BRIEF REPORT

A Simple Characterization of Assignment Mechanisms on Set Constraints

Ping Zhan[1]

Abstract

We consider the problem of allocating divisible/indivisible goods to agents according to agents' ordinal preferences. Hashimoto et al. [15] provided a nonalgorithmic and axiomatic characterization of well-studied *probabilistic serial* (PS) mechanism. Recently, Fujishige et al. [12] generalized the PS mechanism where goods are enlarged from a fixed set to a family of sets which is a polytope defined by a system of linear inequalities associated with submodular functions. The above *extended PS* (EPS) greatly improved the flexibility of allocations. Based on these two results, in this paper, we investigate the nonalgorithmic and axiomatic characterization of EPS. We show that the EPS rule is the *only* mechanism satisfying the *ordinal fairness* and a newly defined *non-wastefulness*. The submodularity plays a crucial role in our arguments.

Keywords Probabilistic serial · Submodular functions · Ordinal fairness · Non-wastefulness

1 Introduction

We study the problem of allocating divisible/indivisible goods among agents. Various studies have been conducted on this subject. Over the last decade, many studies treated fair assignments over indivisible goods; see a recent survey paper [1] for more details. However, after a seminal paper by Bogomolnaia and Moulin [6] introducing the PS mechanism, various extensions and characterizations of PS have been proposed. Refer to recent papers and references therein [2, 3] for more details, and see [14] for general problems.

✉ Ping Zhan
    zhan@edogawa-u.ac.jp

[1] Department of Communication and Business, Edogawa University, 474 Komaki,
     Nagareyama 270-0198, Chiba, Japan

Springer

The following two types of research on PS are related to our work:

1. Recently, Fujishige et al. [12, 13, 18] proposed an extended PS (EPS) mechanism which enlarged the points (i.e., the goods with fixed quotas) to polytopes of good resources called base polyhedra defined in Sect. 2.
2. Approximately 10 years after Bogomolnaia and Moulin's paper, a completely nonalgorithmic characterization of PS was presented by Hashimoto et al. [15].

In this paper, we generalize the characterization of PS by Hashimoto et al. [15] to the EPS allocation mechanism [12]. Precisely:

We used two concepts, *ordinal-fairness* and *non-wastefulness* to characterize EPS. We keep the same definition of *ordinal fairness* as the one given in [15], and redefined the *non-wastefulness* in a simple form. Therefore, the difference between PS and EPS can be viewed as the difference between the two non-wasteful concepts. Submodularity, the property of tight sets given in Proposition 1 plays a crucial role in the proofs associated with the non-wastefulness.

Although we assume a submodularity in our problem, the resource space is a polytope defined by a system of linear inequalities including many special cases. The results provide possibilities for further research, e.g., Doğan et al. [9] showed that some of their results used the ordinal-fairness. Refer to our papers [11–13, 18] for what the generalization means about EPS.

Here are some related works of literature.

Bogomolnaia and Moulin [6] proposed the well-studied PS mechanism. A fundamental property of the assignments obtained using the PS mechanism is the ordinal efficiency with respect to agents' preferences, along with envy-freeness and weak strategy-proofness. Since then, PS has been extended or generalized to various settings, see, e.g., [2, 3, 14].

We enlarged the fixed quotas of goods to polytopes such that the quotas of goods vary [12, 18], and we also include the layer structure in [7], as pointed in [12, 18]. Comparing the allocation problems in [6, 15], our setting is multi-unit demands with submodular constraints on goods, which change the weak-strategyproof into a weaker concept called Nash equilibrium; see [12] for more details.

Furthermore, e.g., Chatterji and Li [8] treated the assignments of bundles. The mechanism is equivalent to PS under a restricted domain. Preferences are defined over bundles of goods with fixed quota for each good, which differs from our setting, polytopes, defined later.

After a decade of elusiveness, as indicated in [5], nonalgorithmic, axiomatic characterization was provided in [15], followed by a leximin characterization given in [4].

Our result is based on that of Hashimoto et al. [15], which is a redefinition of PS. Precisely, Hashimoto et al. provided two characterizations. Meanwhile, our work is a generalization of the first one[1] which includes a new concept *ordinal fairness*

---

[1] The second is characterized by sd-efficient, sd-envy-free, and weakly invariant.

defined later. Besides, the axiomatic characterization proposed in [15] significantly depends on the fixed quotas of goods.

Heo and Yilmaz [17] generalized the characterization in [15] to weak preferences [17]. The mechanism with weak preferences and submodular constraints on goods has been completed by Sano and Zhan [18]. We restrict ourselves to the assignment problem with strict preferences in our characterization.

The following example shows the difference between PS and EPS, what agents can obtain when fixed quotas of goods are released, and related characterization.

Example 1 Consider three agents $\{1, 2, 3\}$ and three goods $\{e_1, e_2, e_3\}$, respectively, and let the quota of each good be 1. Suppose that the agents' preferences are as follows:

$$
\begin{array}{cccc}
1 & e_1 & e_2 & e_3 \\
2 & e_1 & e_2 & e_3 \\
3 & e_3 & e_1 & e_2
\end{array}
$$

i.e., agents 1 and 2 prefer e_1 to e_2 to e_3, and agent 3 prefers e_3 to e_1 to e_2. Here is an assignment:

$$
P = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}
\begin{pmatrix}
\frac{1}{2} & \frac{1}{2} & 0 \\
\frac{1}{2} & \frac{1}{2} & 0 \\
0 & 0 & 1
\end{pmatrix}
\qquad (1)
$$

where each entry $P(i, j)$ of P means the proportion of good e_j that agent i obtained.

