continuous function* over the range of $\widetilde{Y}_i$ and the set $\left\{\widetilde{Y}_i : T\left(a_i, b_i, \widetilde{Y}_i\right) \geq 0\right\}$ of feasible outputs is a *compact set.*[5]

Definition 2.1.3 The *LPF* gives the maximum amount of output $\widetilde{Y}_i$ that can be produced in the set B_j for given levels of L and K, i.e.,

$$Y_i := \max_{\widetilde{Y}_i \in B_j}\{\widetilde{Y}_i : T\left(a_i, b_i, \widetilde{Y}_i\right) \geq 0\} = f\left(a_i, L, b_i, K\right)$$

Definition 2.1.4 For a certain time t, the level of $\widetilde{Y}_i$ does not influence the exogenous level of knowledge; then, we define the *technology frontier* (*TF*) as the set of combinations of technology pairs $\left(a_i, b_i\right)$ and the level of knowledge N such that

$$H_N\left(a_i, b_i\right) = \left\{\left(a_i, b_i\right) : T\left(a_i, b_i, N\right) = 0\right\}.$$

Definition 2.1.1 and Proposition 2.1.1 imply that for given levels of L and K, the ATF has embedded in it the following microeconomic information: (1) the *availability* of labor- and capital-saving technological innovations a_i and b_i; (2) the existing *substitutability or complementarity* between the technological innovations and their relationships; and (3) the relationship among technological innovations and output through the LPF for given inputs in the economy. Thus, we can consistently obtain an LPF and a TF from our ATF for the representative firm that chooses an optimal technology pair $\left(a_i, b_i\right)$ from B_j. As opposed to the previous literature, we derive the LPF and the TF from a unique primitive, the ATF. In contrast, in the literature, it has been assumed that the LPF and TF already exist. Our model has a different nature because we do not need to use extreme value theory (probability theory) to provide

5 $$\left\{{{\widetilde{Y}}_{i} : T(a}_{i}, {b}_{i},{\widetilde{Y}}_{i})\ge 0\right\}$$ is a compact set, as it is *a closed interval and bounded* from above and below ($${\widetilde{Y}}_{i}\ge 0$$, and (iv) of Assumption 2.1.1).

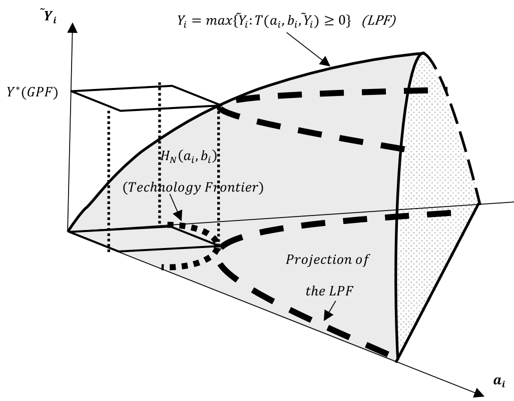

Fig. 2 Derivation of the LPF, TF, and GPF from the augmented transformation function

a microfounded model. Then, the GPF is obtained by maximizing the former subject to the latter.

Now, for a given level of knowledge N and levels of L and K, we can also define a *global production function* (GPF) [see, e.g., (Growiec 2008a; Jones 2005)] for the representative firm as the maximum amount of output Y_i^* that results from solving the problem of maximizing the LPF subject to the TF (as in Fig. 2). That is,

$Y_i^*(L, K, N) := \max_{a_i, b_i} f\left(a_i, b_i; L, K\right)$ subject to $H_N\left(a_i, b_i\right)$.

Proposition 2.1.3 *The GPF $Y_i^*(L, K, N)$ satisfies the following properties: monotone, nondecreasing in inputs and level of knowledge, quasiconcave, and satisfying $Y_i^*(0,0,0) = 0$.*

Proof See Appendix A.1.

Definition 2.1.5 Given the GPF, $Y_i^*(L, K, N)$, that satisfies the properties of Proposition 2.1.2, we can define the set $B_j^{''}$ as follows:

$$B_j^{''} = \left\{(a_i, L, b_i, K, \widetilde{Y}_i, N_i) | \widetilde{Y}_i \leq Y_i^*(L, K, N); T\left(a_i, L, b_i, K, \widetilde{Y}_i, N_i\right) \geq 0\right\}.$$

Proposition 2.1.4 *If $Y_i^*(L, K, N)$ satisfies the conditions in Proposition 2.1.4, the set $B_j^{''}$ defined in 2.1.5 satisfies the properties of a proper augmented feasible production set in Assumption 2.1.1.*

Proof See Appendix A.1.

Note that there exists a duality defined among B_j, the ATF and the GPF.

Probabilistic Setup: Linking the Deterministic Microfounded ATF with its Corresponding Probability Function

Now, we construct a probabilistic version of our augmented transformation function, which can be achieved using Sklar's theorem. Specifically, we construct a joint distribution function of $a_i, b_i, \widetilde{Y}_i$, and N_i for given levels of L and K by using a multivariate copula, which allows us to link the *availability* and *substitutability* microparameters to their probabilistic counterparts (location and correlation probabilistic parameters).

Assumption 2.2.1 There exists a random vector $\left(g_a\left(a_i\right), g_b\left(b_i\right), g_{\widetilde{Y}}\left(\widetilde{Y}_i\right), g_N\left(N_i\right)\right)$ into a probability space $\left(\Omega_i, \mathcal{F}_i, P_i\right)$ (Spanos, 1986) that includes monotone transformations of both labor- and capital-saving technological innovations $\left(a_i, b_i\right)$, output $\widetilde{Y}_i$, and the level of knowledge N_i, which are available to the j-th firm at a given point in time for given levels of L and K.

Let $\left(g_a(a), g_b(b), g_{\widetilde{Y}}\left(\widetilde{Y}\right), g_N(N)\right)$ be *a realization* of this vector. Then, this assumption implies that there exist marginal distribution functions U, V, W, and X that describe its probabilistic behavior. It also implies that there exists a joint distribution of the random vector $\left(g_a\left(a_i\right), g_b\left(b_i\right), g_{\widetilde{Y}}\left(\widetilde{Y}_i\right), g_N\left(N_i\right)\right)$, which is determined by a copula $C(U, V, W, X)$ (Nelsen, 2006). Notably, our setup is general enough that it allows for more than one substitutability parameter that might determine the shape of the GPF. First, we define a 4-dimensional copula as follows:

Definition 2.2.1 *A* 4*-dimensional copula* of the random vector $\left(g_a(a), g_b(b), g_{\widetilde{Y}}\left(\widetilde{Y}\right), g_N(N)\right)$ is a continuous and nondecreasing real-valued function C, parametrized with the set of parameters Θ, from $\updownarrow^4 = [0,1] \times [0,1] \times [0,1] \times [0,1]$ to $\updownarrow = [0,1]$ that links the marginal distributions $U = U\left(g_a(a);\Theta_a\right), V = V\left(g_b(b);\Theta_b\right)$, $W = W\left(g_{\widetilde{Y}}\left(\widetilde{Y}\right);\Theta_{\widetilde{Y}}\right)$, and $X = X\left(g_N(N);\Theta_N\right)$ together by using Sklar's theorem to obtain the joint distribution.

$$
\begin{aligned}
&F\left(g_a(a), g_b(b), g_{\widetilde{Y}}\left(\widetilde{Y}\right), g_N(N);\Theta\right) \\
&\quad = C\left(U\left(g_a(a);\Theta_a\right), V\left(g_b(b);\Theta_b\right), W\left(g_{\widetilde{Y}}\left(\widetilde{Y}\right);\Theta_{\widetilde{Y}}\right), X\left(g_N(N);\Theta_N\right);\Theta\right);
\end{aligned}
$$

where $\Theta_a, \Theta_b, \Theta_{\widetilde{Y}}$, and Θ_N are the margins' parameter sets and where Θ is the copula's parameter set that describes the probabilistic relationship, i.e., the probabilistic dependence structure, between the involved random variables.