Next, suppose that the resource of goods is formalized as[2]

$$
x_P(e_i) + x_P(e_j) \le 2 \qquad \text{for } i, j \in \{1, 2, 3\},
$$
$$
x_P(e_1) + x_P(e_2) + x_P(e_3) \le 3,
$$

where $x_p \in \mathbb{R}^3$ is a vector, and each component of $x_P(e_i)\,(i \in \{1, 2, 3\})$ is the column sum of P. We can obtain the following assignment:

$$
P = \begin{array}{c} 1 \\ 2 \\ 3 \end{array}
\begin{pmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\qquad (2)
$$

Although the total resource is 3 in both cases, the resource associated with (2) is more flexible than fixed quota 1. Agents 1 and 2 obtain 1, more portion of their

---

[2] This can be formulated as a submodular rank function as follows:

$$
\rho(X) = \begin{cases} 2 & \text{if } |X| = 1 \\ |X| & \text{if } |X| > 1 \end{cases} \quad (\forall X \subseteq \{e_1, e_2, e_3\}).
$$

.

most preferred good e_1 in (2), comparing $\frac{1}{2}$ in (1). Assignments of (1) and (2) are computed from PS and EPS, respectively.

The rest of this article is organized as follows. Section 2 introduces the model, notations, and preliminary results. Section 3 describes the EPS mechanism. Section 4 gives the definitions of non-wastefulness and ordinal fairness. The main theorem is presented in Sect. 5. Section 6 gives the concluding remarks.

2 The Model

Let $N = \{1, 2, \ldots, n\}$ be a set of *agents* and E be a set of *goods*. Each good $e \in E$ is considered as a type of goods and the number of available goods e can be more than one. Each agent $i \in N$ wants to obtain a certain amount of goods, denoted by $d(i) \in \mathbb{Z}_{>0}$. The vector $\mathbf{d} = (d(i) \mid i \in N) \in \mathbb{Z}_{>0}^N$ is called the *demand (upper bound) vector*.[3]

Suppose that each agent $i \in N$ has an ordinal *preference*

$$L^i : \; e_1^i \succ_i \cdots \succ_i e_m^i, \tag{3}$$

where $\{e_1^i, \ldots, e_m^i\} = E$. Let $\mathcal{L}$ denote the profile of preferences L^i $(i \in N)$.

The resource space of goods is defined as a polytope $\{x \in \mathbb{R}_{\geq 0}^m \mid Ax \leq \mathbf{b}\}$, where A is a nonnegative $k \times m$ matrix, and $\mathbf{b}$ is a k-dimensional vector. By a slight abuse of notation, we also use $(A, \mathbf{b})$ to denote a set of solutions induced by a system of linear inequalities.

For a given economy $(N, E, \mathcal{L}, \mathbf{d}, (A, \mathbf{b}))$, denoted by **RA**, a *random assignment*, also called an *expected allocation*, is a nonnegative matrix $P \in \mathbb{R}_{\geq 0}^{N \times E}$ satisfying

(i) $\sum_{e \in E} P(i, e) \leq d(i)$ for all $i \in N$
(ii) $\sum_{i \in N} P^i \in \{x \in \mathbb{R}^m \mid Ax \leq \mathbf{b}\}$,

where P^i is the ith row vector of P, representing the probability of goods allocated to the agent i.

Note that $\sum_{i \in N} P^i$ is a vector $x_P \in \mathbb{R}^m$, each element of x_P associated with a column sum of P, denoted as $x_P(e)$ for $e \in E$. Hence, we have $\sum_{i \in N} P^i = (x_P(e) \mid e \in E) = x_P$.

Next, we give a further definition of $(A, \mathbf{b})$.

Given a set function $\rho : 2^E \to \mathbb{R}$, the family of sets of goods is a polytope $\mathbf{P}(\rho) \subset \mathbb{R}^E$ satisfying

$$\mathbf{P}(\rho) = \{x \in \mathbb{R}^E \mid x(X) \leq \rho(X), \; \forall X \subseteq E\}. \tag{4}$$

where $x(X) = \sum_{e \in X} x(e)$.

---

[3] Refer to Remark 1 after (6), the definition B(ρ).

To get desirable properties, we assume that ρ satisfies the following *submodular inequalities*

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y), \ \forall X, Y \in 2^E, \tag{5}$$

and ρ is non-decreasing, i.e., $\rho(X) \leq \rho(Y)$ if $X \subseteq Y$ with $\rho(\emptyset) = 0$. Polytope $\mathbf{P}(\rho)$ associated with a pair (E, ρ) called a *submodular system* satisfying these conditions is a *polymatroid* and ρ is a *rank function*. Note that polymatroid $\mathbf{P}(\rho) \subset \mathbb{R}^E_{\geq 0}$. See [10] for more details about submodular functions.

Next, assume that goods are in high demand and the resource is completely allocated. In other words, the total goods assigned are on the maximal vectors of $\mathbf{P}(\rho)$ satisfying

$$\mathrm{B}(\rho) = \{x \in \mathbb{R}^E \mid x \in \mathbf{P}(\rho), x(E) = \rho(E)\}. \tag{6}$$

We call $\mathrm{B}(\rho)$ in (6) the *base polyhedron* of the submodular system (E, ρ), and $\mathrm{B}(\rho) \neq \emptyset$.

In what follows, we treat assignment problem $\mathbf{RA} = (N, E, \mathcal{L}, \mathbf{d} = (d(i) \mid i \in N), (E, \rho))$.

Remark 1 We have the following relation between rank function ρ and base polyhedron $\mathrm{B}(\rho)$ [10, 12].

$$\rho(X) = \max\{x(X) \mid x \in \mathrm{B}(\rho)\} \qquad (\forall X \subseteq E). \tag{7}$$

By the assumption of demand upper bound $\mathbf{d}$ satisfying $\sum_{i \in N} d(i) \geq \rho(E)$, from (6) and (7), we obtain output $x_P \in \mathrm{B}(\rho)$ while maximizing $\mathbf{d}$ in $\mathbf{P}(\rho)$ during the execution of EPS defined in the next section.

Here are some economic properties related to submodularity. If we rewrite (5) as

$$\rho(X) - \rho(X \cap Y) \geq \rho(X \cup Y) - \rho(Y), \ \forall X, Y \in 2^E, \tag{8}$$

inequalities (8) is called "diminishing return" or economies of scale/scope. Property (8) allows goods with higher preferences to share more portion of total goods than those with lower preferences since more preferred goods were chosen earlier, as shown in Example 1 of the introduction, where submodularity acts on goods. When assigning lectures to professors, submodularity implies higher specialities bringing higher effects [16].

The following proposition is fundamental in the theory of submodular functions and optimizations.

Proposition 1 *Given a vector $x \in \mathbf{P}(\rho)$ and $X, Y \subseteq E$, if we have $x(X) = \rho(X)$ and $x(Y) = \rho(Y)$, then $x(X \cup Y) = \rho(X \cup Y)$ and $x(X \cap Y) = \rho(X \cap Y)$. That is, the x-tight sets are closed with respect to the set union and intersection.*

For any $e \in E$, define χ_e to be the unit vector in $\mathbb{R}^E$ such that $\chi_e(f) = 1$ if $f = e$ and $\chi_e(f) = 0$ if $f \in E \setminus \{e\}$.

For each $x \in P(x)$, by Proposition 1, there exists a unique maximal x-tight set, denoted by sat(x), which is equal to the union of all tight sets for x. The function sat: $P(\rho) \to 2^E$ is called the *saturation function*.

Later, we use the following notation in the newly non-wastefulness. For $x \in B(\rho)$ and $e \in \text{sat}(x)$, define

$$\text{dep}(x, e) = \{e' \in E \mid \exists \alpha > 0 : x + \alpha(\chi_e - \chi_{e'}) \in B(\rho)\}, \tag{9}$$

which is called the *dependence function* for (E, ρ). It is known that [10, Chap. 2]

$$\text{dep}(x, e) = \bigcap_{e \in X \subseteq E} \{X \mid x(X) = \rho(X)\}. \tag{10}$$

Using Proposition 1, we have $x(\text{dep}(x, e)) = \rho(\text{dep}(x, e))$.

We will repeat chains in the sequel. It is essential to know [10, 12, Sec. 7] that for $\forall x \in B(\rho)$, the family of tight sets $\{X \subseteq E \mid x(X) = \rho(X)\}$ is completely determined by a maximal chain:

$$\hat{\mathcal{C}} : S_0 = \emptyset \subset \hat{S}_1 \subset \cdots \subset \hat{S}_p = E, \tag{11}$$

where $x(\hat{S}_i) = \rho(\hat{S}_i)$ for $i = 0, 1, \ldots, p$.

Let $\hat{T}_i = \hat{S}_i \setminus \hat{S}_{i-1}$ for $i = 1, \ldots, p$, then $\{\hat{T}_1, \ldots, \hat{T}_p\}$ form a partition of E. By the maximal chain, we mean that for each $X \subseteq E$ with $x(X) = \rho(X)$, we have $X = \hat{T}_{\ell_1} \cup \cdots \cup \hat{T}_{\ell_k}, 1 \leq \ell_1 \leq \cdots \leq \ell_k \leq p$.

In the following, we write one element set $\{e\}$ as e for simplicity.

3 Extended PS Mechanism

Fix a random assignment problem $\mathbf{RA} = (N, E, \mathcal{L}, \mathbf{d}, (E, \rho))$. A *mechanism* is a mapping from $\mathbf{RA}$ into allocation $P \in \mathbb{R}_{\geq 0}^{N \times E}$ meeting some requirements.

As mentioned earlier, EPS is an extension of the standard PS mechanism of [6], which was proposed by Fujishige et al., and for more details, the reader can refer to [12].

Recall that for each $i \in N$, agent i's preference is given by L^i in (3), and $\mathcal{L} = (L^i \mid i \in N)$. Based on the collection (a multiset) of the first (most favorite) elements e_1^i for all agents $i \in N$, define a nonnegative integral vector $b(\mathcal{L}) \in \mathbb{Z}_{\geq 0}^E$ by

$$b(\mathcal{L}) = \sum_{i \in N} d(i) \chi_{e_1^i}, \tag{12}$$

where we may have $e_1^i = e_1^j$ for distinct $i, j \in N$ and $d(i)$ is the integral demand upper bound of agent $i \in N$.