Theorem 2.2.1 *Using Assumptions 2.1.1 and 2.2.1 and assuming a nonempty set $B_j \neq \{0\}$, for a particular realization $\left(g_a(a), g_b(b), g_{\widetilde{Y}}\left(\widetilde{Y}\right), g_N(N)\right)$ of the random vector $\left(g_a\left(a_i\right), g_b\left(b_i\right), g_{\widetilde{Y}}\left(\widetilde{Y}_i\right), g_N\left(N_i\right)\right)$[6] and a given exogenous level of knowledge N, we find that the following monotonically increasing transformation of our ATF*[7]

$$T_C\left(g_a(a), g_b(b), g_{\widetilde{Y}}\left(\widetilde{Y}\right), g_N(N)\right) = \frac{T\left(g_a(a), g_b(b), g_{\widetilde{Y}}\left(\widetilde{Y}\right), g_N(N)\right)}{maxT\left(g_a(a), g_b(b), g_{\widetilde{Y}}\left(\widetilde{Y}\right), g_N(N)\right)}$$

is the deterministic version of the following 4-copula function:

$$\begin{aligned} &C\left(g_a(a), g_b(b), g_{\widetilde{Y}}\left(\widetilde{Y}\right), g_N(N)\right) \\ &\quad = C\left(U\left(g_a(a);\Theta_a\right), V\left(g_b(b);\Theta_b\right), W\left(g_{\widetilde{Y}}\left(\widetilde{Y}\right);\Theta_{\widetilde{Y}}\right), X\left(g_N(N);\Theta_N\right);\Theta\right), \end{aligned}$$

where $U\left(g_a(a);\Theta_a\right), V\left(g_b(b);\Theta_b\right)$, and $X\left(g_N(N);\Theta_N\right)$ are distributions of the monotonically increasing transformations of a, b, and N, respectively, and $W\left(g_{\widetilde{Y}}\left(\widetilde{Y}\right);\Theta_{\widetilde{Y}}\right)$ is the distribution of a monotonically decreasing transformation of $\widetilde{Y}$. In addition, by using Sklar's theorem, there exists a 4-dimensional distribution function with margins U, V, W, and X given by

$$F\left(g_a(a), g_b(b), g_{\widetilde{Y}}\left(\widetilde{Y}\right), g_N(N);\Theta\right) = C\left(g_a(a), g_b(b), g_{\widetilde{Y}}\left(\widetilde{Y}\right), g_N(N)\right) \tag{1}$$

which is a probabilistic version of our deterministic augmented transformation function $T_C\left(g_a(a), g_b(b), g_{\widetilde{Y}}\left(\widetilde{Y}\right), g_N(N)\right)$ and has the properties defined in Assumption 2.1.1.

Proof See Appendix A.1.

Thus, as shown in Fig. 1, we have constructed a probabilistic version of our transformation function not only by building a particular copula with margins for $g_a(a), g_b(b), g_{\widetilde{Y}}\left(\widetilde{Y}\right)$, and $g_N(N)$ but also by selecting specific values for the copula's parameter set. These parameters must be consistent with the theoretical parameters of the transformation function. Once we have a probabilistic version of the

[6] Notice that this is the general case derived from our setup. A particular and relevant case is the one in which Y is deterministic, assuming that we have a realization of the random variables a_i, b_i and N_i, for a given L and K. Therefore, in this particular case, a trivariate copula on the transformation of a_i, b_i and N_i can be used. We thank an anonymous referee for useful comments on this issue.

[7] The maximum in $T\left(a_i, b_i, \widetilde{Y}_i, N\right)$ exists because, as N is a fixed exogenous variable, there is a limit for the size of knowledge at time t. Then, as L and K are fixed, there exists a bounded set of feasible outputs that can be attained by this firm.

transformation function, we can derive the LPF and the TF from it by following the procedure indicated in the next corollary.

Corollary 2.2.1 *The LPF and the TF can be obtained by using Definition 2.2.1 and Eq. (1), as follows:*

(a) *The functional form of a deterministic LPF can be obtained as follows. By marginalizing $C\left(a, b, \widetilde{Y}, N\right)$ with respect to the random variable N_i $(X = 1)$*[8]*, using the continuous and nondecreasing transformations h_a and h_b of a_i, and b_i, respectively, and a decreasing transformation $h_{\widetilde{Y}}$ of $\widetilde{Y}_i$,*[9] *assuming specific values for the copula dependence parameter set Θ, fixing the probability $F_{abY}\left(h_a(a), h_b(b), h_{\widetilde{Y}}\left(\widetilde{Y}\right);\Theta\right) = C_{ab\widetilde{Y}}\left(a, b, \widetilde{Y}\right)$ at the level P_0 in Eq. (1), and rearranging the resulting equation, the following result is obtained:*

$$Y = W_Y^{-1}\left(C_W^{-1}\left(U\left(a;L, \Theta_a\right), V\left(b;K, \Theta_b\right);\Theta\right), P_0\right);$$

where C_W^{-1} and W_Y^{-1} are inverse transformations of C and W[10] *with respect to W and Y, respectively.*

(b) *The deterministic TF can be obtained by marginalizing $C\left(a, b, \widetilde{Y}, N\right)$ with respect to the random variable $\widetilde{Y}_i$ $(W = 1)$,*[11] *assuming continuous and decreasing transformations q_a and q_b and an increasing transformation q_N*[12] *of the variables a_i, b_i, and N_i, respectively, taking specific values for the copula parameter set Θ, fixing the probability $F_{abN}\left(q_a(a), q_b(b), q_N(N);\theta\right) = C_{abN}(a, b, N)$ at a constant level P_1, and rearranging the related equation in terms of a and b as follows:*

$$H(a, b) = X_N^{-1}\left(C_X^{-1}\left(U\left(q_a(a);\Theta_a\right), V\left(q_b(b);\Theta_b\right)\right), P_1\right) = A_N,$$

where C_X^{-1} and X_N^{-1} are inverse transformations of C and X[13] *with respect to X and A_N, respectively, and A_N is a constant because N is fixed.*

Proof See Appendix A.1.

A relevant case is one in which we assume that joint distribution is given by a generalized Archimedean copula family, with a probabilistic dependence parameter set Θ, as shown below.

8 We then obtain the marginalized copula $C_{ab\widetilde{Y}}\left(a, b, \widetilde{Y}\right) = C\left(a, b, \widetilde{Y}, 1\right)$ (Nelsen 2006).

9 In the LPF, $g_a(a)$, $g_b(b)$, and $g_{\widetilde{Y}}\left(\widetilde{Y}\right)$ are denoted as $h_a(a)$, $h_b(b)$, and $h_{\widetilde{Y}}\left(\widetilde{Y}\right)$, respectively.

10 These inverse transformations exist if C_W and W_Y are strictly monotone.

11 Then, we obtain the marginalized copula $C_{abN}(a, b, N) = C(a, b, 1, N)$.

12 In the TF, $g_a(a)$, $g_b(b)$, and $g_{\widetilde{Y}}\left(\widetilde{Y}\right)$ are denoted as $q_a(a)$, $q_b(b)$, and $q_N(N)$, respectively.

13 These inverse transformations exist if C_X and X_N are strictly monotone.

Proposition 2.2.1 *Based on Theorem 2.1.1 and Corollary 2.1.1, the shape of the LPF and its elasticity of substitution σ are determined not only by the substitutability parameter set Θ, i.e., the copula's dependence parameter set in the probabilistic space but also by the parameters that reflect the availability of technological innovations in the economy Θ_a and Θ_b, i.e., the margins' parameters in the probabilistic space. In addition, if the marginalized copula C_{abN} that represents the TF belongs to the family of generalized Archimedean copulas, the substitutability parameter set Θ can be mapped to the dependence parameter set through a statistical correlation measure $\mu = \mu(\Theta)$ such as either Kendall's tau or Spearman's rho.*

Proof See Appendix A.1.

Corollary 2.2.2 *Based on Proposition 2.2.1, let C be the copula that represents our transformation function. If the marginalized copula function $C_{ab}\left(U, V;\Theta_{abN}\right) = C_{abN}\left(U, V, 1;\Theta_{abN}\right)$, which determines the joint distribution of a and b, belongs to a generalized Archimedean family (with generator ψ); then, we can write the transformation function as*

$$T_C\left(a, b, \widetilde{Y}, N\right) = C\left(C_{ab}\left(U, V;\Theta_{abN}\right), W\left(g_{\widetilde{Y}}\left(\widetilde{Y};\Theta_{\widetilde{Y}}\right)\right), X\left(g_N(N);\Theta_N\right);\Theta\right)$$
$$= C\left(\psi\left[\psi^{-1}\left(U\left(a, L;\Theta_a\right)\right) + \psi^{-1}\left(V\left(b, K;\Theta_b\right)\right)\right], W\left(g_{\widetilde{Y}}\left(\widetilde{Y};\Theta_{\widetilde{Y}}\right)\right), X\left(g_N(N);\Theta_N\right);\Theta\right).$$

Additionally, if the derivatives of the functions U and V are separable as

$$U'\left(a, L;\Theta_a\right) = U_1\left(a;\Theta_a\right)U_2(L;\Theta) \text{ and } V'\left(b, K;\Theta_b\right) = V_1\left(b;\Theta_b\right)V_2(K;\Theta),$$

then the elasticity of substitution σ depends only on Θ through a dependence measure μ; i.e., $\sigma = r(\mu(\Theta))$, where r is a real function.

Additionally, if the marginalized copula $C_{ab}\left(U, V;\Theta_{abN}\right)$ is described by a single dependence parameter $\theta\left(\Theta_{abN} = \{\theta\}\right)$, then θ comes out as the substitutability parameter of the derived LPF and constitutes the unique link between the elasticity of substitution $\sigma = \sigma(\theta)$ and the probabilistic correlation measure $\mu = \mu(\theta)$, as follows:

(a) $\sigma = \sigma\left(m^{-1}(\mu)\right)$, where $\mu = m(\theta)$;

(b) $\mu = \mu\left(n^{-1}(\sigma)\right)$, where $\sigma = n(\theta)$.

Proof See Appendix A.1.

The construction of the transformation function in Corollary 2.2.1 is a relevant case in production theory.

Next, as we have deterministic versions of the LPF and the TF, we can obtain the GPF from the firm's technology choice problem (Jones and Manuelli, 1990):

$$\max_{a,b} Y = W_Y^{-1}\left(C_W^{-1}\left(U\left(a;L,\Theta_a\right),V\left(b;K,\Theta_b\right);\Theta\right),P_0\right), \tag{2}$$

such that $H(a,b) = X_N^{-1}\left(C_X^{-1}\left(U\left(q_a(a);\Theta_a\right),V\left(q_b(b);\Theta_b\right)\right),P_1\right) = A_N$, where N represents the given level of knowledge at time t.

The solution to (2) yields different closed functional forms of the textbook GPFs, depending on the following values contained in our suitable monotonic transformations of the margins U, V, W, and X: the substitutability parameters, the associated copula parameter set Θ; the availability parameters Θ_a, Θ_b, and Θ_N; the output parameters $\Theta_{\tilde{Y}}$; and the associated tail-thickness parameters.

Proposition 2.2.2 *Given Assumption 2.1.1 and the resulting microfounded LPF and TF, the firm's technology choice problem (2) has a unique solution if and only if the LPF and the TF are tangential at the point* $[a^*\left(L,K;\Theta_a,\Theta_b,\Theta_{\tilde{Y}},\Theta_N,\Theta\right)$ *,* $b^*(L,K;\Theta_a,\Theta_b,\Theta_{\tilde{Y}},\Theta_N,\Theta)]$*, which can occur only if the LPF is convex and the TF is concave or if both are convex but the LPF is more sharply curved than the TF in the feasible augmented technology set for a given L and K. We obtain the following general GPF* $Y^* = Y^*\left(L,K,N;\Theta_a,\Theta_b,\Theta_{\tilde{Y}},\Theta_N,\Theta\right)$ *by substituting the optimal values of* a^* *and* b^* *into the LPF.*

Proof See Appendix A.1.

Note that as the GPF inherits the parameters of the LPF and the TF, the elasticity of substitution σ_{GPF} in the GPF is determined by the substitutability parameter set Θ, i.e., the copula's dependence parameter set in the probabilistic space and by the availability parameter set Θ_a, Θ_b, $\Theta_{\tilde{Y}}$, and Θ_N, i.e., the margins' parameters set in the probabilistic space. In other words,

$$\sigma_{GPF} = \sigma_{GPF}\left(U(a^*,L),V(b^*,K);\Theta_a,\Theta_b,\Theta_{\tilde{Y}},\Theta_N,\Theta\right).$$

This setup is microfounded because if we wish to obtain a GPF, we assume a general ATF by describing the *substitutability or complementarity* between the technological innovations and their relationship with the output level. Next, we can associate this microeconomic structure with a particular probability distribution (i.e., a copula) with its margins and connect it to the data with observables (i.e., patents and output), without having data on L and K, to validate the empirical predictions of the model. Specifically, from Corollary 2.2.2, we can derive formulas that relate the substitutability between the innovations and the relationship between the factors and output to a particular probabilistic dependence measure, such as Kendall's tau correlation coefficient, and the availability parameters to tail-thickness parameters.

Illustration of How to Derive a Nested CES GPF and its Distribution Function Based on an Augmented Transformation Function

Next, we use our setup to derive a novel nested CES GPF that inherits its properties from a well-behaved ATF, augmented by labor- and capital-saving technological innovations. The derived nested CES GPF is determined both theoretically and probabilistically by the parameters θ and δ, which are defined in the parameter space Θ. Here, θ describes the substitutability (dependence) between the two types of technological innovations, whereas δ characterizes the combined effect of the pair of technological innovations on the output. This GPF is general enough to derive textbook GPFs (Leontief, Cobb–Douglas, CES, and linear). We build the probabilistic version of the transformation function (ATF), T, which is useful to link the microparameters of our nested CES GPF with its probabilistic counterparts and approximate its empirical shape. Finally, such a link allows us to describe a method to compute the microparameters of the GPF by simply estimating two Kendall's tau correlation coefficients: (1) between the two types of technological innovations and (2) between the combined pair of technological innovations and output.

Assumption 3.1 There exists an augmented transformation function

$$T = A_0\left(A - B\left[\left(a\gamma_L\right)^{-\alpha\theta}L^{-\theta} + \left(b\gamma_K\right)^{-\beta\theta}K^{-\theta}\right]^{\frac{\delta}{\theta}}N^{-\epsilon\delta}\right)^{\frac{1}{\eta\delta}} - \widetilde{Y}$$

that satisfies Definition 2.1.1 and Proposition 2.1.1, where $\left(a, L, b, K, \widetilde{Y}, N\right) \gg 0$, which is described by the substitutability parameters θ and δ and the availability parameters α and β for output $\widetilde{Y}$ and fixed levels of L, K, and N. In addition, based on Theorem 2.2.1, we can construct a probabilistic version of this transformation function T (which is the basis to build a nested CES function) by using the following nested 4-variate copula[14]:

$$C(U, V, W, X) = max\left\{\left(W^{-\delta} + \left[max\left\{\left[U^{-\theta}+V^{-\theta} - 1\right]^{-\frac{1}{\theta}}, 0\right\}X\right]^{-\delta} - 1\right)^{-\frac{1}{\delta}}, 0\right\} \tag{3}$$

This copula describes the joint behavior of the random variables $a_i, b_i, \widetilde{Y}_i$, and N_i through the dependence parameters δ and θ, whose domains are defined on $[-1, \infty]$. If we marginalize with respect to N_i $(X = 1)$ in Eq. (3), we obtain the trivariate copula that describes the joint behavior of the random variables a_i, b_i, and $\widetilde{Y}_i$. Similarly,

[14] In the technology choice literature, no attempt has been made to derive a setup that leads to an estimable empirical model by constructing a link between the microeconomic parameters and the statistical parameters, as we propose here. This is a straightforward method to estimate the GPF for any sector of the economy by using data on the output and technological innovations (i.e., patents) at the firm level.

if we marginalize with respect to $Y_i(W = 1)$ in Eq. (3), we obtain the following trivariate copula that describes the joint behavior of the random variables a_i, b_i, and N_i.

Next, let us assume that the labor- and capital-saving technological innovations, knowledge level, and output are random draws from the marginal Pareto distributions (Growiec 2008a; Jones 2005); then, the functional form is given by

$$F\left(Z;\gamma_Z,\zeta\right) = P(Z \le z) = 1 - \left(\frac{z}{\gamma_Z}\right)^{-\zeta}, \text{where } 0 < \gamma_Z \le z \text{and } \zeta > 0.$$

Using Corollary 2.2.1 and Assumption 3.1 and assuming consistent transformations $h_a\left(a_i\right)$, $h_b(b_i)$, and $h_{\widetilde{Y}}\left(Y_i\right)$ of the random variables a_i, b_i, and Y_i, respectively, and replacing $\widetilde{Y}_i$ with Y_i, we obtain a trivariate probabilistic version of the LPF

$$P\left(Y_i > Y, a_i > \frac{1}{a}, b_i > \frac{1}{b}\right) = \left[\left(\frac{Y}{\gamma_Y}\right)^{\delta\eta} + \left[\left(\gamma_a a\right)^{-\alpha\theta} + \left(\gamma_b b\right)^{-\beta\theta} - 1\right]^{\frac{\delta}{\theta}} - 1\right]^{-\frac{1}{\delta}}. \tag{4}$$

Corollary 2.2.1 guarantees that by using transformations $q_a\left(a_i\right)$, $q_b(b_i)$, and $q_N\left(N_i\right)$ of the random variables a_i, b_i, and N_i, respectively, we can obtain a probabilistic version of the TF as

$$P\left(a_i > a, b_i > b, N_i \le N\right) = \left[\left(\frac{a}{\gamma_a}\right)^{\alpha\theta} + \left(\frac{b}{\gamma_b}\right)^{\beta\theta} - 1\right]^{-1/\theta} \left(\gamma_N N\right)^{\varepsilon}. \tag{5}$$

Note that random variables a_i and b_i are strongly correlated through the copula parameter θ in the innovative process. Then, they are mutually correlated with Y_i when we consider the full production process through the parameter δ and are independent of N. These relations are statistically and microeconomically meaningful.