During the execution of the following algorithm, the current lists L^i ($i \in N$) may get shorter because of removal of exhausted goods. Also note that S_p is the set of types of goods saturated at stage p.

Algorithm I Extended Probabilistic Serial Algorithm (EPS) [3, 12]

Input: A random assignment problem $\mathbf{RA} = (N, E, \succ, \mathbf{d}, (E, \rho))$.

Output: An expected allocation $P \in \mathbb{R}_{\geq 0}^{N \times E}$.

Step 0:   For each $i \in N$ put $x^i \leftarrow \mathbf{0} \in \mathbb{R}^E$ (the zero vector), and put $S_0 \leftarrow \emptyset$.

Put $p \leftarrow 1$, $x^* \leftarrow \mathbf{0}$, and $\lambda_0 \leftarrow 0$.

Step 1:   For current (updated) $\mathcal{L} = (L^i \mid i \in N)$, using $b(\mathcal{L})$ in (12), compute

$$\lambda_p = \max\{t \geq \lambda_{p-1} \mid x^* + (t - \lambda_{p-1})b(\mathcal{L}) \in \mathrm{P}(\rho)\}. \tag{13}$$

For each $i \in N$ put $x^i \leftarrow x^i + (\lambda_p - \lambda_{p-1})d(i)\chi_{e_1^i}$.

Put $x^* \leftarrow x^* + (\lambda_p - \lambda_{p-1})b(\mathcal{L})$ and $S_p \leftarrow \mathrm{sat}(x^*)$.

Step 2:   Put $T \leftarrow S_p \setminus S_{p-1}$.

Update L^i by removing all elements of T from current L^i $(i \in N)$.

Step 3:   If $\rho(S_p) < \rho(E)$, then put $p \leftarrow p + 1$ and go to Step 1.

Otherwise $(\rho(S_p) = \rho(E))$ put $P(i, e) \leftarrow x^i(e)$ for all $i \in N$ and $e \in E$.

Return P.

When Algorithm I terminates, we obtain a chain:

$$\mathcal{C}: \quad \emptyset = S_0 \subset S_1 \subset \cdots S_p = E \tag{14}$$

satisfying $\rho(S_t) = x_P(S_t)$ for all $0 \leq t \leq p$. Chain (14) is a subchain of the maximal chain of x_P defined in (11).

Remark 2 Recall that agent $i \in N$ can be viewed as $d(i)$ subagents, and each subagent eats goods at unit speed. As in paper [6], the parameter t in (13) can be considered as time. The monotonic assumption of ρ implies that the agents eat at nonnegative consecutive time intervals.

Fix a preference profile $\mathcal{L} = (L^i \mid i \in N)$. Given two assignments P and Q, recall that $P^i = (P(i, e) \mid i \in N)$ and $Q^i = (Q(i, e) \mid i \in N)$. We say that P^i *stochastically dominates* (sd) Q^i, denoted by $P^i \succeq_i^{\mathrm{sd}} Q^i$, if $\sum_1^\ell P(i, e_\ell^i) \geq \sum_1^\ell Q(i, e_\ell^i)$ for all $\ell = 1, \ldots, m$. We say that Q is *stochastically dominated* by P if we have $P^i \succeq_i^{\mathrm{sd}} Q^i$ for all $i \in N$ and $P \neq Q$. An assignment P is *ordinally efficient* if P is not stochastically dominated by any other allocation.

An allocation P is normalized sd-envy-free if for all $i, j \in N$, $\frac{1}{d(i)}P^i \succeq_i^{sd} \frac{1}{d(j)}P^j$.

Theorem 2 (Theorems 5.1 and 5.2 [12]) *Algorithm I computes an expected allocation that is ordinally efficient and normalized envy-free.*

Example 2 Suppose $N = \{1, 2, 3, 4\}$ and $E = \{a, b, c, g\}$. The preference profile is given as follows,

$$\begin{array}{l} 1 \;\; a \succ_1 b \succ_1 c \succ_1 g \\ 2 \;\; a \succ_2 c \succ_2 b \succ_2 g \\ 3 \;\; a \succ_3 c \succ_3 g \succ_3 b \\ 4 \;\; b \succ_4 a \succ_4 g \succ_4 c \end{array} \tag{15}$$

The demand upper bound vector is given by $\mathbf{d} = (4, 2, 1, 1) \in \mathbb{Z}_{>0}^N$, and the submodular function ρ on E is defined by

$$\rho(X) = \begin{cases} 4|X| & \text{if } |X| \leq 2 \\ 8 & \text{if } |X| > 2 \end{cases} \quad (\forall X \subseteq E). \tag{16}$$

Each good in E can be interpreted as a shop in a tenant, where the value of ρ represents the area. Here, The total area is eight. To keep diversity, the area of each shop is at most four, while the popular shop can also get larger portions of the area.

The following matrix P is an assignment obtained using EPS.