Lemma 3.1 *Deterministic versions of the LPF and TF can be derived by using Assumption 3.1 and Corollary 2.2.1 for given levels of L and K. First, by fixing probability level P_0 in Eq. (4), defining $\gamma_a^{-\alpha\theta} \equiv \gamma_L^{-\alpha\theta}\left(1 + L^{-\theta}\right)and\gamma_b^{-\beta\theta} \equiv \gamma_K^{-\beta\theta}\left(1 + K^{-\theta}\right)$ and choosing $\left(a\gamma_L\right)^{-\alpha\theta} + \left(b\gamma_K\right)^{-\beta\theta} = 1$,*[15] *we obtain the following CES LPF:*

$$Y = \gamma_Y\left(A - \left[\left(a\gamma_L\right)^{-\alpha\theta} L^{-\theta} + \left(b\gamma_K\right)^{-\beta\theta} K^{-\theta}\right]^{\frac{\delta}{\theta}}\right)^{\frac{1}{\eta\delta}}. \tag{6}$$

Furthermore, by fixing the probability level P_1 in Eq. (5), we obtain the TF

[15] Expressions $\varphi = \left(a\gamma_L\right)^{-\alpha\theta} > 0$ and $1 - \varphi = \left(b\gamma_K\right)^{-\beta\theta} > 0$, which are known as the technological change indices of a CES function, are shaped by the availability parameters (tail indices α, β) and the substitutability parameters (copula´s dependence parameters θ and δ). In addition, Eq. (6) describes a general nested CES production function, as it might consider nonlinear and linear relationships between technological innovations and the factors of production L and K.

$$H(a,b) = \left(\frac{a}{\gamma_a}\right)^{\alpha\theta} + \left(\frac{b}{\gamma_b}\right)^{\beta\theta} = A_N \tag{7}$$

where $A = 1 + P_0^{-\delta} > 1, A_N = 1 + P_1^{-\theta}\left(\gamma_N N\right)^{\epsilon\theta}$, *and* $-1 \leq \delta, \theta \leq \infty$.

Proof See Appendix A.2.2.

Next, the firm's technology choice problem consists of maximizing (6) given (7); then, its solution is given in the next lemma.

Lemma 3.2 *Based on Proposition 2.2.1 and Eqs. (6) and (7), the producer's optimization problem has a unique solution* (a^*, b^*) *that is substituted into the LPF to obtain the following optimal CES GPF*:

$$Y^*(L,K,N) = \gamma_Y\left(A - A_N^{-\frac{\delta}{\theta}}\left[\gamma_L^{-\alpha\theta}\sqrt{1+L^\theta}L^{-\theta} + \gamma_K^{-\beta\theta}\sqrt{1+K^\theta}K^{-\theta}\right]^{\frac{2\delta}{\theta}}\right)^{\frac{1}{\eta\delta}} \tag{8}$$

Proof See Appendix A.2.2.

Relationships Between the Microeconomic and Probabilistic Models

Note that the copula dependence parameters δ and θ come out directly as the microeconomic parameters that determine the shape of the nested CES GPF in Eq. (7). These two parameters also describe the dependence between the output and the combined pair of innovations δ and between the innovations themselves θ. Thus, the set of microparameters (θ,δ) constitutes a link between their corresponding copulas, which are defined on the probability space $(\Omega, \mathcal{F}, P)$, and the deterministic TF and the LPF, which are defined on the augmented feasible plans set B_j. To obtain a simpler interpretation of the LPF and TF, from a probabilistic point of view, the dependence parameters δ and θ can be mapped to the interval [−1,1] by using two dependence measures: the dependence between a_i and b_{I} and the dependence between $\left(a_i, b_{\mathrm{i}}\right)$ and Y_i.

As we use a nested Clayton copula with generator $\psi(t)$ for the production model, we represent the copula as $C(U,V,W) = \psi_2^{-1}[\psi_2(W) + \psi_2(\psi_1^{-1}\left[\psi_1(U) + \psi_1(V)\right])]$, with correlation parameters θ and δ. Then, the dependence measure Kendall's tau can be obtained by using the following formula

$$\tau_c = 1 + 4\int_0^1 \frac{\psi(t)}{\psi'(t)}dt \tag{9}$$

Lemma 3.1.1 *Based on Assumption 3.1, Lemma 3.1, and Eqs. (4), (5), and (6), we can link the parameter* σ_{Yab}*, which describes the relationship between the output (Y) and the effect of both technological innovations* ($\left[U^{-\theta}+V^{-\theta} - 1\right]^{-\frac{1}{\theta}}$), *with the* *parameter δ through Kendall's tau correlation coefficient -1 $\leq \tau_{wuv} \leq 1$, and the elasticity of substitution σ_{ab} between a_i and b_I in the LPF with the substitutability parameter θ through Kendall's tau correlation coefficient $-1 \leq \tau_{uv} \leq 1$. Thus, we can obtain empirical estimates of the correlations τ_{uv} and τ_{wuv} by using data on labor- and capital-saving technological innovations (i.e., patents) and output data to compute the numerical values of the parameters δ and θ and the elasticities $\sigma_{ab} = \frac{1}{1+\theta}$ and $\sigma_{Yab} = \frac{1}{1-\eta\delta}$. Conversely, if we assume values of σ_{ab} and σ_{Yab}, we can obtain estimates of the correlation between the two types of innovations τ_{uv} and between output and innovations τ_{wuv}.*

Then, we have the following relationships*:*

(a) Elasticity of substitution $\sigma_{ab} = \frac{1-\tau_{uv}}{1+\tau_{uv}}$, where $\theta = \frac{2\tau_{uv}}{1-\tau_{uv}}$.
(b) Dependence measure *(Kendall's tau)* $\tau_{uv} = \frac{1-\sigma_{ab}}{1+\sigma_{ab}}$, where $\sigma_{KL} = \frac{1}{1+\theta}$.
(c) Elasticity of substitution $\sigma_{Yab} = \frac{1-\tau_{wuv}}{1-(1+2\eta)\tau_{wuv}}$, where $\delta = \frac{2\tau_{wuv}}{1-\tau_{wuv}}$.
(d) Dependence measure between output and innovations *(Kendall's tau)*$\tau_{wuv} = \frac{\delta}{2+\delta}$.

Proof See Appendix A.2.2.

Note that as we know the joint distribution of a_i, b_i and Y_i defined in Eq. (4), we can also recover the values of τ_{uv}, τ_{wuv}, σ_{ab}, and σ_{Yab} by using the formulas in *(a)*, *(b)*, *(c)*, and *(d)*. The link between σ_{ab} and τ_{uv}, i.e., between the probabilistic space $(\Omega, \mathcal{F}, P)$ and the set of augmented production plans B_j, is the substitutability parameter θ, with $[-1, \infty]$ as its range of values. Thus, we can vary the values of the parameter θ to obtain all the nested classic textbook production functions. In addition, the shape of the GPF and its elasticity of substitution depend not only on the substitutability parameters, i.e., the copula's dependence parameters and statistically approximated by Kendall's tau correlation coefficient but also on the availability parameters, i.e., functional forms of the margins and the values of their location parameters.

Note that the link between σ_{KL} and τ_c is the substitutability parameter θ. The range of values of the substitutability parameter in our transformation function is $[-1, \infty]$*;* thus, we can vary the values of the parameter θ to obtain all the classic production functions. We confirm that the shape of the GPF and its elasticity of substitution depend not only on the substitutability parameter, which is reflected by the copula dependence parameter but also on the availability parameters, i.e., the functional forms of the margins and the values of their parameters. Table 1 shows a summary of the derived GPFs when the LPF and the TF share the same set of parameter values $-1 \leq \theta \leq \infty$. Note that specific values of θ are associated with different values of the dependence measure τ_c, the elasticity of substitution σ_{KL}, and the shape of the GPF.[16]

[16] The cases presented in this table are the ones in which the LPF and GPF have the same shape. However, there are cases where the LPF and GPFs are not of the same class, which includes the cases developed by (Acemoglu 2003; Growiec 2008a; Jones 2005).

Table 1 GPFs derived from a transformation function ($-1 \leq \theta \leq \infty$); *Source*: Author calculations

Substitutability parameter $\boldsymbol{\theta}$	Kendall's Tau	Elasticity of substitution (LPF)	Global production function (GPF)
$-1 < \theta < \infty, \theta \neq 0$	$-1 < \tau < 1, \tau \neq 0$	$0 < \sigma < \infty, \sigma \neq 1$	CES-type function
$\theta = 0$	$\tau = 0$	$\sigma = 1$	Cobb−douglas
$\theta \to \infty$	$\tau = 1$	$\sigma = 0$	Leontief
$\theta = -1$	$\tau = -1$	$\sigma \to \infty$	Perfect substitutes

Conclusion

The shape of the GPF is obtained from a unique ATF that reflects the structure of the set of production plans augmented by labor- and capital-saving technological innovations, which is the primitive datum in the theory. We show that the shape of the GPF is determined by all the relations among technological innovations and output and the way they are combined, rather than by the shape of two primitives, the LPF and the TF. By mapping the ATF to its corresponding copula, using Sklar's theorem, we ensure that the probabilistic model of the GPF has the same structure as the production set and, consequently, we can consistently link the elasticities and the correlations. Thus, we provide micro foundations for the GPF exclusively based on the microeconomic model, and we develop a parallel probabilistic model with similar properties. This approach allows us to relax the restrictive assumptions on the functional forms of the primitives, including the probability models often used to determine the shape of the TF, which impose strong restrictions on the shape of the calculated GPFs. We could also propose an ATF and its corresponding production set that gives rise to GPFs with different substitutability properties (i.e., an ATF with Leontief or Cobb−Douglas properties).

Appendix A.1: Proofs of the General Setup

Proof of Proposition 2.1.1 First, monotonicity holds because T is nondecreasing in a_i, b_i, and N_i and nonincreasing in $\widetilde{Y}_i$. Next, to prove concavity with respect to a_i, b_i, and N_i, we let $z_0 = \left(a_{i0}, L_{i0}, b_{i0}, K_{i0}, \widetilde{Y}_i, N_{i0}\right)$ and $z_1 = \left(a_{i1}, L_{i1}, b_{i1}, K_{i1}, \widetilde{Y}_i, N_{i1}\right)$ be two augmented production plans in B_j. By Assumption 1, $\omega z_0 + (1-\omega)z_1$ is also in B_j for all λ ∈ [0,1], given that B_{j} is a convex set. Thus,

$$
\begin{aligned}
&T\left(\lambda z_0 + (1-\lambda)z_1\right) - \lambda T\left(z_0\right) + (1-\lambda)T\left(z_1\right) \\
&\quad = \max_{\lambda\widetilde{Y}_{io}+(1-\lambda)\widetilde{Y}_{i1}} \left\{ \lambda\widetilde{Y}_{io} + (1-\lambda)\widetilde{Y}_{i1} \,:\, \lambda z_0 + (1-\lambda)z_1 \in B_j \right\} \\
&\qquad - \lambda\left[\max_{\widetilde{Y}_{io}}\left\{ \widetilde{Y}_{io} \,:\, z_0 \in B_j \right\}\right] - (1-\lambda)\left[\max_{\widetilde{Y}_{i1}}\left\{ \widetilde{Y}_{i1} | z_1 \in B_j \right\}\right] \geq 0.
\end{aligned}
$$

Because we have more options for $\widetilde{Y}_{io}$ and $\widetilde{Y}_{i1}$ to optimize, then

$$T\left(\lambda z_0 + (1-\lambda)z_1\right) \geq \lambda \mathrm{T}\left(z_0\right) + (1-\lambda)\mathrm{T}\left(z_1\right),$$

and the transformation function is concave.

To prove continuity from above, let us fix $\widetilde{Y}_i$ and a given decreasing sequence $\{(a_l, L_l, b_l, K_l, N_l)\}_{l\in\mathbb{N}}$ that goes to $\left(a_i, L_i, b_i, K_i N_i\right)$ such that $\left(a_l, L_l, b_l, K_l N_l\right) \geq \left(a_i, L_i, b_i, K_i N_i\right)$ for all $l \in \mathbb{N}$. From (i) in Assumption 2.1.1, let $z_i = \left(a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i\right)$ such that $\widetilde{Y}_i = \underset{\widetilde{Y}_i}{max}\left\{\widetilde{Y}_i : (a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i) \in B_j\right\}$. Then, $\left(a_l, L_l, b_l, K_l, N_l\right) \geq \left(a_i, L_i, b_i, K_i N_i\right)$ and $z_l = \left(a_l, L_l, b_l, K_l, \widetilde{Y}_i, N_l\right) \in B_j$, which implies that $T\left(z_l\right) \geq T\left(z_i\right) = 0$. Since T is a monotone function, $\{T\left(a_l, L_l, b_l, K_l, \widetilde{Y}_i, N_l\right)\}_{l\in\mathbb{N}}$ converges to $T\left((a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i\right)$.

Finally, let $z_0 = \left(a_i, L_i, b_i, K_i, \widetilde{Y}_{i0}, N_i\right)$ and $z_1 = \left(a_i, L_i, b_i, K_i, \widetilde{Y}_{i1}, N_i\right)$ be two augmented production plans in B_j. By Assumption 2.1.1, $\omega z_0 + (1-\omega)z_1$ is also in B_j for all $\omega \in [0,1]$. Then,

$$T\left(\omega z_0 + (1-\omega)z_1\right) = T\left(a_i, L_i, b_i, K_i, \omega\widetilde{Y}_{i0} + (1-\omega)\widetilde{Y}_{i1}, N_i\right) \leq \max\left\{T\left(z_0\right), T\left(z_1\right)\right\}$$

which implies that T is quasiconcave in $\widetilde{Y}_i$. If $\widetilde{Y}_{i0} \leq \widetilde{Y}_i$, then from (iii) in Assumption 2.1.1, $\mathrm{T}\left(a_i, L_i, b_i, K_i, \widetilde{Y}_{i0}, N_i\right) \geq T\left(a_i, L_i, b_i, K_i, \widetilde{Y}_{i1}, N_i\right)$, which implies that T is non-increasing in $\widetilde{Y}_i$. □.

Proof of Proposition 2.1.3 First, the monotonicity and nondecreasing characteristics in inputs and the level of knowledge properties are inherited by the ATF $T(a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i)$. Then,

$$Y^*(0,0,0) = \left[\underset{a_i,b_i}{max}\, f\left(a_i, b_i; 0,0\right) \quad \text{subject to} \quad H_0\left(a_i, b_i\right)\right] = 0$$

because by definition of the ATF, $f\left(a_i, 0, b_i, 0\right) = \underset{\widetilde{Y}_i \in B_j}{max}\{\widetilde{Y}_i : T\left(a_i, 0, b_i, 0, \widetilde{Y}_i\right) \geq 0\}$ $= 0$ (property (ii) in Assumption 2.1.1). Finally, to prove cuasiconcavity, let $z_0 = \left(a_i, L_{i0}, b_i, K_{i0}, N_{i0}\right)$ and $z_1 = \left(a_i, L_{i1}, b_i, K_{i1}, N_{i1}\right)$, such that $z_{0\widetilde{Y}} = \left(a_i, L_{i0}, b_i, K_{i0}, \widetilde{Y}_i, N_{i0}\right)$ and $z_{1\widetilde{Y}} = \left(a_i, L_{i1}, b_i, K_{i1}, \widetilde{Y}_i, N_{i1}\right)$ are in B_j for all$\omega \in [0,1]$. Then, $\omega z_{0\widetilde{Y}} + (1-\omega)z_{1\widetilde{Y}}$ is also in B_j for all$\omega \in [0,1]$. Then, by quasiconcavity of$f\left(a_i, b_i; 0,0\right)$, we have that

$$
\begin{aligned}
&Y^*\left(\omega z_0 + (1-\omega)z_1\right) \\
&\quad = Y^*\left(a_i, \omega L_{i0} + (1-\omega)L_{i1}, b_i, \omega K_{i0} + (1-\omega)K_{i1}, \omega N_{i0} + (1-\omega)N_{i1}\right) \\
&\quad \le max\left\{Y^*\left(z_0\right), Y^*\left(z_1\right)\right\}.
\end{aligned}
$$