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{cccc} a & b & c & g \\ \begin{pmatrix} \frac{16}{7} & \frac{12}{7} & 0 & 0 \\ \frac{8}{7} & 0 & \frac{6}{7} & 0 \\ \frac{4}{7} & 0 & \frac{3}{7} & 0 \\ 0 & \frac{4}{7}+\frac{3}{7} & 0 & 0 \end{pmatrix} \end{array} . \tag{17}$$

After Step 1, agents 3 and 4 get $\frac{4}{7}$ of their best goods, agent 2 gets $2 \times \frac{4}{7}$, and agent 1 gets $4 \times \frac{4}{7}$. At Step 2, they get the remaining goods. The (maximal) set of goods exhausted at steps 1 and 2 of EPS are $\{a\}$ and $\{a, b, c, g\}$, respectively.

Using the same B(ρ) in (16), if we change the preference profile (15) by swapping goods a and g while keeping the others, we obtain an assignment P as follows,

$$P = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{cccc} a & b & c & g \\ \begin{pmatrix} 0 & \frac{12}{7} & 0 & \frac{16}{7} \\ 0 & 0 & \frac{6}{7} & \frac{8}{7} \\ 0 & 0 & \frac{3}{7} & \frac{4}{7} \\ 0 & 1 & 0 & 0 \end{pmatrix} \end{array} . \tag{18}$$

Now good g, instead of a, shares a portion 4 from the total resources 8 to sufficiently satisfy agents' preferences.

4 Non-wastefulness and Ordinal Fairness

Two concepts, namely, non-wastefulness and ordinal fairness, are essential for characterizing the EPS mechanism introduced in Sect. 3.

Let $\mathbf{RA} = (N, E, \mathcal{L}, \mathbf{d}, (E, \rho))$.

Definition 1 A random assignment P is *non-wasteful* at $\mathcal{L}$ if $\forall (i, e) \in N \times E$ such that $P(i, e) > 0$, we have:

$$e \notin \text{dep}(x_P, e') \quad \forall e' >_i e. \tag{19}$$

Recall that $\text{dep}(x_P, e')$ is also a well-define x_P-tight set, i.e., $x_P(\text{dep}(x_P, e')) = \rho(\text{dep}(x_P, e'))$ as indicated in (10).

Compared with the case of PS where each good $e \in E$ with a fixed quota q_e, the non-wastefulness at $\mathcal{L}$ is $\forall e' >_i e$ such that $P(i, e) > 0$, we have $x_P(e') = q_{e'}$. Our definition is also simple despite the bind being a set function.

To understand definition (19) further, let us define a notation on triple (L^i, e, P^i) as

$$\overline{W}(L^i, e, P^i) \equiv \bigcup_{e' >_i e} \text{dep}(x_P, e'). \tag{20}$$

Then, for any good $e \in E$ with $P(i, e) > 0$, we can see that all $e' \in E$ satisfying $e' >_i e$ are in a saturated set $\overline{W}(L^i, e, P^i)$ which does not include good e from following facts.

  (i) $e \notin \overline{W}(L^i, e, P^i)$,
  (ii) $\{e' \mid e' >_i e\} \subseteq \overline{W}(L^i, e, P^i))$,
  (iii) $x_P(\overline{W}(L^i, e, P^i)) = \rho(\overline{W}(L^i, e, P^i))$

We have: (i) follows from the definitions of non-wasteful and $\overline{W}$. (ii) follows from $e' \in \text{dep}(x_P, e')$. Since $\text{dep}(x_P, e')$ is a x_P-tight set, we obtain (iii) by Proposition 1. All $\overline{W}(L^i, e, P^i)$ form a chain of x_P-tight sets with $P(i, e) > 0$ because this is an inductive property on preferences for each pair $(i, e) \in N \times E$.

Proposition 3 *Algorithm I computes an expected allocation $\widehat{P}$ that is non-wasteful.*

Proof Suppose that $\widehat{P}$ is not non-wasteful. Then, by the definition, there exists $i \in N$, $e \in E$, and $\widehat{P}(i, e) > 0$ such that

$$e \in \text{dep}(x_{\widehat{P}}, e') \quad \forall e' >_i e. \tag{21}$$

By the definition of dep function (9), there exists $\epsilon > 0$, and the vector $x_{\widehat{P}} + \epsilon(\chi_{e'} - \chi_e) \in \text{B}(\rho)$. Then, we can obtain a new assignment $\widehat{P}'$ by replacing $\widehat{P}'(i, e) = \widehat{P}(i, e) - \epsilon$, $\widehat{P}'(i, e') = \widehat{P}(i, e') + \epsilon$, and keeping other entries the same as those of $\widehat{P}$. Assignment $\widehat{P}'$ sd-dominates $\widehat{P}$. This contradicts the ordinal efficiency of EPS by Theorem 2.

Recall that $\text{dep}(x_{\hat{P}}, e')$ is a well-defined $x_{\hat{P}}$-tight set by Proposition 1 because of submodularity. $\qquad \square$

Let $U(L^i, e) \equiv \{e' \in E \mid e' \succeq_i e\}$ be the *upper contour set of good e at $\succ_i$*. For an assignment P, let $F(L^i, e, P^i)$ be the probability that agent i is allocated at least as good as e under P^i which satisfies

$$F(L^i, e, P^i) = \frac{1}{d(i)} \sum_{e' \in U(L^i, e)} P(i, e'), \quad \forall (i, e) \in N \times E, \tag{22}$$

we call $F(L^i, e, P^i)$ agent *i's (normalized) surplus at e under P^i*.

Definition 2 An assignment P is *ordinally fair*[4] at $\mathcal{L}$ if $\forall i, j \in N$, $e \in E$ such that $P(i, e) > 0$, we have:

$$F(L^i, e, P^i) \le F(L^j, e, P^j). \tag{23}$$

As mentioned in the introduction, ordinal fairness in Definition 2 is the same as the one in [15], which defines a fair property. From (23), we obtain that for any ordinally fair assignment, agents must share the same surplus value if they have a positive probability on the same good.[5]

Example 2, continues For assignment (17), we have $F(L^1, a, P^1) = F(L^2, a, P^2) = F(L^3, a, P^3) = 4/7$. All of the remaining $F(L^i, e, P^i)$ are 1.

The following Proposition 4 follows from [15]. It gives a relation with the chain (14) of EPS, and we provide its proof for completeness.

Proposition 4 *Algorithm I computes an expected allocation $\hat{P}$ that is ordinally fair.*

Proof For the chain (14) obtained after Algorithm I, denote $\Delta S_t = S_t \setminus S_{t-1}$ ($1 \le t \le p$).

Suppose $P(i, e) > 0$, $e = e_1^i \in \Delta S_t$ at Step t for agent i, then each agent j has $e_1^j = e' \succeq_j e$ in current L^j. We have $e' \in \Delta S_t$ or $e' \in S_{t+\ell}$, where $\ell > 1$. In the former case, we have both $e, e' \in \Delta S_t$. Hence, $F(L^i, e, \hat{P}^i) = F(L^j, e, \hat{P}^j)$. In the latter case, we have

$$F(L^i, e, \hat{P}^i) < F(L^j, e', \hat{P}^j) \le F(L^j, e, \hat{P}^j), \tag{24}$$

where the first inequality from $e \in \Delta S_t$ and $e' \in S_{t+\ell}$, the second one from $e' \succeq_j e$. Both cases mean

---

[4] According to [15], "ordinal fairness encompasses Pareto efficiency and envy-freeness ... with perfectly divisible goods."

[5] Heo and Yilmaz [17] indicate that non-wastefulness and ordinal fairness are two independent properties, i.e., there are assignments satisfying only one of them, respectively, and their setting is a special case of ours.

$$F(L^i, e, \widehat{P}^i) \le F(L^j, e, \widehat{P}^i), \ \forall e \in \Delta S_t \ (t \ge 1), \forall i, j \in N. \tag{25}$$

Since t, i, and j are arbitrary, and S_t, ΔS_t, and $S_{t+\ell}$ ($\ell > 1$) include all goods, i.e., (25) is satisfied by $\forall e \in E$ and $\forall i, j \in N$. Hence, $\widehat{P}$ is ordinally fair. □

Example 2 continues Assignment (17) can only have two values of $F(L^i, e, P^i)$. It is easy to check that Proposition 4 is satisfied since (17) is obtained using the EPS. The smaller values of $F(L^i, a, P^i)$ with $P(i, a) > 0$ are equal.

5 Main Results

We now state our main characterization of the EPS mechanism. The proof follows closely to that of Theorem 1 given by [15]. However, we give the proofs related to non-wastefulness for completeness.

Theorem 5 *A mechanism is ordinally fair and non-wasteful if and only if it is EPS.*

The "if" part of Theorem 5 has been proved by Propositions 3 and 4. Hence, we will show the converse. The main difference from [15] is the non-wastefulness associated with submodularity.

We need some notations.

Fix an assignment P. For all $e \in E$, define

$$\pi(e) = \min_{i \in N} F(\succ_i, e, P^i), \tag{26}$$

we get different values $\pi_1 < \cdots < \pi_p$ of π with $1 \le p \le m$. Grouping goods satisfying

$$T_s(e) = \{e \in E \mid \pi(e) = \pi_s\}, \tag{27}$$

we obtain a partition $\{T_1, \ldots, T_p\}$ of E. Moreover, let $\mathbf{S}_s = T_1 \cup \cdots \cup T_s$ with $\mathbf{S}_0 = \emptyset$.

For each $\bar{e}_s \in E \setminus \mathbf{S}_{s-1}$, define

$$N_s(\bar{e}_s) = \{i \in N \mid \bar{e}_s \succeq_i e', \forall e' \in E \setminus \mathbf{S}_{s-1}\} \tag{28}$$

as the set of agents who prefer $\bar{e}_s$ most in the complement of set $\mathbf{S}_{s-1}$. Additionally, if $\bar{e}_s \in T_s$, we denote above $\bar{e}_s$ as $\bar{e}_{T_s}$.

Lemma 6 is about the ordinal fairness, and its proof is similar to that of Theorem 1 (Steps 1 and 2) [15].

Lemma 6 *If P is an ordinally fair assignment, we have*:

$$P(k, \bar{e}_{T_s}) = 0 \quad if \ k \notin N_s(\bar{e}_{T_s}), \tag{29}$$

$$F(\succ_i, \bar{e}_{T_s}, P^i) = \pi(\bar{e}_{T_s}), \quad \forall i \in N_s(\bar{e}_{T_s}). \tag{30}$$

Note that there is no corresponding part of Lemma 7 in the paper [15] since the non-wastefulness is simple for each good with a fixed quota.