□

Proof of Proposition 2.1.4 (i) As $0 \in B_j^{''}$, $B_j^{''} \neq \emptyset$, and is closed by definition (due to $\left(a_i, L, b_i, K, \widetilde{Y}_i, N_i\right) \geq 0$, $\left(a_i, L, b_i, K, \widetilde{Y}_i, N_i\right) \geq 0$, and $T\left(a_i, L, b_i, K, \widetilde{Y}_i, N_i\right) \geq 0$). (ii) Convexity fits because T and Y_i are concave. (iii) Fits because T is monotone. Property (iv) holds because $0 \le \widetilde{Y}_i^{\prime} \le \widetilde{Y}_i$ implies $0 \le \widetilde{Y}_i^{\prime} \le \widetilde{Y}_i \le Y^*(L, K, N)$, and by monotonicity of T. (v) Given a_i, L_i, b_i, K_i, and N_i, the set $\{\widetilde{Y}_i|(a_i, L_i, b_i, K_i, \widetilde{Y}_i, N_i) \in B_j^{''}\}$ is bounded above by $Y_i^*(L, K, N)$. □

Proof of Theorem 2.2.1 $T_C\left(a, b, \widetilde{Y}, N\right)$ is a well-defined function since B_j has at least an interior point z. Then, $T(z) > 0$, which implies that $max\ T\left(a_i, b_i, \widetilde{Y}_i, N\right) > 0$. In addition, by definition, $0 \le T_C\left(a, b, \widetilde{Y}, N\right) \le 1$, which is the range of a proper copula function. Next, we show that $T_C\left(a, b, \widetilde{Y}, N\right)$ satisfies the conditions to be represented as a copula function (see Nelsen (2006)). From (ii) in Assumption 1, $T_C\left(0, b, \widetilde{Y}, N\right) = T_C\left(a, 0, \widetilde{Y}, N\right) = 0$, $T_C\left(a, b, max\ \widetilde{Y}, N\right) = 0$ holds because $\left(a, L, b, K, max\ \widetilde{Y}, N\right)$ lies on the boundary of B_j. These two properties are equivalent to the fact that the proposed transformation function is grounded like a copula function. $T_C(a, max\ b, 0, N), T_C(max\ a, b, 0, N)$, $T_C(a, b, 0, max\ N)$, and $T_C\left(max\ a, max\ b, \widetilde{Y}, max\ N\right)$ have the properties of the marginal distribution of $a_i, b_i, \widetilde{Y}_i$, and N_i. Additionally, $T_C(0, max\ b, 0{,}0) = T_C(max\ a, 0{,}0{,}0) = T_C(0{,}0, 0, max\ N) = T_C\left(max\ a, max\ b, max\ \widetilde{Y}, max\ N\right) = 0$ and $T_C(max\ a, max\ b, 0, max\ N) = 1$, which ensures that the cumulative probability over the entire domain is equal to 1 as in a well-defined distribution function. T_c inherits the properties of being continuous from above and nondecreasing for $g_a(a)$, $g_b(b)$, and $g_{\widetilde{Y}}\left(\widetilde{Y}\right)$ from T. We show that the T_C-volume is non-negative for the 3-variated case, which naturally extends to the 4-variated copula. Then, let us assume $a_0 \le a_1, b_0 \le b_1$, and $\widetilde{Y}_0 \le \widetilde{Y}_1$. We define the 3-box

$$
\mathrm{M} = \left[a_0, a_1\right] \times \left[b_0, b_1\right] \times \left[\widetilde{Y}_0, \widetilde{Y}_1\right].
$$

Given that $g_a(a)$ and $g_b(b)$ are nondecreasing and $g_{\widetilde{Y}}\left(\widetilde{Y}\right)$ is nonincreasing, the T_C-volume of the 4-box B is given by

$$T_C(M) = T_C\left(a_1, b_1, \widetilde{Y}_0\right) - T_C\left(a_1, b_0, \widetilde{Y}_0\right) - T_C\left(a_0, b_1, \widetilde{Y}_0\right) + T_C\left(a_0, b_0, \widetilde{Y}_0\right)$$
$$-T_C\left(a_1, b_1, \widetilde{Y}_1\right) + T_C\left(a_1, b_0, \widetilde{Y}_1\right) + T_C\left(a_0, b_1, \widetilde{Y}_1\right) - T_C\left(a_0, b_0, \widetilde{Y}_1\right).$$

Given that T_C is concave, for all $\mu \in [0,1]$, we have

$$T_C\left[\mu\left(a_1, b_0, \widetilde{Y}_l\right) + (1-\mu)\left(a_0, b_1, \widetilde{Y}_l\right)\right] \geq \mu T_C\left(a_1, b_0, \widetilde{Y}_l\right) + (1-\mu)T_C\left(a_0, b_1, \widetilde{Y}_l\right);$$

In particular, when $\mu = \frac{1}{2}$, we can define $T_{\frac{l}{2}} = \frac{1}{2}[T_C\left(a_1, b_0, \widetilde{Y}_l\right) + T_C\left(a_0, b_1, \widetilde{Y}_l\right)]$.

From the concavity of T_C, we have $T_{\frac{l}{2}} - T_C\left(a_0, b_0, \widetilde{Y}_l\right) \leq T_C\left(a_1, b_1, \widetilde{Y}_l\right) - T_{\frac{l}{2}}$, which implies that

$$2T_{\frac{l}{2}} = T_C\left(a_1, b_0, \widetilde{Y}_l\right) + T_C\left(a_0, b_1, \widetilde{Y}_l\right) \leq T_C\left(a_1, b_1, \widetilde{Y}_l\right) + T_C\left(a_0, b_0, \widetilde{Y}_l\right).$$

Therefore, we have that the volume

$$T_C^l = T_C\left(a_1, b_1, \widetilde{Y}_l\right) - T_C\left(a_1, b_0, \widetilde{Y}_l\right) - T_C\left(a_0, b_1, \widetilde{Y}_l\right) + T_C\left(a_0, b_0, \widetilde{Y}_l\right) \geq 0.$$

T_C is nondecreasing in $\widetilde{Y}_l$, $T_C^0 \leq T_C^1$, and $T_C(M) = T_C^1 - T_C^0 \geq 0$, which implies that the transformation function T_C behaves as a 3-increasing function and is similar to a 3-copula (and then the 4-copula) function. □

Proof of Corollary 2.2.1 The proof easily follows by applying the steps described in this corollary.

Proof of Proposition 2.2.1 Since the LPF and the TF depend on the substitutability parameter set Θ; the LPF depends on the availability parameters in the economy, Θ_a ,Θ_b, and Θ_N; and the TF depends on Θ_a,Θ_b, and $\Theta_{\widetilde{Y}}$, the shape of the resulting GPF depends on these 5 sets of parameters. The marginal technical rate of substitution (MRTS) between L and K $MRTS\left(U\left(a^*, L;\Theta_a\right), V\left(b^*, K;\Theta_b\right), \Theta_{\widetilde{Y}}, \Theta\right)$ also depends on these parameter sets; then, the elasticity of substitution of the GPF is defined by

$$\sigma\left(U(a^*, L), V(b^*, K), \Theta_a, \Theta_b, \Theta_{\widetilde{Y}}, \Theta_N, \Theta\right)$$
$$= \left(\frac{dLn\left[MRTS\left(U\left(a^*, L;\Theta_a\right), V\left(b^*, K;\Theta_b\right), \Theta_{\widetilde{Y}}, \Theta_N;\Theta\right)\right]}{dLn\left(\frac{K}{L}\right)}\right)^{-1}.$$

In addition, based on Ida et al. (2014), any measure of dependence μ based on the copula C of a generalized Archimedean copula should be of the form $\mu = \iint_{I^2} f(u, v, C(u, v;\Theta))dC(u, v;\Theta)$, where $f = f(u, v, C)$ is an appropriate smooth positive function. Then, the dependence measure only depends on the copula's parameter set, i.e., $\mu = \mu(\Theta)$. □

Proof of Corollary 2.2.2 Based on Proposition 2.1.2 and the definition of a generalized Archimedean copula, we can denote the transformation function as

$$
\begin{aligned}
T(a,L,b,K,Y,N) &= C\left(U\left(a,L;\Theta_a\right),V\left(b,K;\Theta_b\right),W\left(g_{\widetilde{Y}}\left(Y;\Theta_{\widetilde{Y}}\right)\right),X\left(g_N(N);\Theta_N\right);\Theta\right)\\
&= C\left(\psi\left[\psi^{-1}\left(U\left(a,L;\Theta_a\right)\right)+\psi^{-1}\left(V\left(b,K;\Theta_b\right)\right)\right],W\left(g_{\widetilde{Y}}\left(Y;\Theta_{\widetilde{Y}}\right)\right),X\left(g_N(N);\Theta_N\right);\Theta\right).
\end{aligned}
$$