Lemma 7 *If P is a non-wasteful assignment, then*

$$x_P(\mathbf{S}_s) = \rho(\mathbf{S}_s) \quad \forall s = 1, \dots, p. \tag{31}$$

Proof We prove it by induction. This is true for $\mathbf{S}_0 = \emptyset$. Assuming that it is true for $\forall t < s$, we show that it is true for s.

First, we show

$$\mathrm{dep}(x_P, e_s) \subseteq \mathbf{S}_s \quad \forall e_s \in T_s. \tag{32}$$

For a contradiction, suppose $\exists \hat{e} \in \mathrm{dep}(x_P, e_s)$ such that $\hat{e} \in E \setminus \mathbf{S}_s$. Let $\hat{e} \in T_t$ with $t > s$.

If there exists $i \in N$ such that $P(i, \hat{e}) > 0$ and $e_s \succ_i \hat{e}$, by the definition of non-wastefulness, we have $\hat{e} \notin \mathrm{dep}(x_P, e_s)$, which is a contradiction to the assumption. Otherwise, for all $P(i, \hat{e}) > 0$, we have $\hat{e} \succ_i e_s$. By the definition of (22) and (26), we obtain another contradiction of $t > s$.

Since $e_s \in \mathrm{dep}(x_P, e_s)$ and from (32), we obtain

$$T_s \subseteq \bigcup_{e_s \in T_s} \mathrm{dep}(x_P, e_s) \subseteq \mathbf{S}_s. \tag{33}$$

From definition $T_s \cup \mathbf{S}_{s-1} = \mathbf{S}_s$ and (33), we get

$$\mathbf{S}_{s-1} \bigcup_{e_s \in T_s} \mathrm{dep}(x_P, e_s) = \mathbf{S}_s. \tag{34}$$

Now, by the inductive assumption $x_P(\mathbf{S}_{s-1}) = \rho(\mathbf{S}_{s-1})$, $x_P(\mathrm{dep}(x_P, e_s)) = \rho(\mathrm{dep}(x_P, e_s))$, and Proposition 1, we have (31), this ends the proof. □

From Lemmas 6 and 7, we have: If P is an ordinally fair and non-wasteful assignment, then

$$\sum_{\bar{e}_{T_s} \in T_s} \sum_{j \in N} P(j, \bar{e}_{T_s}) = \sum_{\bar{T}_{e_s} \in T_s} \sum_{j \in N_s(\bar{e}_{T_s})} P(j, \bar{e}_{T_s}) = \rho(\mathbf{S}_s) - \rho(\mathbf{S}_{s-1}), \quad 1 \le s \le p. \tag{35}$$

The rest of the proof of Theorem 5 can be shown in a very similar way as Steps 4 and 5 in [15]. (Arguments that do not use non-wastefulness can be extended, these are omitted.)

Let $\widehat{P}$ represent the random assignment obtained from EPS. We prove it by induction.

Suppose that for $\forall t < s$, **(i)** $\forall \bar{e}_{T_t} \in T_t$ and $\forall i \in N_t(\bar{e}_{T_t})$, we have $F(L^i, \bar{e}_{T_t}, P^i) = F(L^i, \bar{e}_{T_t}, \widehat{P}^i) = \pi(\bar{e}_{T_t})$, **(ii)** $\forall \bar{e}_{T_t} \in T_t$ and $\forall k \notin N_t(\bar{e}_{T_t})$, we have $P(k, \bar{e}_{T_e}) = \widehat{P}(k, \bar{e}_{T_e}) = 0$. The inductive assumption holds trivially for $s = 1$. We prove that they hold for Step s and thus $P = $ EPS.

By the same arguments as that given in [15], we can show at Step s of EPS,

$$F(L^i, \bar{e}_{T_s}, \widehat{P}^i) \geq \pi(\bar{e}_{T_s}) \quad \forall \bar{e}_{T_s} \in T_s, \ \forall i \in N_s(\bar{e}_{T_s}). \tag{36}$$

Now, we show that for each $\bar{e}_{T_s} \in T_s$ and $j \in N_s(\bar{e}_{T_s})$, we have $\pi(\bar{e}_{T_s}) = F(L^j, \bar{e}_{T_s}, \widehat{P}^j)$ and $\widehat{P}(k, \bar{e}_{T_s}) = 0 \ \forall k \notin N_s(\bar{e}_{T_s})$. Proving the first claim is sufficient since $\bar{e}_{T_s}$ cannot be allocated to agent k in this case.

Suppose for a contradiction that there exists $\bar{e}_s \in T_s$ and $i \in N_s(\bar{e}_{T_s})$, $F(L^i, \bar{e}_{T_s}, P^i) = \pi(\bar{e}_{T_s}) < F(L^i, \bar{e}_{T_s}, \widehat{P}^i)$. Then,

$$
\begin{aligned}
\sum_{\bar{e}_{T_s} \in T_s} \sum_{j \in N_s(\bar{e}_{T_s})} P(j, \bar{e}_{T_s}) &= \sum_{\bar{e}_{T_s} \in T_s} \sum_{j \in N_s(\bar{e}_{T_s})} \left\{ F(L^j, \bar{e}_{T_s}, P^j) - \sum_{e' >_j \bar{e}_{T_s}} P(j, e') \right\} \\
&< \sum_{\bar{e}_{T_s} \in T_s} \sum_{j \in N_s(\bar{e}_{T_s})} \left\{ F(L^j, \bar{e}_{T_s}, \widehat{P}^j) - \sum_{e' >_j \bar{e}_{T_s}} \widehat{P}(j, e') \right\} \\
&= \sum_{\bar{e}_{T_s} \in T_s} \sum_{j \in N_s(\bar{e}_{T_s})} \widehat{P}(j, \bar{e}_{T_s}) = \rho(\mathbf{S}_s) - \rho(\mathbf{S}_{s-1}),
\end{aligned}
$$