The marginal rate of technical substitution (MRTS) is given by

$$
MRTS=\frac{\frac{\partial T}{\partial K}}{\frac{\partial T}{\partial L}}=\frac{C'(a,L,b,K,Y,N;\Theta)\psi'(a,L,b,K;\Theta)\left[\psi^{-1}\right]'(v;\Theta)V_1\left(b;\Theta_b\right)V_2(K;\Theta)}{C'(a,L,b,K,Y,N;\Theta)\psi'(a,L,b,K;\Theta)\left[\psi^{-1}\right]'(u;\Theta)U_1\left(a;\Theta_a\right)U_2(L;\Theta)},
$$

As u and v are simple variables, $\left[\psi^{-1}\right]'(u;\Theta)=\left[\psi^{-1}\right]'(v;\Theta)$, which reduces to

$$
MRTS=\frac{V_1\left(b;\Theta_b\right)V_2(K;\Theta)}{U_1\left(a;\Theta_a\right)U_2(L;\Theta)}.
$$

Thus, we take the derivative with respect to $Ln\left(\frac{K}{L}\right)$, and we obtain the elasticity of substitution, which is given by

$$
\sigma=\left(\frac{dLn[MRTS]}{dLn\left(\frac{K}{L}\right)}\right)^{-1}=\left(\frac{d\left[Ln\left(\frac{V_1\left(b;\Theta_b\right)}{U_1\left(a;\Theta_a\right)}\right)+Ln\left(\frac{V_2(K;\Theta)}{U_2(L;\Theta)}\right)\right]}{dLn\left(\frac{K}{L}\right)}\right)^{-1}=\left(\frac{dLn\left(\frac{V_2(K;\Theta)}{U_2(L;\Theta)}\right)}{dLn\left(\frac{K}{L}\right)}\right)^{-1}.
$$

Finally, suppose that $\Theta_{abN}=\{\theta\}$. If $\mu=m(\theta)$ has an inverse (m^{-1}), then we obtain $\theta=m^{-1}(\mu)$ and $\sigma=\sigma\left(m^{-1}(\mu)\right)$. Similarly, if $\sigma=n(\theta)$ has an inverse, then $\mu=\mu(\theta)=\mu\left(n^{-1}(\sigma)\right)$. □

Proof of Proposition 2.2.2 Let $Y=f(a,b;L,K)$ be the LPF and $H(a,b)=A_N$ be the TF. Both are defined in the (a,b) space for given values of$L$,K, and$N$. A competitive firm maximizes $\widetilde{Y}$ over the set of feasible technology pairs$D=\left\{(a,b):T_C\left(a,b,\widetilde{Y},N\right)\geq 0\right\}$. Since we have shown that the transformation function T_C is represented by a copula in the probability space with two increasing and nondecreasing margins of a and$b$, respectively, the derived deterministic LPF $f(a,b;L,K)$ also increases over the compact set D and attains its maximum at the TF by (i) and (iii) of Assumption 1. Then, the maximization of the $LPFY_i=f\left(a_i,b_i;L,K\right)=\max_{B_j}\{\widetilde{Y}_i:T_C\left(a_i,b_i,\widetilde{Y}_i\right)\geq 0\}$, which is subject to the TF $H\left(a_i,b_i\right)=\left\{\left(a_i,b_i\right):T_C\left(a_i,b_i,N_i\right)=0\right\}$ occurs when they coincide over the TF, with the unique solution(a^*,b^*) when these curves are tangent at this point, which can occur only if the LPF is convex and the TF is concave or the LPF is sharper than the TF in the feasible augmented technology set.

In addition, given that the LPF and TF are represented by the distribution functions $F\left(h_a(a),h_b(b),h_{\widetilde{Y}}(Y);\Theta_a,\Theta_b,\Theta_{\widetilde{Y}},\Theta\right)$ and $F\left(q_a(a),q_b(b),q_N(N);\Theta_a,\Theta_b,\Theta_N,\Theta\right)$, respectively, they are determined not only by the substitutability parameter set Θ but also by the parameters Θ_a, Θ_b, $\Theta_{\tilde{Y}}$, and Θ_N. Then, once we maximize the LPF, which is subject to the TF, the optimal values a^* and b^* also depend on Θ, Θ_a, Θ_b, $\Theta_{\tilde{Y}}$, and Θ_N.

Since we obtain the GPF by substituting a^* and b^* into the LPF, we have

$$Y^* = Y^*\left(L, K, N, U(a^*, L), V(b^*, K); \Theta, \Theta_a, \Theta_b, \Theta_{\tilde{Y}}, \Theta_N\right).$$

□

Appendix A.2: Proofs for the Microfounded CES Model

A.2.1. Proofs of Random Variable Transformations

Here, we show how to obtain the margins of the transformed random variables used in our examples. For a Pareto distributed random variable c_I, its density function is given by

$$f_{c_i}(c) = \rho \frac{\gamma_c^\rho}{c^{\rho+1}}, \, where \; c \geq \gamma_c > 0.$$

If we use the transformation $Z = -c_i$, its density is

$$f_Z(z) = f_{c_i}(-z)\left|\frac{d(-z)}{dz}\right| = \rho \frac{\gamma_c^\rho}{(-z)^{\rho+1}};$$

Therefore, we have the resulting marginal probability

$$P\left(c_i > c\right) = P(-Z > c) = P(Z < -c) = F_Z(-c) = \int_{-\infty}^{-c} \rho \frac{\gamma_c^\rho}{(-u)^{\rho+1}} du = \left(\frac{c}{\gamma_c}\right)^{-\rho}.$$

However, using the transformation $R = \frac{1}{c_i}$, we have

$$f_R(r) = f_{c_i}\left(\frac{1}{r}\right)\left|\frac{d\left(\frac{1}{r}\right)}{dr}\right| = \rho \frac{\gamma_c^\rho}{\left(\frac{1}{r}\right)^{\rho+1}}\left|-\frac{1}{r^2}\right| = \rho \gamma_c^\rho r^{\rho-1},$$

and the resulting marginal probability (where $0 < \gamma_c < \frac{1}{c}$) is given by

$$P\left(c_i > \frac{1}{c}\right) = P\left(\frac{1}{R} > \frac{1}{c}\right) = P(R < c) = F_R(c) = \int_0^c \rho \gamma_c^\rho u^{\rho-1} du = \gamma_c^\rho c^\rho. \square$$

We use the results from Appendix A.2.1 for the transformed Pareto random variables $h_a(a_i) = \frac{1}{a_i}, h_b(b_i) = \frac{1}{b_i}$ and $h_{\tilde{Y}}\left(Y_i\right) = -Y_i$ to obtain (4) by substituting their marginal distributions into the nested Clayton copula (3). Similarly, we use the following Pareto random variables $q_a(a_i) = -a_i$, $q_b(b_i) = -b_i$, and $q_N(N_i) = \frac{1}{N_i}$ to obtain (5).

A.2.2. Model Proofs

Proof of Lemma 3.1 Fixing the probability $P_0 = P\left(a_i > \frac{1}{a}, b_i > \frac{1}{b}, Y_i > Y\right)$ in Eq. (4) and solving for $\left(\frac{Y}{\gamma_Y}\right)^{\eta}$, we have

$$\left(\frac{Y}{\gamma_Y}\right)^{\delta\eta} = 1 + P_0^{-\delta} - \left[\left(a\gamma_a\right)^{-\alpha\theta} + \left(b\gamma_b\right)^{-\beta\theta} - 1\right]^{\frac{\delta}{\theta}}.$$

Now, we define $\gamma_a = \gamma_L\left(1 + L^{-\theta}\right)^{-\frac{1}{\alpha\theta}}$ and $\gamma_b = \gamma_K\left(1 + K^{-\theta}\right)^{-\frac{1}{\beta\theta}}$ and substitute them into the previous equation to obtain

$$\begin{aligned} Y &= \gamma_Y\left(1 + P_0{}^{-\delta} - \left[\left[a\gamma_L\left(1 + L^{-\theta}\right)^{-\frac{1}{\alpha\theta}}\right]^{-\alpha\theta} + \left[b\gamma_K\left(1 + K^{-\theta}\right)^{-\frac{1}{\beta\theta}}\right]^{-\beta\theta} - 1\right]^{\frac{\delta}{\theta}}\right)^{\frac{1}{\delta\eta}} \\ &= \gamma_Y\left(1 + P_0{}^{-\delta} - \left[(a\gamma_L)^{-\alpha\theta}L^{-\theta} + (b\gamma_K)^{-\beta\theta}K^{-\theta} + (a\gamma_L)^{-\alpha\theta} + (b\gamma_K)^{-\beta\theta} - 1\right]^{\frac{\delta}{\theta}}\right)^{\frac{1}{\delta\eta}}. \end{aligned}$$