where the strict inequality follows from (36), Lemma 6, the supposition, and $P(j, \bar{e}_{T_s}) = \widehat{P}(j, \bar{e}_{T_s})$ for all $\bar{e}_{T_s} \in T_s, j \in N_s(\bar{e}_{T_s})$ by the inductive assumption. Therefore,

$$\sum_{t \leq s} \sum_{\bar{e}_{T_t} \in T_t} \sum_{j \in N_t(\bar{e}_{T_t})} P(j, \bar{e}_{T_t}) < \rho(T_1 \cup \cdots \cup T_s) = \rho(\mathbf{S}_s),$$

this contradicts $\sum_{i \in N} P^i \in B(\rho)$ if $s = p$, violates Lemma 7. Hence, P coincides with $\widehat{P}$. $\qquad \square$

6 Concluding Remarks

We considered the problem of allocating a family of good sets to agents using agents' ordinal preferences.

– We provide a new definition of *non-wastefulness* given in (19), despite the constraints are defined on the sets of goods. From Lemma 6, and the *ordinal fairness* in (23), we obtain the original form of submodular constraints on sets shown in Lemma 7. We adapted the arguments by Hashimoto et al. [15], and submodularity plays a central role in our setting.
– The main result is given in Theorem 5, a simple nonalgorithmic redefinition, or characterization of the EPS mechanism developed by Fujishige et al. [12]. The EPS mechanism greatly increases the flexibility of assignments.

The leximin maximization, a characterization given in [4], can also be extended to the EPS mechanism for a random assignment with a family of good sets, as shown in [11, 18].

Recently, the related simultaneous eating mechanism (which is a mechanism presented in Sect. 3 with different agents' eating speed) is also generalized on polytopes [3]. The unified characterization here will provide a clue for further research in these assignment problem.

Availability of Data and Material Data sharing does not apply to this article as no datasets were generated or analyzed during the current study.

Code Availability It does not apply to this article as no code is used in this study.

Declarations

Ethics Approval Ethics do not apply to this article as no ethic issue is included in this study.

Consent to Participate It does not apply to this article since no participation was involved in this study.

Consent for Publication The author declares consent to publication.

Conflict of Interest The author declares no competing interests.

References

1. Amanatidis G, Aziz H, Birmpas G, Filos-Ratsikas A, Li B, Moulin H, Voudouris A, Wu X (2022) Fair division of indivisible goods: a survey. Preprint at https://arxiv.org/abs/2202.07551
2. Aziz H, Brandl F (2021) The vigilant eating rule: a general approach for probabilistic economic design with constraints. http://arxiv-export-lb.library.cornell.edu/pdf/2008.08991
3. Balbuzanov I (2022) Constrained random matching. J Econ The 203:105472
4. Bogomolnaia A (2015) Random assignment: redefining the serial rule. J Econ The 158:308–318
5. Bogomolnaia A, Heo EJ (2012) Probabilistic assignment of objects: characterizing the serial rule. J Econ The 147:2072–2082
6. Bogomolnaia A, Moulin H (2001) A new solution to the random assignment problem. J Econ The 100:295–328
7. Budish Che YK, Kojima K, Milgrom P (2013) Designing random allocation mechanisms: theory and applications. Amer Econ Rev 103:178–200
8. Chatterji S, Liu P (2020) Random assignments of bundles. J Math Econ 87:15–30

9. Doğan B, Doğan S, Yildiz K (2018) A new ex-ante efficiency criterion and implications for the probabilistic serial mechanism. J Econ The 175:178–200

10. Fujishige S (2005) Submodular functions and optimization, 2nd edn. Elsevier, Amsterdam

11. Fujishige S, Sano Y, Zhan P (2016) A solution to the random assignment problem with a matroidal family of goods. RIMS Preprint, RIMS-1852, Kyoto Univ

12. Fujishige S, Sano Y, Zhan P (2018) The random assignment problem with submodular constraints on goods. ACM Trans Econ Comput 6:28. https://doi.org/10.1145/3175496

13. Fujishige S, Sano Y, Zhan P (2019) Submodular optimization views on the random assignment problem. Math Program 178(1–2):585–501

14. Guo X, Sikdar S, Wang H, Xia L, Cao Y, Wang H (2021) Probabilistic serial mechanism for multi-type resource allocation. Autonomous Agents and Multi-Agent Systems (35) article 15

15. Hashimoto T, Hirata D, Kesten O, Kurino M, Ünver MU (2014) Two axiomatic approaches to the probabilistic serial mechanism. The Econ 9:253–277

16. Heo EJ (2014) Probabilistic assignment problem with multi-unit demands: a generalization of the serial rule and its characterization. J Math Econ 54:40–47

17. Heo EJ, Yilmaz Ö (2015) A characterization of the extended serial correspondence. J Math Econ 59:102–110

18. Sano Y, Zhan P (2021) Extended random assignment mechanisms on a family of good sets. SN Oper Res Forum 2:52