We change the scale to obtain $\left(a\gamma_L\right)^{-\alpha\theta} + \left(b\gamma_K\right)^{-\beta\theta} = 1$, and finally, we obtain

$$Y = \gamma_Y\left(A - \left[\left(a\gamma_L\right)^{-\alpha\theta}L^{-\theta} + \left(b\gamma_K\right)^{-\beta\theta}K^{-\theta}\right]^{\frac{\delta}{\theta}}\right)^{\frac{1}{\eta\delta}}, \text{ where } A = 1 + P_0{}^{-\delta}.$$

To derive the TF, we fix the probability $P_1 = P\left(a_i > a, b_i > b, N_i \le N\right)$ in Eq. (5) and simplify the equation to *obtain*

$$H(a,b) = \left(\frac{a}{\gamma_a}\right)^{\alpha\theta} + \left(\frac{b}{\gamma_b}\right)^{\beta\theta} = 1 + P_1^{-\theta}\left(\gamma_N N\right)^{\varepsilon\theta} = A_N.$$

□

Proof of Lemma 3.2 We define the following Lagrange multiplier function:

$$L = \gamma_Y\left(A - \left[\left(a\gamma_L\right)^{-\alpha\theta}L^{-\theta} + \left(b\gamma_K\right)^{-\beta\theta}K^{-\theta}\right]^{\frac{\delta}{\theta}}\right)^{\frac{1}{\eta\delta}} - \lambda\left[\left(\frac{a}{\gamma_a}\right)^{\alpha\theta} + \left(\frac{b}{\gamma_b}\right)^{\beta\theta} - A_N\right].$$

Then, we combine the first-order conditions to obtain $b^{2\beta\theta} = \frac{1+L^{\theta}}{1+K^{\theta}}a^{2\alpha\theta}$. Substituting the previous equation into the constraint and solving for $a^{\alpha\theta}$, we have

$$a^{\alpha\theta} = \frac{\gamma_L^{\alpha\theta}\gamma_K^{\beta\theta}L^\theta K^\theta A_N}{\gamma_K^{\beta\theta}K^\theta\left(1+L^\theta\right)+\gamma_L^{\alpha\theta}L^\theta\sqrt{\left(1+L^\theta\right)\left(1+K^\theta\right)}}.$$

Similarly, we derive the expression for the optimal value of $b^{\beta\theta}$

$$b^{\beta\theta} = \frac{\gamma_L^{\alpha\theta}\gamma_K^{\beta\theta}L^\theta K^\theta A_N}{\gamma_K^{\beta\theta}K^\theta\sqrt{\left(1+L^\theta\right)\left(1+K^\theta\right)}+\gamma_L^{\alpha\theta}L^\theta(1+K)^\theta}.$$

Then, we substitute these optimal values into the LPF as

$$Y^* = \gamma_Y\left(A - A_N^{-\frac{\delta}{\theta}}\left[\gamma_L^{-\alpha\theta}\sqrt{1+L^\theta}L^{-\theta} + \gamma_K^{-\beta\theta}\sqrt{1+K^\theta}K^{-\theta}\right]^{\frac{2\delta}{\theta}}\right)^{\frac{1}{\eta\delta}}.$$

□

Proof of Lemma 3.1.1 Consider the copulas $C(W,Z) = max\left\{\left(W^{-\delta}+Z^{-\delta}-1\right)^{-\frac{1}{\delta}},0\right\}$, where $Z = max\left\{\left[U^{-\theta}+V^{-\theta}-1\right]^{-\frac{1}{\theta}},0\right\}$ and $C(U,V) = max\left\{\left[U^{-\theta}+V^{-\theta}-1\right]^{-\frac{1}{\theta}},0\right\}$. Then, by using the generators $\psi(t)=\frac{1}{\delta}\left(t^{-\delta}-1\right)$ and $\psi(t)=\frac{1}{\theta}\left(t^{-\theta}-1\right)$ in the next formula $\tau_c = 1+4\int_0^1\frac{\psi(t)}{\psi'(t)}dt$ and applying simple algebra, we obtain $\tau_{wuv}=\frac{\delta}{2+\delta}$ and $\tau_{uv}=\frac{\theta}{\theta+2}$, respectively. However, using the formula $\sigma=\left(\frac{dA}{dB}\right)\left(\frac{B}{A}\right)$, we calculate the elasticity of substitution of transformation and local production functions.

First, we define $Z=\left[\left(a\gamma_L\right)^{-\alpha\theta}L^{-\theta}+\left(b\gamma_K\right)^{-\beta\theta}K^{-\theta}\right]^{\frac{1}{\eta\theta}}$ from the transformation function and obtain the rate of substitution (RS) as RS $=\varphi\left(\frac{Z}{Y}\right)^{\eta\delta-1}$; then, we obtain the derivative

$$\frac{dLn|RS|}{dLn\left(\frac{Y}{Z}\right)} = \frac{d}{dLn\left(\frac{Y}{Z}\right)}\left(Ln(\varphi)+(1-\eta\delta)Ln\left(\frac{Y}{Z}\right)\right) = 1-\eta\delta.$$

Thus, we obtain the result $\sigma_{Yab}=\left(\frac{dLn|RS|}{dLn\left(\frac{Y}{Z}\right)}\right)^{-1}=\frac{1}{1-\eta\delta}$. Second, the technical rate of substitution (TRS) is given by TRS $=\frac{\varphi}{1-\varphi}\left(\frac{K}{L}\right)^{\theta+1}$; then, we obtain the derivative

$$\frac{dLn|TRS|}{dLn\left(\frac{K}{L}\right)} = \frac{d}{dLn\left(\frac{K}{L}\right)}\left(Ln\left(\frac{\varphi}{1-\varphi}\right)+(1+\theta)Ln\left(\frac{K}{L}\right)\right) = (1+\theta).$$

Thus, we obtain the result $\sigma_{KL} = \left(\frac{dLn|TRS|}{dLn\left(\frac{K}{L}\right)} \right)^{-1} = \frac{1}{1+\theta}$.

Finally, to obtain the identities, we rewrite δ and θ as functions of σ_{Yab}, σ_{KL}, τ_{wuv} and τ_{uv} to obtain $\delta = \frac{2\tau_{wuv}}{1-\tau_{wuv}}$ and $\theta = \frac{2\tau_{uv}}{1-\tau_{uv}}$. □

Appendix B: General Definitions and Theorems

Definition B.1 A 4-dimensional copula is a function C from $I^4 = [0,1] \times [0,1] \times [0,1] \times [0,1]$ to $I = [0,1]$ that satisfies the following properties:

(1) For every $(U, V, W.X) \in I^4$, if at least one of such variables is zero, then $C(U, V, W, X) = 0$;
(2) $C(1,1,1,N) = N$; $C(1,1,W,1) = W$; $C(1,V,1,1) = V$, and $C(U,1,1,1) = U$;
(3) For every $\boldsymbol{r}, \boldsymbol{s} \in I^4$ such that $\boldsymbol{r} \leq \boldsymbol{s}$, $V_C(M) = \sum sgn(\boldsymbol{c})F(\boldsymbol{c}) \geq 0$;

where $M = \left[r_1, s_1\right] \times \left[r_2, s_2\right] \times \left[r_3, s_3\right] \times \left[r_4, s_4\right]$ and the sum is over all the vertices $c = \left(c_1, c_2, c_3, c_4\right)$ of M (each c_k is equal to either r_k or s_k) and $sgn(\boldsymbol{c})$ is given by

$$sgn(\boldsymbol{c}) = \begin{cases} 1, & if\ c_k = r_k \text{ for an even number } of\ k's, \\ -1, & if\ c_k = r_k \text{ for an odd number } of\ k's. \end{cases}$$

Theorem (*Sklar's Theorem*) *Let C be a 4-copula and U, V, W, and X be the marginal distributions of the random variables $a_i, b_i\ \widetilde{Y}_i$, and N_i respectively. Then, there exists a 4-dimensional distribution function F with margins U, V, and W defined by*

$$F\left(a, b, \widetilde{Y}, N\right) = C\left(U(a), V(b), W\left(\widetilde{Y}\right), X(N); \Theta\right) \tag{B.1}$$

Conversely, let F be a 4-dimensional joint distribution function with margins U, V, W, and X. Then, for all $\left(a, b, \widetilde{Y}, N\right)$ in $\overline{\mathbb{R}}^4$ there exists a 4-dimensional copula C defined by (B.1). If U, V, W, and X are all continuous, then C is unique; otherwise, C is uniquely determined on its range $Ran(U) \times Ran(V) \times Ran(W) \times Ran(X)$.

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